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OSCILLATIONS
R-MODULE

BASIC CONCEPTS

1. **Periodic Motion:** Any motion that repeats itself at regular intervals of time is called periodic motion.
   - **Examples:**
     - Motion of hands of a clock
     - Motion of planets
     - Motion of electron around the nucleus
     - Motion of a ball in a bowl
   - In a periodic motion, force is always directed towards a fixed point which may or may not be on the path of motion.
2. **Oscillatory Motion:** The motion of a body is said to be oscillatory or vibratory or harmonic motion if it moves back and forth (to and fro) about a fixed point after regular intervals of time.
   - **Examples:**
     - Motion of the pendulum of a wall clock
     - Vibration of the wire of Sitar
     - Motion of the prongs of a tuning fork
   - Every oscillatory motion is periodic but every periodic motion need not be oscillatory. Circular motion is a periodic motion, but it is not oscillatory motion.
3. **Some Terms Related to Periodic Motion:**
   - **Time Period:** The time taken by a particle to complete one oscillation is called its time period. Its SI unit is second. It is denoted by T.
   - **Frequency:** It is defined as the number of oscillations per second. It is denoted by the symbol ν.
   - The relation between ν and T is

     $$T = \frac{1}{\nu}$$
   - The SI unit of frequency is hertz (Hz).

     $$1 \text{ Hz} = 1 \text{ oscillation per second} = 1\text{s}^{-1}$$
   - **Angular Frequency:** The rate of change of phase angle of a particle with respect to time is defined as its angular frequency.

     $$\omega = 2\pi\nu = \frac{2\pi}{T}$$

     The SI unit of ω is rad s^{-1}.
   - **Displacement:** It is the distance of oscillating particle from the mean position at any instant. Displacement variable is measured as a function of time and it can have both positive and negative values. Displacement can be represented by a mathematical function of time. In case of periodic motion, the displacement can be given by

     $$x = f(t) = A \sin \omega t \text{ or } x = f(t) = A \cos \omega t$$

     where A is called amplitude.

- The first relation is valid when the time is measured from the mean position and the second relation is valid when the time is measured from the extreme position of the particle executing SHM along a straight line path.

- **Phase:** The phase of vibrating particle at any instant gives the state of the particle with regard to its position and the direction of motion at that instant. It is denoted by ϕ.
- **Phase Difference:** The phase difference between two vibrating particles tells the lack of harmony in the vibrating states of the two particles at any instant.

4. **Periodic Function:** Any function that repeat itself at regular intervals of time is called a periodic function.

- A function $f(t)$ is said to be periodic with period T,

$$f(t) = f(t + T)$$

- Any periodic function can be expressed as a superposition of sine and cosine functions of different time periods with suitable coefficients.
- $\sin\omega t$, $\cos\omega t$, and $\sin\omega t + \cos\omega t$ are the periodic functions with a period $\frac{2\pi}{\omega}$ and $e^{-\omega}$ and $\log_e(\omega t)$ are non–periodic functions.

5. **Simple Harmonic Motion (SHM):** When a particle moves in a straight line to and fro about its mean position such that the restoring force acting on the particle is directly proportional to its displacement from mean position and directed towards mean position, the motion of the particle is said to be simple harmonic motion.

$$f \propto -x$$

and $$x = A\cos(\omega t + \phi_0)$$

- **Velocity in SHM:** It is the rate of change of displacement of the particle at any instant. It is given by

$$v = \frac{dx}{dt} = \frac{d}{dt}[A\cos(\omega t + \phi_0)]$$

$$= -\omega A\sin(\omega t + \phi_0) = -\omega\sqrt{A^2 - x^2}$$

  - The maximum value of velocity is called velocity amplitude v_m of the motion.

- **Acceleration in SHM:** It is the rate of change of velocity of the particle at any instant. It is given by

$$a = \frac{dv}{dt} = \frac{d}{dt}[-\omega A\sin(\omega t + \phi_0)]$$

$$= -\omega^2 A\cos(\omega t + \phi_0) = -\omega^2 x$$

i.e., $$a \propto -x$$

  - The maximum value of acceleration of particle is called acceleration amplitude (a_m).

6. **Energy in Simple Harmonic Motion:** When a particle of mass m executes SHM, then at a displacement x from mean position, the particle possesses kinetic and potential energy.

Kinetic energy, $$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t + \phi)$$

$$= \frac{1}{2}m\omega^2(A^2 - x^2)$$

Kinetic energy of a particle executing SHM is periodic with period $T = \frac{2\pi}{\omega}$. It is zero at extreme positions and maximum at mean position.

The potential energy of a particle in SHM is given by

Potential energy, $$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

$$= \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi)$$

- Potential energy of a particle executing simple harmonic motion is also periodic with period $T = \frac{2\pi}{\omega}$, being zero at the mean position and maximum at the extreme displacements.

Total energy,

$$E = K + U$$
$$= \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi) + \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$$
$$= \frac{1}{2}m\omega^2A^2 = 2\pi^2 m\upsilon^2 A^2$$

In SHM, total energy remains constant at all instants and at all displacements. It depends upon the mass, amplitude and frequency of vibration of the particle.

In SHM, at the mean position total energy is in the form of its kinetic energy and at the extreme positions, total energy is in the form of its potential energy.

7. **Oscillation of Mass-Spring System:** When a mass m is attached to a massless spring and pulled downwards, it executes SHM. If l is extension in the spring on attaching mass m and k is its force constant, then time period of SHM executed by the spring

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{m}{k}}$$

- **Springs connected in series:** If two springs of spring constants k_1 and k_2 are connected in series, then the spring constant k of the combination is given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k = \frac{k_1k_2}{k_1 + k_2}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1k_2}}$$

- **Springs connected in parallel:** If two springs of spring constants k_1 and k_2 are connected in parallel, then the spring, constant k of combination is

$$k = k_1 + k_2$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

8. **Simple Pendulum:** The motion of simple pendulum swinging through small angles is approximately simple harmonic. The periodic oscillation is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Where l is the length of the pendulum and g is the acceleration due to gravity.

- Time period of a simple pendulum is independent of mass, shape and material of bob and it is also independent of the amplitude of oscillation, provided it is small.
- Time period of a simple pendulum depends on l as $T \propto \sqrt{l}$, so the graph between T and l will be a parabola while between T and l will be a straight line.
- Time period of a simple pendulum depends on acceleration due to gravity as $T \propto (1/\sqrt{g})$. With increase in g, T will decrease or vice versa.

9. **Undamped and Damped Oscillations:**

Undamped Oscillations: The oscillations of body free from any external effect are called undamped or free oscillations. In practice the free oscillations are not possible. The frequency of free oscillations of the body is called its natural frequency.

Damped Oscillations: When the body oscillates, some friction is always present. Due to which the energy of oscillations decreases continuously and the amplitude of vibrations decreases continuously. These oscillations are called damped oscillations.

- If damping is very large, the oscillations die out very soon. For example a simple pendulum oscillates in air, but not in water.
- The energy of the system executing damped oscillations will go on decreasing with time but the oscillations of the system remain periodic.
- The damping forces are active in the oscillating system which are generally the frictional or viscous forces.

10. Forced Oscillations and Resonance:

- **Forced Oscillations :** When an external periodic force is applied on oscillating body, the body, after a short time, begins to oscillate with the frequency of external periodic force. Such oscillations are called forced oscillations.
- **Resonance :** If the frequency of external periodic force is equal to the natural frequency of the body, the amplitude of oscillations becomes quite large. Such forced vibrations are called resonant vibrations and the phenomenon is said to be **resonance.**
- Due to presence of damping, the resonant frequency is slightly less than the natural frequency of the body.

NCERT TEXTBOOK QUESTIONS

Q. 1. Which of the following examples represent periodic motion ?

(*a*) A swimmer completing one (return) trip from one bank of a river to the other and back.

(*b*) A freely suspended bar magnet displaced from its N-S direction and released.

(*c*) A hydrogen molecule rotating about its centre of mass.

(*d*) An arrow released from a bow.

Ans. The periodic motions are (*b*) and (*c*).

Q. 2. Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion ?

(*a*) The rotation of earth about its axis

(*b*) Motion of an oscillatory mercury column in a U-tube

(*c*) Motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point

(*d*) General vibrations of a polyatomic molecule about its equilibrium position

Ans. (*a*) Motion of earth about its axis is periodic but not SHM.

(*b*) Motion of an oscillatory mercury column is SHM.

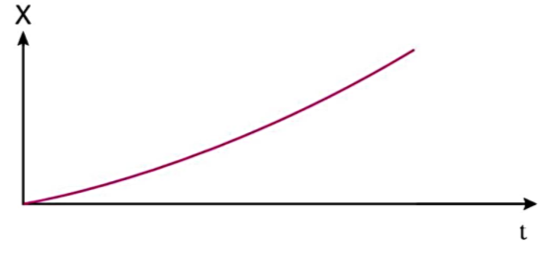
(*c*) Motion of a ball bearing inside a smooth curved bowl is SHM.

(*d*) General vibrations of a polyatomic molecule are superposition of SHMs of different frequencies, so it is periodic but not SHM.

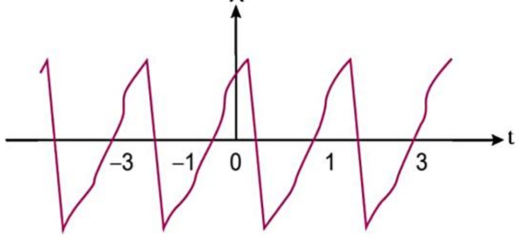
Thus, (*b*) and (*c*) are SHMs, while (*a*) and (*d*) are periodic motions.

Q. 3. Figure depicts four *x-t* plots for linear motion of a particle. Which of the plots represent periodic motion ? What is the period of motion (in case of periodic motion)?

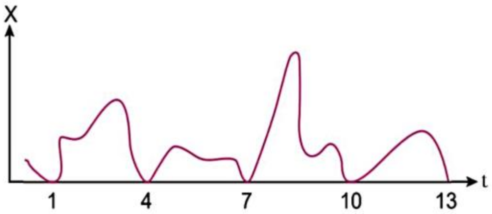
(*a*)



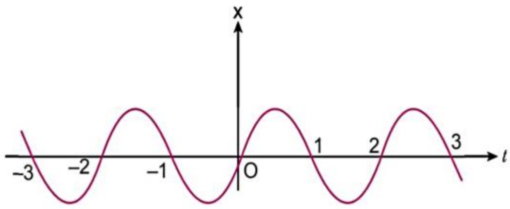
(*b*)



(*c*)



(*d*)



Ans. (*b*) and (*d*) are periodic motions, each has a period 2 s

(*a*) is not periodic because it does not repeat at all.

(*c*) is not periodic because it does repeat entire motion, it repeats merely at one position which is not enough for periodic motion.

Q. 4. Which of the following functions of time represent (*i*) simple harmonic, (*ii*) periodic but not simple harmonic, and (*iii*) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant).

(*a*) $\sin \omega t - \cos \omega t$ **(*b*)** $\sin^3 \omega t$

(*c*) $3\cos\left(\frac{\pi}{4} - 2\omega t\right)$ **(*d*)** $\cos \omega t + \cos 3\omega t + \cos 5\omega t$

(*e*) $\exp(-\omega^2 t^2)$ **(*f*)** $1 + \omega t + \omega^2 t^2$

Ans. (*a*) For a SHM, acceleration ∝ – displacement, *i.e.*, $\vec{a} \propto -\vec{x}$

Here $x = \sin \omega t - \cos \omega t$

$$\Rightarrow \quad \text{Velocity, } v = \frac{dx}{dt} = \omega \cos \omega t + \omega \sin \omega t$$

$$\text{Acceleration, } a = \frac{d^2x}{dt^2} = -\omega^2 \sin \omega t + \omega^2 \cos \omega t$$

$$= -\omega^2(\sin \omega t - \cos \omega t) = -\omega^2 x$$

Therefore, the function $(\sin \omega t - \cos \omega t)$ represents SHM with time period, $T = \frac{2\pi}{\omega}$.

(*b*) The function $\sin^3 \omega t$ represents periodic motion with a period $\frac{2\pi}{\omega}$ but the motion is not SHM (because acceleration, $\vec{a}$ is not proportional to $-\vec{x}$).

(*c*) The function $3\cos\left(\frac{\pi}{4} - 2\omega t\right)$ represents SHM with a period $T = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$ (since here $\vec{a} \propto -\vec{x}$).

This may be seen as follows:

$$x = 3\cos\left(\frac{\pi}{4} - 2\omega t\right)$$

Velocity, $\frac{dx}{dt} = 6\,\omega \sin\left(\frac{\pi}{4} - 2\omega t\right)$

Acceleration, $\frac{d^2x}{dt^2} = -12\,\omega^2 \cos\left(\frac{\pi}{4} - 2\omega t\right) = -4\omega^2 x$

$\vec{a} \propto -\vec{x}$, so motion is SHM, angular frequency = 2ω

$$T = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$$

(*d*) The given function $f(t) = \cos\omega t + \cos 3\omega t + \cos 5\omega t$ is the sum of three independent SHM motions, but the function itself is not SHM but it is periodic. The periods of three independent simple harmonic motions are $\frac{2\pi}{\omega}, \frac{2\pi}{3\omega}, \frac{2\pi}{5\omega}$

The period of whole motion is $T = \frac{2\pi}{\omega}$

Thus, the function is periodic but not SHM.

(*e*) $f(t) = \exp(-\omega^2 t^2)$ is not a periodic function.

(*f*) $f(t) = 1 + \omega t + \omega^2 t^2$

When $t \to \infty$, $f(t) \to \infty$. This is not periodic function.

Q. 5. A particle is in linear simple harmonic motion between two points *A* and *B*, 10 cm apart. Take the direction from *A* to *B* as the positive direction and give the signs of velocity, acceleration and force on the particle when it is

(*a*) at the end *A*,

(*b*) at the end *B*,

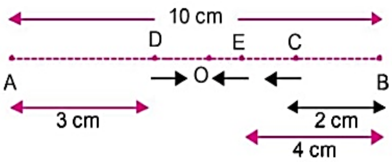
(*c*) at the mid-point of *AB* going towards *A*,

(*d*) at 2 cm away from *B* going towards *A*,

(*e*) at 3 cm away from *A* going towards *B*, and

(*f*) at 4 cm away from *B* going towards *A*.

Ans. In SHM acceleration and force are directed towards origin.



(*a*) At end *A*, velocity is zero, acceleration and force both have positive direction (from *A* to *O*).

(*b*) At end *B*, velocity is zero, acceleration and force both are negative (directed from *B* to *O*).

(*c*) At point *O* which is mid point of *AB*, when particle goes towards *A* velocity is negative, acceleration and force both are 0 (zero).

(*d*) At point *C* which is 2 cm away from *B* and particle going towards *A*, velocity, acceleration and force all are negative.

(*e*) At a point *D* which is 3 cm away from *A* and particle going towards *B*, velocity, acceleration and force all are positive.

(*f*) At a point *E*, which is a distance 4 cm away from *B* and particle going towards *O*, velocity, acceleration and force all are negative.

Q. 6. Which of the following relationships between the acceleration *a* and the displacement *x* of a particle involve simple harmonic motion?

(*a*) $a = 0.7x$ | **(*b*) $a = -200x^2$**

(*c*) $a = -10x$ | **(*d*) $a = 100x^2$**

Ans. The necessary and sufficient condition of SHM is $\vec{a} \propto -\vec{x}$.

This condition is satisfied only in (*c*); hence only the relation $a = -10x$ represents SHM.

Q. 7. The motion of a particle executing simple harmonic motion is described by the displacement function,

$$x(t) = A\cos(\omega t + \phi).$$

If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle? The angular frequency of the particle is $\pi\ s^{-1}$. If instead of the cosine function, we choose the sine function to describe the SHM: $x = B\sin(\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions?

Ans. Given equation is

$$x(t) = A\cos(\omega t + \phi) \quad \ldots(i)$$

Velocity, $$v(t) = \frac{dx(t)}{dt} = -A\omega\sin(\omega t + \phi) \quad \ldots(ii)$$

Given, at $t = 0$, $x = 1$ and v = ω cm/s

Also, angular frequency, $\omega = \pi s^{-1}$

At $t = 0$, $v = \pi$ cm/s.

Putting $t = 0$, $x = 1$ cm in equation (i), we get

$$A\cos\phi = 1 \quad \ldots(iii)$$

Now putting $t = 0$, $v = \pi$ cm/s, $\omega = \pi s^{-1}$ in equation (ii)

$$\pi = -A\pi\sin\phi$$

$$\Rightarrow \quad A\sin\phi = -1 \quad \ldots(iv)$$

Dividing equation (iv) by equation (iii)

$$\tan\phi = -1$$

$$\phi = \tan^{-1}(-1)$$

$$= 2\pi - \frac{\pi}{4} = \mathbf{\frac{7\pi}{4}}$$

and from equation (iii), $A = \dfrac{1}{\cos\phi} = \dfrac{1}{\cos\frac{7\pi}{4}} = \sqrt{2}$ cm = **1.41 cm**

Thus amplitude, $A = \sqrt{2}$ cm = **1.41 cm** and initial phase angle = $\mathbf{\frac{7\pi}{4}}$

Now taking the sine function

$$x = B\sin(\omega t + \alpha) \quad \ldots(v)$$

Velocity, $$v = \frac{dx}{dt} = B\omega\cos(\omega t + \alpha) \quad \ldots(vi)$$

Putting $t = 0$, $x = 1$ cm in equation (v)

$$1 = B\sin\alpha \Rightarrow B = \frac{1}{\sin\alpha} \quad \ldots(vii)$$

Putting $t = 0$, $v = \omega$ cm/s$=\pi$ cm/s in equation (v)

$$\pi = B\pi\cos\alpha \Rightarrow B\cos\alpha = 1 \Rightarrow B = \frac{1}{\cos\alpha} \quad \ldots(viii)$$

Comparing equation (vii) and equation ($viii$), $\sin\alpha = \cos\alpha \Rightarrow \alpha = \frac{\pi}{4} = \mathbf{45°}$

$$B = \frac{1}{\sin\alpha} = \frac{1}{\sin\pi/4} = \sqrt{2} \text{ cm} = \mathbf{1.41\ cm}$$

Thus amplitude **1.41 cm** and initial phase $= \frac{\pi}{4} = \mathbf{45°}$

Q. 8. A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body?

Ans. Spring constant, $k = \left(\frac{F}{x}\right) = \frac{\text{Maximum force}}{\text{Maximum extension}}$

Here maximum weight = 50 kgf = 50g

Maximum extension, x = 20 cm = 0.20 m

$$k = \frac{50g}{0.20} = \frac{50 \times 9.8}{0.20} = 2450 \text{ N/m}$$

Time period, $T = 2\pi\sqrt{\frac{m}{k}}$

Mass suspended, $m = \frac{kT^2}{4\pi^2}$

Here T = 0.6 s, $\therefore m = \frac{2450 \times (0.6)^2}{4 \times (3.14)^2} = 22.36$ kg

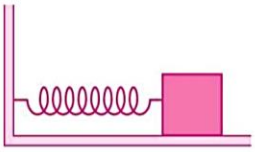
Weight of body = mg = 22.36 × 9.8 = **219.13 N**

Q. 9. A spring having a spring constant 1200 Nm^{-1} is mounted on a horizontal table as shown in Fig. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released. Determine

(*i*) the frequency of oscillations,

(*ii*) maximum acceleration of the mass and

(*iii*) maximum speed of the mass.



Ans. (*i*) Given $k = 1200 \text{ Nm}^{-1}$, m = 3 kg

Angular frequency, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3}} = 20$ rad/s

Frequency of oscillations, $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2 \times 3.14} \times 20 = \frac{20}{6.28}$ = **3.18 s^{-1}**

(*ii*) Acceleration of mass, $a = -\omega^2 y$

If A is amplitude or maximum displacement from mean position

$$a_{max} = \omega^2 A$$

Given amplitude, $A = 2 \text{ cm} = 2 \times 10^{-2}$ m

$$a_{max} = (20)^2 \times 2 \times 10^{-2} = \mathbf{8 \text{ ms}^{-2}}$$

(*iii*) Speed of mass, $v = \omega\sqrt{A^2 - y^2}$

For maximum speed, y = 0 (mean position)

Maximum speed; $v_{max} = \omega A = 20 \times 2 \times 10^{-2}$ = **0.40 ms^{-1}**

Q. 10. In Q. 9, let us take the position of mass when the spring is unstretched as x = 0, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch (t = 0), the mass is

(*a*) at the mean position,

(*b*) at the maximum stretched position, and

(*c*) at the maximum compressed position.

In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

Ans. Here amplitude, $A = 2.0$ cm

Angular frequency, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3}} = 20$ rad/s

Equation of motion of SHM having initial phase ϕ is

$$x = A \sin(\omega t + \phi) \qquad ...(i)$$

(*a*) At $t = 0, x = 0$; from (*i*)

$\therefore \quad 0 = A \sin\phi \Rightarrow \phi = 0$

$\therefore$ Equation of x versus t is

$x = A \sin \omega t$ or $x = \mathbf{2.0 \sin 20}\boldsymbol{t}$

(*b*) At $t = 0$ the body is at maximum stretched position *i.e.*, at $t = 0, x = +A;$ from equation (*i*)

$\therefore \quad A = A \sin\phi \Rightarrow \sin\phi = 1$ or $\phi = \frac{\pi}{2}$

Equation of x versus t is

$$x = A = \sin\left(\omega t + \frac{\pi}{2}\right) = A \cos \omega t$$

$\Rightarrow \quad x = \mathbf{2.0 \cos 20}\boldsymbol{t}$

(*c*) At $t = 0$ the body is at maximum compressed position *i.e.*, at $t = 0, x = -A;$ from equation (*i*)

$\therefore \quad -A = A \sin\phi \Rightarrow \sin\phi = -1$ or $\phi = \frac{3}{2}\pi$

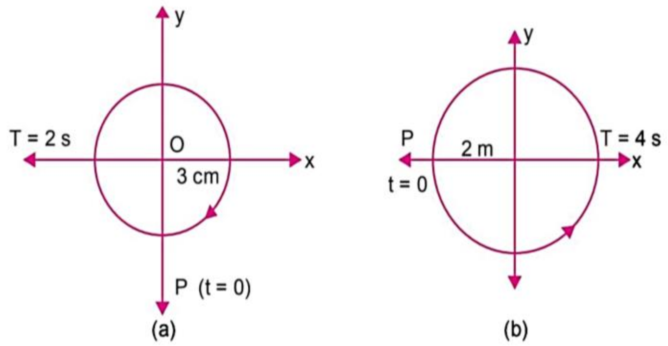
Equation of x versus t is

$$x = A \sin\left(\omega t + \frac{3\pi}{2}\right) = -A \cos \omega t$$

or $\Rightarrow \quad x = \mathbf{-2.0 \cos 20}\boldsymbol{t}$

Obviously, the functions $x(t)$ neither differ in amplitude nor in frequency; they differ only in initial phase.

Q. 11. The given figures correspond to two circular motions.



The radius of the circle, the period of revolution, the initial position and the sense of revolution (*i.e.*, clockwise or anti-clockwise) are indicated on each figure. Obtain the corresponding simple harmonic motions of the *x*-projection of the radius vector of the revolving particle *P*, in each case.

Ans. The equation of SHM of x projection of radius vector is

$$x = A \cos(\omega t + \phi) \qquad ...(i)$$

(*a*) Here A = maximum x-projection of radius vector = 3 cm

$$T = 2 \text{ s}$$

$\therefore$ Angular frequency, $\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$ rad/s

As particle is moving clockwise, at $t = 0, \phi = +\frac{\pi}{2}$;

Therefore, equation (*i*) becomes

$x = 3\cos\left(\pi t + \frac{\pi}{2}\right)$ or $x = \mathbf{-3\sin\pi t}$, where x is in cm.

(*b*) Here $A = 2$ m, $T = 4$ s

Angular frequency, $\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$ rad s^{-1}

As particle is moving anti-clockwise and the x-projection is at extreme left, $\phi = -\pi$.

$\therefore$ Equation (*i*) becomes

$$x = 2\cos\left(\frac{\pi}{2}t - \pi\right)$$

$x = \mathbf{-2\cos\frac{\pi}{2}t}$, where x is in metre.

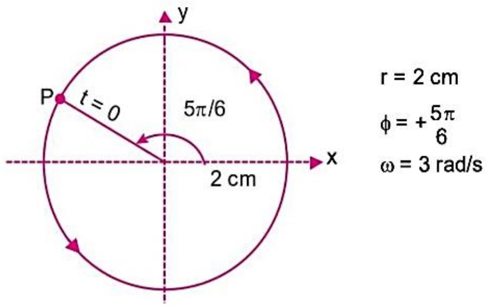
Q. 12. Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t = 0) position of the particle, the radius of the circle and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anti-clockwise in every case : (x in cm and t is in s).

(*a*) $x = -2\sin\left(3t + \frac{\pi}{3}\right)$ **(*b*)** $x = \cos\left(\frac{\pi}{6} - t\right)$

(*c*) $x = 3\sin\left(2\pi t + \frac{\pi}{4}\right)$ **(*d*)** $x = 2\cos\pi t$

Ans. (*a*) $x = -2\sin\left(3t + \frac{\pi}{3}\right)$

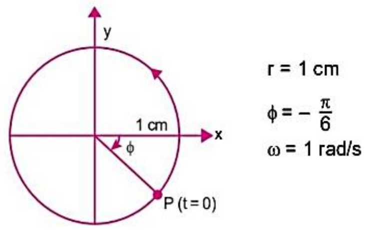
This may be expressed as $x = 2\cos\left(3t + \frac{\pi}{3} + \frac{\pi}{2}\right)$ or $x = 2\cos\left(3t + \frac{5\pi}{6}\right)$



Accordingly the reference circle is shown in figure.

(*b*) $x = \cos\left(\frac{\pi}{6} - t\right)$

or $x = \cos\left(t - \frac{\pi}{6}\right)$



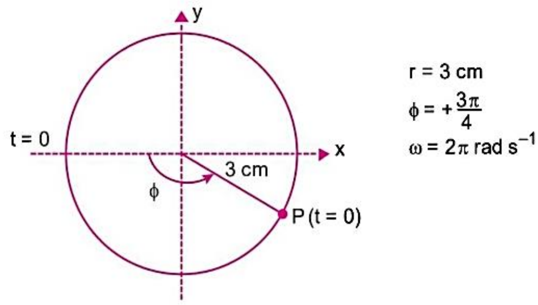
Comparing with equation $x = A\cos(\omega t + \phi)$, we have

A = 1cm, ω = 1 rad/s, $\phi = -\frac{\pi}{6}$

(*c*) $x = 3\sin\left(2\pi t + \frac{\pi}{4}\right)$

$$= -3\cos\left(2\pi t + \frac{\pi}{4} + \frac{\pi}{2}\right)$$

$$= -3\cos\left(2\pi t + \frac{3\pi}{4}\right)$$



The negative sign shows that motion starts on the negative side of x-axis.

Comparing with standard equation

$$x = A\cos(\omega t + \phi)$$

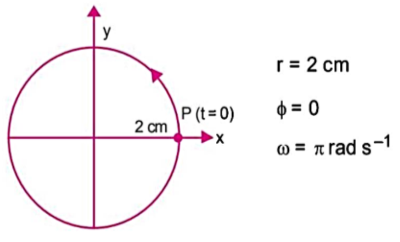
We have $A = 3$ cm, $\omega = 2\pi$ rad/s, $\phi = \frac{3\pi}{4}$

(*d*) $x = 2\cos \pi t$

Comparing with standard equation

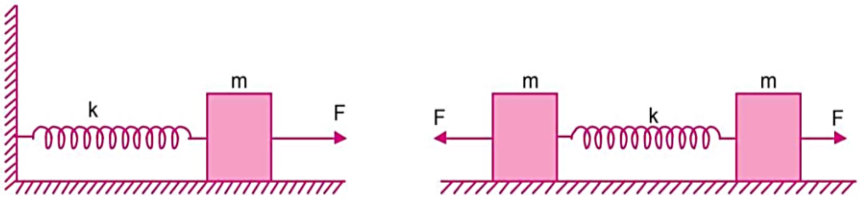
$$x = A\cos(\omega t + \phi)$$

$$A = 2\text{ cm}, \omega = \pi \text{ rad s}^{-1}, \phi = 0$$



(d)

Q. 13. Figure (*a*) shows a spring force constant *k* clamped rigidly at one end and a mass *m* attached to its free end. A force *F* applied at the free end stretches the spring. Fig. (*b*) shows the same spring with both ends free and attached to a mass *m* at either end. Each end of the spring in Fig. (*b*) is stretched by the same force *F*.



(*a*) (*b*)

(*a*) What is the maximum extension of the spring in the two cases?

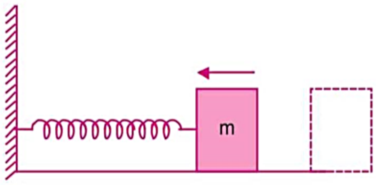
(*b*) If the mass in Fig. (*a*) and the two masses in Fig. (*b*) are released, what is the period of oscillation in each case?

Ans. (*a*) Maximum extension (x_{max}) of spring in Fig. (*a*) is given by

$$F = kx_{max} \text{ or } x_{max} = \frac{F}{k}$$

In Fig. (*b*), one force acts as a support to maintain the system at rest on the floor; while extension is done to the other force, therefore, again maximum extension $x_{max} = \frac{F}{k}$.

(*b*) In Fig. (*a*), if mass *m* is released and *x* is instantaneous displacement of mass, then restoring force



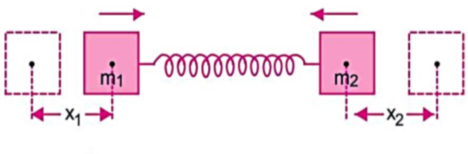
$$F = -kx$$

$$\Rightarrow \quad m\frac{d^2x}{dt} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Comparing with standard equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

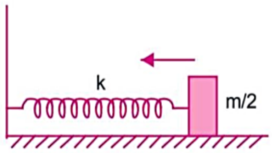


the angular frequency, $\omega = \sqrt{\frac{k}{m}}$

Time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

In Fig. (*b*) when two masses are released; then by theorem of centre of mass, each (equal) mass is displaced by same amount in opposite direction.

$\equiv$



So, total stretching of spring = $2x$

Restoring force on each mass = $-k.2x$

Equations of motion of each mass is

$$\frac{d^2x}{dt^2} = -k.(2x)$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{2k}{m}x$$

Comparing with standard equation $\frac{d^2x}{dt^2} = -\omega^2 x$,

We have angular frequency, $\omega = \sqrt{\frac{2k}{m}}$

Time period, $T = \frac{2\pi}{\omega} = \mathbf{2\pi\sqrt{\frac{m}{2k}}}$

Remark : Such two body problems may be reduced to one body problem, by fixing one mass and placing a fictitious mass, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ given by at the location of other mass, so time period, $T = 2\pi\sqrt{\frac{\mu}{k}}$

Here $m_1 = m_2 = m$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m}{2}$$

$$T = 2\pi\sqrt{\frac{m/2}{k}} = 2\pi\sqrt{\left(\frac{m}{2k}\right)}$$

Q. 14. The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed?

Ans. Amplitude of piston, $A = \frac{1.0}{2} = 0.5$ m

Given angular frequency, $\omega = 200 \text{ rad/min} = \frac{200}{60}$ rad/s

Maximum speed, $v_{max} = \omega A = \left(\frac{200}{60}\right) \times 0.5 =$ **1.67 m/s**

Q. 15. The acceleration due to gravity on the surface of moon is 1.7 ms^{-2}. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s? (g on the surface of earth is 9.8 ms^{-2})

Ans. Time period of simple pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}$$

For given value of L, $\quad T \propto \frac{1}{\sqrt{g}}$

If g_m and g_e are accelerations due to gravity and T_m and T_e the time periods of simple pendulum on the surface of moon and earth respectively, then

$$\frac{T_m}{T_e} = \sqrt{\frac{g_e}{g_m}} \Rightarrow T_m = \sqrt{\frac{g_e}{g_m}}T_e$$

Given $g_m = 1.7$ ms^{-2}, $g_e = 9.8$ ms^{-2}, $T_e = 3.5$ s

$$T_m = \sqrt{\frac{9.8}{1.7}} \times 3.5 \text{ s} = \mathbf{8.4\ s}$$

Q. 16. Answer the following questions:

(a) Time period of a particle in SHM depends on the force constant k and mass m of the particle:

$T = 2\pi\sqrt{\frac{m}{k}}$. A simple pendulum executes SHM approximately. Why then is the time period of a pendulum independent of the mass of the pendulum?

(b) The motion of a simple pendulum is approximately simple harmonic for small angle oscillations. For larger angles of oscillation, a more advanced analysis shows that T is greater than $2\pi\sqrt{\frac{l}{g}}$. Think of a qualitative argument to appreciate this result.

(c) A man with a wristwatch on his hand falls from the top of a tower. Does the watch give correct time during the free fall?

(d) What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity?

Ans. (a) Time period in SHM, $T = 2\pi\sqrt{\frac{m}{k}}$.

In the case of a simple pendulum the force constant is itself proportional to mass and hence the time period, $T = 2\pi\sqrt{\frac{m}{k}}$ becomes independent of mass.

(b) In the case of a simple pendulum, the restoring force is $F = -mg\sin\theta$.

The formula, $T = 2\pi\sqrt{\frac{l}{g}}$ has been derived by assuming $\sin\theta \simeq \theta$. This is true only for small θ. But actually $\sin\theta = -\frac{\theta^3}{\theta!} + \frac{\theta^5}{\theta!} + \ldots\ldots$

Clearly, for large angles $\sin\theta < \theta$; so restoring force $F = -mg\sin\theta = mg\,\theta$. (for small angles) becomes less than $mg\,\theta$ for large angles. This is equivalent to reducing the effective value of g if we stick to same treatment for finding the value of time period; hence the time period, $T = 2\pi\sqrt{\frac{l}{g_{eff}}}$ will become greater than $T = 2\pi\sqrt{\frac{l}{g}}$.

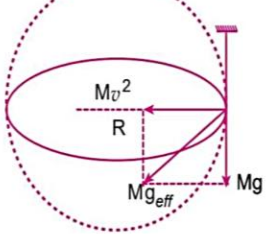
(c) The working of wristwatch depends on spring-action and its time period, $T = 2\pi\sqrt{\frac{m}{k}}$ is independent of acceleration due to gravity, therefore, for a freely falling wristwatch, though the effective value of becomes zero, but the time period of watch remains same, so the wristwatch gives correct time.

(d) In a freely falling cabin, effective value of g is zero, so frequency $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ becomes zero.

Q. 17. A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?

Ans. When bob of pendulum is suspended in a car, moving on circular track, it experiences an additional horizontal centripetal force $\frac{Mv^2}{R}$,

so effective force will be $\sqrt{(Mg)^2 + \left(\frac{Mv^2}{R}\right)^2}$.



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Effective acceleration due to gravity,

$$Mg_{eff} = \sqrt{(Mg)^2 + \left(\frac{Mv^2}{R}\right)^2}$$

$$\therefore \qquad g_{eff} = \sqrt{(g)^2 + \left(\frac{v^2}{R}\right)^2}$$

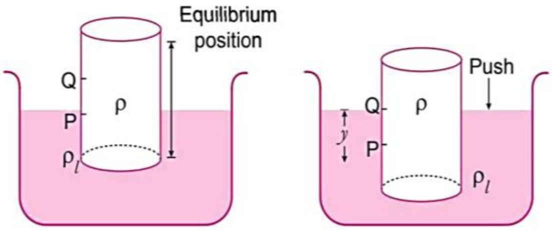
Time period, $T = 2\pi\sqrt{\frac{l}{g_{eff}}} = \mathbf{2\pi\sqrt{\{l/(g^2 + v^4/R^2)\}}}$

Q. 18. A cylindrical piece of cork of base area A and height h floats in a liquid of density ρ_l. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}$$

where ρ is density of cork. (Ignore damping due to viscosity of the liquid). [HOTS]

Ans. In equilibrium, weight of the cork is balanced by the upthrust of the liquid. Let the cork be slightly depressed through distance y from the equilibrium position and left to itself. It begins to oscillate under the restoring force,



F = Net upward force = Weight of liquid coloumn of height y

or $\qquad F = -Ay\rho_l g = -A\rho_l g\, y$

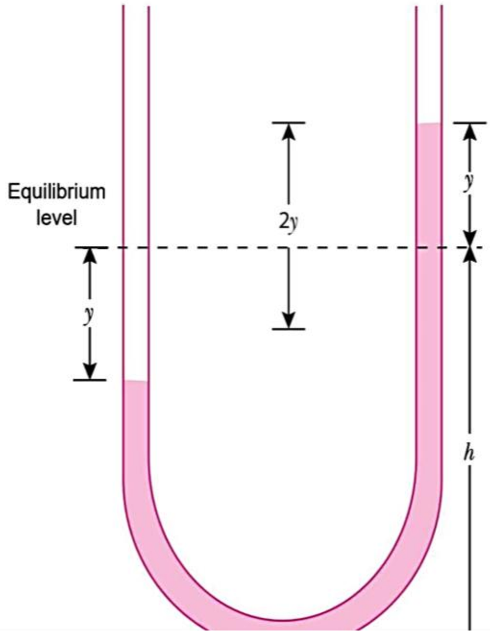
i.e., $\qquad F \propto -y$

Negative sign shows that F and y are in opposite directions. Hence the cork executes SHM with force constant, $k = A\rho_l g$

also, mass of cork = $A\rho h$

$\therefore$ Period of oscillation of the cork is $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{A\rho h}{A\rho_l g}} = \mathbf{2\pi\sqrt{\frac{\rho h}{\rho_l g}}}$

Q. 19. One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion. [HOTS]



Ans. Initially, suppose the U-tube of cross-section A contains liquid of density ρ upto height h. Then mass of the liquid in the U-tube is

$$m = \text{Volume} \times \text{Density} = A \times 2h \times \rho$$

If the liquid in one arm is depressed by distance y, it rises by the same amount in the other arm. If left to itself, the liquid begins to oscillate under the restoring force,

F = Weight of liquid coloumn of height $2y$

$F = -A \times 2y \times \rho \times g = -2A\rho g y$

i.e., $\qquad F \propto -y$

Thus, –ve sign shows the force on the liquid is proportional to displacement and acts in the opposite direction. Hence, the liquid in the U-tube executes SHM with force constant $k = 2A\rho g$

The time-period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{A \times 2h \times \rho}{2A\rho g}} = 2\pi\sqrt{\frac{h}{g}}$$

If l is the length of the liquid coloumn, then

$$l = 2h \text{ and } T = \mathbf{2\pi}\sqrt{\frac{l}{2g}}$$

Q. 20. You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of the spring constant *k* and assuming that each wheel supports 750 kg.

Ans. Total weight is supported by four springs, each of spring constant k,

so $$\frac{1}{4}Mg = kx$$

$$\Rightarrow \quad k = \frac{Mg}{4x} = \frac{3000 \times 10}{4 \times 0.15}\text{N/m} = \mathbf{5 \times 10^4\ Nm^{-1}}$$

Q. 21. A body describes simple harmonic motion with an amplitude of 5 cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (*a*) 5 cm (*b*) 3 cm (*c*) 0 cm.

Ans. Acceleration, $a = -\omega^2 y$

Velocity, $v = \omega(A^2 - y^2)$

Given, Amplitude, $A = 5$ cm,

$$\text{Time period}, T = 0.2\text{ s}, \omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\,\pi \text{ rad s}^{-1}$$

(*a*) When $y = 5$ cm,

Acceleration, $a = -\omega^2 y = -(10\pi)^2 \times 5 \text{ cm s}^{-2}$

$= -500\,\pi^2 \text{ cm s}^{-2} = \mathbf{-5\pi^2\ ms^{-2}}$

Velocity, $v = \omega\sqrt{(A^2 - y^2)} = \omega\sqrt{(5^2 - 5^2)} = \mathbf{0}$

(*b*) When $y = 3$ cm,

Acceleration, $a = -\omega^2 y = -(10\pi)^2 \times 3 \text{ cm s}^{-2}$

$= -300\,\pi^2 \text{ cm s}^{-2} = \mathbf{-3\pi^2\ ms^{-2}}$

Velocity, $v = \omega\sqrt{(A^2 - y^2)} = 10\pi\sqrt{(5^2 - 3^2)} = 10\pi \times 4 \text{ cm s}^{-2} = \mathbf{0.4\pi\ ms^{-1}}$

(*c*) When $y = 0$ cm,

$a = -\omega^2 y = -\omega^2 \times 0 = \mathbf{0}$

$v = \omega\sqrt{A^2 - y^2} = 10\pi\sqrt{(5)^2 - 0}$

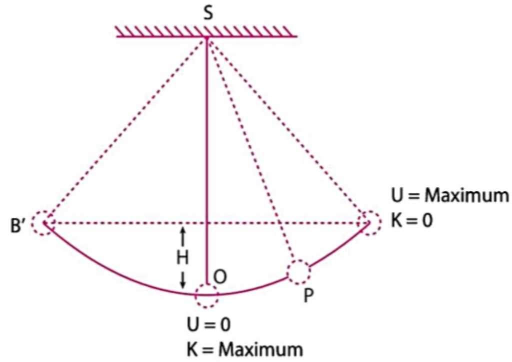
$= 50\,\pi \text{ cm s}^{-1} = \mathbf{0.50\,\pi\ ms^{-1}}$

CASE-BASED QUESTIONS

1. Read the following paragraph and answer the questions.

Energy Transformation in SHM

We have seen that oscillating body has kinetic energy and potential energy both. These energies vary with displacement of body from mean position. The kinetic energy is maximum at mean position and zero at extreme position, while potential energy is maximum at extreme position and minimum at mean position. The total energy is the sum of kinetic energy and potential energy and remains constant throughout the motion.



(*i*) How is the frequency of oscillations related with the frequency of change in the K.E and P.E of the body in SHM?

(*ii*) How much is K.E for displacement equal to one-forth the amplitude?

(*iii*) On what factors does the energy of harmonic oscillator depends?

(*iv*) What is the frequency of total energy of a particle in SHM?

Answers:

(*i*) P.E or K.E completes two vibrations in a time during which SHM completes one vibration or the frequency of P.E or K.E is double than that of SHM.

(*ii*) Given, $x = \dfrac{A}{4}$

So, K.E $= \dfrac{1}{2}m\omega^2(A^2 - x^2) = \dfrac{1}{2}m\omega^2\left[A^2 - (A/4)^2\right]$

$$= \frac{1}{2}\times\frac{15}{16}[m\omega^2A^2] = \mathbf{\frac{15}{16}(K.E)_{max}}$$

It is $\dfrac{15}{16}$th of maximum K.E.

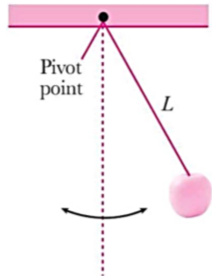
(*iii*) The energy of a harmonic oscillator depends on its: (*a*) mass (m), (*b*) frequency (ν) and (*c*) amplitude A *i.e.*, $E = 2\pi^2 m\nu A^2$.

(*iv*) The frequency of total energy of particle in SHM is zero because it remains constant during oscillations.

2. Read the following paragraph and answer the questions.

The Simple Pendulum

If you hang an apple at the end of a long thread fixed at its upper end, and then set the apple swinging back and forth a small distance, you easily see that the apple's motion is periodic. Is it, in fact, simple harmonic motion? To anwer, we consider a simple pendulum, which consists of a particle of mass m suspended from one end of an unstretchable, massless string of length L that is fixed at the other end, as in fig. The bob is free to swing back and forth in the plane of the page, to the left and right of a vertical line through the pendulum's pivot point.

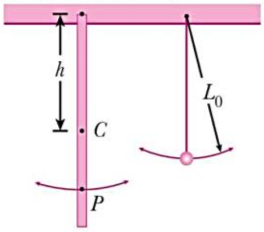


(*i*) A simple pendulum of length l has a bob of mass m which is moving on the circular arc of angle θ in a vertical plane. A sphere, also of mass m is placed at the end of the arc. What momentum will be transferred to the sphere by the moving bob?

(ii) If on going up a hill from earth's surface, the value of g decreases by 5%, then what change must be made in the length of the pendulum of a clock in order to get correct time ?

(iii) A simple pendulum executing SHM is falling freely along with the support. Will its time period change ?

(iv) A meter stick swings about a pivot point at one end, at distance from its centre of mass shown in figure. What is its period of oscillation?



Answers:

(i) At the end of the arc, the kinetic energy of bob will be minimum or zero or we can say that the momentum of the bob will be zero at the end of circular arc. So, the bob will not transfer any momentum to the sphere.

(ii) $T = 2\pi\sqrt{\frac{l}{g}}$, in order to keep T constant, l should also be decreased by 5%.

(iiii) For simple pendulum, $T = 2\pi\sqrt{\frac{l}{g}}$. The pendulum will not oscillate since effective value of g will be zero and time period will become infinite.

(iv) Here the stick is not a simple pendulum because its mass is not concentrated in a bob at the end opposite to the pivot point, so the stick is a physical pendulum. We can treat the stick as a uniform rod of length L and mass m.

Now, I = moment of inertia of stick $= \frac{1}{3}mL^2$

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mg\left(\frac{1}{2}L\right)}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$= 2\pi\sqrt{\frac{2\times1.00}{(3)\times9.8}} = \mathbf{1.64\ s}$$

3. Read the following paragraph and answer the questions.

Resonance

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called resonance. Resonance in mechanical systems can be destructive.

A company of soldiers once destroyed a bridge by marching across it in step; the frequency of their steps was close to a natural vibration frequency of the bridge, and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge.

(i) The soldiers marching on a suspended bridge are advised to go out of steps. Why?

(ii) What is the main differences between forced oscillations and resonance?

(iii) Why is the amplitude of forced oscillation very small when the frequency of the external force is different from the natural frequency of the body?

(iv) A sonometer wire is vibrating in resonance with a tuning fork. Keeping the tension applied same, the length of the wire is doubled. Under what conditions, the tuning fork will still be in resonance and why?

Answers:

(*i*) The soldiers marching on a suspended bridge are advised to go out of steps because in such a case the frequency of marching steps matches with natural frequency of the suspended bridge and hence resonance takes place, as a result amplitude of oscillation increases enormously which may lead to the collapsing of bridge.

(*ii*) The frequency of external periodic force is different from the natural freqeuncy of the oscillator in case of forced oscillation but in resonance two frequencies are equal.

(*iii*) The amplitude of forced oscillation depends upon the frequency of applied force and the natural frequency of the body. When the difference between this frequencies is large, the amplitude of forced oscillation is very small.

(*iv*) On doubling the length of sonometer wire (keeping tension same), the frequency of sonometer wire is halved; so sonometer wire will still vibrate in resonance with the tuning fork.

SHORT ANSWER QUESTIONS–I (2 marks)

Q. 1. A block of mass 2 kg is attached to a spring of spring constant 50 Nm^{-1}. The block is pulled to a distance of 5 cm from the mean position at $x = 0$ on a horizontal frictionless surface from rest at $t = 0$. Write the expression for its displacement at any time t.

Ans. Amplitude, A (maximum displacement from mean position) = 5 cm

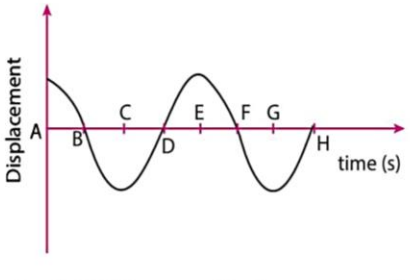
Angular frequency, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{2}} = 5$ rad/s

Equation of SHM of mass is $x = A \sin(\omega t + \phi)$.

The condition at $t = 0$ gives $\phi = 0$

$\therefore$ Equation of SHM is $x = 5 \sin 5t$.

Q. 2. Displacement versus time curve for a particle executing SHM is shown in figure. Identify the points marked at which (*i*) velocity of the oscillator is zero, (*ii*) speed of the oscillator is maximum. [*NCERT Exemplar*]



Ans. In displacement-time graph of SHM, zero displacement values correspond to mean position, where velocity of the oscillator is maximum. Whereas the crest and troughs represent amplitude position, where displacement is maximum and velocity of the oscillator is zero.

(*i*) The points A, C, E, G lie at mean positions (maximum displacement, $y = A$). Hence the velocity of the oscillator is zero.

(*ii*) The points B, D, F, H lie at mean position (zero displacement, $y = 0$). We know the speed is maximum at mean position.

Q. 3. The frequency of oscillation of a mass m suspended by a spring is ν_1. If the spring is cut into two equal halves and the same mass is attached to one half, then the frequency becomes ν_2. Determine the ratio $\frac{\nu_2}{\nu_1}$.

Ans. Frequency of mass - spring system, $\nu_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

When spring is out into two equal halves, the spring constant of each half becomes $2k$ $\left(\text{since } k \propto \frac{1}{l}\right)$ therefore, $\nu_2 = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

$$\frac{\nu_2}{\nu_1} = \sqrt{\mathbf{2}}$$

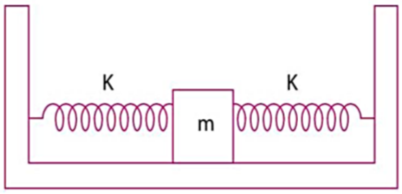
Q. 4. What are the two basic characteristics of a simple harmonic motion? [*NCERT Exemplar*]

Ans. The two basic characteristics of a simple harmonic motion are:

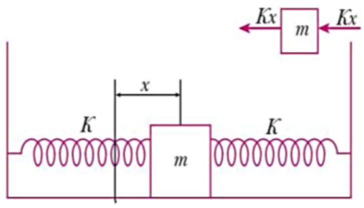
(*i*) Acceleration is directly proportional to displacement.

(*ii*) The direction of acceleration is always towards the mean position, that is opposite to displacement.

Q. 5. Two identical springs of spring constant K are attached to a block of mass m and to fixed supports as shown in figure. When the mass is displaced from equilibrium position by a distance x towards right, find the restoring force. [*NCERT Exemplar*]



Ans. When mass is displaced from equilibrium position by a distance x towards right, the right spring gets compressed by x developing a restoring force (Kx) towards left on the block. The left spring is stretched by an amount x developing a restoring force (Kx) left on the block.



$F_1 = -Kx$ (for left spring) and $F_2 = -Kx$ (for right spring)

Restoring force, $F = F_1 + F_2 = -2Kx$

$\therefore \quad |F| = 2Kx$ towards left

Q. 6. What is the ratio of maximum acceleration to the maximum velocity of a simple harmonic oscillator? [*NCERT Exemplar*]

Ans. Let us write the displacement equation for the SHM; $x = A\sin(\omega t + \phi)$.

Velocity of the particle, $v = \frac{dx}{dt} = \frac{d}{dt}(A\sin(\omega t + \phi)) = A\omega\cos(\omega t + \phi)$

Maximum velocity $|v|_{max} = A\omega$

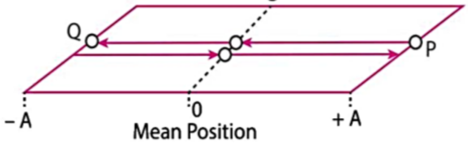
Now acceleration, $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t + \phi)$

Maximum acceleration $|a|_{max} = \omega^2 A$

$$\frac{|v|_{max}}{|a|_{max}} = \frac{\omega A}{\omega^2 A} = \frac{1}{\omega} \quad \Rightarrow \quad \frac{\boldsymbol{a}_{\mathbf{max}}}{\boldsymbol{v}_{\mathbf{max}}} = \boldsymbol{\omega}$$

Q. 7. What is the ratio between the distance travelled by the oscillator in one time period and amplitude? [*NCERT Exemplar*][HOTS]

Ans. In the diagram shown, a particle is executing SHM between P and Q. The particle starts from mean position 'O' moves to amplitude position 'P', then particle turn back and moves from 'P' to 'Q'. Finally the particle turns back again and return to mean position 'O'. In this way the particle completes one oscillation in one time period.



Total distance travelled while it goes from $O \rightarrow P \rightarrow O \rightarrow Q \rightarrow O = OP + PO + OQ + QO$

Amplitude $= OP = A$

Hence, ratio of distance and amplitude $= 4A/A = \mathbf{4}$

Q. 8. The bob of a simple pendulum is in the form of a hollow sphere filled with water. If a fine hole is made in the bottom such that water emerges slowly from the hole, what will be the effect on the time period ? [HOTS]

Ans. For simple pendulum, $T = 2\pi\sqrt{\frac{l}{g}}$. When water emerges slowly from the hole, initially its centre of gravity becomes lower, till the bob is half-empty and then it begins to go up. So, effective length of pendulum increases, becomes maximum and then decreases to initial value. Therefore, time period first increases, becomes maximum and then decreases and attains initial value.

Q. 9. Out of two clocks on the earth, one is controlled by a pendulum and the other by a spring. If both the clocks be taken on the moon, then will the clocks show correct time on moon? [HOTS]

Ans. The spring clock will show correct time, while pendulum clock will run slow. The reason is that for spring clock both force constant (k) and mass (m) remain unchanged on moon, therefore time period, $T = 2\pi\sqrt{\frac{m}{k}}$ will remain unchanged, while the value of g on moon is less than that on earth, therefore for pendulum clock time period, $T = 2\pi\sqrt{\frac{l}{g}}$ will increase.

Q. 10. Two bodies A and B of equal masses are suspended from two separate massless springs of spring constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal, what will be the ratio of amplitude of vibration of A to that of B?

Ans. Let A_1 and A_2 be the amplitude of A and B.

Maximum velocity $v_{max} = A\omega$ where $\omega = \sqrt{\frac{k}{M}}$.

$$A_1\omega_1 = A_2\omega_2$$

$$A_1\sqrt{\frac{k_1}{M}} = A_2\sqrt{\frac{k_2}{M}} \Rightarrow \frac{A_1}{A_2} = \sqrt{\frac{\boldsymbol{k_2}}{\boldsymbol{k_1}}}$$

Q. 11. A body of weight 0.5 kg suspended from a spring, increases the length of spring by 5.0 cm. This body is pulled slightly downward and released, the body executes simple harmonic motion. Find the force constant of spring and time period of motion. (g = 9.8 m/s^2)

Ans. The force constant, $k = \frac{Mg}{l_0}$

Here $M = 0.5\text{ kg}, l_0 = 5.0\text{ cm} = 5.0\times10^{-2}\text{ m}$

$$k = \frac{0.5\times9.8}{5.0\times10^{-2}} = \mathbf{98\ N/m}$$

Time period, $T = 2\pi\sqrt{\left(\frac{M}{k}\right)} = 2\times3.14\sqrt{\left(\frac{0.5}{98}\right)} = \frac{6.28}{14} = \mathbf{0.448\ s}$

Q. 12. The equation of motion of a particle executing simple harmonic motion (SHM) is $a = -bx$ where a is the acceleration of the particle and x is the displacement from the mean position and b is a constant. What is the time period of the particle?

Ans. Equation of SHM is $a = -bx \quad \Rightarrow \quad \frac{x}{a} = \frac{1}{b}$ (numerically)

$\therefore$ Time period $T = 2\pi\sqrt{\frac{\text{displacement }(x)}{\text{acceleration }(a)}} = 2\pi\sqrt{\frac{1}{b}} = \frac{\mathbf{2\pi}}{\sqrt{\boldsymbol{b}}}$

Q. 13. A particle is executing SHM. The amplitude of motion is 0.01 m and its frequency is 60 Hz. Calculate the maximum acceleration of the particle.

Ans. Acceleration, $a = \omega^2 y$

Maximum displacement, $y = y_{max} = A = 0.01\text{ m}$

$\therefore$ Maximum acceleration, $a_{max} = \omega^2 A = (2\pi\nu)^2 A$

Here, $\nu = 60$ Hz

$$a_{max} = (2\pi \times 60)^2 \times 0.01 = 144\,\pi^2 \text{ m/s}^2 = 144 \times (3.14)^2 = \mathbf{1419.8\ m/s^2}$$

Q. 14. The time taken by a simple pendulum to complete 100 vibrations is 49 s in Mumbai and 50 s in Poona. Calculate the ratio of accelerations due to gravity in Mumbai and Poona.

Ans. As length of pendulum is same, we have

$$T_1 = 2\pi\sqrt{\frac{l}{g_1}} \text{ and } T_2 = 2\pi\sqrt{\frac{l}{g_2}}$$

Dividing we get, $\dfrac{T_1}{T_2} = \sqrt{\dfrac{g_2}{g_1}}$

Squaring we get, $\dfrac{g_1}{g_2} = \left(\dfrac{T_2}{T_1}\right)^2$

Given $T_1 = \dfrac{49}{100}$ s, $T_2 = \dfrac{50}{100}$ s

$$\frac{g_1}{g_2} = \frac{(50/100)^2}{(49/100)^2} = \left(\frac{50}{49}\right)^2 = \frac{2500}{2401} \approx \mathbf{1.041}$$

Q. 15. Show that for a particle executing SHM, velocity and displacement have a phase difference of $\pi/2$. *[NCERT Exemplar]*

Ans. Let the displacement equation of SHM

$$x = A\cos\omega t$$

Velocity $v = \dfrac{dx}{dt} = A\omega(-\sin\omega t) = -A\omega\sin\omega t$

$\Rightarrow \quad v = A\omega\cos\left(\dfrac{\pi}{2} + \omega t\right)$

Now, phase of displacement $\phi_1 = \omega t$

Phase of velocity $\phi_2 = \dfrac{\pi}{2} + \omega t$

$\therefore$ Difference in phase of velocity to that of phase of displacement

$$\Delta\phi = \phi_2 - \phi_1 = \left(\frac{\pi}{2} + \omega t\right) - (\omega t) = \boldsymbol{\frac{\pi}{2}}$$

Q. 16. Find the displacement of a simple harmonic oscillator at which its P.E. is half of the maximum energy of the oscillator. *[NCERT Exemplar]*

Ans. Let us assume that the required displacement where P.E. is half of the maximum energy of the oscillator be x.

The potential energy of the oscillator at this position,

$$\text{P.E.} = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 \quad ...(i)$$

Maximum energy of the oscillator = Maximum potential energy = Total energy

$$\text{T.E.} = \frac{1}{2}m\omega^2A^2 \quad ...(ii)$$

Where, A = amplitude of motion

We are given, P.E. $= \dfrac{1}{2}$T.E.

$$\frac{1}{2}m\omega^2x^2 = \frac{1}{2}\left[\frac{1}{2}m\omega^2A^2\right]$$

$$x^2 = \frac{A^2}{2} \text{ or } x = \sqrt{\frac{A^2}{2}} = \pm\frac{\boldsymbol{A}}{\sqrt{\mathbf{2}}}$$

SHORT ANSWER QUESTIONS–II (3 marks)

Q. 1. A simple pendulum is suspended in a stationary lift and has a time period T. What will be the effect on its time period if (*i*) lift goes up with an acceleration 'a', (*ii*) lift moves down with an acceleration 'a', (*iii*) lift moves up or down with uniform speed?

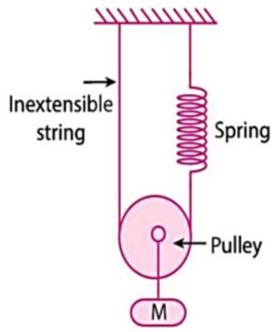
Ans. (*i*) Time period decreases since $g' = g + a$ and $T = 2\pi\sqrt{\left\{\frac{l}{g+a}\right\}}$.

(*ii*) Time period increases since $g' = g - a$ and $T = 2\pi\sqrt{\left\{\frac{l}{g-a}\right\}}$.

(*iii*) Time period will remain unchanged since

$g' = g$ and $T = 2\pi = \sqrt{\frac{l}{g}}$.

Q. 2. Find the time period of mass M when displaced from its equilibrium position and then released for the system as shown in figure. [*NCERT Exemplar*][HOTS]



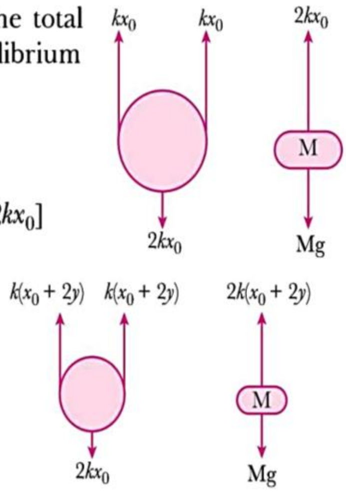
Ans. For observing oscillation, we have to displace the block slightly beyond equilibrium position and find the acceleration due to the restoring force.

Let in the equilibrium position, the spring has extended by an amount x_0.

Tension in the spring = kx_0

For equilibrium of the mass M, $Mg = 2kx_0$

Let the mass, M be pulled through a distance y and then released. But, string is inextensible, hence the spring alone will contribute the total extension $y + y = 2y$, to lower the mass down by y from initial equilibrium mean position x_0. So, net extension in the spring = $x_0 + 2y$.



From free body diagram of the block,

$$Mg - 2k(x_0 + 2y) = Ma$$

$$Mg - 2kx_0 - 4ky = Ma \quad \Rightarrow \quad Ma = -4ky \; [\because Mg = 2kx_0]$$

$$a = -\left(\frac{4k}{M}\right)y$$

k and M being constant, $a \propto -x$. Hence, motion is SHM.

Comparing the above acceleration expression with standard SHM equation

$a = \omega^2 y$, we get

$$\omega^2 = \frac{4k}{M} \Rightarrow \omega = \sqrt{\frac{4k}{M}}$$

Time period, $T = \frac{2\pi}{\omega} = \mathbf{2\pi\sqrt{\frac{M}{4k}}}$

Q. 3. Which of the following functions of time represent (*a*) simple harmonic motion and (*b*) periodic but not simple harmonic motion?
(*i*) $\sin \omega t - \cos \omega t$ (*ii*) $\sin^2 \omega t$. Find the period in each case.

Ans. (*i*) Let $x(t) = \sin \omega t - \cos \omega t = \sin \omega t - \sin\left(\frac{\pi}{2} - \omega t\right)$

Using the formula,

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\left(\frac{A-B}{2}\right), \quad \text{we get}$$

$$x(t) = 2\cos\frac{\pi}{4}\sin\left(\omega t - \frac{\pi}{4}\right) = 2\times\frac{1}{\sqrt{2}}\sin\left(\omega t - \frac{\pi}{4}\right)$$

$$\Rightarrow \qquad x(t) = \sqrt{2}\sin\left(\omega t - \frac{\pi}{4}\right)$$

This function represents SHM having period $T = \frac{2\pi}{\omega}$ and initial phase $-\frac{\pi}{4}$ or $\frac{7\pi}{4}$.

(*ii*) Let $x(t) = \sin^2\omega t = \frac{1}{2}(1 - \cos 2\omega t)$

velocity, $v = \frac{dx}{dt} = \omega \sin 2\omega t$

acceleration, $a = \frac{dv}{dt} = \frac{d}{dt}(\omega \sin 2\omega t) = 2\omega^2 \cos^2\omega t$

As acceleration is not proportional to (–*x*) so the function $x(t) = \sin^2\omega t$ does not represent SHM. It represents periodic motion of period, $T = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$.

Q. 4. The displacement equation for a particle executing simple harmonic motion is

$y = 0.2 \sin 50\pi(t + 0.01)$ metre

where y is the displacement at instant t. Calculate the amplitude, time period, maximum velocity, initial phase and initial displacement.

Ans. The displacement of the particle is

$$y = 0.2\sin 50\pi(t + 0.01) \text{ metre} \qquad ...(i)$$

The general equation of SHM is

$$y = A\sin(\omega t + \phi) \qquad ...(ii)$$

Comparing (*i*) and (*ii*), we get

Amplitude $A = \mathbf{0.2\ m}$

$\omega = 50\pi$

$\therefore$ Periodic time, $T = \frac{2\pi}{\omega} = \frac{2\pi}{50\pi} = \mathbf{0.04\ s}$

Maximum velocity : $v_{max} = \omega A = 50\pi \times 0.2 = 10 \times 3.14 = \mathbf{31.4\ m/s}$

Initial phase (*i.e.*, at $t = 0$), $\phi = 50\pi \times 0.01 = \mathbf{0.5\pi}$

Displacement of particle at the start (*i.e.*, at $t = 0$) is obtained by putting $t = 0$ in equation (*i*) *i.e.*, initial displacement

$$y = 0.2\sin 50\pi(t + 0.01)$$

$$= 0.2\sin 50\pi \times 0.01 = 0.2\sin\frac{\pi}{2} = \mathbf{0.2\ m}$$

Q. 5. A simple pendulum has a time period = 4 s. Now its length is changed, and effective length = 4 m. What should be its length so that it may complete 15 oscillations in 30 s?

Ans. If initial length of pendulum is L, then time period

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad ...(i)$$

If new length is L' then new time period

$$T' = 2\pi\sqrt{\frac{L'}{g}} \qquad ...(ii)$$

Dividing equation (*ii*) by (*i*), we get

$$\frac{T'}{T} = \sqrt{\frac{L'}{L}} \quad \text{or} \quad \left(\frac{T'}{T}\right)^2 = \frac{L'}{L}$$

$\therefore$ $$L' = \left(\frac{T'}{T}\right)^2 L$$

Given $T = 4$ s, $L = 4$ m

$$T' = \frac{30}{15} = 2 \text{ s}$$

$$L' = \left(\frac{2}{4}\right)^2 \times 4 = \mathbf{1\ m}$$

Q. 6. The time period of a body executing SHM is 2 seconds. If displacement of particle be 0 at $t = 0$, then after how much time will its displacement be half of its amplitude?

Ans. The standard equation of SHM is

$$y = A\sin(\omega t + \phi) \qquad ...(i)$$

If displacement $y = 0$ at $t = 0$, then equation (*i*) gives

$$0 = A\sin\phi \text{ or } \sin\phi = 0 \qquad \therefore \phi = 0$$

Hence equation (*i*) becomes

$$y = A\sin\omega t = A\sin\frac{2\pi t}{T} \qquad \left[\because \omega = \frac{2\pi}{T}\right]$$

Here $y = \frac{A}{2}, T = 2$ seconds

$$\therefore \frac{A}{2} = A\sin\frac{2\pi t}{2} \text{ or } \sin\pi t = \frac{1}{2}$$

or $$\sin\pi t = \sin\left(30° \times \frac{\pi}{180°}\right) \Rightarrow \sin\pi t = \sin\frac{\pi}{6}$$

$$\pi t = \pi/6$$

$\Rightarrow$ $$t = \mathbf{\frac{1}{6}\ s}$$

Q. 7. A 5 kg collar is attached to a spring of spring constant 500 Nm^{-1}. It slides without friction over a horizontal rod. The collar is displaced from its equilibrium position by 10.0 cm and released. Calculate

(*i*) the period of oscillation,

(*ii*) the maximum speed and

(*iii*) the maximum acceleration of the collar.

Ans. Given mass m=5 kg, spring constant $k = 500\ Nm^{-1}$, amplitude $A = 10.0$ cm $= 0.10$ cm

(*i*) Time period, $T = 2\pi\sqrt{\frac{m}{k}}$

$$= 2 \times 3.14\sqrt{\frac{5.0}{500}} = \frac{2 \times 3.14}{10} = 0.628 \text{ s} = \mathbf{0.63\ s} \text{ (upto two significant figures)}$$

(*ii*) Maximum speed, $v_{max} = A\omega$

Angular frequency,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{5.0}} = 10 \text{ rad/s}$$

$$v_{max} = 0.10 \times \sqrt{100} = \mathbf{1.0\ ms^{-1}}$$

This speed occurs at equilibrium position ($x = 0$).

(*iii*) Maximum acceleration, $a_{max} = \omega^2 A = (10)^2 \times 0.10 = \mathbf{10\ ms^{-2}}$

It occurs at the extremities.

Q. 8. A body of mass m is attached to one end of a massless spring which is suspended vertically from a fixed point. The mass is held in hand so that the spring is neither stretched nor compressed. Suddenly the support of hand is removed. The lowest position attained by the mass during oscillations is 4 cm below the point, where it was held in hand. *[NCERT Exemplar]*

(*i*) What is the amplitude of oscillations?

(*ii*) What is the frequency of oscillator?

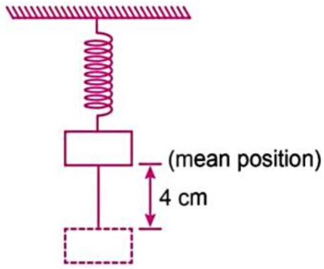
Ans. (*i*) In equilibrium position, let spring be stretched through y_0, then $mg = ky_0$ ...(*i*)

When mass is released, the gravitational P.E of mass is converted into elastic P.E of spring.

$$\frac{1}{2}kl^2 = mgl$$

Using equation (*i*), $\frac{1}{2}\left(\frac{mg}{y_0}\right)l^2 = mgl \Rightarrow y_0 = \frac{l}{2} = \frac{4\text{ cm}}{2} = 2\text{ cm}$

That is equilibrium position is at a distance 2 cm below the initial position.



Amplitude of motion, $A = l - y_0 = 4\text{ cm} - 2\text{ cm} = \mathbf{2\text{ cm}}$

(*ii*) Frequency of mass-spring system, $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$$= \frac{1}{2\pi}\sqrt{\frac{mg/y_0}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{y_0}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{9.8}{2\times10^{-2}}}$$

$$= \frac{1}{2\times3.14}\sqrt{490} = \frac{22.14}{6.28} = \mathbf{3.52\text{ Hz}}$$

Q. 9. A body of mass 0.50 kg suspended by an ideal spring oscillates up and down. The amplitude of oscillations is 0.50 m and periodic time is 1.57 s. Determine (*i*) maximum speed of body (*ii*) maximum kinetic energy (*iii*) total energy and (*iv*) force constant of the spring.

Ans. Here, $m = 0.50$ kg, $A = 0.50$ m and $T = 1.57$ s

(*i*) Maximum speed of the body executing SHM is given by

$$v_{max} = \omega A = \frac{2\pi}{T}A = \frac{2\times3.14}{1.57}\times0.50 = \mathbf{2\text{ m/s}}$$

(*ii*) Maximum kinetic energy, $K_{max} = \frac{1}{2}mv^2_{max}$

$$= \frac{1}{2}\times0.50\times(2)^2 = \mathbf{1\text{ J}}$$

(*iii*) Total energy = maximum kinetic energy = **1 J**

(*iv*) Time period, $T = 2\pi\sqrt{\frac{m}{k}}$ or $T^2 = 4\pi^2\frac{m}{k}$

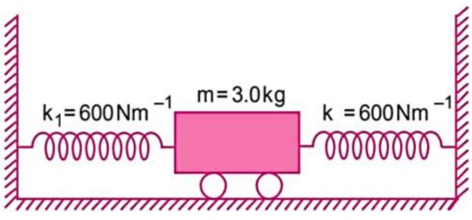
$\therefore$ Force constant, $k = \frac{4\pi^2 m}{T^2} = \frac{4\times(3.14)^2\times0.50}{(1.57)^2} = \mathbf{8\text{ N/m}}$

Q. 10. A trolley of mass 3.0 kg as shown in figure, is connected to two springs, each of spring constant 600 N/m. If the trolley is displaced from its equilibrium position by 5.0 cm and released, what is

(*i*) the time period of ensuring oscillations and

(*ii*) the maximum speed of the trolley ?

(*iii*) How much energy is dissipated as heat by the time the trolley comes to rest due to damping forces?



Ans. (*i*) When two springs of force constants k_1 and k_2 are attached to the trolley of mass m and the trolley is displaced to one side and released, the trolley begins to execute SHM. Let at any instant the displacement of the trolley be x (towards right), then restoring forces in the springs will be k_1x and k_2x (towards left), so net restoring force

$$F = -k_1x - k_2x = -(k_1 + k_2)x$$

Clearly effective force constant is $k_{eff} = k_1 + k_2$

Time period, $$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

Here $k_1 = k_2 = 600\ \text{Nm}^{-1}, m = 3.0\ \text{kg}$

$$T = 2\times 3.14\sqrt{\frac{3.0}{600+600}} = \frac{2\times 3.14}{20} = \mathbf{0.314\ s}$$

(*ii*) Maximum speed of trolley, $v_{max} = \omega A$, where A is amplitude.

Angular frequency, $\omega = \frac{2\pi}{T} = \frac{2\times 3.14}{0.314} = 20$ rad/s

Amplitude, $A = 5.0\ \text{cm} = 5.0\times 10^{-2}$ m

$$v_{max} = 20\times 5.0\times 10^{-2}\ \text{ms}^{-1} = \mathbf{1.0\ ms^{-1}}$$

(*iii*) When trolley comes the rest, the total energy is dissipated as heat.

Total energy of system, $E = \frac{1}{2}m\omega^2A^2$

$$= \frac{1}{2}\times 3.0\times(20)^2\times(5\times 10^{-2})^2 = \mathbf{1.5\ J}$$

Q. 11. Draw a graph to show the variation of P.E, K.E and total energy of a simple harmonic oscillator with displacement. [*NCERT Exemplar*]

Ans. The potential energy (P.E) of a simple harmonic oscillator is

$$\text{P.E} = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 \qquad ...(i)$$

When, P.E is plotted against displacement x, we will obtain a parabola.

When $x = 0$, P.E = 0

When $x = \pm A$, P.E = maximum

$$= \frac{1}{2}m\omega^2A^2$$

The kinetic energy (K.E) of a simple harmonic oscillator K.E = $\frac{1}{2}mv^2$

But velocity of oscillator $v = \omega\sqrt{A^2 - x^2}$

$\Rightarrow$ $$\text{K.E} = \frac{1}{2}m[\omega\sqrt{A^2-x^2}]^2$$

or $$\text{K.E} = \frac{1}{2}m\omega^2(A^2 - x^2) \qquad ...(ii)$$

This is also parabola, if we plot K.E against displacement x.

i.e., K.E = 0 at $x = \pm$ A

and K.E = $\frac{1}{2}m\omega^2A^2$ *at* $x = 0$

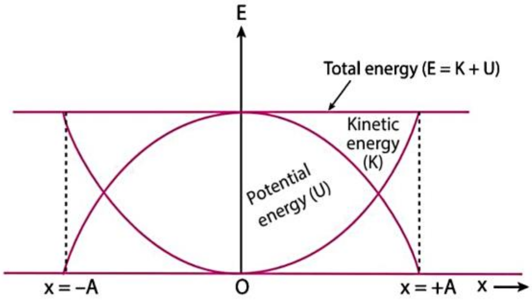
Now, total energy of the simple harmonic oscillator = P.E + K.E

[using equation (*i*) and (*ii*)]

$$= \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2(A^2 - x^2)$$

$$= \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2A^2 - \frac{1}{2}m\omega^2x^2$$

$$\text{T.E} = \frac{1}{2}m\omega^2 A^2 = \text{constant}$$



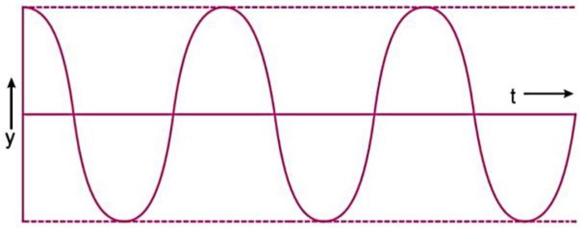
We see that T.E is constant and independent of x.

Plotting under the above guidelines K.E, P.E and T.E versus displacement x-graph as follows:

Note: From the graph we note that potential energy or kinetic energy completes two vibrations in a time during which SHM completes one vibration. Thus the frequency of potential energy or kinetic energy is double than that of SHM.

Q. 12. What are free and damped oscillations? Give examples.

Ans. **Free oscillations:** The oscillations of a body which remain unaffected by external forces are called free oscillations (or vibrations). The frequency of free oscillations is called the natural frequency of the body.



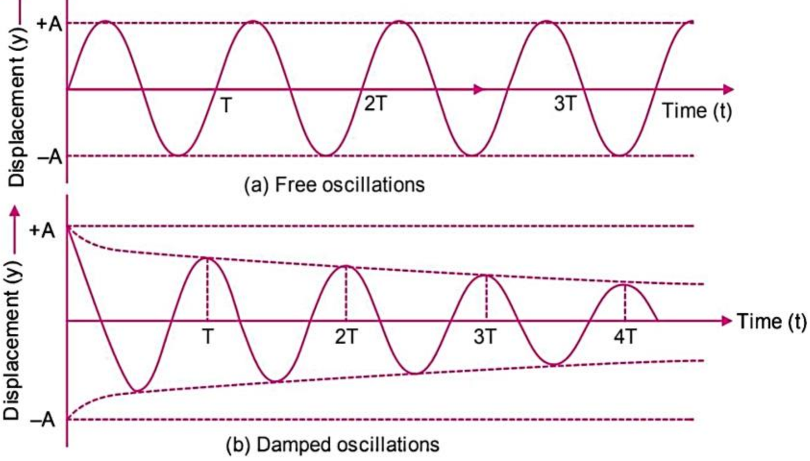
The examples of free oscillations are

(*i*) The oscillations of a simple pendulum in vacuum. The natural frequency of these vibrations depends on length of pendulum and the acceleration due to gravity 'g' at that place.

(*ii*) The oscillations of prongs of a tuning fork in vacuum. The natural frequency of these oscillations depends on the length and thickness of prongs and also on the elasticity of metal of prongs.

(*iii*) The vibrations of string of sonometer or sitar in vacuum.

Damped oscillations: As energy $\propto$ (amplitude)^2, therefore amplitude of oscillations decreases slowly due to presence of frictional or damping forces. Such oscillations are called damped oscillations. The continuous decrease in amplitude of oscillations due to energy dissipation is called the damping.



(a) Free oscillations

(b) Damped oscillations

Examples of Damped oscillations :

(*i*) Oscillations of simple pendulum in air

(*ii*) Oscillations of prongs of tuning fork in air

(*iii*) Oscillations of wire of sonometer or sitar in air

LONG ANSWER QUESTIONS (5 marks)

Q. 1. (*i*) Write some important characteristics of simple harmonic motion.

(*ii*) Derive the equation of motion of a simple harmonic motion.

Ans. (*i*) The characteristics of simple harmonic motion are :

(*a*) The motion of a body is in a straight line to and fro about a fixed point. This fixed point is called the equilibrium position.

(*b*) The force acting on a body is always directed towards fixed point and is directly proportional to its displacement from that point.

If displacement of body from fixed point is y and the force on the body is F, then for simple harmonic motion,

$$F \propto -y \text{ or } F = -ky$$

This is necessary and sufficient condition for simple harmonic motion in mathematical form.

(*ii*) The necessary and sufficient condition for simple harmonic motion is that the restoring force (F) is proportional to displacement (y).

i.e., $$F \propto -y \text{ or } F = -ky \quad \ldots(i)$$

where k is a constant.

The acceleration 'a' of particle is given by

$$a = \frac{F}{m} = -\frac{k}{m}y$$

Substituting $$\frac{k}{m} = \omega^2 \quad \ldots(ii)$$

we get $$a = -\omega^2 y \quad \ldots(iii)$$

As acceleration is rate of change of velocity of particle

$$a = \frac{dv}{dt}$$

$$\frac{dv}{dt} = -\omega^2 y$$

$$\frac{dv}{dy}\cdot\frac{dy}{dt} = -\omega^2 y$$

$\Rightarrow$ $\frac{dv}{dy}\cdot v = -\omega^2 y$ (since velocity $v = \frac{dy}{dt}$)

$\Rightarrow$ $v\,dv = -\omega^2 y\,dy$

Integrating, we get

$$\int v\,dv = -\omega^2 \int y\,dy + C, \quad \text{where } C = \text{constant of integration}$$

$$\frac{v^2}{2} = -\omega^2\frac{y^2}{2} + C \quad \ldots(iv)$$

Now at $y = y_0$, $v = v_0$ (say); then equation (*iv*) gives

$$\therefore \quad \frac{v_0^2}{2} = -\frac{\omega^2 y_0^2}{2} + C \quad \Rightarrow \quad \frac{v_0^2}{2} + \frac{\omega^2 y_0^2}{2} = C$$

In view of this, equation (*iv*) gives

$$\frac{v^2}{2} = -\frac{\omega^2 y^2}{2} + \frac{v_0^2}{2} + \frac{\omega^2 y_0^2}{2} \quad \text{or} \quad v^2 = v_0^2 + \omega^2 y_0^2 - \omega^2 y^2$$

$$v = \sqrt{v_0^2 + \omega^2 y_0^2 - \omega^2 y^2} \quad \text{or} \quad v = \omega\sqrt{y_0^2 + \frac{v_0^2}{\omega^2} - y^2}$$

Substituting $y_0^2 + \frac{v_0^2}{\omega^2} = A^2$, we get

$$v = \omega\sqrt{A^2 - y^2} \quad \ldots(v)$$

As $v = \frac{dy}{dt}$

$\therefore \quad \frac{dy}{dt} = \omega\sqrt{A^2 - y^2}$

or $$\frac{dy}{\sqrt{A^2 - y^2}} = \omega dt$$

Integrating, $\int \frac{dy}{\sqrt{A^2 - y^2}} = \omega \int dt + \phi$ where ϕ is constant of integration.

$\Rightarrow \quad \sin^{-1}\left(\frac{y}{A}\right) = \omega t + \phi \quad \ldots(vi)$

or $\frac{y}{A} = \sin(\omega t + \phi)$

$\Rightarrow \quad \mathbf{y = A \sin(\omega t + \phi)}$

This is equation of motion of simple harmonic motion where $(\omega t + \phi)$ is phase of particle at time t.

Q. 2. ***(i)*** **Define amplitude, periodic time, frequency and phase with reference to oscillatory motion.**

(ii) **Obtain the time-displacement curve of simple harmonic motion.**

Ans. *(i)* **Amplitude:** The maximum displacement of particle executing SHM is called 'amplitude'. As from equation of motion of SHM, $y = A \sin(\omega t + \phi)$, maximum value of $\sin(\omega t + \phi)$ is 1, therefore, from equation, maximum displacement, $y_{max} = A$. Thus A is amplitude of motion.

Periodic Time: The time taken by a particle executing SHM to complete one oscillation is called the periodic time. It is denoted by T.

Periodic time, $$T = \frac{2\pi}{\omega}$$

Frequency : The number of oscillations completed per second by particle executing SHM is called the frequency. It is denoted by v.

Frequency $$\nu = \frac{1}{T} = \frac{\omega}{2\pi}.$$

Phase : The position and direction of motion of a vibrating particle is different at different instants. The instantaneous position and direction of motion of the vibrating particle is expressed by a physical quantity called the phase. The general equation of simple harmonic motion is

$$y = A \sin(\omega t + \phi)$$

The quantity $(\omega t + \phi)$ is called the phase of the particle. At $t = 0$, the phase is ϕ, it is called the initial phase.

If at any instant, the two vibrating particles pass through their equilibrium positions in same direction, they are said to be in same phase.

(ii) Equation of SHM is

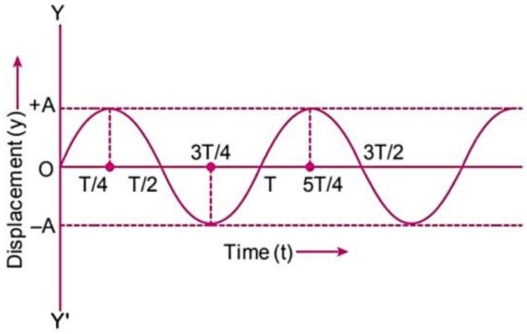
$$y = A \sin \omega t$$

Substituting $\omega = \frac{2\pi}{T}$, we get

$$y = A \sin \frac{2\pi t}{T}$$

From this equation, we can find displacements of particle in different times and plot the graph with time on x-axis and displacement on y-axis. The displacements at different times are shown in following table :

Time (t)	0	$T/4$	$T/2$	$3T/4$	T	$5T/4$	$3T/2$	...
Displacement (y)	0	A	0	$-A$	0	A	0	...



As the graph is a curve of equation $y = A\sin\frac{2\pi t}{T}$, therefore, the curve is called the sine curve.

Q. 3. (*i*) Derive an expression for the velocity, acceleration and time period of a particle executing SHM.

(*ii*) What do you mean by restoring force and force constant?

Ans. (*i*) The general equation of SHM is

$$y = A\sin(\omega t + \phi) \quad ...(i)$$

(*a*) Velocity : Differentiating equation (*i*) with respect to time t, we get

$$\text{velocity, } v = \frac{dy}{dt} = A\omega\cos(\omega t + \phi) \quad ...(ii)$$

$$= A\omega\sqrt{1-\sin^2(\omega t + \phi)}$$

$$= A\omega\sqrt{1-\left(\frac{y}{A}\right)^2}$$

i.e., $$\boldsymbol{v = \omega\sqrt{A^2 - y^2}} \quad ...(iii)$$

(*b*) Acceleration : Differentiating equation (*ii*) again with respect to time t, we get

acceleration, $a = \frac{dv}{dt} = A\omega\{-\omega\sin(\omega t + \phi)\} = -A\omega^2\sin(\omega t + \phi)$. Using (*i*), we get,

acceleration, $\boldsymbol{a = -\omega^2 y}$ ...(*iv*)

(*c*) Time Period : If T is time period of particle executing SHM, then

$$T = \frac{2\pi}{\omega}$$

The acceleration of particle, executing SHM at displacement y

$$a = -\omega^2 y$$

i.e., $$\omega^2 = -\frac{a}{y} = \frac{a}{y} \quad \text{(numerically)}$$

$\therefore$ $$\omega = \sqrt{\left(\frac{a}{y}\right)}$$

$\therefore$ $$T = \frac{2\pi}{\sqrt{a/y}} = 2\pi\sqrt{\left(\frac{y}{a}\right)}$$

i.e., $$\boldsymbol{T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}} \quad ...(v)$$

(*ii*) When particle executing SHM is in its equilibrium position, the force on the particle is zero. Therefore, the particle may remain at rest in equilibrium position. But if the particle is displaced a little from its equilibrium position, a force begins to act on the particle. This force is always directed towards the equilibrium position and tends to bring the particle to equilibrium position. This force is called the restoring force. In general, the restoring force means the internal force acting on the particle.

The relation between displacement and acceleration is $\vec{a} = -\omega^2\vec{y}$

$\therefore$ Force, $\vec{F} = m\vec{a} = -m\omega^2\vec{y}$ ...(*i*)

As $m\omega^2$ is a constant,

$\therefore$ $\vec{F} \propto -\vec{y}$

That is **the restoring force is proportional to its displacement from equilibrium position.**

Equation (*i*) may be written as

$$\vec{F} = -k\vec{y} \quad ...(ii)$$

where $k = m\omega^2$ ...(*iii*)

is a constant, called the force constant or spring factor.

From equation (*iii*), $\omega^2 = \frac{k}{m}$ or $\omega = \sqrt{\frac{k}{m}}$ ...(*iv*)

If T is time period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$ using equation (*iv*)

$$\therefore \quad \boldsymbol{T = 2\pi\sqrt{\left(\frac{m}{k}\right)}}$$

This is the relation of time period in terms of force constant (k) and mass (m) of particle.

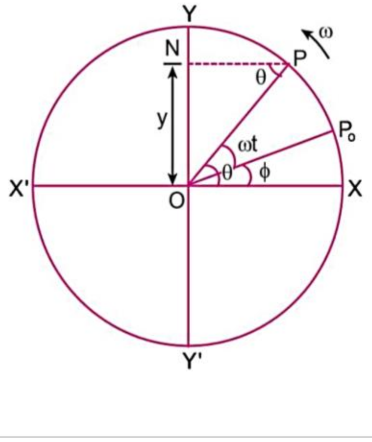
Q. 4. Prove that simple harmonic motion may be regarded as the projection of uniform circular motion along the diameter of the circle. Derive an expression for the displacement of a particle in simple harmonic motion.

Ans. Consider a particle of mass m to be rotating with uniform angular velocity ω in a circular path of radius A. Clearly the motion of the particle is periodic, but not simple harmonic.

When the particle is at P then the foot of perpendicular PN drawn from P on diameter YY' of circle is at point N, when particle is at Y the foot of perpendicular is also at Y, when particle reaches X' the foot of perpendicular moves from Y to O, when particle reaches Y' the foot of particle moves to Y', when particle is at X the foot reaches to point O. When particle returns again to P the foot again reaches N.

Clearly the foot of perpendicular dropped from instantaneous positions of particle moving with uniform speed along a circle on any diameter YY' is oscillatory motion.

The time interval in which the particle completes one revolution, the foot of perpendicular also completes one oscillation. This time interval is the time period.



The displacement of foot of perpendicular (N) when particle at P is $ON = y$,

The force on the particle towards centre O is, $F = m\omega^2 A$

The component of this force along direction OY,

$$F_y = -m\omega^2 A\sin\theta$$

Negative sign shows that the direction of force F_y is towards O.

From ΔOPN, $\sin\theta = \frac{ON}{OP} = \frac{y}{A}$

$\therefore$ $F_y = -m\omega^2 A.\frac{y}{A}$ ...(*i*)

As m and ω are constants, we may write

$$m\omega^2 = \text{a constant } k \text{ (say)}$$

$\therefore$ $$F_y = -ky$$

or $$F_y \propto -y \qquad ...(ii)$$

Thus the force acting on foot of perpendicular (N) is proportional to displacement y from mean position and its direction is towards mean position. This is necessary and sufficient condition for simple harmonic motion. Therefore, the vibratory motion of foot of perpendicular is simple harmonic. Thus, the motion of a particle on a uniform circular path is periodic but not simple harmonic, while the motion of the foot of perpendicular drawn from the instantaneous positions of the particle on any diameter is simple harmonic motion (SHM).

Suppose the initial position of the particle moving along a circular path with uniform angular velocity ω is at P_0 (*i.e.*, at $t = 0$, particle is at P_0). Then

$$\angle P_0OX = \phi \text{ (say)}$$

After time t, the particle is at point P

$$\angle POP_0 = \omega t$$

The projection of perpendicular drawn from P on diameter YY' is N

The displacement of projection N from equilibrium position O

$$y = ON = OP \sin(\angle NPO) = OP \sin\theta$$
$$= OP \sin(\angle POX)$$

$(\because \angle NPO = \angle POX = \theta)$

$$y = A\sin(\omega t + \phi)\,\text{s}$$

This is the general equation of SHM.

Q. 5. Derive the expressions for the kinetic energy and potential energy of a simple oscillator. Hence show that the total energy is conserved in SHM. Draw graphs for
(*i*) energy verses displacement and (*ii*) energy verses time.

Ans. **Total energy in SHM:** The energy of a harmonic oscillator is partly kinetic and partly potential. When a body is displaced from its equilibrium position by doing work upon it, it acquires potential energy. When the body is released, it begins to move back with velocity, thus acquiring kinetic energy.

Kinetic energy: At any instant, the displacement of a particle executing SHM is given by

$$x = A\cos(\omega t + \phi_0)$$

$\therefore$ $$\text{Velocity, } v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi_0)$$

Hence, kinetic energy of the particle at any displacement x is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi_0) \qquad ...(i)$$

But $$A^2\sin^2(\omega t + \phi_0) = A^2[1 - \cos^2(\omega t + \phi_0)]$$
$$= A^2 - A^2\cos^2(\omega t + \phi_0) = A^2 - x^2 \qquad ...(ii)$$

From equation (*i*) and (*ii*)

$$K = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$$

where $$k = m\omega^2$$

Potential energy: When the displacement of a particle from its equilibrium position is x, the restoring force acting on it is

$$F = -kx$$

If we displace the particle further through a small distance dx, then work done against the restoring force is given by

$$dW = -F\,dx = +kx\,dx$$

The total work done in moving the particle form mean position ($x = 0$) to displacement x is given by

$$W = \int dW = \int_0^x kx\,dx = k\left[\frac{x^2}{2}\right]_0^x = \frac{1}{2}kx^2$$

This work done against the restoring force is stored as the potential energy of the particle. Hence potential energy of a particle at displacement x is given by

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi_0).$$

Total energy: At any displacement x, the total energy of a harmonic oscillatory is given by

$$E = K + U = \frac{1}{2}k(A^2 - x^2) + \frac{1}{2}kx^2$$

or

$$E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2 = 2\pi^2 m\nu^2A^2 \qquad [\because \omega = 2\pi\nu]$$

Thus the total mechanical energy of a harmonic oscillator is independent of time or displacement. Hence in the absence of any frictional force, the total energy of a harmonic oscillator is conserved.

Obviously, the total energy of particle in SHM is

(*a*) directly proportional to mass m of the particle,

(*b*) directly proportional to the square of its frequency ν, and

(*c*) directly proportional to the square of its vibrational amplitude A.

Graphical representation: At the mean position, $x = 0$

Kinetic energy, $K = \frac{1}{2}k(A^2 - 0^2) = \frac{1}{2}kA^2$

Potential energy, $U = = \frac{1}{2}k(0^2) = 0$

Hence at the mean position, the energy is all kinetic.

At the extreme position, $x = \pm A$

Kinetic energy, $K = \frac{1}{2}k(A^2 - A^2) = 0$

Potential energy, $U = = \frac{1}{2}kA^2$

Hence at the two extreme positions, the energy is all potential.

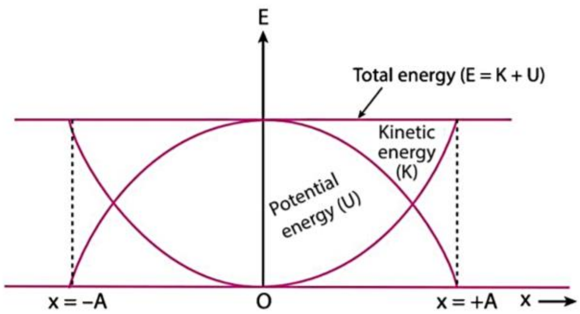
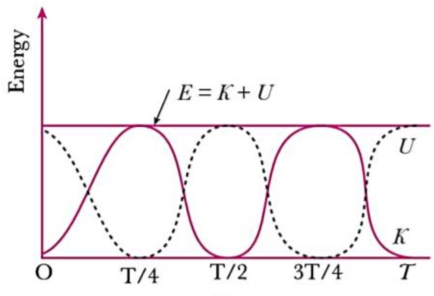


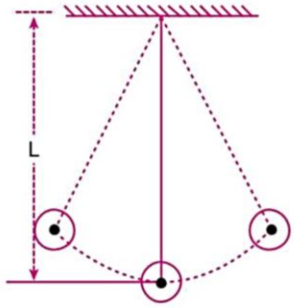
Figure (*i*) shows the variations of kinetic energy K, potential energy U and total energy E with displacement x. The graphs for K and U are parabolic while that for E is a straight line parallel to the displacement axis. At $x = 0$, the energy is all kinetic and for $x = \pm A$, the energy is all potential.



(ii)

Figure (*ii*) shows the variations of energies K, U and E of a harmonic oscillator with time t. Clearly, twice in each cycle, both kinetic and potential energies assume their peak values. Both of these energies are periodic functions of time, the time period of each being $T/2$.

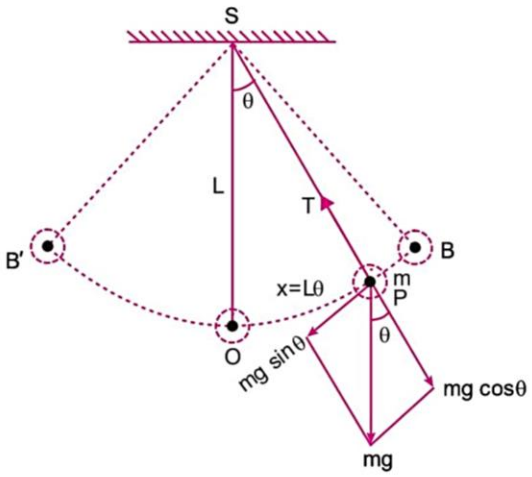
Q. 6. Show that for small oscillations, the motion of a simple pendulum is simple harmonic. Derive an expression for its time period.



Ans. A simple pendulum is a heavy point mass suspended from one end of a weightless, perfectly elastic and inextensible string whose other end is tied to a rigid support. This is definition of an ideal simple pendulum. In practice a simple pendulum is formed by suspending a small solid sphere of a metal (*e.g.*, copper, brass) from one end of a light cotton string, whose other end is tied to a rigid support. The rigid support to which the upper end of string is tied is called the point of suspension. The distance of point of suspension from the centre of gravity of the bob is called the length of the simple pendulum.

i.e., Length of pendulum, L = length of string including hook (L_0) + radius of bob (r).

When the bob is displaced slightly to one side from its mean position, then the bob executes oscillatory motion about its mean position.



Expression for Time Period: Let SO be the vertical position of pendulum where S is point of suspension and O is the mean position of bob. Let m be the mass and L the length of pendulum. The bob is displaced a little on one side (say from O to B) and released, the bob performs SHM about mean position O along the circular arc BOB'. The maximum displacement of bob is from O to B.

Suppose P is the instantaneous position of the bob. At this instant its string makes an angle θ with the vertical. The forces acting on the bob are

(*i*) weight of bob, mg acting vertically downward,

(*ii*) tension T in the string.

The weight mg may be resolved into two components : radial component and tangential component. The radial component is $mg\cos\theta$ and the tangential component is $mg\sin\theta$ (towards the mean position). The tension T and radial component $mg\cos\theta$ are oppositely directed, therefore, net force towards centre of circular arc is $T - mg\cos\theta$. This provides the necessary centripetal force for motion along the circular arc. If v is the speed of bob at position P, then

$$T - mg\cos\theta = \frac{mv^2}{L} \qquad \ldots(i)$$

This is the condition of equilibrium of bob along the radial direction.

The tangential component $mg\sin\theta$ provides the restoring torque about the point of suspension (S). This torque tends to bring the bob towards its mean position and is given by

$$\tau = -(mg\sin\theta).L \qquad \ldots(ii)$$

where negative sign shows that the torque and angular displacement θ are oppositely directed.

For rotational motion of bob

$$\tau = I\alpha \qquad \ldots(iii)$$

where I is moment of inertia about the point of suspension and α is angular acceleration.

From equations (*ii*) and (*iii*), we get

$$I\alpha = -(mg\sin\theta)L$$

or

$$\alpha = -\frac{mgL}{I}\sin\theta \qquad \ldots(iv)$$

This equation may represent SHM if θ is small, we have

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$$

If θ is small, then sin θ = θ, so equation (*iv*) takes the form $\alpha = -\frac{mgL}{I}\theta$ ...(*v*)

Clearly, $\alpha \propto -\theta$

That is angular acceleration of bob is directly proportional to the angular displacement. This is necessary and sufficient condition for SHM, hence the motion of bob, for small angular displacement, is simple harmonic. Standard equation of SHM is

$$\alpha = -\omega^2\theta \quad \text{...(vi)}$$

Comparing equation(*v*) and (*vi*), we have

Angular frequency, $\omega = \sqrt{\frac{mgL}{I}}$

Time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgL}}$ ...(*vii*)

As whole mass of simple pendulum is concentrated in the mass of bob, therefore, moment of inertia of simple pendulum about point of suspension

$$I = mL^2 \quad \text{...(viii)}$$

Substituting this value in equation (*vii*), we get

$$T = 2\pi\sqrt{\frac{mL^2}{mgL}}$$

and $$\boldsymbol{T = 2\pi\sqrt{\frac{L}{g}}}$$

This is expression for time period of simple pendulum.