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Comprehensive Topic Coverage: The study module comprehensively covers the Straight Line in Coordinate Geometry, spanning slope, intercepts, equations, parallel and perpendicular lines, distance formula, and more – aligning with the IIT-JEE Class 11 curriculum.

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COORDINATE(2-D)GEOMETRY

STRAIGHT LINE AND INTRO

01

MATHEMATICS

IIT-JEE-XI

Chapter Highlights

Distance Formula, Section Formulae, Area of a Triangle, Condition for Collinearity of Three Points, Stair Method for Finding the Area, Area of a Quadrilateral, Area of a Polygon, Stair method, Locus, Translation of Axes, Rotation of Axes, Reflection (Image) of a Point, General Equation of a Straight Line, Slope of a Line, Intercept of a Line on the Axes, Equation of a Straight Line in Various Forms, Reduction of the General Equation to Different Standard Forms, Angle between Two Intersecting Lines, Condition for Two Lines to be Coincident, Parallel, Perpendicular or Intersecting, Equation of a Line Parallel to a Given line, Equation of a Line Perpendicular to a Given line, Point of Intersection of Two Given Lines, Concurrent Lines, Position of Two Points Relative to a Line, Length of Perpendicular from a Point on a Line, Distance between Two Parallel Lines, Equations of Straight Lines Passing Through a Given Point and Making a Given Angle with a Given Line, Reflection on the Surface, Image of a Point with Respect to a Line, Equations of the Bisectors of the Angles between Two Lines, Equations of Lines Passing Through the Point of Intersection of Two Given Lines, Standard Points of a Triangle, Orthocentre, Coordinates of Nine Point Circle.

DISTANCE FORMULA

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

Fig. 18.1

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

REMARK

- Distance is always positive. Therefore, we often write PQ instead of $|PQ|$.
- Distance MN between two points $M(x_1, 0)$ and $N(x_2, 0)$ on the x-axis is $|x_2 - x_1|$. Similarly, the distance between two points $M(x_1, y_1)$ and $x(x_2, y_1)$, (which lie on a line parallel to x-axis) is $|x_2 - x_1|$.
- Distance AB between two points $A(0, y_1)$ and $B(0, y_2)$ on the y-axis is $|y_2 - y_1|$. Similarly, the distance between two points $A(x_1, y_1)$ and $B(x_1, y_2)$, (which lie on a line parallel to y-axis) is $|y_2 - y_1|$.
- Distance between the origin $O(0, 0)$ and the point $P(x, y)$ is $OP = \sqrt{x^2 + y^2}$.

SOLVED EXAMPLES

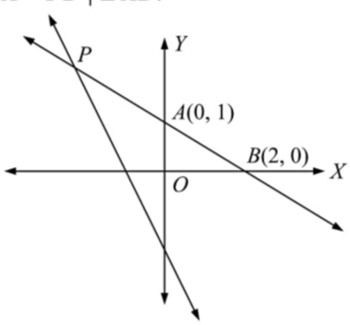
1. Consider the point $A \equiv (0, 1)$ and $B \equiv (2, 0)$. Let P be a point on the line $4x + 3y + 9 = 0$. Coordinates of the point P such that $|PA - PB|$ is maximum, are

(A) $\left(-\frac{84}{5}, \frac{13}{5}\right)$ (B) $\left(-\frac{12}{5}, \frac{17}{5}\right)$

(C) $\left(-\frac{6}{5}, \frac{17}{5}\right)$ (D) none of these

Solution: (A)

We have, $|PA - PB| \le AB$.



Thus, for $|PA - PB|$ to be maximum, point A, B and P must be collinear. The equation of line AB is

$$x + 2y = 2$$

Solving it with the given line, we get $P \equiv \left(-\frac{84}{5}, \frac{13}{5}\right)$

2. A ladder of length 'a' rests against the floor and a wall of a room. If the ladder begins to slide on the floor, then the locus of its middle point is

(A) $x^2 + y^2 = a^2$ (B) $2(x^2 + y^2) = a^2$
(C) $x^2 + y^2 = 2a^2$ (D) $4(x^2 + y^2) = a^2$

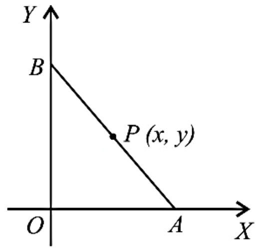
Solution: (D)

Let AB be the ladder. Let the cross section of the floor and wall be taken as the coordinates axes. Let $P(x, y)$ be the mid-point of AB whose locus is required. Then the coordinates of A and B are $(2x, 0)$ and $(0, 2y)$ respectively.

Given, $AB = a$

$\Rightarrow \sqrt{(2x-0)^2 + (0-2y)^2} = a \Rightarrow 4x^2 + 4y^2 = a^2$

or $4(x^2 + y^2) = a^2$, which is the required locus.



3. The point $(2t^2 + 2t + 4, t^2 + t + 1)$ lies on the line $x + 2y = 1$ for

(A) all real values of t
(B) some real values of t
(C) $t = \frac{-4 \pm \sqrt{7}}{8}$
(D) none of these

Solution: (D)

The point $(2t^2 + 2t + 4, t^2 + t + 1)$ lies on the line $x + 2y = 1$ if $(2t^2 + 2t + 4) + 2(t^2 + t + 1) = 1$

i.e., $4t^2 + 4t + 5 = 0$

Here, discriminent $= 16 - 4 \times 4 \times 5 = -64 < 0$.

$\therefore$ No real value of t is possible.

Hence, the given point cannot lie on the line.

4. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is

(A) square (B) circle
(C) straight line (D) two intersecting lines

Solution: (A)

If α and β are the lengths of perpendiculars, then $|\alpha| + |\beta| = 1$ (given), whose graph is a square.

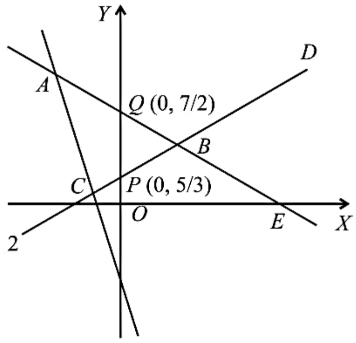
5. The condition to be imposed on β so that $(0, \beta)$ lies on or inside the triangle having sides $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$ is

(A) $0 < \beta < \frac{5}{3}$ (B) $0 < \beta < \frac{7}{2}$
(C) $\frac{5}{3} \le \beta \le \frac{7}{2}$ (D) none of these

Solution: (C)

Clearly point $(0, \beta)$ lies on y-axis.

Drawing the graph of the three straight lines, we see that $Q \equiv \left(0, \frac{7}{2}\right)$ and $P \equiv \left(0, \frac{5}{3}\right)$



Therefore, the point $(0, \beta)$ lies on or inside ΔABC, when

$$\frac{5}{3} \le \beta \le \frac{7}{2}$$

6. A straight line L with negative slope passes through the point (8, 2) and intersects the positive coordinate axes at points P and Q. As L varies the absolute minimum value of $OP + OQ$ is (O is origin).

(A) 12 (B) 14
(C) 18 (D) 20

Solution: (C)

The equation of the line is

$$(y - 2) = m(x - 8),\ m < 0$$

The coordinates of P and Q are $P\left(8 - \frac{2}{m}, 0\right)$ and $Q(0, 2 - 8m)$.

Therefore, $OP + OQ = 8 - \frac{2}{m} + 2 - 8m$

$$= 10 + \frac{2}{-m} + 8(-m)$$

$$\ge 10 + 2\sqrt{\frac{2}{-m} \times 8(-m)} = 18$$

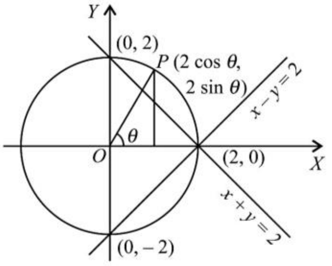
Thus, absolute minimum value of $OP + OQ = 18$.

7. If the point $(2 \cos\theta, 2 \sin\theta)$ does not fall in that angle between the lines $y = |x - 2|$ in which the origin lies then θ belongs to

(A) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (B) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(C) $(0, \pi)$ (D) none of these

Solution: (B)

Clearly the point ($2\cos\theta$, $2\sin\theta$) lies on the circle



$$x^2 + y^2 = 4$$

The two lines represented by the equation $y = |x - 2|$ are $y = x - 2$ and $y = 2 - x$.

From the figure, θ can be vary from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

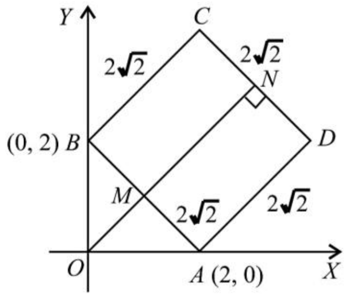
8. On the portion of the straight line $x + y = 2$ which is intercepted between the axes, a square is constructed away from the origin, with this portion as one of its side. If p denotes the perpendicular distance of a side of this square from the origin, then the maximum value of p is

(A) $\sqrt{2}$ (B) $2\sqrt{2}$

(C) $3\sqrt{2}$ (D) $4\sqrt{2}$

Solution: (C)

$p = ON = OM + MN = \perp$ distance from



O to the line $AB + AD$

$$= \frac{2}{\sqrt{2}} + 2\sqrt{2} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$$

9. If the points $P_i\left(\frac{a_i^3}{a_i - 1}, \frac{a_i^2 - 3}{a_i - 1}\right), i = 1, 2, 3$ are collinear and a_1, a_2 and a_3 are distinct real numbers, then

(A) $a_1a_2a_3 - \Sigma a_1a_2 - 3\Sigma a_1 = 0$

(B) $a_1a_2a_3 + \Sigma a_1a_2 - 3\Sigma a_1 = 0$

(C) $a_1a_2a_3 - \Sigma a_1a_2 + 3\Sigma a_1 = 0$

(D) $-a_1a_2a_3 + \Sigma a_1a_2 - 2\Sigma a_1 = 0$

Solution: (C)

Let the given points lie on the line $lx + my + n = 0$

$$\therefore \quad l\left(\frac{a_i^3}{a_i - 1}\right) + m\left(\frac{a_i^2 - 3}{a_i - 1}\right) + n = 0$$

$$\Rightarrow \quad la_i^3 + ma_i^2 + na_i - 3m - n = 0$$

$$\therefore \quad \sum a_1 = -\frac{m}{l}, \sum a_1a_2 = \frac{n}{l} \text{ and } a_1a_2a_3 = \frac{3m + n}{l}$$

$$\therefore \quad a_1a_2a_3 + 3\sum a_1 - \sum a_1a_2 = 0$$

10. Let $ax + by + c = 0$ be a variable straight line, where a, b and c are first, third and seventh terms of an increasing A.P. Then, the variable straight line always passes through a fixed point which lies on

(A) $x^2 + y^2 = 4$ (B) $x^2 + y^2 = 13$

(C) $y^2 = 2x$ (D) $2x + 3y = 9$

Solution: (B)

Let d be the common difference of A.P., then

$b = a + 2d$ and $c = a + 6d$. Clearly, $(b - a) \times 3 = c - a$

$\Rightarrow \quad 2a - 3b + c = 0$

Thus, the straight line $ax + by + c = 0$ passes through the point $(2, -3)$ which also satisfies $x^2 + y^2 = 13$.

11. If a, b, c form a G.P., then twice the sum of the ordinates of the points of intersection of the line $ax + by + c = 0$ and the curve $x + 2y^2 = 0$ is

(A) $\frac{b}{a}$ (B) $\frac{c}{a}$

(C) $\frac{a}{c}$ (D) $\frac{a}{b}$

Solution: (A)

Let a, b, c be in G.P. with common ratio r.

Then, $b = ar$ and $c = ar^2$.

So, the equation of the line is $ax + by + c = 0$

$\Rightarrow \quad ax + ary + ar^2 = 0 \quad \Rightarrow \quad x + ry + r^2 = 0$

This line cuts the curve $x + 2y^2 = 0$

Eliminating x, we get $2y^2 - ry + r^2 = 0$

If the roots of the quadratic equation are y_1 and y_2, then

$$y_1 + y_2 = \frac{r}{2} \quad \Rightarrow \quad 2(y_1 + y_2) = r = \frac{b}{a} = \frac{c}{b}.$$

12. If a, b, c are in A.P., then the straight line $ax + by + c = 0$ will always pass through a fixed point whose coordinates are

(A) $(-1, -2)$ (B) $(1, 2)$

(C) $(-1, 2)$ (D) $(1, -2)$

Solution: (D)

Since a, b, c are in A.P., $\therefore 2b = a + c$

or $\quad a - 2b + c = 0$

$\Rightarrow$ The line $ax + by + c = 0$ passes through the point $(1, -2)$.

SECTION FORMULAE

1. The coordinates of the point $P(x, y)$ dividing the line segment joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ *internally* in the ratio $m_1 : m_2$, are given by

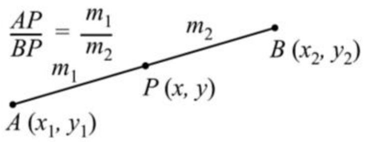


Fig. 18.2

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2},\ y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

NOTE

- To remember the formula it is helpful to note that m_1 is multiplied by the coordinate 'away from it' and similarly is m_2 and the sum is then divided by $m_1 + m_2$. Thus the above result can be remembered with the help of the figure given below.

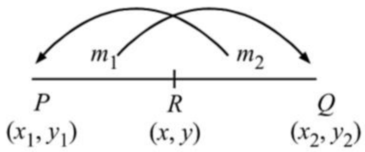


Fig. 18.3

- The coordinates of the point $P(x, y)$, dividing the line segment joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ *externally* in the ratio $m_1:m_2$, are given by

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2},\ y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

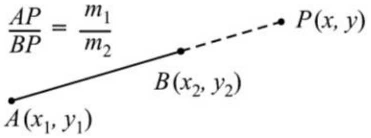


Fig. 18.4

- The coordinates of the mid-point of the line segment joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

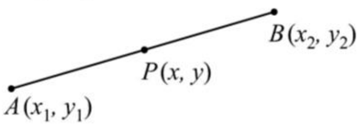


Fig. 18.5

TRICK(S) FOR PROBLEM SOLVING

The coordinates of any point on a line joining the two points A and B are given by $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$. Such a point divides the given line in the ratio $\lambda : 1$. If λ is positive, then the point divides internally and if λ is negative, then the point divides externally.

SOLVED EXAMPLES

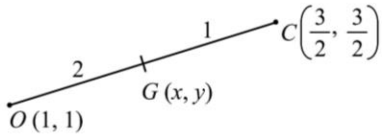
13. If a triangle has its orthocentre at (1, 1) and circumcentre at $\left(\frac{3}{2}, \frac{3}{4}\right)$, then the coordinates of the centroid of the triangle are

(A) $\left(\frac{4}{3}, -\frac{5}{6}\right)$ (B) $\left(\frac{4}{3}, \frac{5}{6}\right)$

(C) $\left(-\frac{4}{3}, \frac{5}{6}\right)$ (D) $\left(-\frac{4}{3}, -\frac{5}{6}\right)$

Solution: (B)

Since the centroid divides the line joining the orthocentre and circumcentre in the ratio 2 : 1 internally, therefore, if the centroid is (x, y), then



$$x = \frac{2 \cdot \frac{3}{2} + 1 \cdot 1}{2 + 1} = \frac{4}{3} \quad \text{and} \quad y = \frac{2 \cdot \frac{3}{4} + 1 \cdot 1}{2 + 1} = \frac{5}{6}.$$

$\therefore$ Coordinates of centroid are $\left(\frac{4}{3}, \frac{5}{6}\right)$.

14. A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line $x = 3$, then the coordinates of the other vertices are

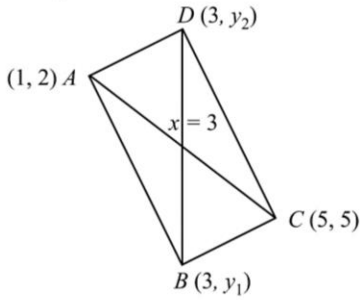
(A) (3, –1), (3, –6) (B) (3, 1), (3, 5)

(C) (3, 2), (3, 6) (D) (3, 1), (3, 6)

Solution: (D)

Let $A \equiv (1, 2)$ and $C \equiv (5, 5)$. Since the vertices B and D lie on the line $x = 3$, therefore, let $B \equiv (3, y_1)$ and $D \equiv (3, y_2)$.

Since AC and BD bisect each other, so they have same middle point



i.e., $\frac{y_1 + y_2}{2} = \frac{2 + 5}{2}$

or $y_1 + y_2 = 7$ (1)

Also, $BD^2 = AC^2$

$\Rightarrow$ $(y_1 - y_2)^2 = (1 - 5)^2 + (2 - 5)^2 = 25$

or $y_1 - y_2 = \pm 5$ (2)

Solving (1) and (2), we get $y_1 = 6, y_2 = 1$

or $\quad y_1 = 1, y_2 = 6$

Thus, the other vertices of the rectangle are (3, 1) and (3, 6).

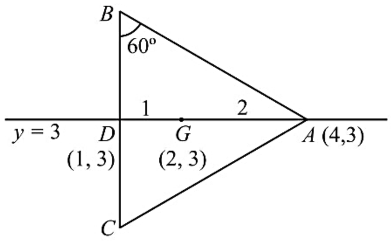
15. If the centroid and a vertex of an equilateral triangle are (2, 3) and (4, 3) respectively, then the other two vertices of the triangle are

(A) $(1, 3\pm\sqrt{3})$ (B) $(2, 3\pm\sqrt{3})$

(C) $(1, 2\pm\sqrt{3})$ (D) $(2, 2\pm\sqrt{3})$

Solution: (A)

G being the centroid, divides AD in the ratio 2 : 1.



Since $AG = 2$, $\therefore$ $GD = 1$,

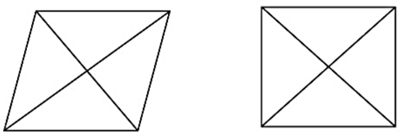
$\therefore$ Coordinates of D, using section formula, are $D(1, 3)$.

Now $AD = 1 + 2 = 3$, $\therefore$ $\tan 60° = \dfrac{3}{BD}$ $\Rightarrow$ $BD = \sqrt{3}$.

$\therefore$ $B \equiv (1, 3+\sqrt{3})$ and $C \equiv (1, 3-\sqrt{3})$.

NOTE

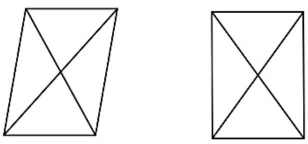
The given figure is a



parallelogram if | rectangle if

Fig. 18.6

parallelogram if	rectangle if
(a) Opposite sides are equal	(a) Opposite sides are equal
(b) Diagonals are unequal	(b) Diagonals are equal
(c) Diagonals bisect each other	(c) Diagonals bisect each other.



rhombus if | square if

Fig. 18.7

rhombus if	square if
(a) All four sides are equal	(a) All four sides are equal
(b) Diagonals are unequal	(b) Diagonals are equal
(c) Diagonals bisect each other at right angles.	(c) Diagonals bisect each other at right angles.

SOLVED EXAMPLE

16. The diagonals of a parallelogram $PQRS$ are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then $PQRS$ must be a

(A) rectangle (B) square

(C) cyclic quadrilateral (D) rhombus

Solution: (D)

Since the product of slopes of the diagonals is –1, therefore, the diagonals are at right angles. Hence, $PQRS$ is a rhombus.

AREA OF A TRIANGLE

The area of ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by:

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

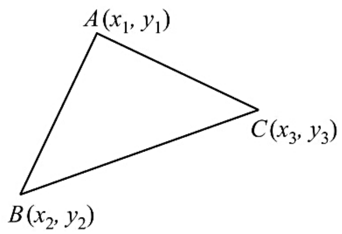


Fig. 18.8

TRICK(S) FOR PROBLEM SOLVING

- Area of a triangle is always taken as positive.
- If area of a triangle is given, then use ± sign.

CONDITION FOR COLLINEARITY OF THREE POINTS

The points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ will be collinear (i.e., will lie on a straight line) if the area of the triangle, assumed to be formed by joining them is zero.

i.e., $\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)] = 0$

or $[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)] = 0$,

which can also be written in the form

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

Sign of an area: An area will be considered to be +ve, if in going round the boundary, it always lies to the left i.e., if the order of description of the boundary curve is anti-clockwise. It will be regarded –ve otherwise.

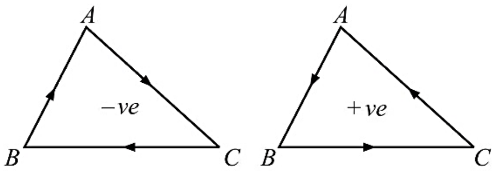


Fig. 18.9

STAIR METHOD FOR FINDING THE AREA

- Write the coordinates of the vertices taken in order in two columns. At the end, repeat the coordinates of the first vertex.
- Mark the arrow-heads as indicated. Each arrow-head shows the product.
- The sign of the product remains the same for downward arrows while it changes for an upward arrow.
- Divide the result by 2.

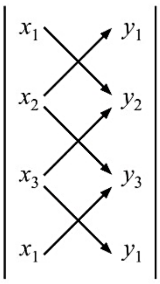


Fig. 18.10

Thus, $\Delta = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$

TRICK(S) FOR PROBLEM SOLVING

- In an equilateral triangle

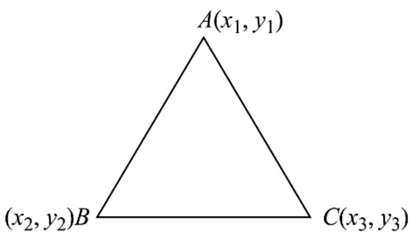


Fig. 18.11

(i) having sides a, area is $\frac{\sqrt{3}}{4}a^2$

(ii) having length of perpendicular as 'p', area is $\frac{p^2}{\sqrt{3}}$.

- If $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the equations of three sides of a triangle, then the area of triangle is given by

$$\Delta = \left| \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} \right|$$

- Area of the rhombus formed by $ax \pm by \pm c = 0$ is $\left|\frac{2c^2}{ab}\right|$.
- Area of the parallelogram formed by the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + d_2 = 0$ is

$$\left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|$$

SOLVED EXAMPLES

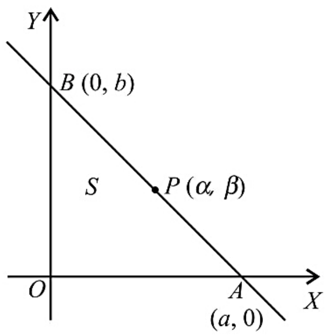
17. Through the point $P(\alpha, \beta)$, where $\alpha\beta > 0$ the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is drawn so as to form with coordinate axes a triangle of area S. If $ab > 0$, then the least value of S is

(A) $\alpha\beta$ (B) $2\alpha\beta$
(C) $4\alpha\beta$ (D) none of these

Solution: (B)

The equation of the given line is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (1)$$



This line cuts x-axis and y-axis at $A(a, 0)$ and $B(0, b)$ respectively.

Since area of $\Delta OAB = S$ (Given)

$\therefore \left|\frac{1}{2}ab\right| = S$ or $ab = 2S$ ($\because ab > 0$) (2)

Since the line (1) passes through the point $P(\alpha, \beta)$

$\therefore \quad \frac{\alpha}{a}+\frac{\beta}{b}=1 \quad$ or $\quad \frac{\alpha}{a}+\frac{a\beta}{2S}=1 \qquad$ [Using (2)]

or $\qquad a^2\beta - 2aS + 2\alpha S = 0.$

Since a is real, $\therefore\ 4S^2 - 8\alpha\beta S \ge 0$

or $4S^2 \ge 8\alpha\beta S$ or $S \ge 2\alpha\beta \left(\because S = \frac{1}{2}ab > 0 \text{ as } ab > 0\right)$

Hence the least value of $S = 2\alpha\beta$.

18. $P(3, 1)$, $Q(6, 5)$ and $R(x, y)$ are three points such that the angle RPQ is a right angle and the area of ΔRPQ = 7, then the number of such points R is

(A) 0 (B) 1
(C) 2 (D) 4

Solution: (C)

Since the angle RPQ is a right angle,

$\therefore$ slope of RP × slope of $PQ = -1$

$$\Rightarrow \frac{1-y}{3-x}\times\frac{5-1}{6-3} = -1 \Rightarrow 3x+4y=13 \qquad (1)$$

Also, area of $\Delta RPQ = 7$

$$\Rightarrow \frac{1}{2}\left|\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 6 & 5 & 1 \end{vmatrix}\right| = 7$$

$$\Rightarrow \frac{1}{2}[x\ (1-5) - y(3-6) + 1(15-6)] = \pm 7$$

$\Rightarrow\ -4x + 3y + 9 = \pm 14 \quad \Rightarrow \quad -4x + 3y = 5 \qquad (2)$

and $\qquad -4x + 3y = -23 \qquad (3)$

Solving Eq. (1) and (2) and (1) and (3), we get two different coordinates of the point R. So, there are two such points R.

19. If two vertices of an equilateral triangle have integral coordinates then the third vertex will have

(A) coordinates which are irrational
(B) atleast one coordinate which is irrational
(C) coordinates which are rational
(D) coordinates which are integers

Solution: (B)

Let the vertices of the equilateral triangle be (x_1, y_1), (x_2, y_2) and (x_3, y_3). If none of x_i and y_i $(i = 1, 2, 3)$ are irrational, then

$$\text{area of } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \text{retional}.$$

But the area of an equilateral triangle

$$= \frac{\sqrt{3}}{4}(\text{side})^2 = \text{irrational}$$

Thus, the two statements are contradictory. Therefore, both the coordinates of the third vertex cannot be rational.

20. Let $P(2, -4)$ and $Q(3, 1)$ be two given points. Let $R\ (x, y)$ be a point such that $(x-2)\ (x-3) + (y-1)\ (y+4) = 0$. If area of ΔPQR is $\frac{13}{2}$, then the number of possible positions of R are

(A) 2 (B) 3
(C) 4 (D) none of these

Solution: (A)

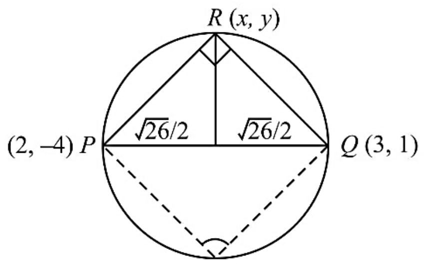
We have

$$(x-2)\ (x-3) + (y-1)\ (y+4) = 0$$

$$\Rightarrow \left(\frac{y+4}{x-2}\right)\times\left(\frac{y-1}{x-3}\right) = -1$$

$$\Rightarrow \quad RP \perp RQ \text{ or } \angle PRQ = \frac{\pi}{2}$$

$\therefore$ The point R lies on the circle whose diameter is PQ.



Now, area of $\Delta PQR = \frac{13}{2}$

$$\Rightarrow \frac{1}{2}\times\sqrt{26}\times(\text{altitude}) = \frac{13}{2}$$

$$\Rightarrow \text{altitude} = \frac{\sqrt{26}}{2} = \text{radius}$$

$\Rightarrow$ there are two possible positions of R.

21. The base of a triangle lies along the line $x = a$ and is of length a. The area of the triangle is a^2, if the vertex lies on the line

(A) $x = 0$ (B) $x = -a$
(C) $x = 3a$ (D) $x = -3a$

Solution: (B, C)

Let h be the height of the triangle.

Since, the area of the triangle is a^2

$$\therefore \frac{1}{2}\times a\times h = a^2 \quad \Rightarrow \quad h = 2a$$

Since the base lies along the line $x = a$, the vertex lies on the line parallel to the base at a distance $2a$ from it. So, the required lines are

$$x = a \pm 2a \text{ i.e., } x = -a \quad \text{or} \quad x = 3a$$

22. If a, c, b are three terms of a G.P., then the line $ax + by + c = 0$

(A) has a fixed direction
(B) always passes through a fixed point

(C) forms a triangle with the axes whose area is constant

(D) always cuts intercepts on the axes such that their sum is zero.

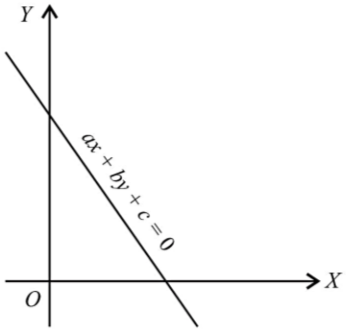
Solution: (C)

Since a, c, b are in G.P.,

$\therefore \qquad c^2 = ab \qquad (1)$

The area of the triangle

$$= \frac{1}{2} \times \left(-\frac{c}{a}\right) \times \left(-\frac{c}{b}\right)$$



$$= \frac{1}{2} \times \frac{c^2}{ab} = \frac{1}{2} \qquad \text{[Using (1)]}$$

= constant

23. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(A) lie on a straight line

(B) lie on an ellipse

(C) lie on a circle

(D) are vertices of a triangle

Solution: (A)

Let $\frac{x_2}{x_1} = \frac{x_3}{x_2} = r$ and $\frac{y_2}{y_1} = \frac{y_3}{y_2} = r$

$\Rightarrow \quad x_2 = x_1 r,\ x_3 = x_1 r^2,\ y_2 = y_1 r$ and $y_3 = y_1 r^2$.

We have,

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 r & y_1 r & 1 \\ x_1 r^2 & y_1 r^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ 0 & 0 & 1-r \\ 0 & 0 & 1-r \end{vmatrix}$$

(Applying $R_3 \to R_3 - rR_2$ and $R_2 \to R_2 - rR_1$) $= 0$

($\because$ R_2 and R_3 are identical)

Thus, (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a straight line.

24. Two vertices of a triangle are (2, –1) and (3, 2) and third vertex lies on the line $x + y = 5$. If the area of the triangle is 4 units then third vertex is

(A) (0, 5) or (4, 1) (B) (5, 0) or (1, 4)

(C) (5, 0) or (4, 1) (D) (0, 5) or (1, 4)

Solution: (B)

Let $A \equiv (2, -1)$ and $B \equiv (3, 2)$.

Let the third vertex be $C(\alpha, \beta)$.

Then, $\qquad \alpha + \beta = 5$ (given) $\qquad (1)$

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ \alpha & \beta & 1 \end{vmatrix} = \pm 4 \qquad \text{(given)}$$

$\Rightarrow \qquad \beta - 3\alpha = 1 \qquad (2)$

or $\qquad \beta - 3\alpha = -15 \qquad (3)$

Solving (1) and (2), we get, $\alpha = 1, \beta = 4$

Solving (1) and (3), we get, $\alpha = 5, \beta = 0$

Thus, the third vertex is either (5, 0) or (1, 4).

AREA OF A QUADRILATERAL

The area of a quadrilateral, whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$, is

$$= \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}\right]$$

NOTE

The rule for writing down the area of a quadrilateral is the same as that of a triangle.

Thus, area of quadrilateral with vertices (x_r, y_r), $r = 1, 2, 3, 4$ is

$$\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)]$$

AREA OF A POLYGON

- The area of a polygon of n sides with vertices $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, ..., $A_n(x_n, y_n)$ is

$$= \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right]$$

STAIR METHOD

Repeat first coordinates one time in last. For down arrow use positive sign and for up arrow use negative sign.

$$\therefore \text{ Area of polygon} = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1\\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1\end{vmatrix}\right|$$

$$= \frac{1}{2}\{[(x_1y_2 + x_2y_3 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_1)]\}$$

SOLVED EXAMPLES

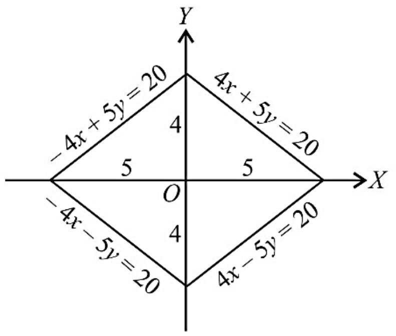
25. The area of the region enclosed by $4|x| + 5|y| \le 20$ is

(A) 10 (B) 20
(C) 40 (D) none of these

Solution: (C)

The four lines enclosing the given region are $4x + 5y = 20$, $4x - 5y = 20$, $-4x + 5y = 20$ and $-4x - 5y = 20$.

Clearly, the four lines form a rhombus having diagonals of length 10 and 8.



$\therefore$ Required area $= \frac{1}{2} \times 10 \times 8 = 40.$

26. Let $S_1, S_2, \ldots$ be squares such that for each $n \ge 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 cm^2?

(A) 7 (B) 8
(C) 9 (D) 10

Solution: (B, C, D)

Let a be the side of the square, then diagonal $d = a\sqrt{2}$.

Given : $a_n = \sqrt{2}a_{n+1}$

$$\Rightarrow a_{n+1} = \frac{a_n}{\sqrt{2}} = \frac{a_{n-1}}{(\sqrt{2})^2} = \frac{a_{n-2}}{(\sqrt{2})^3} = \cdots = \frac{a_1}{(\sqrt{2})^n}$$

$$\therefore a_{n+1} = \frac{a_1}{(\sqrt{2})^n} \Rightarrow a_n = \frac{a_1}{(\sqrt{2})^{n-1}} = \frac{10}{2^{\frac{n-1}{2}}}$$

Now, area of $S_n < 1 \Rightarrow a_n^2 < 1 \Rightarrow \frac{100}{2^{n-1}} < 1$

$\Rightarrow 2^n > 200 > 2^7 \Rightarrow n > 7$

$\therefore n = 8, 9, 10$

LOCUS

The locus of a moving point is the path traced by it under certain geometrical condition or conditions.

For example, if a point moves in a plane under the geometrical condition that its distance from a fixed point O in the plane is always equal to a constant quantity a, then the curve traced by the moving point will be a circle with centre O and radius a. Thus, locus of the point is a circle with centre O and radius a.

WORKING RULE TO FIND THE LOCUS OF A POINT

- Let the coordinates of the moving point P be (h, k).
- Using the given geometrical conditions, find the relation between h and k. This relation must contain only h, k and known quantities.
- Express the given relation in h and k in the simplest form and then put x for h and y for k. The relation, thus obtained, will be the required equation of the locus of (h, k).

TRANSLATION OF AXES

Sometimes a problem with a given set of axes can be solved more easily by translation of axes. The translation of axes involves the shifting of the origin to a new point, the new axes remaining parallel to the original axes.

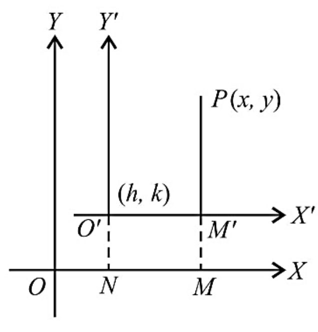


Fig. 18.12

Let OX, OY be the original axes and O' be the new origin. Let coodinates of O' referred to original axes, i.e., OX, OY be (h, k).

Let $O'X'$ and $O'Y'$ be drawn parallel to and in the same direction as OX and OY respectively. Let P be any point in the plane having coordinates (x, y) referred to old axes and (X, Y) referred to new axes. Then,

$$x = OM = ON + NM = ON + O'M'$$
$$= h + X = X + h \quad \text{or} \quad X = x - h$$

and
$$y = MP = MM' + M'P = NO' + M'P$$
$$= k + Y = Y + k. \quad \text{or} \quad Y = y - k$$

Thus, the point whose coordinates were (x, y) has now the coordinates $(x - h, y - k)$.

TRICK(S) FOR PROBLEM SOLVING

If origin is shifted to point (h, k) without rotation of axes, then new equation of curve can be obtained by putting $x + h$ in place of x *and* $y + k$ in place of y

ROTATION OF AXES

Rotation of Axes without Changing the Origin

Let OX, OY be the original axes and OX', OY' be the new axes obtained by rotating OX and OY through an angle θ in the anticlockwise sense. Let P be any point in the plane having coordinates (x, y) with respect to axes OX and OY and (x', y') with respect to axes OX' and OY'. Then,

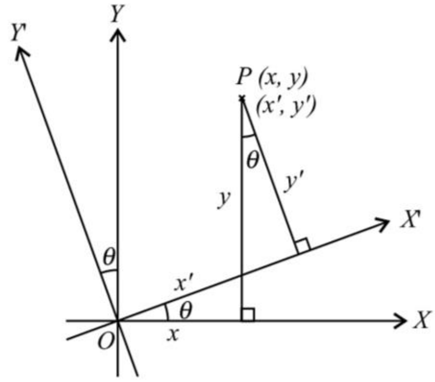


Fig. 18.13

$$x = x'\cos\theta - y'\sin\theta; \; y = x'\sin\theta + y'\cos\theta$$

and
$$x' = x\cos\theta + y\sin\theta; \quad y' = -x\sin\theta + y\cos\theta$$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

	x↓	y↓
x'→	$\cos\theta$	$\sin\theta$
y'→	$-\sin\theta$	$\cos\theta$

Change of Origin and Rotation of Axes If origin is changed to $O'(h, k)$ and axes are rotated about the new origin O' by angle θ in the anticlockwise sense such that the new co-ordinates of $P(x, y)$ become (x', y'), then the equations of transformation will be

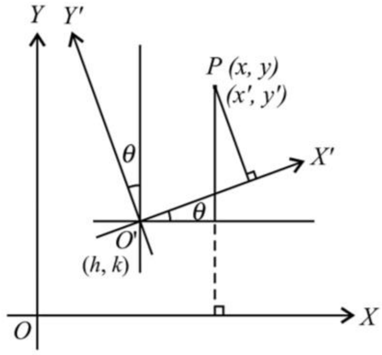


Fig. 18.14

$$x = h + x'\cos\theta - y'\sin\theta$$

and
$$y = \beta + x'\sin\theta + y'\cos\theta$$

REFLECTION (IMAGE) OF A POINT

Let (x, y) be any point, then its image with respect to

(i) x axis is $(x, -y)$ (ii) y-axis is $(-x, y)$
(iii) origin is $(-x, -y)$ (iv) line $y = x$ is (y, x)

TRICK(S) FOR PROBLEM SOLVING

- If area is a rational number. Then the triangle cannot be equilateral.
- If two opposite vertices of a rectangle are (x_1, y_1) and (x_2, y_2), then its area is $|(y_2 - y_1)(x_2 - x_1)|$.
- If two opposite vertices of a square are $A(x_1, y_1)$ and $C(x_2, y_2)$, then its area is

$$= \frac{1}{2}AC^2 = \frac{1}{2}[(x_2 - x_1)^2 + (y_2 - y_1)^2]$$

SOLVED EXAMPLES

27. The image of the point $(3, -8)$ under the transformation $(x, y) \to (2x + y, 3x - y)$ is

(A) $(-2, 17)$ (B) $(2, 17)$
(C) $(-2, -17)$ (D) $(2, -17)$

Solution: (A)

Let (x_1, y_1) be the image of the point (x, y) under the given transformation.

Then, $x_1 = 2x + y = 2(3) - 8 = -2$

and $y_1 = 3x - y = 3(3) - (-8) = 17$

Hence, the image is $(-2, 17)$.

28. Without changing the direction of coordinates axes, origin is transferred to (α, β) so that the linear terms in the equation $x^2 + y^2 + 2x - 4y + 6 = 0$ are eliminated. The point (α, β) is

(A) (–1, 2) (B) (1, –2)
(C) (1, 2) (D) (–1, –2)

Solution: (A)

The given equation is

$$x^2 + y^2 + 2x - 4y + 6 = 0 \quad (1)$$

Putting $x = x' + \alpha$, $y = y' + \beta$ in (1), we get

$x'^2 + y'^2 + x'(2\alpha + 2) + y'(2\beta - 4) + (\alpha^2 + \beta^2 + 2\alpha - 4\beta + 6) = 0$

To eliminate linear terms, we should have $2\alpha + 2 = 0$ and $2\beta - 4 = 0$

$\Rightarrow \quad \alpha = -1$ and $\beta = 2$

$\therefore \quad (\alpha, \beta) \equiv (-1, 2)$

29. Let P be the image of the point (–3, 2) with respect to x-axis. Keeping the origin as same, the coordinate axes are rotated through an angle 60° in the clockwise sense. The coordinates of point P with respect to the new axes are

(A) $\left(\frac{2\sqrt{3}-3}{2}, -\frac{(3\sqrt{3}+2)}{2}\right)$

(B) $\left(\frac{2\sqrt{3}-3}{2}, \frac{3\sqrt{3}+2}{2}\right)$

(C) $\left(-\frac{(2\sqrt{3}-3)}{2}, \frac{3\sqrt{3}+2}{2}\right)$

(D) none of these

Solution: (A)

Since P is the image of the point (–3, 2) with respect to x-axis, therefore, the coordinates of P are (–3, –2).

Let (x', y') be the coordinates of P with respect to new axes. Then,

$x' = x \cos \alpha + y \sin \alpha = -3 \cos(-60°) - 2 \sin(-60°)$

$= -\frac{3}{2} + \sqrt{3} = \frac{2\sqrt{3}-3}{2}.$

$y' = -x \sin \alpha + y \cos \alpha = 3 \sin(-60°) - 2 \cos(-60°)$

$= -\left(\frac{3\sqrt{3}+2}{2}\right)$

$\therefore$ Coordinates of P are $\left[\frac{2\sqrt{3}-3}{2}, -\left(\frac{3\sqrt{3}+2}{2}\right)\right]$.

30. A ray of light travelling along the line $x + \sqrt{3}y = 5$ is incident on the x-axis and after refraction it enters the other side of the x-axis by turning $\frac{\pi}{6}$ away from the x-axis. The equation of the line along which the refracted ray travels is

(A) $x + \sqrt{3}y - 5\sqrt{3} = 0$

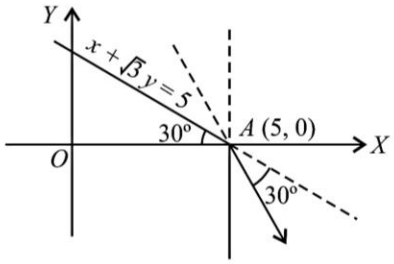
(B) $x - \sqrt{3}y - 5\sqrt{3} = 0$

(C) $\sqrt{3}x + y - 5\sqrt{3} = 0$

(D) $\sqrt{3}x - y - 5\sqrt{3} = 0$

Solution: (C)

The refracted ray passes through the point (5, 0) and makes an angle 120° with positive direction of x-axis



$\therefore$ The equation of the refracted ray is $(y - 0) = \tan 120° (x - 5)$

$\Rightarrow \quad y = -\sqrt{3}(x - 5)$ or $\sqrt{3}x + y - 5\sqrt{3} = 0$

31. A line L has intercepts a and b on the coordinate axes. When the axes are rotated through an angle, keeping the origin fixed, the same line L has intercepts p and q. Then,

(A) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$

(B) $\frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2} - \frac{1}{q^2}$

(C) $\frac{1}{a^2} + \frac{1}{b^2} = 2\left(\frac{1}{p^2} + \frac{1}{q^2}\right)$

(D) none of these

Solution: (A)

Since the line L has intercepts a and b on the coordinate axes, therefore its equation is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (1)$$

When the axes are rotated, its equation with respect to the new axes and same origin will become

$$\frac{x}{p} + \frac{y}{q} = 1 \quad (2)$$

In both the cases, the length of the perpendicular from the origin to the line will be same.

$$\therefore \quad \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2}}}$$

or $$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$$

which is the required relation.

GENERAL EQUATION OF A STRAIGHT LINE

An equation of the form: $ax + by + c = 0$, where a, b, c are any real numbers not all zero, always represents a straight line.

Equation of a straight line is always of first degree in x and y.

SLOPE OF A LINE

If a line makes an angle $\theta\left(\theta \neq \frac{\pi}{2}\right)$ with the positive direction of x-axis, the slope or gradient of that line is usually denoted by m, i.e., $\tan\theta = m$.

TRICK(S) FOR PROBLEM SOLVING

- The slope of a line parallel to x-axis = 0 and perpendicular to x-axis is undefined.
- If three points A, B, C are collinear, then slope of AB = slope of BC = slope of AC.
- If a line is equally inclined to the axes, then it will make an angle of 45° or 135° with x-axis (i.e., positive direction of x-axis) and hence its slope will be tan 45° or tan 135° = ± 1.
- Slope of the line joining two points (x_1, y_1) and (x_2, y_2) is given as

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}.$$

- Slope of the line $ax + by + c = 0$, $b \neq 0$ is $-\frac{a}{b}$, i.e.,

$$\frac{(\text{Coefficient of } x)}{(\text{Coefficient of } y)}.$$

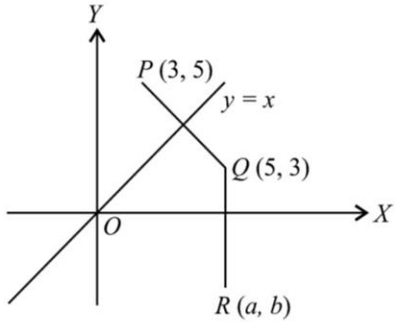
SOLVED EXAMPLES

32. The image of the point $P(3, 5)$ with respect to the line $y = x$ is the point Q and the image of Q with respect to the line $y = 0$ is the point $R(a, b)$, then (a, b)

(A) (5, 3)
(B) (5, –3)
(C) (–5, 3)
(D) (–5, –3)

Solution: (B)

Let (x_1, y_1) be the image of the point $P(3, 5)$ with respect to the line $y = x$. Then, $x_1 = 5$, $y_1 = 3$.



∴ $Q = (5, 3)$

Since the image of the point $Q(5, 3)$ w.r.t. the line $y = 0$ is (a, b).

∴ $a = 5$ and $b = -3$

∴ $(a, b) = (5, -3)$

INTERCEPT OF A LINE ON THE AXES

1. **Intercept of a line on x-axis** If a line cuts x-axis at a point $(a, 0)$, then a is called the intercept of the line on x-axis. $|a|$ is called the length of the intercept of the line on x-axis. Intercept of a line on x-axis may be positive or negative.
2. **Intercept of a line on y-axis** If a line cuts y-axis at a point $(0, b)$, then b is called the intercept of the line on y-axis and $|b|$ is called the length of the intercept of the line on y-axis. Intercept of a line on y-axis may be positive or negative.

Equations of Lines Parallel to Axes

Equation of x-axis The equation of x-axis is $y = 0$.

Equation of y-axis The equation of y-axis is $x = 0$.

Equation of a line parallel to y-axis The equation of the straight line parallel to y-axis at a distance 'a' from it on the positive side of x-axis is $x = a$.

If a line is parallel to y-axis, at a distance a from it and is on the negative side of x-axis, then its equation is $x = -a$.

Equation of a line parallel to x-axis The equation of the straight line parallel to x-axis at a distance b from it on the positive side of y-axis is $y = b$.

If a line is parallel to x-axis, at a distance b from it and is on the negative side of y-axis, then its equation is $y = -b$.

SOLVED EXAMPLES

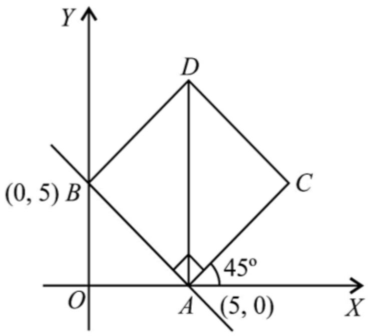
33. A square is constructed on the portion of the line $x + y = 5$ which is intercepted between the axes, on the side of the line away from origin. The equations to the diagonals of the square are

(A) $x = 5$, $y = -5$
(B) $x = 5$, $y = 5$

(C) $x = -5, y = 5$
(D) $x - y = 5, x - y = -5$

Solution: (B)

Clearly, the equations of the two diagonals are $x = 5$ and $y = 5$.



34. If a straight line cuts intercepts from the axes of coordinates the sum of the reciprocals of which is a constant k, then the line passes through the fixed point

(A) (k, k) (B) $\left(\frac{1}{k}, \frac{1}{k}\right)$

(C) $(k, -k)$ (D) $(-k, k)$

Solution: (B)

Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (1)$$

Its intercepts on x-axis and y-axis are a and b respectively.

Given: $\frac{1}{a} + \frac{1}{b} = k$

$$\Rightarrow \frac{1}{ak} + \frac{1}{bk} = 1 \quad \text{or} \quad \frac{1/k}{a} + \frac{1/k}{b} = 1 \quad (2)$$

From (2) it follows that the line (1) passes through the fixed point $\left(\frac{1}{k}, \frac{1}{k}\right)$.

EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS

Slope-Intercept Form

The equation of a straight line whose slope is m and which cuts an intercept c on the y-axis is given by

$$y = mx + c.$$

If the line passes through the origin, then $c = 0$ and hence the equation of the line will become $y = mx$.

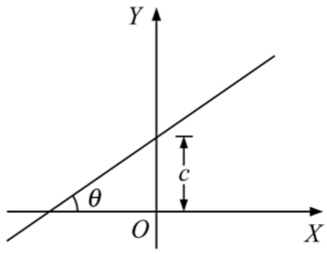


Fig. 18.15

Point-Slope Form

The equation of a straight line passing through the point (x_1, y_1) and having slope m is given by

$$(y - y_1) = m(x - x_1).$$

Two-Point Form

The equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

SOLVED EXAMPLES

35. In the above problem, coordinates of the point P such that $|PA - PB|$ is minimum are

(A) $\left(-\frac{9}{20}, -\frac{12}{5}\right)$ (B) $\left(\frac{3}{20}, \frac{12}{5}\right)$

(C) $\left(\frac{9}{20}, -\frac{12}{5}\right)$ (D) $\left(-\frac{9}{20}, \frac{12}{5}\right)$

Solution: (A)

The minimum value of $|PA - PB|$ is zero which is attained if $PA = PB$, i.e., P must lie on the perpendicular bisector of AB.

The equation of perpendicular bisector of AB is

$$y - \frac{1}{2} = 2(x - 1) \quad \text{or} \quad y = 2x - \frac{3}{2}$$

Solving it with the given line, we get $P \equiv \left(-\frac{9}{20}, -\frac{12}{5}\right)$

36. Given the system of straight line $a(2x + y - 3) + b(3x + 2y - 5) = 0$, the line of the system farthest form the point $(4, -3)$ has the equation

(A) $3x - 4y + 1 = 0$ (B) $4x + 3y - 5 = 0$
(C) $7x - y + 4 = 0$ (D) none of these

Solution: (A)

The given system of lines pass through (1, 1)

So, the required line is the through (1, 1) and perpendicular to the join of (1, 1) and (4, –3).

$\therefore$ The equation of line is $\frac{y-1}{x-1} = \frac{3}{4}$, i.e., $3x - 4y + 1 = 0$

37. The image of the point (–8, 12) with respect to the line mirror $4x + 7y + 13 = 0$ is

(A) (16, –2) (B) (–16, 2)
(C) (16, 2) (D) (–16, –2)

Solution: (D)

Equation of the given line is

$$4x + 7y + 13 = 0 \quad (1)$$

Let $Q(\alpha, \beta)$ be the image of the point $P(-8, 12)$ w.r.t. line (1).

Then, $PQ \perp$ line (1) and $PC = CQ$.

Equation of the line PC is

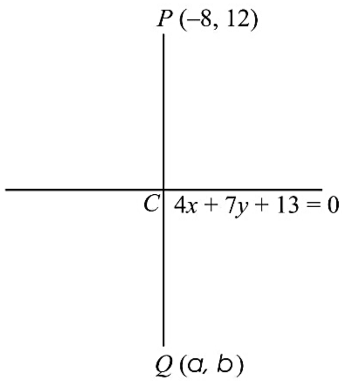
$$(y - 12) = \frac{7}{4}(x + 8)$$

[PC is $\perp$ to the line (1) and passes through (–8, 12)]

or $$7x - 4y + 104 = 0 \quad (2)$$

Solving Eq. (1) and (2), we get

$$x = -12 \text{ and } y = 5. \quad \therefore C \equiv (-12, 5)$$



Since C is mid-point of PQ,

$\therefore \quad -12 = \frac{\alpha - 8}{2}$ and $5 = \frac{\beta + 12}{2}$

$\Rightarrow \quad \alpha = -16$ and $\beta = -2$

$\therefore \quad Q \equiv (-16, -2)$.

Intercept Form

The equation of a straight line which cuts off intercepts a and b on x-axis and y-axis respectively is given by

$$\frac{x}{a} + \frac{y}{b} = 1$$

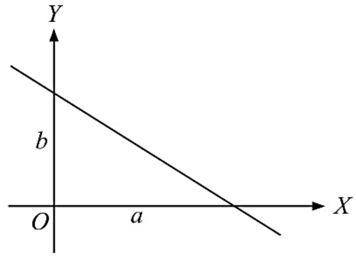


Fig. 18.16

SOLVED EXAMPLES

38. Through the point (1, 1), a straight line is drawn so as to form with coordinate axes a triangle of area S. The intercepts made by the line on the coordinate axes are the roots of the equation

(A) $x^2 - |S|\,x + 2\,|S| = 0$
(B) $x^2 + |S|\,x + 2\,|S| = 0$
(C) $x^2 - 2\,|S|\,x + 2\,|S| = 0$
(D) none of these

Solution: (C)

If a, b are the intercepts made by the line, then the equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$.

Since it passes through (1, 1), $\therefore \quad \frac{1}{a} + \frac{1}{b} = 1$

$$\Rightarrow \quad \frac{a+b}{ab} = 1 \quad (1)$$

Also, area of the triangle made by the straight line on the coordinate axes is S

$$\therefore \quad \frac{1}{2}ab = |S| \text{ i.e., } ab = 2\,|S| \quad (2)$$

So, by (1), $\quad a + b = 2\,|S| \quad (3)$

From (2) and (3), the intercepts a and b are the roots of the equation $x^2 - 2\,|S|\,x + 2\,|S| = 0$.

39. A line passing through the point $P(4, 2)$, meets the x-axis and y-axis at A and B respectively. If O is the origin, then locus of the centre of the circum circle of ΔOAB is

(A) $x^{-1} + y^{-1} = 2$ (B) $2x^{-1} + y^{-1} = 1$
(C) $x^{-1} + 2y^{-1} = 1$ (D) $2x^{-1} + 2y^{-1} = 1$

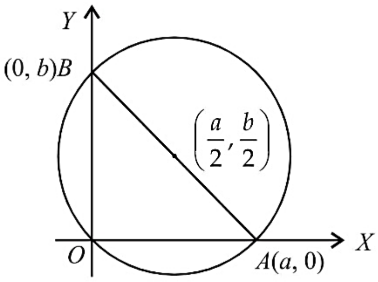
Solution: (B)

Let the coordinates of A and B be $(a, 0)$ and $(0, b)$ respectively.

Then, equation of line AB is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Since, it passes through the point $P(4, 2)$



$$\therefore \quad \frac{4}{a}+\frac{2}{b}=1. \qquad (1)$$

Now, centre of the circumcircle of $\Delta OAB = \left(\frac{a}{2}, \frac{b}{2}\right)$.

So, Eq. (1) can be written in the form $\frac{2}{a/2}+\frac{1}{b/2}=1$

$\therefore$ locus of circumcentre is $\frac{2}{x}+\frac{1}{y}=1$ or $2x^{-1}+y^{-1}=1$

40. If the equal sides AB and AC (each equal to a) of a right angled isosceles triangle ABC be produced to P and Q so that $BP \times CQ = AB^2$, then the line PQ always passes through the fixed point

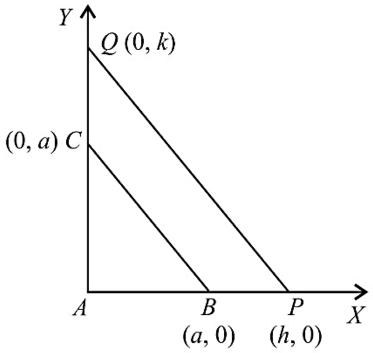
(A) $(a, 0)$ (B) $(0, a)$
(C) (a, a) (D) none of these

Solution: (C)

We take A as the origin and AB and AC as x-axis and y-axis respectively.

Let $AP = h, AQ = k.$

Equation of the line PQ is



$$\frac{x}{h}+\frac{y}{k}=1 \qquad (1)$$

Given, $BP \times CQ = AB^2$

$\Rightarrow \quad (h-a)(k-a)=a^2$

$\Rightarrow \quad hk - ak - ah + a^2 = a^2 \quad$ or $\quad ak + ha = hk$

or $$\frac{a}{h}+\frac{a}{k}=1 \qquad (2)$$

From (2), it follows that line (1) i.e., PQ passes through the fixed point (a, a).

Normal Form (or Perpendicular Form)

The equation of a straight line upon which the length of the perpendicular from the origin is p and the perpendicular makes an angle α with the positive direction of x-axis is given by

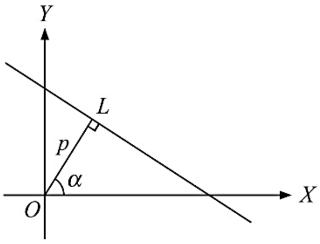


Fig. 18.17

$$x \cos \alpha + y \sin \alpha = p.$$

In normal form of equation of a straight line p is always taken as positive and α is measured from positive direction of x-axis in anti-clockwise direction between 0 and 2π.

Parametric Form (or Symmetric Form)

The equation of a straight line passing through the point (x_1, y_1) and making an angle θ with the positive direction of x-axis is given by

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

where r is the distance of the point (x, y) from the point (x_1, y_1).

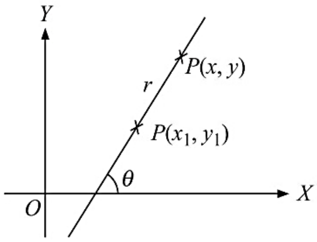


Fig. 18.18

TRICK(S) FOR PROBLEM SOLVING

The coordinates (x, y) of any point P on the line at a distance r from the point $A(x_1, y_1)$ can be taken as

$$(x_1 + r\cos\theta, y_1 + r\sin\theta)$$

or $$(x_1 - r\cos\theta, y_1 - r\sin\theta)$$

where the line is inclined at an angle θ with x-axis

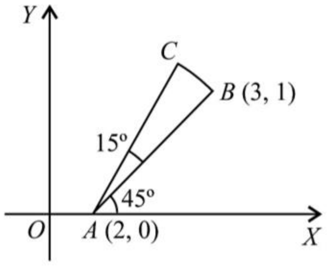
SOLVED EXAMPLES

41. A line joining two points $A(2, 0)$ and $B(3, 1)$ is rotated about A in anticlockwise direction through an angle 15°. If B goes to C in the new position, then the coordinates of C are

(A) $\left(2, \sqrt{\frac{3}{2}}\right)$ (B) $\left(2, -\sqrt{\frac{3}{2}}\right)$

(C) $\left(2+\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$ (D) none of these

Solution: (C)



Slope of line

$$AB = \frac{0-1}{2-3} = 1 = \tan 45°$$

$\therefore\ \angle BAX = 45°$.

Given $\angle CAB = 15°$.

$\therefore\ \angle CAX = 60°$.

$\therefore$ Slope of line $AC = \tan 60° = \sqrt{3}$.

Now, line AC makes an angle of 60° with positive direction of x-axis and

$$AC = AB = \sqrt{(3-2)^2+(1-0)^2} = \sqrt{2}$$

$\therefore$ Coordinates of C are $(2+\sqrt{2}\cos 60°, 0+\sqrt{2}\sin 60°)$

i.e., $\left(2+\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$

42. P is a point on either of the two lines $y-\sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. The coordinates of the foot of the perpendicular from P on the bisector of the angle between them are

(A) $\left[0, \frac{1}{2}(4+5\sqrt{3})\right]$ or $\left[0, \frac{1}{2}(4-5\sqrt{3})\right]$ depending on which line the point P is taken

(B) $\left[0, \frac{1}{2}(4+5\sqrt{3})\right]$

(C) $\left[0, \frac{1}{2}(4-5\sqrt{3})\right]$

(D) $\left[\frac{5}{2}, \frac{5\sqrt{3}}{2}\right]$

Solution: (B)

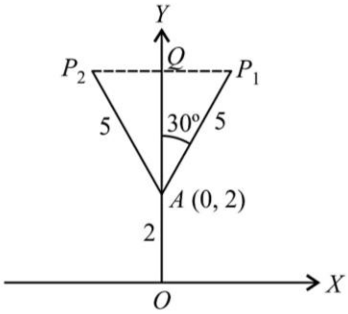
Equation, of two lines are

$$y = \sqrt{3}x+2, \quad \text{if} \quad x \geq 0$$

and

$$y = -\sqrt{3}x+2, \quad \text{if} \quad x \geq 0$$

Clearly $y \geq 2$.



Also, y-axis is the bisector of the angle between the two lines. P_1, P_2 are two points on these lines, at a distance 5 units from A. Q is the foot of the ⊥ from P_1 and P_2 on the bisector (y-axis).

Then, the coordinates of Q are (0, 2 + 5 cos 30º)

$$= \left(0, 2+\frac{5\sqrt{3}}{2}\right) = \left(0, \frac{1}{2}(4+5\sqrt{3})\right)$$

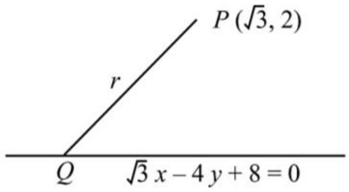
43. If the straight line drawn through the point $P(\sqrt{3}, 2)$ and making an angle $\frac{\pi}{6}$ with the x-axis meets the line $\sqrt{3}x-4y+8=0$ at Q, then the length of PQ is

(a) 4 (b) 5

(c) 6 (d) none of these

Solution: (C)

The given line is



$$\sqrt{3}x-4y+8=0 \qquad (1)$$

Let $PQ = r$

Then, the coordinates of Q are

$$\left(\sqrt{3}+r\cos\frac{\pi}{6}, 2+r\sin\frac{\pi}{6}\right) \text{ or } \left(\sqrt{3}+\frac{\sqrt{3}}{2}r, 2+\frac{r}{2}\right)$$

Since the point Q lies on the given line,

$$\therefore\ \sqrt{3}\left(\sqrt{3}+\frac{\sqrt{3}}{2}r\right)-4\left(2+\frac{r}{2}\right)+8=0$$

$\Rightarrow\ 6+3r-16-4r+16=0$ or $r=6$

Hence, $PQ = 6$

44. A line is drawn from the point $P(\alpha, \beta)$, making an angle θ with the positive direction of x-axis, to meet the line $ax+by+c=0$ at Q. The length of PQ is

(A) $-\dfrac{a\alpha+b\beta+c}{a\cos\theta+b\sin\theta}$ (B) $\left|\dfrac{a\alpha+b\beta+c}{\sqrt{a^2+b^2}}\right|$

(C) $\dfrac{a\alpha+b\beta+c}{a\cos\theta+b\sin\theta}$ (D) none of these

Solution: (A)

Equation of a straight line passing through the point $P(\alpha, \beta)$ and making an angle θ with positive direction of x-axis is

$$\frac{x-\alpha}{\cos\theta}=\frac{y-\beta}{\sin\theta}=r \text{ (say)}$$

Coordinates of any point on this line are

$$(\alpha+r\cos\theta, \beta+r\sin\theta)$$

If it lies on the line $ax + by + c = 0$, then

$$a(\alpha+r\cos\theta)+b(\beta+r\sin\theta)+c=0$$

$$\Rightarrow \quad r=-\frac{a\alpha+b\beta+c}{a\cos\theta+b\sin\theta}$$

Thus, $$PQ=r=-\frac{a\alpha+b\beta+c}{a\cos\theta+b\sin\theta}$$

REDUCTION OF THE GENERAL EQUATION TO DIFFERENT STANDARD FORMS

Slope-Intercept Form: The general form, $Ax + By + C = 0$, of the straight line can be reduced to the form $y = mx + c$ by expressing y as

$$y=-\frac{A}{B}x-\frac{C}{B}=mx+c$$

where $$m=-\frac{A}{B} \text{ and } c=-\frac{C}{B}$$

Thus, slope of the line $Ax + By + C = 0$ is

$$m=\frac{\text{cofficient of } x}{\text{cofficient of } y}=-\frac{A}{B}$$

Intercept Form

The equation $Ax + By + C = 0$ can be reduced to the form $\dfrac{x}{a}+\dfrac{y}{b}=1$ by expressing it as

$$Ax+By=-C$$

or $-\dfrac{A}{C}x-\dfrac{B}{C}y=1$, where $C\neq 0$

or $\dfrac{x}{-\frac{C}{A}}+\dfrac{y}{-\frac{C}{B}}=-1$, which is of the form $\dfrac{x}{a}+\dfrac{y}{b}=1$,

where $a=-\dfrac{C}{A}$ and $b=-\dfrac{C}{B}$ are intercepts on x-axis and y-axis, respectively.

TRICK(S) FOR PROBLEM SOLVING

Intercept of a straight line on x-axis can be found by putting $y = 0$ in the equation of the line and then finding the value of x. Similarly intercept on y-axis can be found by putting $x = 0$ in the equation of the line and then finding the value of y.

Normal Form

To reduce the equation $Ax + By + C = 0$ to the form $x\cos\alpha + y\sin\alpha = p$, first express it as

$$Ax+By=-C \quad (1)$$

CASE 1. If $C < 0$ or $-C > 0$, then divide both sides of Eq. (1) by $\sqrt{A^2+B^2}$, we get

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=-\frac{C}{\sqrt{A^2+B^2}}$$

which is of the form $x\cos\alpha + y\sin\alpha = p$,

where, $\cos\alpha=\dfrac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha=\dfrac{B}{\sqrt{A^2+B^2}}$

and $$p=-\frac{C}{\sqrt{A^2+B^2}}$$

CASE 2. If $C > 0$ or $-C < 0$, then divide Eq. (1) by $-\sqrt{A^2+B^2}$, we get

$$\frac{-A}{\sqrt{A^2+B^2}}x-\frac{B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}}$$

which is of the form $x\cos\alpha + y\sin\alpha = p$,

where $\cos\alpha=-\dfrac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha=-\dfrac{B}{\sqrt{A^2+B^2}}$

and $$p=\frac{C}{\sqrt{A^2+B^2}}$$

ANGLE BETWEEN TWO INTERSECTING LINES

The angle θ between two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is given by

$$\tan\theta=\pm\frac{m_1-m_2}{1+m_1m_2},$$

provided no line is perpendicular to x-axis and the acute angle θ is given by

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

TRICK(S) FOR PROBLEM SOLVING

- If both the lines are perpendicular to x-axis, then the angle between them is 0°.
- If slope of one line is not defined (one of the lines is perpendicular to x-axis and other makes an angle θ with the positive direction of x-axis), then angle between them $= \pi - \theta$.
- The two lines are parallel if and only if $m_1 = m_2$.
- The two lines are perpendicular if and only if $m_1 \times m_2 = -1$.

SOLVED EXAMPLE

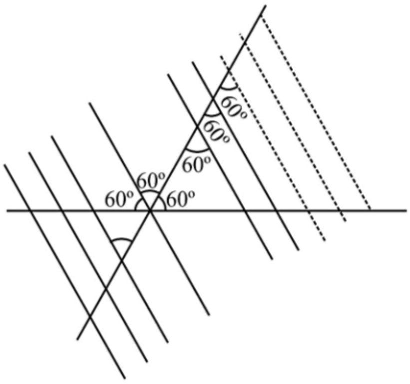
45. Number of equilateral triangles with $y = \sqrt{3}(x-1)+2$ and $y = \sqrt{3x}$ as two of its sides, is

(A) 0 (B) 1
(C) 2 (D) none of these

Solution: (D)

The sides are,

$$y = \sqrt{3}(x-1)+2 \text{ and } y = -\sqrt{3}x$$



The two lines are at an angle of 60° to each other. Now any line parallel to obtuse angle bisector will make equilateral triangle with these lines as its two sides.

CONDITION FOR TWO LINES TO BE COINCIDENT, PARALLEL, PERPENDICULAR OR INTERSECTING

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

1. **Coincident,** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$;
2. **Parallel,** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$;
3. **Perpendicular,** if $a_1a_2 + b_1b_2 = 0$;
4. **Intersecting,** if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ i.e., if they are neither coincident nor parallel.

EQUATION OF A LINE PARALLEL TO A GIVEN LINE

The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + k = 0$, where k is a constant.

WORKING RULE

- Keep the terms containing x and y unaltered.
- Change the constant.
- The constant k is determined from an additional condition given in the problem.

  Thus, the equation of any line parallel to $2x - 3y + 5 = 0$ is $2x - 3y + k = 0$.

SOLVED EXAMPLES

46. The vertices of a ΔOBC are O (0, 0), $B(-3, -1)$ and $C(-1, -3)$. The equation of a line parallel to BC and intersecting sides OB and OC whose distance from the origin is $\frac{1}{2}$, is

(A) $x + y + \frac{1}{\sqrt{2}} = 0$ (B) $x + y - \frac{1}{\sqrt{2}} = 0$
(C) $x + y - \frac{1}{2} = 0$ (D) $x + y + \frac{1}{2} = 0$

Solution: (A)

The equation of line BC is

$$x + y + 4 = 0$$

$\therefore$ Equation of a line parallel to BC is

$$x + y + k = 0$$

This is at a distance $\frac{1}{2}$ from the origin.

$$\therefore \quad \left|\frac{k}{\sqrt{2}}\right| = \frac{1}{2}$$

$$\therefore \quad k = \pm\frac{1}{\sqrt{2}}$$

Since BC and the required line are on the same side of the origin

$$\therefore \quad k = \frac{1}{\sqrt{2}}$$

Hence, the required line is $x + y + \frac{1}{\sqrt{2}} = 0$.

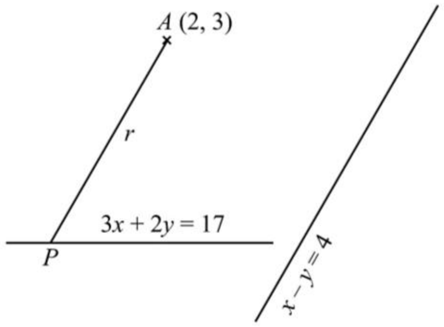
47. The distance of the point (2, 3) from the line $3x + 2y = 17$, measured parallel to the line $x - y = 4$ is

(A) $4\sqrt{2}$ (B) $5\sqrt{2}$
(C) $\sqrt{2}$ (D) none of these

Solution: (C)

Coordinates of any point on the line through (2, 3) and parallel to the line $x - y = 4$, at a distance r, are

$(2 + r\cos\theta, 3 + r\sin\theta)$, where $\tan\theta = 1$.



If this point lies on the line $3x + 2y = 17$,
then $3(2 + r\cos\theta) + 2(3 + r\sin\theta) = 17$

$$\Rightarrow \quad 6+3r\cdot\frac{1}{\sqrt{2}}+6+2r\cdot\frac{1}{\sqrt{2}}=17 \quad \Rightarrow \quad \frac{5}{\sqrt{2}}r=5$$

$$\Rightarrow \quad r=\sqrt{2}$$

EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE

The equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + k = 0$, where k is a constant.

WORKING RULE

- Interchange the coefficients of x and y and change the sign of one of them.
- Change the constant.
- The value of k can be determined from an additional condition given in the problem.

POINT OF INTERSECTION OF TWO GIVEN LINES

Let the two given lines be

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0.$$

Solving these two equations, the point of intersection of the given two lines is given by

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

CONCURRENT LINES

The three given lines are concurrent if they meet in a point.

WORKING RULE TO PROVE CONCURRENCY

Following three methods can be used to prove that the three lines are concurrent:

- Find the point of intersection of any two lines by solving them simultaneously. If this point satisfies the third equation also, then the given lines are concurrent.
- The three lines

  $P \equiv a_1x + b_1y + c_1 = 0$, $Q \equiv a_2x + b_2y + c_2 = 0$,
  $R \equiv a_3x + b_3y + c_3 = 0$ are concurrent if

  $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix} = 0$$

- The three lines $P = 0$, $Q = 0$ and $R = 0$ are concurrent if there exist constants l, m and n, not all zero at the same time, such that

  $$lP + mQ + nR = 0.$$

  This method is particularly useful in theoretical results.

SOLVED EXAMPLE

48. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., then the straight line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$ always passes through a fixed point, that point is

(A) (–1, –2) (B) (–1, 2)
(C) (1, –2) (D) $\left(1, -\frac{1}{2}\right)$

Solution: (C)

Since $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\Rightarrow \quad \frac{1}{a}+\frac{1}{c}=\frac{2}{b} \qquad (1)$$

The given line is $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{b}+\left(\frac{2}{b}-\frac{1}{a}\right)=0 \qquad \text{[Using (1)]}$$

$$\Rightarrow \quad \frac{1}{a}(x-1)+\frac{1}{a}(y+2)=0$$

$\Rightarrow$ The given line passes through the point of intersection of $x - 1 = 0$ and $y + 2 = 0$ i.e., (1, –2) which is a fixed point.

POSITION OF TWO POINTS RELATIVE TO A LINE

Two points (x_1, y_1) and (x_2, y_2) are on the same side or on opposite sides of the line $ax + by + c = 0$ according as the expressions: $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have same sign or opposite signs.

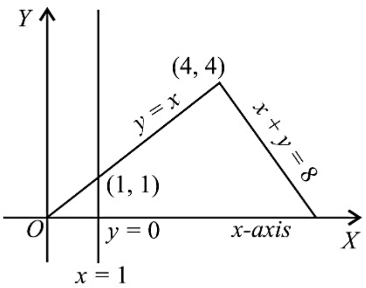
SOLVED EXAMPLES

49. The point $(1, \beta)$ lies on or inside the triangle formed by the lines $y = x$, x-axis and $x + y = 8$, if

(A) $0 < \beta < 1$ (B) $0 \le \beta \le 1$
(C) $0 < \beta < 8$ (D) none of these

Solution: (B)

The point $(1, \beta)$ lies on the line $x = 1$, for all real β. Clearly, from the figure, it will lie on or inside the triangle formed by the given lines if $0 \le \beta \le 1$.



50. Let $P(2, 0)$ and $Q(0, 2)$ be two points and O be the origin. If $A(x, y)$ is a point such that $xy > 0$ and $x + y < 2$, then

(A) A cannot be inside the ΔOPQ
(B) A lies outside the ΔOPQ
(C) A lies either inside ΔOPQ or in the third quadrant
(D) none of these

Solution: (C)

Since $xy > 0$, therefore the point A lies either in the first quadrant or in the third quadrant. Since $x + y < 2$, therefore the point A lies either inside the ΔOPQ or in the third quadrant.

LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

The length of the perpendicular from the point (x_1, y_1) to the line $ax + by + c = 0$ is given by

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Corollary The length of the perpendicular from the origin (0, 0) on the line $ax + by + c = 0$ is

$$\frac{|a \times 0 + b \times 0 + c|}{\sqrt{a^2 + b^2}} = \frac{|c|}{\sqrt{a^2 + b^2}}$$

WORKING RULE

- Make the R.H.S. of the equation of the line zero by transposing every term to L.H.S.
- On the L.H.S., replace x by x_1 and y by y_1.
- Divide by $\sqrt{(\text{coeff. of } x)^2 + (\text{coeff. of } y)^2}$.
- Take the modulus of the expression thus obtained. This will give the length of the perpendicular.

SOLVED EXAMPLES

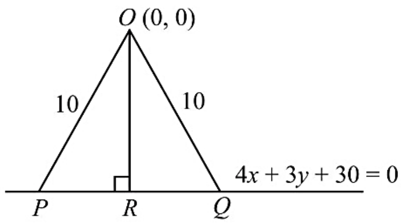
51. If P and Q are two points on the line $4x + 3y + 30 = 0$ such that $OP = OQ = 10$, where O is the origin, then the area of the ΔOPQ is

(A) 48 (B) 16
(C) 32 (D) none of these

Solution: (A)

Let $OR \perp PQ$.

Then, $OR = \dfrac{|4(0) + 3(0) + 30|}{\sqrt{16 + 9}} = \dfrac{30}{5} = 6.$



$\therefore \quad PR = \sqrt{OP^2 - OR^2} = \sqrt{100 - 36}$ and

$PQ = 2PR = 16$

$\therefore \quad \text{Area of } \Delta OPQ = \dfrac{1}{2} \times PQ \times OR$

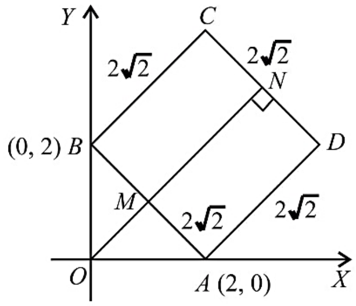
$= \dfrac{1}{2} \times 16 \times 6 = 48$

52. On the portion of the straight line $x + y = 2$ which is intercepted between the axes, a square is constructed away from the origin, with this portion as one of its side. If p denotes the perpendicular distance of a side of this square from the origin, then the maximum value of p is

(A) $\sqrt{2}$ (B) $2\sqrt{2}$
(C) $3\sqrt{2}$ (D) $4\sqrt{2}$

Solution: (C)

$p = ON = OM + MN = \perp$ distance from



O to the line $AB + AD$

$$= \frac{2}{\sqrt{2}} + 2\sqrt{2} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$$

DISTANCE BETWEEN TWO PARALLEL LINES

The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

NOTE

The distance between two parallel lines can also be obtained by taking a suitable point (take $y = 0$ and find x or take $x = 0$ and find y) on one straight line and then finding the length of the perpendicular from this point to the second line.

TRICK(S) FOR PROBLEM SOLVING

- Area of a parallelogram or a rhombus, equations of whose sides are given, can be obtained by using the following formula

$$\text{Area} = \frac{p_1 p_2}{\sin\theta},$$

where $p_1 = DL$ = distance between lines AB and CD,
$p_2 = BM$ = distance between lines AD and BC,
θ = angle between adjacent sides AB and AD.

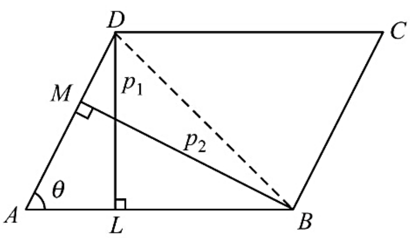


Fig. 18.19

In the case of a rhombus, $p_1 = p_2$. Thus,

- Area of rhombus $= \frac{p_1^2}{\sin\theta}$.

Also, area of rhombus $= \frac{1}{2} d_1 d_2$

where d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus.

SOLVED EXAMPLE

53. The diagonals of the parallelogram whose sides are $lx + my + n = 0$, $lx + my + n' = 0$, $mx + ly + n = 0$, $mx + ly + n' = 0$ include an angle

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$

(C) $\tan^{-1}\left(\frac{l^2 - m^2}{l^2 + m^2}\right)$ (D) $\tan^{-1}\left(\frac{2lm}{l^2 + m^2}\right)$

Solution: (B)

Since the distance between the parallel lines $lx + my + n = 0$ and $lx + my + n' = 0$ is same as the distance between the parallel lines $mx + ly + n = 0$ and $mx + ly + n' = 0$. Therefore, the parallelogram is a rhombus. Since the diagonals of a rhombus are at right angles, therefore the required angle is $\frac{\pi}{2}$.

EQUATIONS OF STRAIGHT LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A GIVEN LINE

The equations of the striaght lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

REFLECTION ON THE SURFACE

Here, IP = Incident Ray
PN = Normal to the surface
PR = Reflected Ray

Then, $\angle IPN = \angle NPR$

Angle of incidence = Angle of reflection

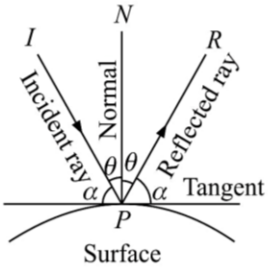


Fig. 18.20

SOLVED EXAMPLE

54. A ray of light is sent along the line which passes through the point (2, 3). The ray is reflected from the point P on x-axis. If the reflected ray passes through the point (6, 4), then the coordinates of P are

(A) $\left(\frac{26}{7}, 0\right)$ (B) $\left(0, \frac{26}{7}\right)$

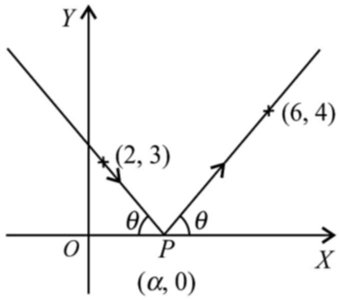
(C) $\left(-\frac{26}{7}, 0\right)$ (D) none of these

Solution: (A)

Let $P \equiv (\alpha, 0)$.

Let the reflected ray makes an angle θ with +ve direction of x-axis, then the incident ray makes angle $(\pi - \theta)$ with positive direction of x-axis.

The slope of the incident ray is



$$= \frac{0-3}{\alpha-2} = \tan(\pi-\theta) \text{ i.e., } \tan\theta = \frac{3}{\alpha-2} \quad (1)$$

The slope of the reflected ray is

$$= \frac{4-0}{6-\alpha} = \tan\theta \text{ i.e., } \tan\theta = \frac{4}{6-\alpha} \quad (2)$$

From Eq. (1) and (2), we get

$$\frac{3}{\alpha-2} = \frac{4}{6-\alpha} \Rightarrow 18 - 3\alpha = 4\alpha - 8$$

$$\Rightarrow 7\alpha = 26 \text{ or } \alpha = \frac{26}{7}$$

$\therefore$ The coordiantes of A are $\left(\frac{26}{7}, 0\right)$.

IMAGE OF A POINT WITH RESPECT TO A LINE

1. **The image of a point with respect to the line mirror.** The image of $A(x_1, y_1)$ with respect to the line mirror $ax + by + c = 0$ be $B(h, k)$ given by,

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

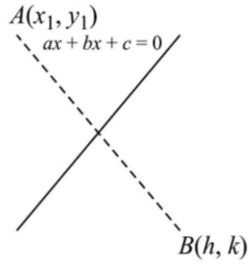


Fig. 18.21

2. **The image of a point with respect to x-axis:** Let $P(x, y)$ be any point and $P'(x', y')$ its image after reflection in the x-axis, then
$x' = x$ and $y' = -y$, ($\because$ O' is the mid point of PP')

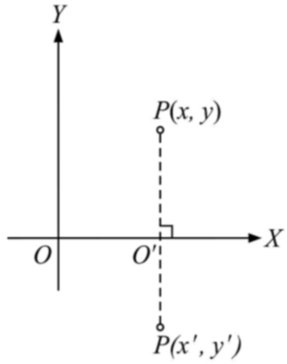


Fig. 18.22

3. **The image of a point with respect to y-axis:** $P(x, y)$ be any point and $P'(x', y')$ its image after reflection in the y-axis, then
$x' = -x$ and $y' = y$ ($\because$ O' is the mid point of PP')

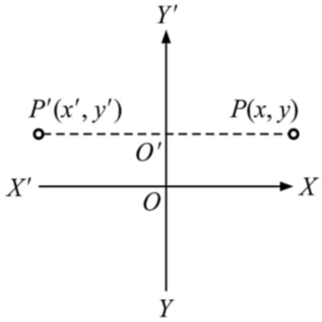


Fig. 18.23

4. **The image of a point with respect to the origin:** Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection through the origin, then
$x' = -x$ and $y' = -y$ ($\because$ O is the mid-point of PP')

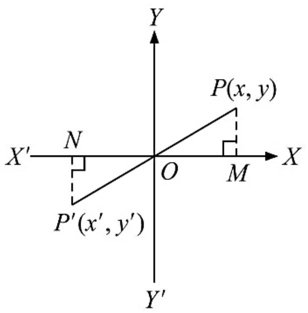


Fig. 18.24

5. **The image of a point with respect to the line $y = x$:** Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection in the line $y = x$, then,
$x' = y$ and $y' = x$ ($\because$ O' is the mid-point of PP')

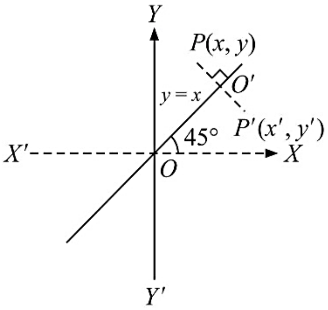


Fig. 18.25

6. **The image of a point with respect to the line $y = x \tan\theta$** : Let $P(x, y)$ be any point and $P'(x', y')$ be its image after reflection in the line $y = x \tan\theta$, then,

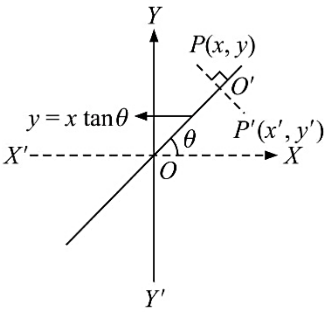


Fig. 18.26

$$x' = x \cos 2\theta + y \sin 2\theta$$
$$y' = x \sin 2\theta - y \cos 2\theta,$$
($\because$ O' is the mid-point of PP')

EQUATIONS OF THE BISECTORS OF THE ANGLES BETWEEN TWO LINES

The equations of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

NOTE

Any point on a bisector is equidistant from the given lines.

Equation of the Bisector of the Acute and Obtuse Angle between Two Lines Let the equations of the two lines be

$$a_1x + b_1y + c_1 = 0 \quad (1)$$
and
$$a_2x + b_2y + c_2 = 0 \quad (2)$$

where $c_1 > 0$ and $c_2 > 0$.
Then the equation

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

is the bisector of the acute or obtuse angle between the lines (1) and (2) according as $a_1a_2 + b_1b_2 < 0$ or > 0.
Similarly, the equation

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = - \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

is the bisector of the acute or obtuse angle between the lines (1) and (2) according as $a_1a_2 + b_1b_2 > 0$ or < 0.

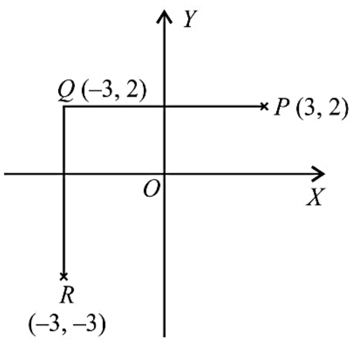
SOLVED EXAMPLE

55. The point (3, 2) is reflected in the y-axis and then moved a distance 5 units towards the negative side of y-axis. The coordinates of the point thus obtained are

(A) (3, –3) (B) (–3, 3)
(C) (3, 3) (D) (–3, –3)

Solution: (D)

Reflection in the y-axis gives the new position as (–3, 2).



When it moves towards the negative side of y-axis through 5 units, then the new position is (–3, 2 – 5) i.e., (–3, –3).

TRICK(S) FOR PROBLEM SOLVING

If $a_1a_2 + b_1b_2 > 0$, then the origin lies in obtuse angle and if $a_1a_2 + b_1b_2 < 0$, then the origin lies in acute angle.

EQUATIONS OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO GIVEN LINES

The equation of any line passing through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0,$$

where k is a parameter. The value of k can be obtained by using one more condition which the required line satisfies.

SOLVED EXAMPLES

56. If a, b, c are three terms of an A.P., then the line $ax + by + c = 0$

(A) has a fixed direction
(B) always passes through a fixed point
(C) always cuts intercepts on the axes such that their sum is zero
(D) forms a triangle with the axes whose area is constant.

Solution: (B)

Let a, b, c be pth, qth and rth terms of an A.P. whose first term is A and common difference is d. The given line is

$ax + by + c = 0$

$\Rightarrow$ $[A + (p-1)d]x + [A + (q-1)d]y + [A + (r-1)d] = 0$

$\Rightarrow$ $A(x + y + 1) + d((p-1)x + (q-1)y + r - 1) = 0$

$\Rightarrow$ The given line passes through the point of intersection of lines $x + y + 1 = 0$ and $(p-1)x + (q-1)y + r - 1 = 0$, which is a fixed point.

57. The line $(p + 2q)x + (p - 3q)y = p - q$ for different values of p and q passes through the fixed point

(A) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (B) $\left(\frac{2}{5}, \frac{2}{5}\right)$
(C) $\left(\frac{3}{5}, \frac{3}{5}\right)$ (D) $\left(\frac{2}{5}, \frac{3}{5}\right)$

Solution: (D)

The equation of the given line can be re-written as

$$p(x + y - 1) + q(2x - 3y + 1) = 0$$

which, clearly, passes through the point of intersection of the lines

$$x + y - 1 = 0 \quad (1)$$

and $$2x - 3y + 1 = 0 \quad (2)$$

for different values of p and q.

Solving (1) and (2), we get the coordinates of the point of intersection as $\left(\frac{2}{5}, \frac{3}{5}\right)$.

58. If $\alpha + \beta + \gamma = 0$, the line $3\alpha x + \beta y + 2\gamma = 0$ passes through the fixed point

(A) $\left(2, \frac{2}{3}\right)$ (B) $\left(\frac{2}{3}, 2\right)$
(C) $\left(-2, \frac{2}{3}\right)$ (D) none of these

Solution: (B)

The given line is $3\alpha x + \beta y + 2\gamma = 0$

$\Rightarrow$ $3\alpha x + \beta y + 2(-\alpha - \beta) = 0$ ($\because \alpha + \beta + \gamma = 0$)

$\Rightarrow$ $\alpha(3x - 2) + \beta(y - 2) = 0$

$\Rightarrow$ the given line passes through the point of intersection of the lines $3x - 2 = 0$ and $y - 2 = 0$ i.e., $\left(\frac{2}{3}, 2\right)$, for all values of α and β.

59. The number of integer values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is

(A) 2 (B) 0
(C) 4 (D) 1

Solution: (A)

The given lines are

$$3x + 4y = 9 \quad (1)$$

and $$y = mx + 1 \quad (2)$$

Solving (1) and (2), we get the x-coordinate of the point of intersection as $x = \frac{5}{4m+3}$.

Since x-coordinate is an integer,

$\therefore$ $4m + 3 = \pm 5$ or $4m + 3 = \pm 1$.

Solving these, only integer values of m are -1 and -2.

$\therefore$ $m = -1, -2$

STANDARD POINTS OF A TRIANGLE

Centroid of a Triangle

The point of intersection of the medians of the triangle is called the centroid of the triangle. The centroid divides the medians in the ratio 2 : 1 (2 from the vertex and 1 from the opposite side).

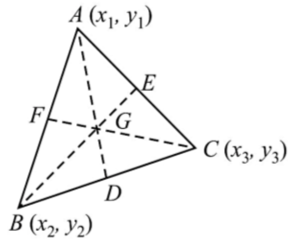


Fig. 18.27

The coordinates of the centroid of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

SOLVED EXAMPLE

60. If the vertices P, Q, R of a ΔPQR are rational points, which of the following points of the ΔPQR is (are) always rational point(s)?

(A) centroid (B) incentre
(C) circumcentre (D) orthocentre

(A rational point is a point both of whose coordinates are rational numbers)

Solution: (A)

Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$; $R \equiv (x_3, y_3)$, where x_i, y_i ($i = 1, 2, 3$) are rational numbers.

Now, the centroid of ΔPQR is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

which is rational point. Incentre, circumcentre and orthocentre depend on sides of the triangle which may not be rational even if vertices are so. For example, for $P(0, 1)$ and $Q(1, 0)$; $PQ = \sqrt{2}$.

Incentre of a Triangle

The point of intersection of the internal bisectors of the angles of a triangle is called the *incentre of the triangle*.

The coordinates of the incentre of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

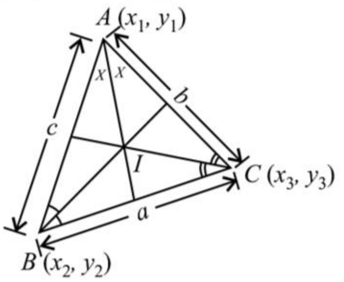


Fig. 18.28

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

TRICK(S) FOR PROBLEM SOLVING

The incentre of the triangle formed by (0, 0), (a, 0) and (0, b) is

$$\left(\frac{ab}{a+b+\sqrt{a^2+b^2}}, \frac{ab}{a+b+\sqrt{a^2+b^2}}\right)$$

Ex-centres of a Triangle A circle touches one side outside the triangle and the other two extended sides then circle is known as excircle.

Let ABC be a triangle then there are three excircles, with three excentres I_1, I_2, I_3 opposite to vertices A, B and C respectively. If the vertices of triangle are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ then

$$I_1 = \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

$$I_2 = \left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

$$I_3 = \left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

Circumcentre The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle is same and this distance is known as the circum-radius of the triangle.

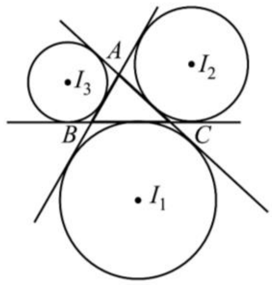


Fig. 18.29

TRICK(S) FOR PROBLEM SOLVING

- The circumcentre of a right angled triangle is the mid point of its hypotenuse.
- The circumcentre of the triangle formed by (0, 0), (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{y_2(x_1^2+y_1^2)-y_1(x_2^2+y_2^2)}{2(x_1y_2-x_2y_1)}, \frac{x_2(x_1^2+y_1^2)-x_1(x_2^2+y_2^2)}{2(x_2y_1-x_1y_2)}\right)$$

WORKING RULE TO PROVE CONCURRENCY

- Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the ΔABC and let circumcentre be $P(x, y)$. Then (x, y) can be found by solving

$$(OA)^2 = (OB)2 = (OC)^2$$

or $$(x-x_1)^2 + (y-y_1)^2 = (x-x_2)^2 + (y-y_2)^2 = (x-x_3)^2 + (y-y_3)^2$$

- Let D, E and F be the mid-points of the sides BC, CA and AB of the ΔABC respectively.

Then, $OD \perp BC$, $OE \perp AC$, $OF \perp AB$.

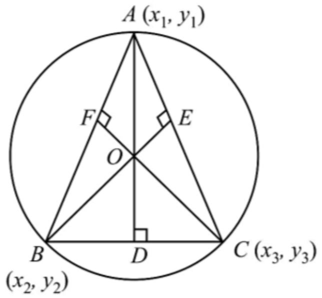


Fig. 18.30

slope of OD × slope of $BC = -1$
slope of OE × slope of $AC = -1$
slope of OF × slope of $AB = -1$

Solving any two of the above equations, we get the circumcentre (x, y).

- (a) If the equations of the three sides of the triangle are given, first of all find the coordinates of the vertices of the triangle by solving the equations of the sides of the triangle taken two at a time.
  (b) Find the coordinates of the middle points of two sides of the triangle.
  (c) Find the equations of the perpendicular bisectors of these two sides and solve them. This will give the coordinates of the circumcentre of the triangle.
- If angles A, B, C and vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ of a ΔABC are given, then its circumcentre is given by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

ORTHOCENTRE

The orthocentre of a triangle is the point of intersection of altitudes.

TRICK(S) FOR PROBLEM SOLVING

- Let O be the orthocentre. Since $AD \perp BC$, $BE \perp CA$ and $CF \perp AB$, then

$$OA \perp BC$$
$$OB \perp CA$$
and
$$OC \perp AB$$

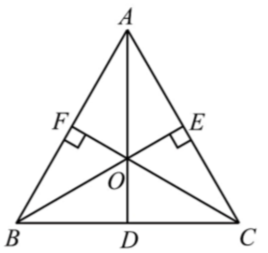


Fig. 18.31

Solving any two of these, we can get coordinates of O.

- (a) Write down the equations of any two sides of the triangle.
  (b) Find the equations of the lines perpendicular to these two sides and passing through the opposite vertices.
  (c) Solve these equations to get the coordinates of the orthocentre.
- If angles A, B and C and vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ of a ΔABC are given, then orthocentre of ΔABC is given by

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

TRICK(S) FOR PROBLEM SOLVING

- If any two lines out of three lines, i.e., AB, BC and CA are perpendicular, then orthocentre is the point of intersection of two perpendicular lines.
- The orthocentre of the triangle with vertices (0, 0), (x_1, y_1) and (x_2, y_2) is

$$\left\{(y_1 - y_2)\left[\frac{x_1x_2 - y_1y_2}{x_2y_1 - x_1y_2}\right], (x_1 - x_2)\left[\frac{x_1x_2 + y_1y_2}{x_1y_2 - x_2y_1}\right]\right\}.$$

- The orthocentre (O), centroid (G) and circum centre (C) of any triangle lie in a straight line and G divides the join of O and C in the ratio 2 : 1.
- In an equilateral triangle, orthocentre, centroid, circumcentre and incentre coincide.

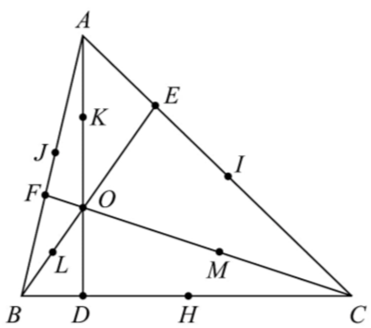


Fig. 18.32

COORDINATES OF NINE POINT CIRCLE

If a circle passes through the feet of perpendiculars (i.e., D, E, F), midpoints of sides BC, CA, AB respectively (i.e., H, I, J) and the midpoints of the line joining the orthocentre O to the angular points A, B, C (i.e., K, L, M), thus the nine points D, E, F, H, I, J, K, L, M, all lie on a circle.

This circle is known as nine point circle and its centre is called the nine-point centre.

IMPORTANT POINTS

- The orthocentre (*O*). Nine point Centre (*N*), Centroid (*G*) and Circumcentre (*C*) all lie in the same striaght line.
- The Nine point centre bisects the join of Orthocentre (*O*) and Circumcentre (*C*)
- The radius of Nine Point Circle is half the radius of Circumcirle.

SOLVED EXAMPLE

61. If the equations of the sides of a triangle are $x + y = 2$, $y = x$ and $\sqrt{3}y + x = 0$, then which of the following is an exterior point of the triangle?

(A) orthocentre (B) incentre
(C) centroid (D) none of these

Solution: (A)

The lines $y = x$ and $\sqrt{3}y + x = 0$, are inclined at 45° and 150°, respectively, with the positive direction of *x*-axis. So, the angle between the two lines is an obtuse angle. Therefore, orthocentre lies outside the given triangle, whereas incentre and centroid lie within the triangle (In any triangle, the centroid and the incentre lie within the triangle).

EXERCISES

Single Option Correct Type

1. If one of the diagonals of a square is along the line $x = y$ and one of its vertices is (3, 0), then its side through this vertex nearer to the origin is given by the equation.

(A) $y - 3x + 9 = 0$
(B) $3y + x - 3 = 0$
(C) $x - 3y - 3 = 0$
(D) $3x + y - 9 = 0$

2. Through the point $P(\alpha, \beta)$, where $\alpha\beta > 0$ the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is drawn so as to form with coordinate axes a triangle of area *S*. If $ab > 0$, then the least value of *S* is

(A) $\alpha\beta$ (B) $2\alpha\beta$
(C) $4\alpha\beta$ (D) none of these

3. A line joining two points *A* (2, 0) and *B* (3, 1) is rotated about *A* in anti-clockwise direction through an angle 15°. If *B* goes to *C* in the new position, then the coordinates of *C* are

(A) $\left(2, \sqrt{\frac{3}{2}}\right)$ (B) $\left(2, -\sqrt{\frac{3}{2}}\right)$

(C) $\left(2 + \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$ (D) none of these

4. *P* is a point on either of the two lines $y - \sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. The coordinates of the foot of the perpendicular from *P* on the bisector of the angle between them are

(A) $\left[0, \frac{1}{2}(4 + 5\sqrt{3})\right]$ or $\left[0, \frac{1}{2}(4 - 5\sqrt{3})\right]$ depending on which line the point *P* is taken

(B) $\left[0, \frac{1}{2}(4 + 5\sqrt{3})\right]$

(C) $\left[0, \frac{1}{2}(4 - 5\sqrt{3})\right]$

(D) $\left[\frac{5}{2}, \frac{5\sqrt{3}}{2}\right]$

5. A string of length 12 units is bent first into a square *PQRS* and then into a right-angled ΔPQT by keeping the side *PQ* of the square fixed and other is one more than its side. Then, the area of *PQRS* equals

(A) ar (ΔPQT) (B) $\frac{3}{2} \cdot$ ar (ΔPQT)
(C) $2 \cdot$ ar (ΔPQT) (D) none of these

6. The condition to be imposed on β so that $(0, \beta)$ lies on or inside the triangle having sides $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$ is

(A) $0 < \beta < \frac{5}{3}$ (B) $0 < \beta < \frac{7}{2}$
(C) $\frac{5}{3} \le \beta \le \frac{7}{2}$ (D) none of these

7. The point $(1, \beta)$ lies on or inside the triangle formed by the lines $y = x$, x-axis and $x + y = 8$, if
(A) $0 < \beta < 1$ (B) $0 \le \beta \le 1$
(C) $0 < \beta < 8$ (D) none of these

8. A ray of light travelling along the line $x + \sqrt{3}y = 5$ is incident on the x-axis and after refraction it enters the other side of the x-axis by turning $\frac{\pi}{6}$ away from the x-axis. The equation of the line along which the refracted ray travels is
(A) $x + \sqrt{3}y - 5\sqrt{3} = 0$
(B) $x - \sqrt{3}y - 5\sqrt{3} = 0$
(C) $\sqrt{3}x + y - 5\sqrt{3} = 0$
(D) $\sqrt{3}x - y - 5\sqrt{3} = 0$

9. A ray of light is sent along the line which passes through the point (2, 3). The ray is reflected from the point P on x-axis. If the reflected ray passes through the point (6, 4), then the coordinates of P are
(A) $\left(\frac{26}{7}, 0\right)$ (B) $\left(0, \frac{26}{7}\right)$
(C) $\left(-\frac{26}{7}, 0\right)$ (D) none of these

10. A line passing through the point P (4, 2), meets the x-axis and y-axis at A and B, respectively. If O is the origin, then locus of the centre of the circum circle of ΔOAB is
(A) $x^{-1} + y^{-1} = 2$ (B) $2x^{-1} + y^{-1} = 1$
(C) $x^{-1} + 2y^{-1} = 1$ (D) $2x^{-1} + 2y^{-1} = 1$

11. If the point $(2\cos\theta, 2\sin\theta)$ does not fall in that angle between the lines $y = |x - 2|$ in which the origin lies then θ belongs to
(A) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (B) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(C) $(0, \pi)$ (D) none of these

12. If the equations of the sides of a triangle are $x + y = 2$, $y = x$ and $\sqrt{3}y + x = 0$, then which of the following is an exterior point of the triangle?
(A) orthocentre (B) incentre
(C) centroid (D) none of these

13. A line is drawn from the point $P(\alpha, \beta)$, making an angle θ with the positive direction of x-axis, to meet the line $ax + by + c = 0$ at Q. The length of PQ is
(A) $-\frac{a\alpha + b\beta + c}{a\cos\theta + b\sin\theta}$ (B) $\left|\frac{a\alpha + b\beta + c}{\sqrt{a^2 + b^2}}\right|$
(C) $\frac{a\alpha + b\beta + c}{a\cos\theta + b\sin\theta}$ (D) none of these

14. If the equal sides AB and AC (each equal to a) of a right-angled isosceles triangle ABC be produced to P and Q so that $BP \cdot CQ = AB^2$, then the line PQ always passes through the fixed point
(A) $(a, 0)$ (B) $(0, a)$
(C) (a, a) (D) none of these

15. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G. P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)
(A) lie on a straight line
(B) lie on an ellipse
(C) lie on a circle
(D) are vertices of a triangle

16. Number of equilateral triangles with $y = \sqrt{3}(x - 1) + 2$ and $y = -\sqrt{3}x$ as two of its sides, is
(A) 0 (B) 1
(C) 2 (D) none of these

17. If the distance of any point $P(x, y)$ from the origin is defined as $d(x, y) = \text{Max.}\{|x|, |y|\}$ and $d(x, y) = k$ (non-zero constant), then the locus of the point P is
(A) a straight line (B) a circle
(C) a parabola (D) none of these

18. If a, b, c form an A. P. with common difference $d\ (\neq 0)$ and x, y, z form a G. P. with common ratio $r (\neq 1)$, then the area of the triangle with vertices (a, x), (b, y) and (c, z) is independent of
(A) b (B) r
(C) d (D) x

19. A line of fixed length 2 units moves so that its ends are on the positive x-axis and that part of the line $x + y = 0$ which lies in the second quadrant. The locus of the mid-point of the line has the equation
(A) $(x + 2y)^2 + y^2 = 1$ (B) $(x - 2y)^2 + y^2 = 1$
(C) $(x + 2y)^2 - y^2 = 1$ (D) none of these

20. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q, respectively. The point O divides the segment PQ in the ratio
(A) 1 : 2 (B) 3 : 4
(C) 2 : 1 (D) 4 : 3

21. Let O be the origin and let $A(2, 0)$, $B(0, 2)$ be two points. If $P(x, y)$ is a point such that $xy > 0$ and $x + y < 2$, then

(A) P lies either inside the triangle OAB or in the third quadrant
(B) P cannot be inside the triangle OAB
(C) P lies inside the triangle OAB
(D) none of these

22. Consider the equation $y - y_1 = m(x - x_1)$. In this equation, if m and x_1 are fixed and different lines are drawn for different values of y^1, then,
(A) the lines will pass through a single point
(B) there will be one possible line only
(C) there will be a set of parallel lines
(D) none of these

23. D is a point on AC of the triangle with vertices $A(2, 3)$, $B(1, -3)$, $C(-4, -7)$ and BD divides ABC into two triangles of equal area. The equation of the line drawn through B at right angles to BD is
(A) $y - 2x + 5 = 0$ (B) $2y - x + 5 = 0$
(C) $y + 2x - 5 = 0$ (D) $2y + x - 5 = 0$

24. If two points $A(a, 0)$ and $B(-a, 0)$ are stationary and if $\angle A - \angle B = \theta$ in ΔABC, the locus of C is
(A) $x^2 + y^2 + 2xy \tan \theta = a^2$
(B) $x^2 - y^2 + 2xy \tan \theta = a^2$
(C) $x^2 + y^2 + 2xy \cot \theta = a^2$
(D) $x^2 - y^2 + 2xy \cot \theta = a^2$

25. The straight line $y = x - 2$ rotates about a point where it cuts the x-axis and becomes perpendicular to the straight line $ax + by + c = 0$. Then, its equation is
(A) $ax + by + 2a = 0$
(B) $ax - by - 2a = 0$
(C) $by + ay - 2b = 0$
(D) $ay - bx + 2b = 0$

26. If the point $P(a^2, a)$ lies in the region corresponding to the acute angle between the lines $2y = x$ and $4y = x$, then
(A) $a \in (2, 6)$ (B) $a \in (4, 6)$
(C) $a \in (2, 4)$ (D) none of these

27. The point (4, 1) undergoes the following three successive transformations
(A) Reflection about the line $y = x - 1$
(B) Translation through a distance 1 unit along the positive x-axis
(C) Rotation through an angle $\frac{\pi}{4}$ about the origin in the anti-clockwise direction.
Then, the coordinates of the final point are
(A) (4, 3) (B) $\left(\frac{7}{2}, \frac{7}{2}\right)$
(C) $(0, 3\sqrt{2})$ (D) (3, 4)

28. A light ray emerging from the point source placed at $P(2, 3)$ is reflected at point 'θ' on the y-axis and then passes through the point $R(5, 10)$. Coordinates of 'Q' are
(A) (0, 3) (B) (0, 2)
(C) (0, 5) (D) none of these

29. The distance between two parallel lines is unity. A point P lies between the lines at a distance a from one of them. The length of a side of an equilateral triangle PQR, vertex Q of which lies on one of the parallel lines and vertex R lies on the other line, is
(A) $\frac{2}{\sqrt{3}} \cdot \sqrt{a^2 + a + 1}$ (B) $\frac{2}{\sqrt{3}} \sqrt{a^2 - a + 1}$
(C) $\frac{1}{\sqrt{3}} \sqrt{a^2 + a + 1}$ (D) $\frac{1}{\sqrt{3}} \sqrt{a^2 - a + 1}$

30. Two points A and B are given. P is a moving point on one side of the line AB such that $\angle PAB - \angle PBA$ is a positive constant 2θ. The locus of the point P is
(A) $x^2 + y^2 + 2xy \cot 2\theta = a^2$
(B) $x^2 + y^2 - 2xy \cot 2\theta = a^2$
(C) $x^2 + y^2 + 2xy \tan 2\theta = a^2$
(D) $x^2 - y^2 + 2xy \cot 2\theta = a^2$.

31. The four points $A(p, 0)$, $B(q, 0)$, $C(r, 0)$ and $D(s, 0)$ are such that p, q are the roots of the equation $ax^2 + 2hx + b = 0$ and r, s are those of equation $a'x^2 + 2h'x + b' = 0$. If the sum of the ratios in which C and D divide AB is zero, then
(A) $ab' + a'b = 2hh'$ (B) $ab' + a'b = hh'$
(C) $ab' - a'b = 2hh'$ (D) none of these

32. The coordinates of a point P on the line $3x + 2y + 10 = 0$ such that $|PA - PB|$ is maximum where A is (4, 2) and B is (2, 4), are
(A) (22, 28) (B) (22, –28)
(C) (–22, 28) (D) (–22, –28)

33. A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the point B, C and D, respectively. If $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, the equation of the line is
(A) $2x + 3y + 22 = 0$ (B) $2x - 3y + 22 = 0$
(C) $3x + 2y + 22 = 0$ (D) $3x - 2y + 22 = 0$

34. $A(0, 0)$, $B(2, 1)$ and $C(3, 0)$ are the vertices of a ΔABC and BD is its altitude. If the line through D parallel to the side AB intersects the side BC at a point K, then the product of the areas of the triangles ABC and BDK is

(A) 1 (B) $\frac{1}{2}$
(C) $\frac{1}{4}$ (D) none of these

35. A line cuts the x-axis at $A\ (7, 0)$ and y-axis at $B\ (0, -5)$. A variable line PQ is drawn $\perp$ to AB cutting the x-axis in P and the y-axis in Q. If AQ and BP intersect at R, then the locus of R is

(A) $x(x-7) + y(y+5) = 0$
(B) $x(x-7) - y(y+5) = 0$
(C) $x(x+7) + y(y-5) = 0$
(D) none of these

36. The point (2, 3) undergoes the following three transformations successively

(i) reflection about the line $y = x$
(ii) translation through a distance 2 units along the positive direction of y-axis
(iii) rotation through an angle of 45º about the origin in the anti-clockwise direction.

The final coordinates of the point are

(A) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (B) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$
(C) $\left(\frac{1}{\sqrt{2}}, -\frac{7}{\sqrt{2}}\right)$ (D) none of these

37. Lines $L_1 = ax + by + c = 0$ and $L_2 = lx + my + n = 0$ intersect at the point P and make an angle θ with each other. The equation of line L different from L_2 which passes through P and makes the same angle θ with L_1 is

(A) $2\ (al + bm)\ (ax + by + c) - (a^2 + b^2)\ (lx + my + n) = 0$
(B) $2\ (al + bm)\ (ax + by + c) + (a^2 + b^2)\ (lx + my + n) = 0$
(C) $2\ (a^2 + b^2)\ (ax + by + c) - (al + bm)\ (lx + my + n) = 0$
(D) none of these

38. The equations of the perpendicular bisector of the sides AB and AC of a $\Delta\ ABC$ are $x - y + 5 = 0$ and $x + 2y = 0$, respectively. If the point A is $(1, -2)$ then the equation of the line BC is

(A) $14x + 23y = 40$ (B) $14x - 23y = 40$
(C) $23x + 14y = 40$ (D) $23x - 14y = 40$

39. The equation of a family of lines is given by $(2 + 3t)\ x + (1 - 2t)\ y + 4 = 0$, where t is the parameter. The equation of a straight line, belonging to this family, at the maximum distance from the point (2, 3) is

(A) $21x + 14y = 0$ (B) $21x - 14y = 0$
(C) $14x - 21y = 0$ (D) none of these

40. $ABCD$ is a square whose vertices A, B, C and D are (0, 0), (2, 0), (2, 2) and (0, 2), respectively. This square is rotated in the X-Y plane with an angle of 30º in anti-clockwise direction about an axis passing through the vertex A. The equation of the diagonal BD of this rotated square is

(A) $\sqrt{3}x + (1-\sqrt{3})y = \sqrt{3}$
(B) $(1+\sqrt{3})x - (1-\sqrt{2}) = 2$
(C) $(2-\sqrt{3})x + y = 2(\sqrt{3}-1)$
(D) none of these

41. The equations of the straight lines passing through $(-2, -7)$ and cutting an intercept of length three units between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$ are

(A) $x+2=0, y+7=\frac{7}{24}(x+2)$
(B) $x-2=0, y+7=-\frac{7}{24}(x+2)$
(C) $x+2=0, y+7=-\frac{7}{24}(x+2)$
(D) $x+2=0, y+7=-\frac{7}{12}(x+2)$

42. The coordinates of the point which is at unit distance from the lines $L_1 \equiv 3x - 4y + 1 = 0$ and $L_2 \equiv 8x + 6y + 1 = 0$ and lies below L_1 and above L_2 are

(A) $\left(\frac{6}{5}, \frac{1}{10}\right)$ (B) $\left(\frac{6}{5}, -\frac{1}{10}\right)$
(C) $\left(\frac{6}{5}, \frac{1}{5}\right)$ (D) $\left(\frac{6}{5}, -\frac{1}{5}\right)$

43. The vertices of a triangle are $A(x_1, x_1 \tan\alpha)$, $B(x_2, x_2 \tan\beta)$ and $C(x_3, x_3 \tan\gamma)$. If the circumcentre of triangle ABC coincides with the origin and $H(a, b)$ be its orthocentre then $\frac{a}{b} =$

(A) $\frac{\cos\alpha + \cos\beta + \cos\gamma}{\cos\alpha \cdot \cos\beta \cdot \cos\gamma}$
(B) $\frac{\sin\alpha + \sin\beta + \sin\gamma}{\sin\alpha \cdot \sin\beta \cdot \sin\gamma}$
(C) $\frac{\tan\alpha + \tan\beta + \tan\gamma}{\tan\alpha \cdot \tan\beta \cdot \tan\gamma}$
(D) $\frac{\cos\alpha + \cos\beta + \cos\gamma}{\sin\alpha + \sin\beta + \sin\gamma}$

44. OX and OY are two coordinate axes. On OY is taken a fixed point P and on OX any point Q. On PQ an equilateral triangle is described, its vertex R being on the side of PQ away from O, then the locus of R will be

(A) straight line (B) circle
(C) ellipse (D) parabola

45. If the vertices of a variable triangle are (3, 4), (5 cos θ, 5 sinθ) and (5 sinθ, –5 cosθ), then the locus of its orthocentre is

(A) $(x + y - 1)^2 + (x - y - 7)^2 = 100$
(B) $(x + y - 7)^2 + (x - y + 1)^2 = 100$
(C) $(x + y - 7)^2 + (x - y - 1)^2 = 100$
(D) $(x + y + 7)^2 + (x + y - 1)^2 = 100$

46. If a right-angled isosceles triangle right-angled at origin has $3x + 4y = 6$ as its base, then the area of the triangle is

(A) 7 (B) $\frac{11}{25}$
(C) $\frac{36}{25}$ (D) $\frac{12}{25}$

47. The line $x + y = 1$ meets x-axis at A and y-axis at B. P is the mid-point of AB. P_1 is the foot of the perpendicular from P to OA; M_1 is that from P_1 to OP; P_2 is that from M_1 to OA and so on. If P_n denotes the foot of the nth perpendicular on OA from M_{n-1}, then OP_n is equal to

(A) $\frac{1}{2^n}$ (B) $\frac{1}{2^{n-1}}$
(C) $\frac{1}{2^{n-2}}$ (D) none of these

48. The line $x + y = a$ meets x-axis at A. A triangle AMN is inscribed in the triangle OAB, O being the origin with right angle at N; M and N lie respectively on OB and AB. If area of ΔAMN is $\frac{3}{8}$ of the area of triangle OAB, then $\frac{AN}{BN}$ is equal to

(A) 3 (B) $\frac{1}{3}$
(C) 2 (D) $\frac{2}{3}$

More than One Option Correct Type

49. Let S_1, S_2, ... be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 square cm?

(A) 7 (B) 8
(C) 9 (D) 10

50. A line which makes an acute angle θ with the positive direction of x-axis is drawn through the point $P(3, 4)$ to meet the line $x = 6$ at R and $y = 8$ at S, then

(A) $PR = 3 \sec \theta$
(B) $PS = 4 \operatorname{cosec} \theta$
(C) $PR + PS = \frac{2(3\sin\theta + 4\cos\theta)}{\sin 2\theta}$
(D) $\frac{9}{(PR)^2} + \frac{16}{(PS)^2} = 1$

51. Straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at A. Points B and C are choosen on these lines such that $AB = AC$. The equation of the line BC passing through the point (1, 2) is

(A) $x + 7y + 13 = 0$ (B) $x - 7y + 13 = 0$
(C) $7x + y - 9 = 0$ (D) none of these

52. The equation of the straight line passing through the point (4, 5) and making equal angles with the two straight lines given by the equations $3x - 4y - 7 = 0$ and $12x - 5y + 6 = 0$, is

(A) $9x - 7y - 1 = 0$
(B) $9x + 7y - 1 = 0$
(C) $7x + 9y - 73 = 0$
(D) $7x + 9y + 73 = 0$

53. Let the algebraic sum of the perpendicular distances from the points $A(2, 0)$, $B(0, 2)$, $C(1, 1)$ to a variable line be zero. Then, all such lines

(A) are concurrent
(B) pass through the fixed point (1, 1)
(C) touch some fixed circle
(D) pass through the centroid of ΔABC

54. The equation of the line passing through the point (2, 3) and making intercept of length 2 units between the lines $y + 2x = 3$ and $y + 2x = 5$, is

(A) $x = 2$ (B) $3x + 4y = 18$
(C) $4x + 3y = 18$ (D) none of these

55. Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point (1, 2) and the vertex A is on the y-axis, then the possible coordinates of A are

(A) (0, 0) (B) $\left(0, \frac{5}{2}\right)$
(C) $\left(0, -\frac{5}{2}\right)$ (D) none of these

56. The equations of two equal sides AB and AC of an isosceles triangle ABC are $x + y = 5$ and $7x - y = 3$, respectively. The equation of the side BC, if the area of ΔABC is 5 units, is

(A) $3x + y - 2 = 0$ (B) $3x + y - 12 = 0$
(C) $x - 3y + 1 = 0$ (D) $x - 3y + 21 = 0$

57. If the equation of the mirror be $2x + y - 6 = 0$ and a ray passing through (3, 10) after being reflected by the mirror passes through (7, 2), then the equations of the incident ray and the reflected ray are

(A) $x + 3y - 13 = 0$ (B) $3x - y + 1 = 0$
(C) $x - 3y + 13 = 0$ (D) $3x + y - 1 = 0$

58. Line $x + 2y = 4$ is translated by 3 units closer to the origin and then rotated by 30° in the clockwise sence about the point where the shifted line cuts the x-axis. If the equation of the line in the new position is $y = m(x + c)$, then

(A) $m = \dfrac{2+\sqrt{3}}{2\sqrt{3}-1}$ (B) $m = \dfrac{2+\sqrt{3}}{1-2\sqrt{3}}$

(C) $c = 3\sqrt{5} - 4$ (D) $c = 4 - 3\sqrt{5}$

Passage Based Questions

Passage 1

In oblique coordinates, the equation $y = mx + c$ represents a straight line which is inclined at an angle

$$\tan^{-1}\left(\frac{m\sin w}{1+m\cos w}\right)$$

to the x-axis, where w is the angle between the axes.

If θ be the angle between two lines $y = m_1x + c_1$ and $y = m_2x + c_2$, w be the angle between the axes, then

$$\tan\theta = \frac{(m_1 - m_2)\sin w}{1+(m_1+m_2)\cos w + m_1m_2}$$

The two given lines are parallel if $m_1 = m_2$.
The two lines are perpendicular if $1 + (m_1 + m_2)\cos w + m_1m_2 = 0$.

59. If the straight lines $y = m_1x + c_1$ and $y = m_2x + c_2$ make equal angles with the axis of x and be not parallel to one another, then $m_1 + m_2 + k\,m_1m_2 \cos w = 0$ where $k =$

(A) 1 (B) 2
(C) –1 (D) –2

60. The axes being inclined at an angle of 30°, the slope of the line which passes through the point (–2, 3) and is perpendicular to the straight line $y + 3x = 6$ is

(A) $\dfrac{3\sqrt{3}-2}{\sqrt{3}-6}$ (B) $\dfrac{3\sqrt{3}+2}{\sqrt{3}-6}$

(C) $\dfrac{3\sqrt{3}-2}{\sqrt{3}+6}$ (D) none of these

61. If $y = x\tan\dfrac{11\pi}{24}$ and $y = x\tan\dfrac{19\pi}{24}$ represent two straight lines at right angles, then the angle between the axes is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$

62. The axes being inclined at an angle of 120º, the tangent of the angle between the two straight lines $8x + 7y = 1$ and $28x - 73y = 101$ is $\tan^{-1}\theta$, where $\theta =$

(A) $\dfrac{30\sqrt{3}}{37}$ (B) $\dfrac{15\sqrt{3}}{37}$

(C) $\dfrac{7\sqrt{3}}{37}$ (D) none of these

Match the Column Type

63.

Column-I	Column-II
I. The diagonals of the parallelogram whose sides are $lx + my + n = 0$, $lx + my + n = 0$, $mx + ly + n = 0$, $mx + ly + n' = 0$ include an angle	(A) $\dfrac{5\pi}{12}$
II. The line $x + y = 2$ turns about the point on it, whose ordinate is equal to abscissa, through an angle θ in the clockwise direction so that its equation becomes $y = 2x - 1$. Then, the value of the angle θ is	(B) $\dfrac{\pi}{12}$

III. The larger of the two angles made with the x-axis of a straight line drawn through (1, 2) so that it intersects $x + y = 4$ at a point distant $\frac{\sqrt{6}}{3}$ from (1, 2) is	(C) $\frac{\pi}{2}$ (D) $\tan^{-1}3$

64.

Column-I	Column-II
I. If the points $A(x, y, z)$, $B(y, z + x)$ and $C(z, x + y)$ are such that $AB = BC$, then x, y, z are in	(A) H.P.
II. If a line through the variable point $A(k + 1, 2k)$ meets the lines $7x + y - 16 = 0$, $5x - y - 8 = 0$, $x - 5y + 8 = 0$ at B, C and D, respectively, then AC, AB and AD are in	(B) G.P.
III. The length of the perpendiculars from the points $(m^2, 2m)$, $(mn, m + n)$ and $(n^2, 2n)$ to the line $x\cos\theta + y\sin\theta = p$, where $p = -\frac{\sin^2\theta}{\cos\theta}$, form a	(C) A.G.P.
IV. If the lines $ax + 12y + 1 = 0$, $bx + 13y + 1 = 0$ and $cx + 14y + 1 = 0$ are concurrent, then a, b, c are in	(D) A.P.

Previous Year's Questions

65. A triangle with vertices (4, 0), (–1, –1), (3, 5) is: **[2002]**
(A) isosceles and right angled
(B) isosceles but not right angled
(C) right angled but not isosceles
(D) neither right angled nor isosceles

66. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is: **[2002]**
(A) $x = -1$ (B) $x = 1$
(C) $x = -\frac{3}{2}$ (D) $x = \frac{3}{2}$

67. The incentre of the triangle with vertices $(1, \sqrt{3})$, (0, 0) and (2, 0) is: **[2002]**
(A) $\left(1, \frac{\sqrt{3}}{2}\right)$ (B) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
(C) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (D) $\left(1, \frac{1}{\sqrt{3}}\right)$

68. Three straight lines $2x + 11y - 5 = 0$, $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$: **[2002]**
(A) form a triangle
(B) are only concurrent
(C) are concurrent with one line bisecting the angle between the other two
(D) none of the above

69. A straight line through the point (2, 2) intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. The equation to the line AB so that the triangle OAB is equilateral, is: **[2002]**
(A) $x - 2 = 0$ (B) $y - 2 = 0$
(C) $x + y - 4 = 0$ (D) none of these

70. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of '$c$' is **[2003]**
(A) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$
(B) $a_1^2 + a_2^2 - b_1^2 - b_2^2$
(C) $\frac{1}{2}(a_1^2 + a_2^2 - b_1^2 - b_2^2)$
(D) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$

71. Locus of centroid of the triangle whose vertices are $(a\cos t, a\sin t)$, $(b\sin t, -b\cos t)$ and (1, 0), where t is a parameter, is **[2003]**
(A) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$
(B) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(C) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$
(D) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$

72. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a triangle ABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line **[2004]**
(A) $2x + 3y = 9$ (B) $2x - 3y = 7$
(C) $3x + 2y = 5$ (D) $3x - 2y = 3$

73. The equation of the straight line passing through the point (4, 3) and making intercepts on the co-ordinate axes whose sum is –1 is **[2004]**
(A) $\frac{x}{2} + \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$
(B) $\frac{x}{2} - \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$

(C) $\frac{x}{2}+\frac{y}{3}=1$ and $\frac{x}{2}+\frac{y}{1}=1$

(D) $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{-2}+\frac{y}{1}=1$

74. If the sum of the slopes of the lines given by $x^2-2cxy-7y^2=0$ is four times their product, then c has the value **[2004]**

(A) 1 (B) –1
(C) 2 (D) –2

75. If one of the lines given by $6x^2-xy+4cy^2=0$ is $3x+4y=0$, then c equals **[2004]**

(A) 1 (B) –1
(C) 3 (D) –3

76. Let P be the point (1, 0) and Q a point on the locus $y^2=8x$. The locus of mid-point of PQ is **[2005]**

(A) $y^2-4x+2=0$ (B) $y^2+4x+2=0$
(C) $x^2+4y+2=0$ (D) $x^2-4y+2=0$

77. The line parallel to the x-axis and passing through the intersection of the lines $ax+2by+3b=0$ and $bx-2ay-3a=0$, where $(a, b)\neq(0, 0)$ is **[2005]**

(A) below the x-axis at a distance of $\frac{3}{2}$ from it

(B) below the x-axis at a distance of $\frac{2}{3}$ from it

(C) above the x-axis at a distance of $\frac{3}{2}$ from it

(D) above the x-axis at a distance of $\frac{2}{3}$ from it

78. If a vertex of a triangle is (1, 1) and the mid-points of two sides through this vertex are (–1, 2) and (3, 2) then the centroid of the triangle is **[2005]**

(A) $\left(-1,\frac{7}{3}\right)$ (B) $\left(\frac{-1}{3},\frac{7}{3}\right)$

(C) $\left(1,\frac{7}{3}\right)$ (D) $\left(\frac{1}{3},\frac{7}{3}\right)$

79. A straight line through the point $A(3, 4)$ is such that its intercept between the axes is bisected at A. Its equation is **[2006]**

(A) $x+y=7$ (B) $3x-4y+7=0$
(C) $4x+3y=24$ (D) $3x+4y=25$

80. The locus of the vertices of the family of parabolas $y=\frac{a^3x^2}{3}+\frac{a^2x}{2}-2a$ is **[2006]**

(A) $xy=\frac{105}{64}$ (B) $xy=\frac{3}{4}$

(C) $xy=\frac{35}{16}$ (D) $xy=\frac{64}{105}$

81. If (a, a^2) falls inside the angle made by the lines $y=\frac{x}{2}$, $x>0$ and $y=3x$, $x>0$, then a belongs to **[2006]**

(A) $\left(0,\frac{1}{2}\right)$ (B) $(3,\infty)$

(C) $\left(\frac{1}{2},3\right)$ (D) $\left(-3,-\frac{1}{2}\right)$

82. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which 'k' can take is given by **[2007]**

(A) {1, 3} (B) {0, 2}
(C) {–1, 3} (D) {–3, –2}

83. Let $P=(-1, 0)$, $Q=(0, 0)$ and $R=(3,3\sqrt{3})$ be three points. The equation of the bisector of the angle PQR **[2007]**

(A) $\sqrt{3}x+y=0$ (B) $x+\frac{\sqrt{3}}{2}y=0$

(C) $\frac{\sqrt{3}}{2}x+y=0$ (D) $x+\sqrt{3}y=0$

84. If one of the lines of $my^2+(1-m^2)xy-mx^2=0$ is a bisector of the angle between the lines $x=0$ and $y=0$, then m is **[2007]**

(A) $-\frac{1}{2}$ (B) –2

(C) 1 (D) 2

85. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept –4. Then a possible value of k is **[2008]**

(A) 1 (B) 2
(C) –2 (D) –4

86. The line L given by $\frac{x}{5}+\frac{y}{b}=1$ passes through the point (13, 32) and the line K which is parallel to L has the equation $\frac{x}{c}+\frac{y}{3}=1$. Then, the distance between L and K is **[2010]**

(A) $\sqrt{17}$ (B) $\frac{17}{\sqrt{15}}$

(C) $\frac{23}{\sqrt{17}}$ (D) $\frac{23}{\sqrt{15}}$

87. The lines L_1: $y - x = 0$ and L_2: $2x + y = 0$ intersect the line L_3: $y + 2 = 0$ at two respective points P and Q. The bisector of the acute angle between L_1 and L_2 intersect L_3 at R. **[2011]**

Statement - 1 : The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.
Statement - 2 : In any triangle, bisector of an angle divides the triangle into two similar triangles.

(A) Statement - 1 is true, Statement-2 is true; Statement - 2 is not a correct explanation for Statement -1
(B) Statement - 1 is true, Statement- 2 is false.
(C) Statement - 1 is false, Statement- 2 is true.
(D) Statement - 1 is true, Statement - 2 is true; Statement - 2 is a correct explanation for Statement -1

88. Equation of the ellipse which passes through the point (– 3, 1), whose axes are the coordinate axes and has eccentricity $\sqrt{\frac{2}{5}}$ is **[2011]**

(A) $5x^2 + 3y^2 - 48 = 0$ (B) $3x^2 + 5y^2 - 15 = 0$
(C) $5x^2 + 3y^2 - 32 = 0$ (D) $3x^2 + 5y^2 - 32 = 0$

89. If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k equals **[2012]**

(A) $\frac{29}{5}$ (B) 5
(C) 6 (D) $\frac{11}{5}$

90. A line is drawn through the point (1, 2) to meet the coordinate axes at points P and Q respectively such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is **[2012]**

(A) $-\frac{1}{4}$ (B) –4
(C) –2 (D) $-\frac{1}{2}$

91. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is **[2013]**

(A) $\sqrt{3}y = x - \sqrt{3}$ (B) $y = \sqrt{3}x - \sqrt{3}$
(C) $\sqrt{3}y = x - 1$ (D) $y = x + \sqrt{3}$

92. The abscissa of the incentre of the triangle that has the coordinates of mid points of its sides as (0, 1) (1, 1) and (1, 0) is **[2013]**

(A) $2 - \sqrt{2}$ (B) $1 + \sqrt{2}$
(C) $1 - \sqrt{2}$ (D) $2 + \sqrt{2}$

93. Let a, b, c and d be non-zero numbers. If the point of intersection of the line $4ax + 2ay + c = 0$ with the line $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes then **[2014]**

(A) $2bc - 3ad = 0$ (B) $2bc + 3ad = 0$
(C) $3bc - 2ad = 0$ (D) $3bc + 2ad = 0$

94. Let PS be the median of the triangle with vertices P (2, 2), Q(6, –1) and R(7, 3). The equation of the line passing through (1, –1) and parallel to PS is **[2014]**

(A) $4x - 7y - 11 = 0$ (B) $2x + 9y + 7 = 0$
(C) $4x + 7y + 3 = 0$ (D) $2x - 9y - 11 = 0$

95. The number of points, having both co-ordinates as integers, which lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0), is: **[2015]**

(A) 861 (B) 820
(C) 780 (D) 901

96. Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a: **[2015]**

(A) straight line parallel to y-axis.
(B) circle of radius 2.
(C) circle of radius 3.
(D) straight line parallel to x-axis.

97. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at (–1, –2), then which one of the following is a vertex of this rhombus? **[2016]**

(A) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$ (B) (–3, –9)
(C) (–3, –8) (D) $\left(\frac{1}{3}, -\frac{8}{3}\right)$

ANSWER KEYS

Single Option Correct Type

1. (B)	2. (B)	3. (C)	4. (B)	5. (B)	6. (C)	7. (B)	8. (C)	9. (A)	10. (B)
11. (B)	12. (A)	13. (A)	14. (C)	15. (A)	16. (D)	17. (A)	18. (A)	19. (A)	20. (B)
21. (A)	22. (B)	23. (A)	24. (D)	25. (D)	26. (C)	27. (C)	28. (C)	29. (B)	30. (D)
31. (A)	32. (C)	33. (A)	34. (B)	35. (A)	36. (B)	37. (A)	38. (A)	39. (D)	40. (C)
41. (C)	42. (B)	43. (D)	44. (A)	45. (B)	46. (C)	47. (A)	48. (A)		

More than One Option Correct Type

49. (B, C, D) **50.** (A, B, C, D) **51.** (B, C) **52.** (A, C) **53.** (A, B, D)
54. (A, B) **55.** (A, B) **56.** (A, B, C, D) **57.** (A, B) **58.** (B, C)

Passage Based Questions

59. (B) **60.** (A) **61.** (B) **62.** (A)

Match the Column Type

63. I $\leftrightarrow$ (C), II $\leftrightarrow$ (D), III $\leftrightarrow$ (A, B), **64.** I $\leftrightarrow$ (D), II $\leftrightarrow$ (A), III $\leftrightarrow$ (B), IV $\leftrightarrow$ (D)

Previous Year's Questions

65. (A) **66.** (D) **67.** (D) **68.** (C) **69.** (B) **70.** (A) **71.** (B) **72.** (A) **73.** (D) **74.** (C)
75. (D) **76.** (A) **77.** (A) **78.** (C) **79.** (C) **80.** (A) **81.** (C) **82.** (C) **83.** (A) **84.** (C)
85. (D) **86.** (C) **87.** (B) **88.** (D) **89.** (C) **90.** (C) **91.** (A) **92.** (A) **93.** (C) **94.** (B)
95. (C) **96.** (B) **97.** (D)

HINTS AND SOLUTIONS

Single Option Correct Type

1. The point (3, 0) does not lie on the diagonal $x = 2y$. Let the equation of a side through the vertex (3, 0) be

$y - 0 = m(x - 3)$

Since the angle between a side and a diagonal of a square is $\frac{\pi}{4}$, we have

$$\pm\tan\frac{\pi}{4} = \frac{m - 1/2}{1 + m(1/2)} = \frac{2m-1}{2+m}$$

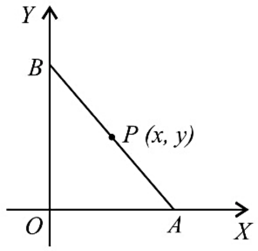
$\Rightarrow \quad m = 3, -1/3$

Thus, the equation of a side through (3, 0) is $y = 3(x - 3)$ or $y = \left(-\frac{1}{3}\right)(x-3)$ and the one nearer to the origin is

$3y + x - 3 = 0$

2. The equation of the given line is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (1)$$



This line cuts x-axis and y-axis at A $(a, 0)$ and B $(0, b)$ respectively.

Since area of $\Delta OAB = S$ (Given)

$$\therefore \quad \left|\frac{1}{2}ab\right| = S \text{ or } ab = 2S(\because ab > 0) \qquad (2)$$

Since the line (1) passes through the point P (α, β)

$$\therefore \quad \frac{\alpha}{a} + \frac{\beta}{b} = 1 \text{ or } \frac{\alpha}{a} + \frac{a\beta}{2S} = 1 \qquad \text{[Using (2)]}$$

or, $a^2\beta - 2aS + 2\alpha S = 0.$

Since a is real, $\therefore$ $4S^2 - 8\alpha\beta S \geq 0$

or, $4S^2 \geq 8\alpha\beta S$ or $S \geq 2\alpha\beta \left(\because \quad S = \frac{1}{2}ab > 0 \text{ as } ab > 0\right)$

Hence, the least value of $S = 2\alpha\beta$.

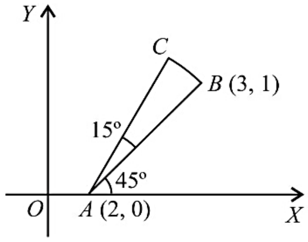
3. Slope of line

$$AB = \frac{0-1}{2-3} = 1 = \tan 45°$$

$\therefore \quad \angle BAX = 45°.$

Given, $\angle CAB = 15°.$

$\therefore \quad \angle CAX = 60°.$



$\therefore$ Slope of line $AC = \tan 60° = \sqrt{3}$.

Now, line AC makes an angle of 60º with positive direction of x-axis and

$AC = AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}$

$\therefore$ Coordinates of C are $(2 + \sqrt{2}\cos 60°, 0 + \sqrt{2}\sin 60°)$

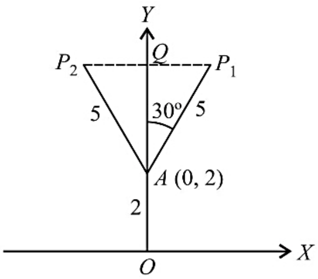
i.e., $\left(2 + \dfrac{1}{\sqrt{2}}, \sqrt{\dfrac{3}{2}}\right)$

4. Equation of two lines are

$$y = \sqrt{3}x + 2, \quad \text{if } x \geq 0$$

and, $$y = -\sqrt{3}x + 2, \quad \text{if } x \leq 0$$

Clearly, $y \geq 2$.



Also, y-axis is the bisector of the angle between the two lines. P_1, P_2 are two points on these lines, at a distance 5 units from A. Q is the foot of the ⊥ from P_1 and P_2 on the bisector (y-axis).

Then, the coordinates of Q are (0, 2 + 5 cos 30º)

$= \left(0, 2 + \dfrac{5\sqrt{3}}{2}\right) = \left(0, \dfrac{1}{2}(4 + 5\sqrt{3})\right)$

5. Side of square = 3 unit

$\Rightarrow$ ar $(PQRS) = (3)^2 = 9$ square unit

One side of ΔPQT is the side PQ of the square i.e., 3 units

The other is one more than its side, i.e., (3 + 1) = 4 units

$\Rightarrow$ ar $(\Delta PQT) = \dfrac{1}{2}(3)(4) = 6$ square unit

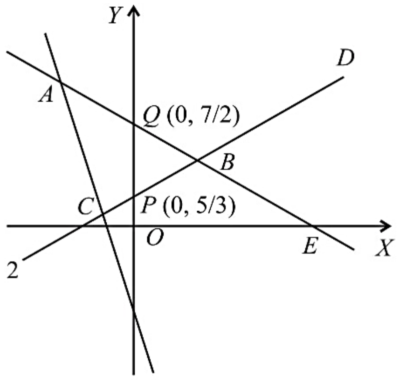
$\therefore$ ar $(PQRS) = \dfrac{3}{2}\{\text{ar}(\Delta PQT)\}$

6. Clearly, point (0, β) lies on y-axis.

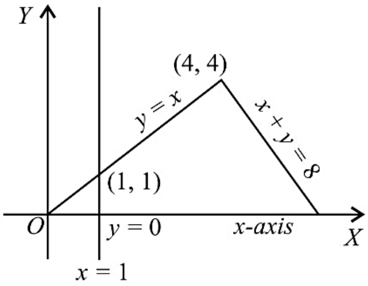
Drawing the graph of the three straight lines, we see that

$Q \equiv \left(0, \dfrac{7}{2}\right)$ and $P \equiv \left(0, \dfrac{5}{3}\right)$.

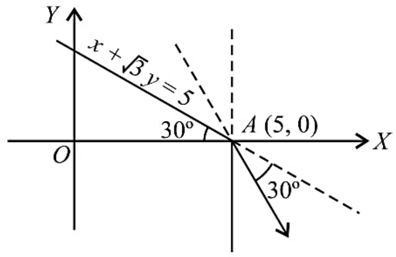
Therefore, the point (0, β) lies on or inside ΔABC, when $\dfrac{5}{3} \leq \beta \leq \dfrac{7}{2}$



7. The point (1, β) lies on the line $x = 1$, for all real β. Clearly, from the figure, it will lie on or inside the triangle formed by the given lines if $0 \leq \beta \leq 1$.



8. The refracted ray passes through the point (5, 0) and makes an angle 120º with positive direction of x-axis



$\therefore$ The equation of the refracted ray is

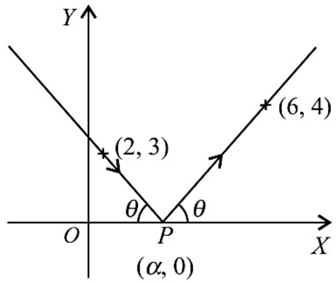
$(y - 0) = \tan 120° (x - 5)$

$\Rightarrow \quad y = -\sqrt{3}(x - 5) \quad$ or $\quad \sqrt{3}x + y - 5\sqrt{3} = 0$

9. Let $P \equiv (\alpha, 0)$.

Let the reflected ray makes an angle θ with positive direction of x-axis, then the incident ray makes angle $(\pi - \theta)$ with positive direction of x-axis.

The slope of the incident ray is



$= \dfrac{0-3}{\alpha-2} = \tan(\pi-\theta)$, i.e., $\tan\theta = \dfrac{3}{\alpha-2}$ (1)

The slope of the reflected ray is

$= \dfrac{4-0}{6-\alpha} = \tan\theta$, i.e., $\tan\theta = \dfrac{4}{6-\alpha}$ (2)

From (1) and (2), we get

$$\frac{3}{\alpha-2} = \frac{4}{6-\alpha} \Rightarrow 18-3\alpha = 4\alpha-8$$

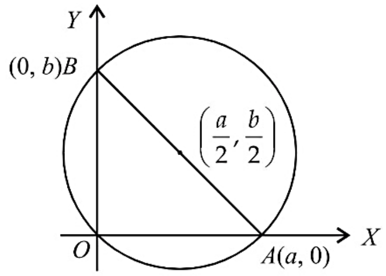
$$\Rightarrow 7\alpha = 26 \text{ or } \alpha = \frac{26}{7}$$

$\therefore$ The coordiantes of P are $\left(\dfrac{26}{7}, 0\right)$.

10. Let the coordinates of A and B be $(a, 0)$ and $(0, b)$, respectively.

Then, equation of line AB is

$$\frac{x}{a} + \frac{y}{b} = 1.$$



Since, it passes through the point P (4, 2)

$\therefore \dfrac{4}{a} + \dfrac{2}{b} = 1.$ (1)

Now, centre of the circumcircle of $\Delta OAB = \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$.

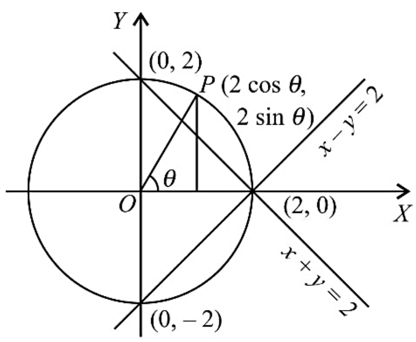
So, equation (1) can be written in the form

$$\frac{2}{a/2} + \frac{1}{b/2} = 1$$

$\therefore$ locus of circumcentre is

$$\frac{2}{x} + \frac{1}{y} = 1 \text{ or } 2x^{-1} + y^{-1} = 1.$$

11. Clearly, the point $(2\cos\theta, 2\sin\theta)$ lie on the circle



$x^2 + y^2 = 4$.

The two lines represented by the equation $y = |x-2|$ are $y = x-2$ and $y = 2-x$.

From the figure, θ can be vary from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

12. The lines $y = x$ and $\sqrt{3}y + x = 0$ are inclined at 45° and 150°, respectively, with the positive direction of x-axis. So, the angle between the two lines is an obtuse angle. Therefore, orthocentre lies outside the given triangle, whereas incentre and centroid lie within the triangle (In any triangle, the centroid and the incentre lie within the triangle).

13. Equation of a straight line passing through the point $P(\alpha, \beta)$ and making an angle θ with positive direction of x-axis is

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r \text{ (say)}$$

Coordinates of any point on this line are

$(\alpha + r\cos\theta, \beta + r\sin\theta)$

If it lies on the line $ax + by + c = 0$, then

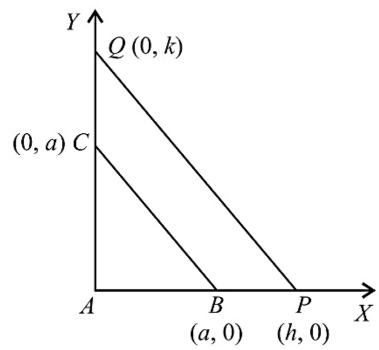
$a(\alpha + r\cos\theta) + b(\beta + r\sin\theta) + c = 0$

$$\Rightarrow r = -\frac{a\alpha + b\beta + c}{a\cos\theta + b\sin\theta}$$

Thus, $PQ = r = -\dfrac{a\alpha + b\beta + c}{a\cos\theta + b\sin\theta}$

14. We take A as the origin and AB and AC as x-axis and y-axis, respectively.

Let $AP = h$, $AQ = k$.



Equation of the line PQ is

$$\frac{x}{h} + \frac{y}{k} = 1 \quad (1)$$

Given, $BP \cdot CQ = AB^2$

$\Rightarrow (h-a)(k-a) = a^2$

$\Rightarrow hk - ak - ah + a^2 = a^2$

or, $ak + ha = hk$

or, $\dfrac{a}{h} + \dfrac{a}{k} = 1$ (2)

From (2), it follows that line (1), i.e., PQ passes through the fixed point **(a, a).**

15. Let $\dfrac{x_2}{x_1} = \dfrac{x_3}{x_2} = r$ and $\dfrac{y_2}{y_1} = \dfrac{y_3}{y_2} = r$

$\Rightarrow x_2 = x_1 r, x_3 = x_1 r^2, y_2 = y_1 r$ and $y_3 = y_1 r^2$.

We have,

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 r & y_1 r & 1 \\ x_1 r^2 & y_1 r^2 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ 0 & 0 & 1-r \\ 0 & 0 & 1-r \end{vmatrix}$$

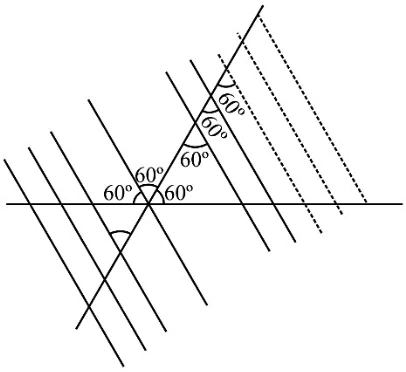
[Applying $R_3 \to R_3 - rR_2$ and $R_2 \to R_2 - rR_1$]

$= 0$ (R_2 and R_3 are identical)

Thus, (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a straight line.

16. The sides are,

$y = \sqrt{3}(x-1)+2$ and $y = -\sqrt{3}x$



The two lines are at an angle of 60° to each other. Now, any line parallel to obtuse angle bisector will make equilateral triangle with these lines as its two sides.

17. We have, $d(x,y) = \text{Max.}\{|x|, |y|\} = k$

If $|x| > |y|$, then $k = |x| \Rightarrow x = \pm k$

If $|y| > |x|$, then $k = |y| \Rightarrow y = \pm k$

Hence, the locus represents a **straight line.**

18. Area of the triangle

$$= \frac{1}{2}\begin{vmatrix} a & x & 1 \\ b & y & 1 \\ c & z & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a & x & 1 \\ b-a & y-x & 0 \\ c-b & z-y & 0 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$]

$= \frac{1}{2}[(b-a)(z-y)-(c-b)(y-x)]$

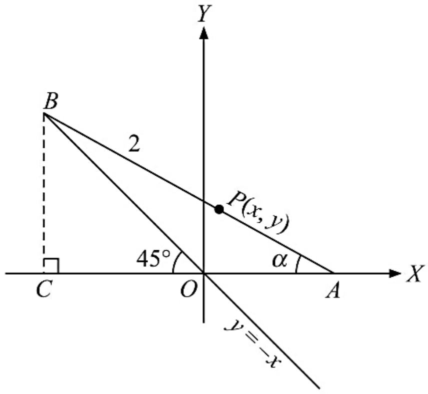
$= \frac{1}{2}d(z-2y+x)$ $\quad [\because b-a = c-b = d]$

$= \frac{1}{2}d(xr^2 - 2xr + x)$ $\quad [\because y = xr \text{ and } z = xr^2]$

$= \frac{1}{2}dx(r-1)^2$, which is independent of b.

19. Let $\angle BAO = \alpha$.

Then, $BC = 2\sin\alpha = CO$ and $CA = 2\cos\alpha$. Therefore, the coordinates of A and B are



$A \equiv (2\cos\alpha - 2\sin\alpha, 0)$

and, $B \equiv (-2\sin\alpha, 2\sin\alpha)$.

If $P(x, y)$ is the mid point of AB, then

$2x = 2\cos\alpha - 4\sin\alpha$ and $2y = 2\sin\alpha$

$\Rightarrow \quad x = \cos\alpha - 2\sin\alpha$ and $y = \sin\alpha$

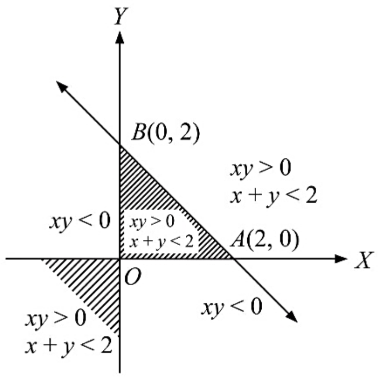
$\Rightarrow \quad \cos\alpha = x + 2y$ and $\sin\alpha = y$

Squaring and adding, we get $(x + 2y)^2 + y^2 = 1$, which is the required locus.

20. It is clear that the lines lie on opposite sides of the origin O. Let the equation of any line through O be $\frac{x}{\cos\theta} = \frac{y}{\sin\theta}$. If $OP = r_1$ and $OQ = r_2$ then the coordinates of P are $(r_1\cos\theta, r_1\sin\theta)$ and that of Q are $(-r_2\cos\theta, -r_2\sin\theta)$.

Since P lies on $4x + 2y = 9$, $2r_1(2\cos\theta + \sin\theta) = 9$ and Q lies on $2x + y + 6 = 0$, $-r_2(2\cos\theta + \sin\theta) = -6$ so that $\frac{r_1}{r_2} = \frac{9}{12} = \frac{3}{4}$ and the required ratio is thus 3 : 4.

21. Since, $xy > 0$, therefore P lies either in the first quadrant or in the third quadrant. The inequality $x + y < 2$ represents all the points below the line $x + y = 2$. Therefore, $xy > 0$ and $x + y < 2$ imply that P lies either inside ΔOAB or in the third quadrant.



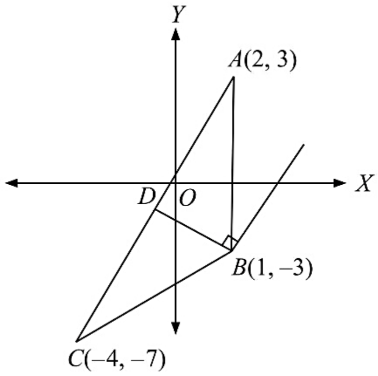
22. Since, $y - y_1 = m(x - x_1)$, where m and x_1 are fixed.

Therefore, the system represents a set of parallel lines of slope m.

Since all these parallel lines have fixed x coordinate (abscissa)

$\therefore$ there will only be one possible line.

23. Since the line BD divides the triangle into two parts of equal area, BD is a median and D is $(-1, -2)$. Slope of $BD = -\frac{1}{2}$



So, the required line is $y + 3 = 2(x - 1)$

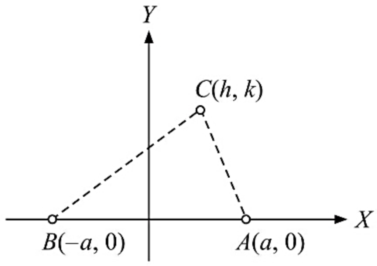
$\Rightarrow y - 2x + 5 = 0$

24. Slope of line $AC = m_{AC} = \frac{k-0}{h-a} = \tan\alpha$

Slope of line $BC = m_{BC} = \frac{k-0}{h+a} = \tan\beta$

Since, $\angle A = \pi - \alpha$ {from figure}

$\therefore \tan A = \tan(\pi - \alpha) = -\tan\alpha = \frac{k}{h-a}$ (1)



Similarly, $\angle B = \beta$

$\therefore \tan B = \tan\beta = \frac{k}{h+a}$ (2)

Also, $A - B = \theta$ {given}

$\therefore \tan(A - B) = \tan\theta$

$$\Rightarrow \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan\theta$$

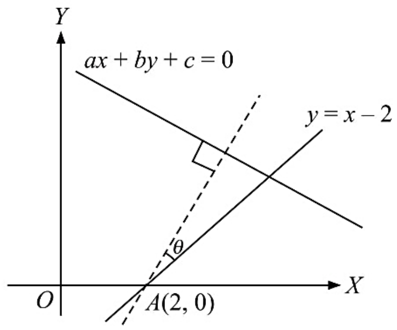
$$\Rightarrow \frac{\left(\frac{-k}{h-a}\right) - \left(\frac{k}{h+a}\right)}{1 + \left(\frac{-k}{h-a}\right)\left(\frac{k}{h+a}\right)} = \tan\theta \text{ \{using (1) and (2)\}}$$

$$\Rightarrow \tan\theta = \frac{-k(h+a) - k(h-a)}{(h^2 - a^2) + (-k^2)} = \frac{-2hk}{h^2 - k^2 - a^2}$$

$\Rightarrow h^2 - k^2 + 2hk \cot\theta = a^2$

$\therefore$ locus is $x^2 - y^2 + 2xy \cot\theta = a^2$

25. Slope of the line in the new position is $\frac{b}{a}$, since it is $\perp$ to the line $ax + by + c = 0$ and it cuts the x-axis at $(2, 0)$. Hence, the required line passes through $(2, 0)$ and its slope is $\frac{b}{a}$. Therefore, its equation is

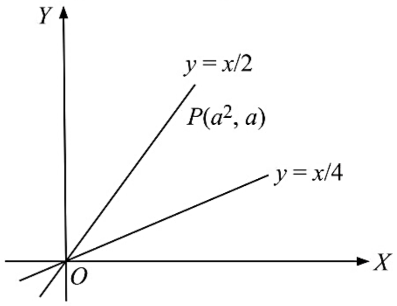


$y - 0 = \frac{b}{a}(x - 2)$

$\Rightarrow ay = bx - 2b$

$\Rightarrow ay - bx + 2b = 0$

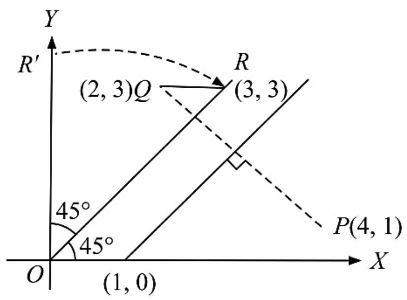
26. We have, $a - \frac{a^2}{4} > 0$ and $a - \frac{a^2}{2} < 0$



$\Rightarrow 0 < a < 4,\ a \in (-\infty, 0) \cup (2, \infty)$

$\Rightarrow a \in (2, 4)$

27. If (α, β) be the image of $(4, 1)$ w.r.t. $y = x - 1$ then $(\alpha, \beta) = (2, 3)$, say point Q.



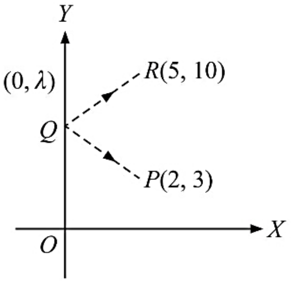
After translation through a distance 1 unit along the positive x-axis of the point.

The coordinates of the final point are $(0, 3\sqrt{2})$.

28. If P_1 be the reflection of P in y-axis then $P_1 \equiv (-2, 3)$

Equation of line P_1R is

$$(y-3)=\frac{10-3}{5+2}(x+2)$$



$\Rightarrow\ y = x + 5$. It meets y-axis at $(0, 5) \Rightarrow Q \equiv (0, 5)$

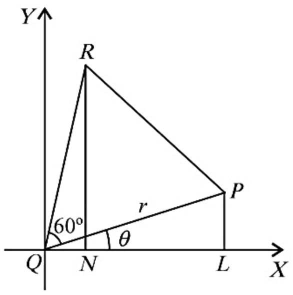
29. Let $PQ = QR = RP = r$

and, $\angle PQX = \theta$, then

$$\angle RQX = \frac{\pi}{3}+\theta.$$

Given, $PL = a,\ RN = 1$.

Now, $a = PL = r \sin\theta$



and, $1 = RN = r\sin\left(\frac{\pi}{3}+\theta\right)$

$$\Rightarrow\ 1 = r\left(\sin\frac{\pi}{3}\cos\theta+\cos\frac{\pi}{3}\sin\theta\right)$$

$$\Rightarrow\ r\left(\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta\right)=1$$

$$\Rightarrow\ r\left(\frac{\sqrt{3}}{2}\sqrt{1-\frac{a^2}{r^2}}+\frac{1}{2}\cdot\frac{a}{r}\right)=1\left(\because \sin\theta=\frac{a}{r}\right)$$

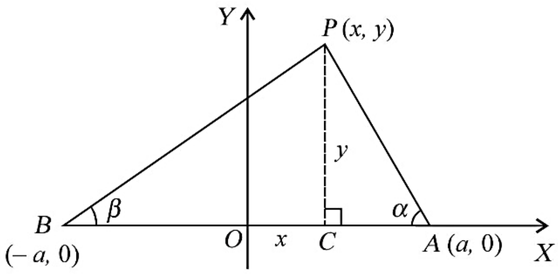
$$\Rightarrow\ \frac{\sqrt{3}}{2}\sqrt{r^2-a^2}=1-\frac{a}{2}$$

$$\text{or}\quad r^2-a^2=\frac{(2-a)^2}{3}=\frac{4+a^2-4a}{3}$$

$$\Rightarrow\ r^2=\frac{4+a^2-4a}{3}+a^2=\frac{4+4a^2-4a}{3}$$

$$\therefore\ r=\frac{2}{\sqrt{3}}\sqrt{a^2-a+1}.$$

30. Let $P(x, y)$ be the moving point whose locus is required. Let $A \equiv (a, 0)$ and $B(-a, 0)$.



Let $\angle PAB = \alpha$ and $\angle PBA = \beta$.

$\therefore$ In ΔPCA, $\tan\alpha = \frac{PC}{CA}=\frac{y}{a-x}$

and in ΔPBC, $\tan\beta = \frac{PC}{BC}=\frac{y}{a+x}$

Given, $\angle PAB - \angle PBA = 2\theta$ (constant)

$\Rightarrow\ \alpha-\beta = 2\theta \Rightarrow \tan(\alpha-\beta)=\tan 2\theta$

$$\Rightarrow\ \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\tan 2\theta$$

$$\Rightarrow\ \frac{\frac{y}{a-x}-\frac{y}{a+x}}{1+\frac{y}{a-x}\cdot\frac{y}{a+x}}=\tan 2\theta$$

$$\Rightarrow\ \frac{2xy}{a^2-x^2+y^2}=\tan 2\theta$$

$\Rightarrow\ x^2-y^2+2xy\cot 2\theta = a^2$,

which is the required locus.

31. Since p, q are the roots of the equation

$ax^2+2hx+b=0$

$$\therefore\ p+q=-\frac{2h}{a}\ \text{and}\ pq=\frac{b}{a} \qquad (1)$$

Also, r, s are the roots of the equation

$a'x^2+2h'x+b'=0$

$$\therefore\ r+s=-\frac{2h'}{a'}\ \text{and}\ rs=\frac{b'}{a'} \qquad (2)$$

Let C divides AB in the ratio $\alpha : 1$.

Then, $r=\frac{\alpha q+p}{\alpha+1}\ \Rightarrow\ \alpha=\frac{p-r}{r-q}$

Let D divides AB in the ratio $\beta : 1$.

Then, $s=\frac{\beta q+p}{\beta+1}\ \Rightarrow\ \beta=\frac{p-s}{s-q}$

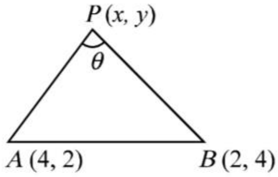
Given, $\alpha+\beta=0$

$$\Rightarrow\ \frac{p-r}{r-q}+\frac{p-s}{s-q}=0 \Rightarrow (p+q)(r+s)-2pq-2rs=0$$

$$\Rightarrow\ \left(-\frac{2h}{a}\right)\left(-\frac{2h'}{a'}\right)-\frac{2b}{a}-\frac{2b'}{a'}=0\ \text{(Using (1) and (2))}$$

$\Rightarrow\ ab'+a'b=2hh'$.

32. Let $P \equiv (x_1, y_1)$ and $\angle APB = \theta$



Then, $\cos\theta = \dfrac{(PA)^2 + (PB)^2 - (AB)^2}{2PA \cdot PB}$

Since $\cos\theta \le 1$

$\Rightarrow \dfrac{(PA)^2 + (PB)^2 - (AB)^2}{2PA \cdot PB} \le 1$

$\Rightarrow (PA)^2 + (PB)^2 - (AB)^2 \le 2\, PA \cdot PB$

$\Rightarrow (PA - PB)^2 \le (AB)^2$

$\Rightarrow |PA - PB| \le |AB| = \sqrt{(4-2)^2 + (2-4)^2}$

$\Rightarrow |PA - PB| \le 2\sqrt{2}$.

$\Rightarrow$ Maximum value of $|PA - PB|$ is $2\sqrt{2}$ when $\theta = 0$

i.e., P lies on the line AB as well as on the given line.

equation of AB is

$y - 2 = \dfrac{4-2}{2-4}(x - 4)$ or $x + y = 6$ (1)

and given line is $3x + 2y + 10 = 0$ (2)

Solving (1) and (2), we get $P \equiv (-22, 28)$.

33. Suppose, the required line has slope $\tan\theta$. Its equation is

$\dfrac{x+5}{\cos\theta} = \dfrac{y+4}{\sin\theta} = r$ (1)

Any point on this line has coordinates

$(-5 + r\cos\theta, -4 + r\sin\theta)$

Its distance from $(-5, -4)$ is r.

Coordinates of points B, C and D are

$B \equiv (-5 + AB\cos\theta, -4 + AB\sin\theta)$

$C \equiv (-5 + AC\cos\theta, -4 + AC\sin\theta)$

$D \equiv (-5 + AD\cos\theta, -4 + AD\sin\theta)$

Points B, C, D lie on the lines

$x + 3y + 2 = 0$ (2)

$2x + y + 4 = 0$ (3)

$x - y - 5 = 0$ (4)

respectively.

$\therefore\ -5 + AB\cos\theta + 3(-4 + AB\sin\theta) + 2 = 0$

$\Rightarrow \dfrac{15}{AB} = \cos\theta + 3\sin\theta$

and, $2(-5 + AC\cos\theta) + (-4 + AC\sin\theta) + 4 = 0$

$\Rightarrow \dfrac{10}{AC} = 2\cos\theta + \sin\theta$

and, $(-5 + AD\cos\theta) - (-4 + AD\sin\theta) - 5 = 0$

$\Rightarrow \dfrac{6}{AD} = \cos\theta - \sin\theta$

Since $\left(\dfrac{15}{AB}\right)^2 + \left(\dfrac{10}{AC}\right)^2 = \left(\dfrac{6}{AD}\right)^2$, we get

$(\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2$

$= (\cos\theta - \sin\theta)^2$

$\Rightarrow 4\cos^2\theta + 9\sin^2\theta + 12\sin\theta\cos\theta = 0$

$\Rightarrow (2\cos\theta + 3\sin\theta)^2 = 0 \Rightarrow 2\cos\theta = -3\sin\theta$

$\Rightarrow \tan\theta = -\dfrac{2}{3}$.

$\therefore$ From (1), the equation of line is

$y + 4 = -\dfrac{2}{3}(x + 5)$

or $2x + 3y + 22 = 0$.

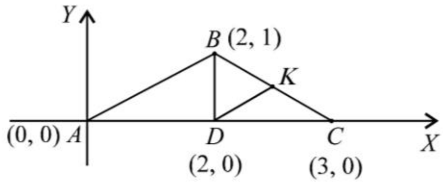
34. Area of ΔABC = Area of ΔABD + Area of ΔBDC

$= \dfrac{1}{2}(2)(1) + \dfrac{1}{2}(1)(1) = \dfrac{3}{2}$.

$\because$ slope of AB = slope of $DK = \dfrac{1}{2}$,

$\therefore$ equation of DK is $y - 0 = \dfrac{1}{2}(x - 2)$ or $2y = x - 2$ (1)

Equation of BC is $y - 0 = -1(x - 3)$ or $y = -x + 3$ (2)



Solving (1) and (2), we get coordinates of k as $\left(\dfrac{8}{3}, \dfrac{1}{3}\right)$.

$\therefore$ Area of $\Delta BDK = \dfrac{1}{2}\left|\begin{vmatrix} 2 & 1 & 1 \\ 2 & 0 & 1 \\ \frac{8}{3} & \frac{1}{3} & 1 \end{vmatrix}\right|$

$= \dfrac{1}{2}\left|2\left(0 - \dfrac{1}{3}\right) - 1\left(2 - \dfrac{8}{3}\right) + 1\left(\dfrac{2}{3} - 0\right)\right| = \dfrac{1}{3}$.

Hence, area of $\Delta ABC \times$ area of $\Delta BDK = \dfrac{3}{2} \times \dfrac{1}{3} = \dfrac{1}{2}$.

35. Equation of the line AB is

$\dfrac{x}{7} + \dfrac{y}{-5} = 1$

$\Rightarrow 5x - 7y - 35 = 0$.

The equation of the variable line PQ perpendicular to AB is $7x + 5y + k = 0$.

$\Rightarrow$ Coordinates of P are $\left(-\dfrac{k}{7}, 0\right)$ and that of Q are $\left(0, -\dfrac{k}{5}\right)$.

Now, equation of the line AQ is $\dfrac{x}{7} + \dfrac{y}{-\frac{k}{5}} = 1$ (1)

and, equation of the line BP is $\dfrac{x}{-\frac{k}{7}} + \dfrac{y}{-5} = 1$ (2)

Now, locus of R, the point of intersection of AQ and BP can be obtained from (1) and (2) by eliminating k from their equations. Equation (1) can be rewritten as

$$\frac{y}{-\frac{k}{5}} = 1 - \frac{x}{7} = \frac{7-x}{7}$$

$$\Rightarrow \quad \frac{-k}{5} = \frac{7y}{7-x} \quad \Rightarrow \quad k = -5\left(\frac{7y}{7-x}\right) \qquad (3)$$

and, equation (2) can be rewritten as $\frac{x}{-\frac{k}{7}} = 1 + \frac{y}{5} = \frac{5+y}{5}$

$$\Rightarrow \quad -\frac{k}{7} = \frac{5x}{5+y} \quad \Rightarrow \quad k = -\frac{35x}{5+y} \qquad (4)$$

From (3) and (4), we get $-\frac{35y}{7-x} = -\frac{35x}{5+y}$

$\Rightarrow \quad y(5+y) = x(7-x)$

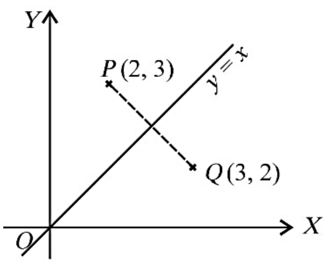
$\Rightarrow \quad x(x-7) + y(y+5) = 0$ is the locus of R.

36. Let Q be the position of the point P (2, 3) after first transformation. Then, $Q \equiv (3, 2)$. Let R be the position of Q after second transformation. Then, $R \equiv (3, 4)$.

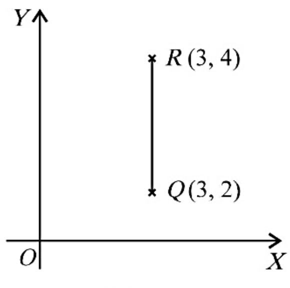
Let S be the new position of R after third transformation. Let $S \equiv (x_1, y_1)$.

If OR makes an angle α with x-axis, then

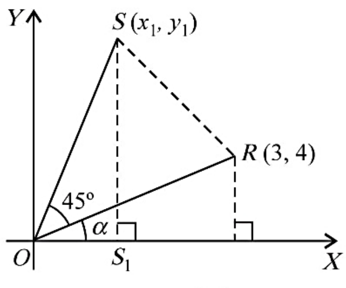
$\tan\alpha = \frac{4}{3}$.



(i)



(ii)



(iii)

$\Rightarrow \quad \sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$. Also, $OR = 5 = OS$.

Now, $\quad x_1 = OS_1 = 5\cos(\alpha + 45°) = \frac{5}{\sqrt{2}}(\cos\alpha - \sin\alpha)$

$$= \frac{5}{\sqrt{2}}\left(\frac{3}{5} - \frac{4}{5}\right) = -\frac{1}{\sqrt{2}}$$

and, $y_1 = SS_1 = 5\sin(\alpha + 45°)$

$$= \frac{5}{\sqrt{2}}(\sin\alpha + \cos\alpha)$$

$$= \frac{5}{\sqrt{2}}\left(\frac{4}{5} + \frac{3}{5}\right) = \frac{7}{\sqrt{2}}.$$

Hence, the coordinates of S are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$.

37. The equation of the line L passing through the intersection of L_1 and L_2 is

$$(ax + by + c) + \lambda(lx + my + n) = 0 \qquad (1)$$

By the given condition, L_1 is the bisector of the angle between L and L_2. Let $Q(\alpha, \beta)$ be any point on L_1.

$\therefore$ Length of perpendicular from $Q(\alpha, \beta)$ on L_2 and L must be equal, thus

$$\frac{l\alpha + m\beta + n}{\sqrt{l^2 + m^2}} = \pm\frac{(a\alpha + b\beta + c) + \lambda(l\alpha + m\beta + n)}{\sqrt{(a + \lambda l)^2 + (b + \lambda m)^2}} \qquad (2)$$

Since $Q(\alpha, \beta)$ lies on L_1 $\therefore$ $a\alpha + b\beta + c = 0 \qquad (3)$

From (2) and (3), we get

$$\frac{1}{\sqrt{l^2 + m^2}} = \pm\frac{\lambda}{\sqrt{(a + \lambda l)^2 + (b + \lambda m)^2}}$$

$\Rightarrow \quad (a + \lambda l)^2 + (b + \lambda m)^2 = \lambda^2(l^2 + m^2)$

$\Rightarrow \quad a^2 + b^2 + 2\lambda al + 2\lambda bm = 0$

$$\Rightarrow \quad \lambda = -\frac{a^2 + b^2}{2(al + bm)} \qquad (4)$$

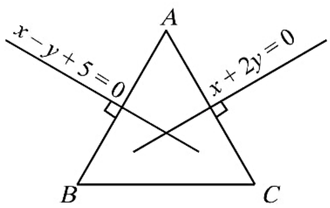
Substituting the value of λ from (4) in (1), we get

$$(ax + by + c) - \frac{a^2 + b^2}{2(al + bm)}(lx + my + n) = 0$$

$\Rightarrow \quad 2(al + bm)(ax + by + c) - (a^2 + b^2)(lx + my + n) = 0$,

which is the required equation of the line L.

38. Let the coordinates of B be (α, β). Since coordinates of A are $(1, -2)$,



$$\therefore \quad \text{the slope of } AB = \frac{\beta + 2}{\alpha - 1} \qquad (1)$$

The equation of the perpendicular bisector of AB is

$$x - y + 5 = 0 \qquad (2)$$

From (1) and (2), we have $\left(\frac{\beta + 2}{\alpha - 1}\right)(1) = -1$

$$\Rightarrow \quad \alpha + \beta + 1 = 0 \qquad (3)$$

Also, the mid point of AB lies on (2),

$$\therefore \quad \left(\frac{\alpha + 1}{2}\right) - \left(\frac{\beta - 2}{2}\right) + 5 = 0$$

$$\Rightarrow \quad \alpha - \beta + 13 = 0 \qquad (4)$$

Solving (3) and (4), we get $\alpha = -7$ and $\beta = 6$.

So, the coordinates of B are $(-7, 6)$.

Similarly, the coordinates of C are $\left(\frac{11}{5}, \frac{2}{5}\right)$.

$\therefore$ The equation of the line BC is

$$y-6=\frac{\frac{2}{5}-6}{\frac{11}{5}+7}(x+7)$$

$\Rightarrow$ $y-6=-\frac{28}{46}\ (x+7) \Rightarrow 23(y-6)+14(x+7)=0$

$\Rightarrow$ $14x+23y=40.$

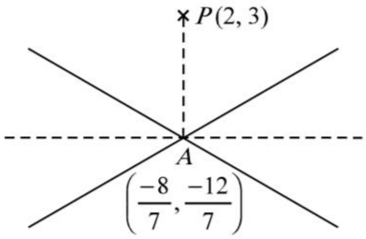
Hence, the equation of the line BC is $14x+23y=40$.

39. We have,

$(2+3t)\,x+(1-2t)\,y+4=0$

$\Rightarrow$ $(2x+y+4)+t\,(3x-2y)=0$

$\Rightarrow$ Every member of the given family of lines passes through the point of intersection of the lines



$2x+y+4=0$ (1)

and, $3x-2y=0$ (2)

Solving (1) and (2), we get

$x=-\frac{8}{7}$ and $y=-\frac{12}{7}$.

So, the point of intersection is $\left(-\frac{8}{7},-\frac{12}{7}\right)$.

The required line passes through the point $\left(-\frac{8}{7},-\frac{12}{7}\right)$ and is $\perp$ to the line joining (2, 3) and $\left(-\frac{8}{7},-\frac{12}{7}\right)$.

Slope of the line joining (2, 3) and $\left(-\frac{8}{7},-\frac{12}{7}\right)$ is

$$=\frac{3+\frac{12}{7}}{2+\frac{8}{7}}=\frac{33}{22}=\frac{3}{2}.$$

$\therefore$ Slope of required line $=-\frac{2}{3}$.

$\therefore$ Equation of the required line is

$$\left(y+\frac{12}{7}\right)=-\frac{2}{3}\left(x+\frac{8}{7}\right)$$

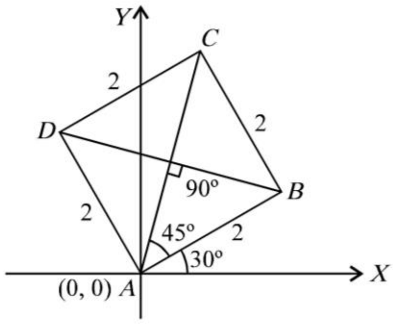
or, $14x+21y+52=0$.

40. Coordinates of vertices B, C and D are

$(2\cos 30°, 2\sin 30°)$,

$(\sqrt{8}\cos 75°, \sqrt{8}\sin 75°)$

and, $(-2\cos 60°, 2\sin 60°)$, respectively.



i.e., $B\equiv(\sqrt{3},1), C\equiv(\sqrt{3}-1,\sqrt{3}+1)$ and $D\equiv(-1,\sqrt{3})$.

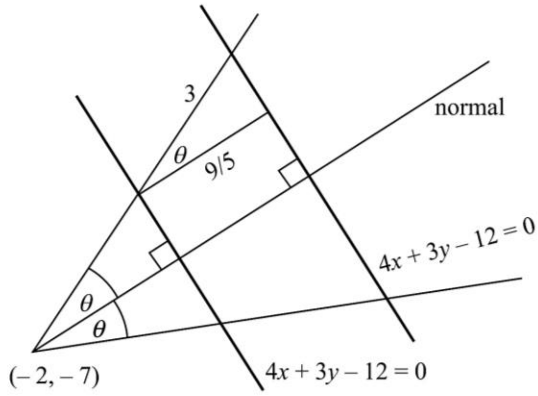
Slope of $BD=\frac{\sqrt{3}-1}{-1-\sqrt{3}}=\frac{(\sqrt{3}-1)^2}{-(\sqrt{3}+1)(\sqrt{3}-1)}=\sqrt{3}-2.$

$\therefore$ Equation of diagonal BD is

$y-1=(\sqrt{3}-2)(x-\sqrt{3})$

$\Rightarrow$ $(2-\sqrt{3})x+y=2(\sqrt{3}-1).$

41. The normal to the given parallel lines cuts an intercept $=\frac{9}{5}$



If θ be the angle that the required line makes with the normal, then

$\cos\theta-\frac{3}{5}\quad\therefore\quad\tan\theta=\frac{4}{3}.$

Let θ be the angle that the required line makes with the normal whose slope is $\frac{3}{4}$. Then, we have

$$\pm\tan\theta=\pm\frac{4}{3}=\frac{m-\frac{3}{4}}{1+\frac{3m}{4}}\quad\Rightarrow\quad m=\infty,\frac{-7}{24}$$

Therefore, the equations of the required lines are

$x+2=0$ and $y+7=\frac{-7}{24}\ (x+2)$.

42. If (x, y) be the coordinates of the required point, then we have

$\frac{|3x-4y+1|}{5}=1$ (1)

and, $\frac{|8x+6y+1|}{10}=1$ (2)

Since (x, y) lies below L_1, therefore

$\frac{3x-4y+1}{-4} < 0$, i.e., $3x - 4y + 1 > 0$

and since it lies above L_2, therefore

$\frac{8x+6y+1}{6} > 0$, i.e., $8x + 6y + 1 > 0$

Removing the mod sign from equations (1) and (2), we have

$+(3x - 4y + 1) = 5$

and, $+(8x + 6y + 1) = 10$

Solving, we get the coordinates of the required point as $\left(\frac{6}{5}, \frac{-1}{10}\right)$.

43. Let R be the radius of the circumcircle and O be the origin, then $AO = \sqrt{x_1^2 + x_2^2 \tan^2 \alpha}$

$\Rightarrow R = x_1 \sec \alpha \Rightarrow x_1 = R\cos\alpha$. Similarly, $x_2 = R\cos\beta$ and $x_3 = R\cos\gamma$

So, the coordinates of vertices are $A(R\cos\alpha, R\sin\alpha)$, $B(R\cos\beta, R\sin\beta)$, $C(R\cos\gamma, R\sin\gamma)$. Hence, the coordinates of centroid G are $\left(\frac{\sum R\cos\alpha}{3}, \frac{\sum R\sin\alpha}{3}\right)$.

Since the orthocentre $H(a, b)$, circumcentre $C(0, 0)$ and the centroid G are collinear, therefore

Slope of OH = Slope of OG

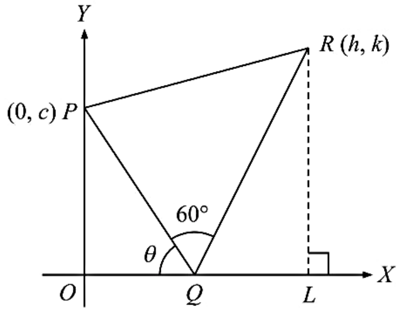
$\Rightarrow \frac{b}{a} = \frac{R(\sin\alpha + \sin\beta + \sin\gamma)}{R(\cos\alpha + \cos\beta + \cos\gamma)}$.

44. Let P be $(0, c)$, c is a constant

From the figure, $h = OL = OQ + QL$

$= c\cot\theta + QR\cos(180 - 60 - \theta)$

$= c\cot\theta + PQ\{\cos 120\cos\theta + \sin 120 \sin\theta\}$



$= c\cot\theta - \frac{PQ}{2}\cos\theta + \frac{\sqrt{3}}{2}PQ\sin\theta$

$= c\cot\theta - \frac{1}{2}c\,\mathrm{cosec}\,\theta\cos\theta + \frac{\sqrt{3}}{2}c\,\mathrm{cosec}\theta\sin\theta \left[\because \sin\theta = \frac{c}{PQ}\right]$

$= \frac{c}{2}(\cot\theta + \sqrt{3})$ (1)

Again, $k = RL = RQ\sin(180 - 60 - \theta)$

$= PQ\left\{\sin\theta.\frac{1}{2} + \cos\theta.\frac{\sqrt{3}}{2}\right\}$

$= c\,\mathrm{cosec}\,\theta\left\{\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right\}$

$= \frac{c}{2}(1 + \sqrt{3}\cot\theta)$ (2)

Eliminating $\cot\theta$ from (1) and (2), we get $k = \sqrt{3}h - c$

$\therefore$ The required locus is $y = \sqrt{3}x - c$ a straight line.

45. Circumcentre of the triangle is (0, 0) and

centroid $\equiv \left(\frac{3 + 5\cos\theta + 5\sin\theta}{3}, \frac{4 + 5\sin\theta - 5\cos\theta}{3}\right)$

$\because$ Centroid divides the join of orthocentre and circumcentre in the ratio, 2 : 1

$\therefore h = 3 + 5\cos\theta + 5\sin\theta$ and

$k = 4 + 5\sin\theta - 5\cos\theta$, where (h, k) represents the orthocentre

$\Rightarrow \sin\theta = \frac{h+k-7}{10}$ and $\cos\theta = \frac{h-k+1}{10}$

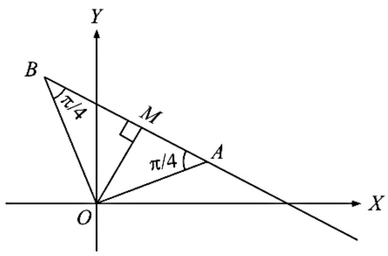
$\Rightarrow (h + k - 7)^2 + (h - k + 1)^2 = 100$

$\therefore$ Locus of orthocentre is $(x + y - 7)^2 + (x - y + 1)^2 = 100$.

46. Let OAB be the right-angled isosceles triangle. Then,

$\angle OAB = \angle OBA = \frac{\pi}{4}$

The perpendicular distance of the line $3x + 4y = 6$ from the origin is



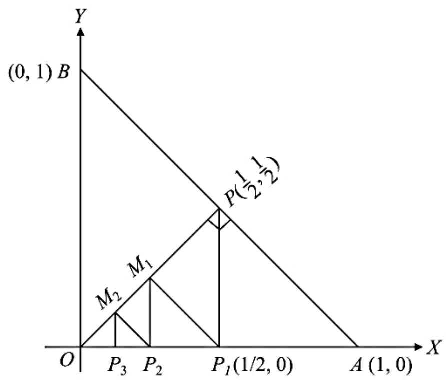
$OM = \frac{6}{\sqrt{3^2+4^2}} = \frac{6}{5}$.

Since, $\angle OAM = \frac{\pi}{4}$, therefore $AM = OM = \frac{6}{5}$.

Hence, area of ΔOAB is $\Delta = OM \times AM = \frac{36}{25}$.

47. P_1 is mid-point of OA, therefore $OP_1 = \frac{1}{2}$

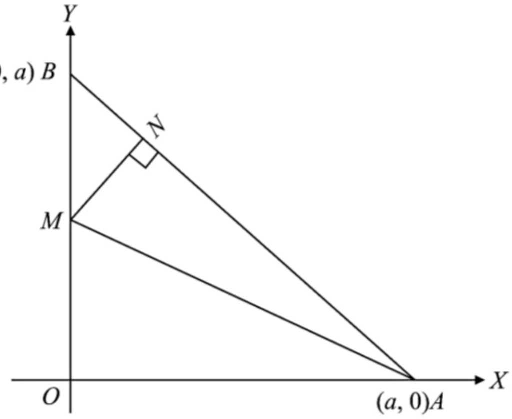
P_2 is mid-point of OP_1, therefore $OP_2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{2^2}$



Proceeding like this, we have

$OP_n = \frac{1}{2^n}$.

48. Let $\frac{AN}{BN} = \lambda$



[putting $x = 0$ in equation (1)]

Then, $N \equiv \left(\frac{a}{1+\lambda}, \frac{a\lambda}{1+\lambda}\right)$

Slope of $AB = -1$

$\therefore$ slope of $MN = +1$

Equation of MN is given by

$$y - \frac{a\lambda}{1+\lambda} = x - \frac{a}{1+\lambda} \qquad (1)$$

Hence, $M \equiv \left(0, \frac{a(\lambda - 1)}{\lambda + 1}\right)$

[putting $x = 0$ in equation (1)]

Area of $\Delta AMN = \frac{1}{2} \times AN \times MN$

$$= \frac{1}{2}\left|\frac{\sqrt{2}a\lambda}{1+\lambda} \times \frac{\sqrt{2}a}{1+\lambda}\right| = \frac{a^2\lambda}{(1+\lambda)^2}$$

Area of $\Delta ABC = \frac{1}{2} \times OA \times OB = \frac{1}{2}a^2$

Given, $\frac{a^2\lambda}{(1+\lambda)^2} = \frac{3}{8} \times \frac{1}{2}a^2$

$\Rightarrow \; 16\lambda = 3(1+\lambda)^2$

$\Rightarrow \; 3\lambda^2 - 10\lambda + 3 = 0 \;\; \Rightarrow \;\; \lambda = \frac{1}{3}, 3$

For $\lambda = \frac{1}{3}$, M lies outside segment OB, hence the only acceptable value for $\lambda = 3$.

More than One Option Correct Type

49. Let a be the side of the square, then diagonal $d = a\sqrt{2}$.

Given, $a_n = \sqrt{2}a_{n+1}$

$\Rightarrow \; a_{n+1} = \frac{a_n}{\sqrt{2}} = \frac{a_{n-1}}{(\sqrt{2})^2} = \frac{a_{n-2}}{(\sqrt{2})^3} = \ldots = \frac{a_1}{(\sqrt{2})^n}$

$\therefore \; a_{n+1} = \frac{a_1}{(\sqrt{2})^n} \;\; \Rightarrow \;\; a_n = \frac{a_1}{(\sqrt{2})^{n-1}} = \frac{10}{2^{\frac{n-1}{2}}}$.

Now, area of $s_n < 1 \;\; \Rightarrow \;\; a_n^2 < 1 \;\; \Rightarrow \;\; \frac{100}{2^{n-1}} < 1$

$\Rightarrow \; 2^n > 200 > 2^7 \;\; \Rightarrow \;\; n > 7$

$\therefore \; n = 8, 9, 10.$

50. The equation of the line in parametric form is

$\frac{x-3}{\cos\theta} = \frac{y-4}{\sin\theta} = r$

Any point on this line is $(3 + r\cos\theta, 4 + r\sin\theta)$

It lies on $x = 6$ if $3 + r\cos\theta = 6 \Rightarrow r = 3\sec\theta$

$\therefore \; PR = 3\sec\theta$

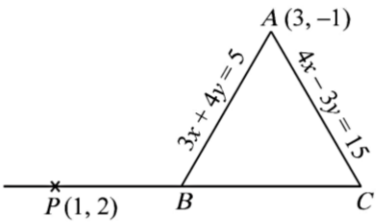
Again, the point lies on $y = 8$ if $4 + r\sin\theta = 8$

$\therefore \; r = 4\operatorname{cosec}\theta$ or $PS = 4\operatorname{cosec}\theta$

51. Let the equations of the lines AB and AC be

$$3x + 4y = 5 \qquad (1)$$

and, $$4x - 3y = 15 \qquad (2)$$



(1) and (2) intersect at $A \equiv (3, -1)$.

[obtained by solving (1) and (2)]

Equation of the line BC, passing through $P(1, 2)$ is

$$y - 2 = m(x - 1) \text{ or } y = m(x-1) + 2 \qquad (3)$$

Solving (2) and (3), we get $C \equiv \left(\frac{21-3m}{4-3m}, \frac{8+11m}{4-3m}\right)$

Solving (1) and (3), we get $B \equiv \left(\frac{4m-3}{4m+3}, \frac{6+2m}{4m+3}\right)$

Since $AC = AB \Rightarrow AC^2 = AB^2$

$\Rightarrow \; \left(\frac{21-3m}{4-3m} - 3\right)^2 + \left(\frac{8+11m}{4-3m} + 1\right)^2$

$= \left(\frac{4m-3}{4m+3} - 3\right)^2 + \left(\frac{6+2m}{4m+3} + 1\right)^2$

$\Rightarrow \; \frac{(9+6m)^2 + (12+8m)^2}{(4-3m)^2}$

$$= \frac{(-8m-12)^2+(6m+9)^2}{(4m+3)^2}$$

$\Rightarrow$ $(100m^2 + 300m + 225)(4m + 3)^2$
$= (100m^2 + 300m + 225)(4 - 3m)^2$

$\Rightarrow$ $25(2m + 3)^2 [(4m + 3)^2 - (4 - 3m)^2] = 0$

$\Rightarrow$ $25(2m + 3)^2 (m + 7)(7m - 1) = 0$

$\Rightarrow$ $m = -\frac{3}{2}, \ -7, \frac{1}{7}.$

When $m = -\frac{3}{2}$, equation (3) becomes $y - 2 = -\frac{3}{2}(x-1)$

or, $3x + 2y - 7 = 0$ which passes through $A(3, -1)$ and hence it cannot be line BC.

When $m = -7$, equation of BC is $y = -7(x - 1) + 2$

or, $7x + y - 9 = 0$

and when $m = \frac{1}{7}$, equation of BC is $y = \frac{1}{7}(x - 1) + 2$ or $x - 7y + 13 = 0$

Hence, equation of BC is $x - 7y + 13 = 0$ or $7x + y - 9 = 0$.

52. The equation of the bisectors of the angles between the given lines are

$$\frac{3x-4y-7}{\sqrt{3^2+(-4)^2}} = \pm\frac{12x-5x+6}{\sqrt{(12)^2+(-5)^2}}$$

$\Rightarrow$ $(39x - 52y - 91) = \pm (60x - 25y + 30)$.

Taking the negative sign,

we get $\quad 99x - 77y - 61 = 0 \quad (1)$

Taking the positive sign,

we get $\quad 21x + 27y + 121 = 0 \quad (2)$

The required lines are the lines through (4, 5) and parallel to the bisectors of the angles between the given lines.

The equation of a line parallel to (1), is

$99x - 77y + k_1 = 0 \quad (3)$

This passes through (4, 5),

$\therefore$ $99(4) - 77(5) + k_1 = 0 \Rightarrow k_1 = -11.$

Putting $k_1 = -11$ in (3), we get

$99x - 77y - 11 = 0$ or $9x - 7y - 1 = 0$.

This is one of the required lines.

The equation of a line parallel to (2), is

$21x + 27y + k_2 = 0 \quad (4)$

This passes through (4, 5),

$\therefore$ $84 + 135 + k_2 = 0 \Rightarrow k_2 = -219.$

Putting $k_2 = -219$ in (4), we get

$21x + 27y - 219 = 0$ or $7x + 9y - 73 = 0$,

which is the other required line.

53. Let the variable line be $ax + by + c = 0 \quad (1)$

Given, $\frac{[(2a+c)+(2b+c)+(a+b+c)]}{\sqrt{a^2+b^2+c^2}} = 0$

$\Rightarrow$ $3a + 3b + 3c = 0$ or $a + b + c = 0$.

So, the equation of the line becomes

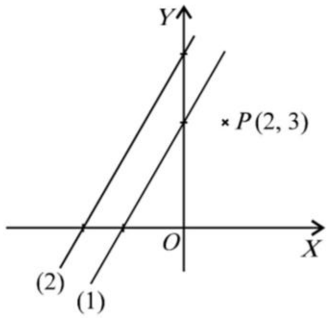
$ax + by - a - b = 0$

or, $a(x - 1) + b(y - 1) = 0$.

$\Rightarrow$ the line passes through the point of intersection of lines $x - 1 = 0$ and $y - 1 = 0$, i.e., the fixed point (1, 1).

So, all such lines are concurrent. Also, (1, 1) is the centroid of the ΔABC.

54. The given lines are



$$2x + y = 3 \quad (1)$$

and, $$2x + y = 5 \quad (2)$$

Equation of any line through $P(2, 3)$ is

$$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r \quad (3)$$

Since line (3) makes an intercept of length 2 units between parallel lines (1) and (2), thus we shall have two values of r as r and $r + 2$ and thus the two corresponding points $(2 + r\cos\theta, 3 + r\sin\theta)$ and $(2 + (r + 2)\cos\theta, 3 + (r + 2)\sin\theta)$ will lie on (1) and (2), respectively.

$\therefore$ $2[2 + (r + 2)\cos\theta] + [3 + (r + 2)\sin\theta] = 5 \quad (4)$

and, $2[2 + r\cos\theta] + [3 + r\sin\theta] = 3 \quad (5)$

Subtracting (5) from (4), we get

$4\cos\theta + 2\sin\theta = 2 \Rightarrow 2\cos\theta + \sin\theta = 1 \quad (6)$

$\Rightarrow$ $2\cos\theta = (1 - \sin\theta) \Rightarrow 4\cos^2\theta = (1 - \sin\theta)^2$

$\Rightarrow$ $4(1 - \sin^2\theta) = (1 - \sin\theta)^2$

$\Rightarrow$ $(5\sin\theta + 3)(1 - \sin\theta) = 0$

$\Rightarrow$ $5\sin\theta + 3 = 0$ or $1 - \sin\theta = 0$

$\Rightarrow$ $\sin\theta = -\frac{3}{5}$ or $\sin\theta = 1$

Putting $\sin\theta = -\frac{3}{5}$ in (6), we get $\cos\theta = \frac{4}{5}$

Putting $\sin\theta = 1$ in (6), we get $\cos\theta = 0$

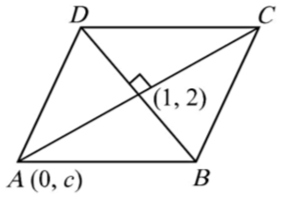
Substituting the values in (3), we get

$$\frac{x-2}{\frac{4}{5}} = \frac{y-3}{-\frac{3}{5}} \text{ and } \frac{x-2}{0} = \frac{y-3}{1}$$

$\Rightarrow$ $3x + 4y = 18$ and $x = 2$.

55. Let the coordinates of A be $(0, c)$. Since (1, 2) is the mid-point of diagonal AC.

$\therefore$ The coordinates of C are $(2, 4 - c)$. Let AD be parallel to $y = x + 2$, therefore, its equation is $y = x + c$. (1)



Let DC be parallel to $y = 7x + 3$, then its equation is

$y = 7x + k$ (2)

Since the point C $(2, 4 - c)$ lies on (2)

$\therefore \quad 4 - c = 7 \times 2 + k \Rightarrow k = -10 - c$ (3)

$\therefore$ Equation of DC will be $y = 7x - 10 - c$ (4)

Solving (1) and (4), the coordinates of D are

$\left(\frac{5+c}{3}, \frac{5+4c}{3}\right)$.

It is given that $ABCD$ is a rhombus.

$\therefore \quad AD = DC \quad \Rightarrow \quad AD^2 = DC^2$

$$\Rightarrow \left(\frac{5+c}{3} - 0\right)^2 + \left(\frac{5+4c}{3} - c\right)^2$$

$$= \left(\frac{5+c}{3} - 2\right)^2 + \left(\frac{5+4c}{3} - 4 + c\right)^2$$

$\Rightarrow \quad (5 + c)^2 + (5 + c)^2 = (c - 1)^2 + (7c - 7)^2$

$\Rightarrow \quad 2(c + 5)^2 = 50(c - 1)^2$

$\Rightarrow \quad 2c^2 + 20c + 50 = 50c^2 - 100c + 50$

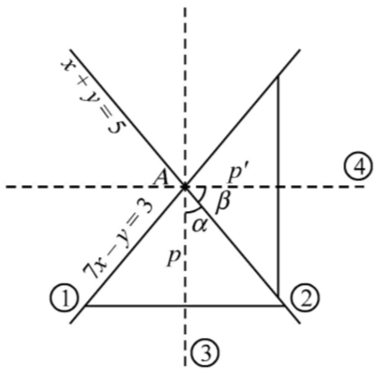
$\Rightarrow \quad 48c^2 = 120c \Rightarrow c = 0$ or $\frac{5}{2}$.

Hence, the possible coordinates of A are $(0, 0)$ or $\left(0, \frac{5}{2}\right)$.

56. The equations of equal sides are

$7x - y = 3$ (1)

and, $x + y = 5$ (2)



Solving (1) and (2), we get

$A \equiv (1, 4)$.

The equations of bisectors of angles of (1) and (2) are

$$\frac{7x - y - 3}{5\sqrt{2}} = \pm \frac{x + y - 5}{\sqrt{2}}$$

or, $7x - y - 3 = \pm (5x + 5y - 25)$

or, $x - 3y + 11 = 0$ (3)

and, $3x + y - 7 = 0$ (4)

Clearly the third side BC is $\perp$ to (3) or (4).

Let α be the angle between (3) and (2), then

$$\tan\alpha = \left|\frac{\frac{1}{3} + 1}{1 - \frac{1}{3}}\right| = 2.$$

Then, base $= 2p \tan\alpha = 4p$.

$\therefore$ Area of $\Delta ABC = 4p \cdot \frac{p}{2} = 5$ or $p^2 = \frac{5}{2}$.

Any line $\perp$ to (3) is $3x + y + \lambda = 0$.

$\perp$ distance from A $(1, 4)$ is $\left|\frac{3 + 4 + \lambda}{\sqrt{10}}\right| = \frac{\sqrt{5}}{\sqrt{2}}$.

$\Rightarrow \quad \lambda + 7 = \pm 5$ or $\lambda = -2, -12$.

$\therefore$ Equation of side BC may be $3x + y - 2 = 0$

or, $3x + y - 12 = 0$.

Suppose, β may be the angle between (4) and (2), then

$$\tan\beta = \left|\frac{-3 + 1}{1 + 3}\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}.$$

Then, the base $= 2p' \tan\beta = p'$.

$\therefore$ Area of $\Delta ABC = \frac{1}{2} p' \times p' = 5$ or $p'^2 = 10$.

Any line $\perp$ to (4) is $x - 3y + \mu = 0$.

$\perp$ distance from A $(1, 4)$ is $\left|\frac{1 - 12 + \mu}{\sqrt{10}}\right| = \sqrt{10}$.

$\Rightarrow \quad \mu - 11 = \pm 10$ or $\mu = 21, 1$.

$\therefore$ Equation of BC may be $x - 3y + 21 = 0$

or, $x - 3y + 1 = 0$.

Hence, all possible equations of BC are

$3x + y - 2 = 0$, $3x + y - 12 = 0$,

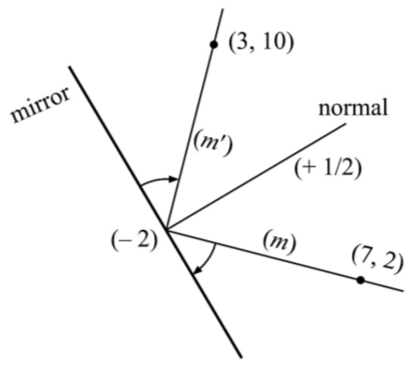
$x - 3y + 1 = 0$, $x - 3y + 21 = 0$.

57. Equation of a straight line passing through the point $(7, 2)$ be

$y - 2 = m(x - 7)$ (1)

and that of a straight line passing through the point $(3, 10)$ be

$y - 10 = m'(x - 3)$ (2)



Since the angle made by the incident and the reflected rays with the mirror must be same, therefore we have

$$\frac{-2-m'}{1-2m'}=\frac{m-(-2)}{1-2m}$$

i.e., $3m + 3m' + 4mm' - 4 = 0$ (3)

Also, since the lines (1), (2) and the mirror will be concurrent, therefore we have

$$\begin{vmatrix} m & -1 & 2-7m \\ m' & -1 & 10-3m' \\ 2 & 1 & -6 \end{vmatrix}=0$$

i.e., $18m - 2m' + 4mm' + 16 = 0$ (4)

Subtracting equation (4) from equation (3), we get

$m' = 3m + 4$ (5)

Now, solving equations (3), (5) we have

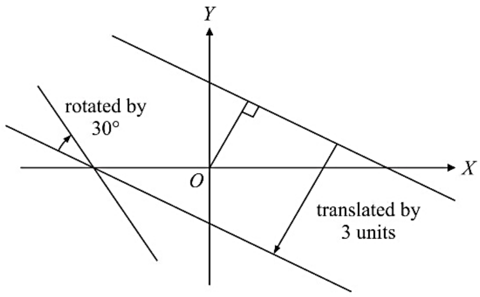
$m=-2,\ -\frac{1}{3}$ and $m'=-2,\ 3$

Since $m = m' = -2$ is not acceptable, therefore equations of the required lines are

$y-2=-\frac{1}{3}(x-7)$ and $y - 10 = 3(x - 3)$.

58. Let equation of the shifted line be

$x + 2y + c = 0$ (1)



Since the distance between the shifted and the original line is given to be 3 units, therefore we have

$\frac{4+c}{\sqrt{5}}=\pm 3$ i.e., $c=+4\pm 3\sqrt{5}$

The given line $x + 2y = 4$ lies above the origin at a distance $\frac{4}{\sqrt{5}}$ units, therefore the line obtained by shifting the given line by 3 units will lie below the origin. Hence, c must be positive.

Thus, the required value of c is $3\sqrt{5}-4$

The shifted line given by equation (1) cuts the x-axis at $(-c, 0)$. If m be the slope of the rotated line, then we have

$$\tan 30° = \frac{-\frac{1}{2}-m}{1-\frac{m}{2}} \Rightarrow m=\frac{-(2+\sqrt{3})}{2\sqrt{3}-1}$$

Passage Based Questions

59. If two lines make equal angles with x-axis and are not parallel, then only possibility is that angles are supplementary, i.e., if one line makes the angle θ with x-axis, the other will make $-\theta$.

$$\tan^{-1}\frac{m_1\sin\omega}{1+m_1\cos\omega}=-\tan^{-1}\frac{m_2\sin\omega}{1+m_2\cos\omega}$$

or, $\frac{m_1\sin\omega}{1+m_2\cos\omega}+\frac{m_2\sin\omega}{1+m_2\cos\omega}=0$

or, $m_1+m_2+2m_1m_2\cos\omega=0$ ($\because \sin\omega\neq 0$).

$\therefore\ k=2$

60. Any line through the point $(-2, 3)$ is

$y - 3 = m(x + 2)$

As the line is perpendicular to the $y + 3x = 6$, we have

$1 + (m - 3)\cos 30° - 3m = 0$ or $m=\frac{3\sqrt{3}-2}{\sqrt{3}-6}$.

61. The equation are given as $y=x\tan\frac{11\pi}{24}$ (1)

and, $y=x\tan\frac{19\pi}{24}$ (2)

If two lines are perpendicular to each other and the axes be inclined at an angle of ω, the condition is $1 + (m + m')\cos\omega + mm' = 0$

Applying the condition

$$1+\left(\tan\frac{11\pi}{24}+\tan\frac{19\pi}{24}\right)\cos\omega+\tan\frac{11\pi}{24}\tan\frac{19\pi}{24}=0$$

or, $\left[\frac{\sin\frac{11\pi}{24}\tan\frac{19\pi}{24}+\sin\frac{19\pi}{24}\cos\frac{11\pi}{24}}{\cos\frac{11\pi}{4}.\cos\frac{19\pi}{4}}\right]\cos\omega$

$$=-\left[\frac{\cos\frac{11\pi}{24}\cos\frac{19\pi}{4}+\sin\frac{19\pi}{4}\sin\frac{11\pi}{4}}{\cos\frac{11\pi}{4}.\cos\frac{19\pi}{4}}\right]$$

or, $\sin\left(\frac{11\pi}{24}+\frac{19\pi}{24}\right)\cos\omega=-\cos\left(\frac{19\pi}{24}-\frac{11\pi}{24}\right)$

or, $\sin\left(\frac{5\pi}{4}\right)\cos\omega=-\cos\left(\frac{\pi}{3}\right)$

or, $\cos\omega=-\frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{5\pi}{4}\right)}=-\frac{\left(\frac{1}{2}\right)}{\left(\frac{-1}{\sqrt{2}}\right)}=\frac{1}{\sqrt{2}}$

or, $\omega=\frac{\pi}{4}$.

62. Lines are given as

$8x + 7y = 1$ (1)

and, $28x - 73y = 101.$ (2)

$\therefore \quad m_1 = -\frac{8}{7}.\ m_2 = \frac{28}{73}$ and $\omega = 120^\circ$ (given)

$$\therefore \quad \tan\theta = \frac{\left(-\frac{8}{7} - \frac{28}{73}\right)\sin 120^\circ}{1 + \left(-\frac{8}{7} + \frac{28}{73}\right)\cos 120^\circ + \left(-\frac{8}{7}\right)\left(\frac{28}{73}\right)}$$

$$= \frac{30\sqrt{3}}{37}.$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{30\sqrt{3}}{37}\right).$$

Match the Column Type

63. I. Since the distance between the parallel lines $lx + my + n = 0$ and $lx + my + n' = 0$ is same as the distance between the parallel lines $mx + ly + n = 0$ and $mx + ly + n' = 0$. Therefore, the parallelogram is a rhombus. Since the diagonals of a rhombus are at right angles, therefore the required angle is $\frac{\pi}{2}$.

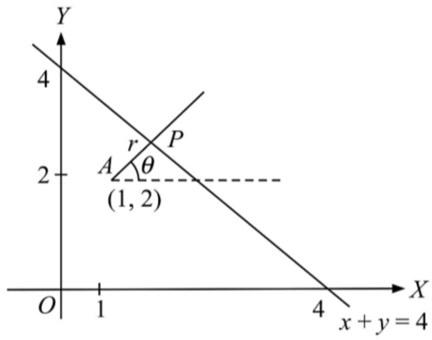
II. $\tan\theta = \left|\frac{2+1}{1-2}\right| = 3 \quad \Rightarrow \quad \theta = \tan^{-1} 3$

III. $AP = r = \frac{\sqrt{6}}{3}$

Equation to $AP : \frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = r$

$\therefore \quad P \equiv (1 + r\cos\theta, 2 + r\sin\theta)$

'P' will satisfy $x + y = 4$



$$\therefore \quad 1 + \frac{\sqrt{6}}{3}\cos\theta + 2 + \frac{\sqrt{6}}{3}\sin\theta = 4$$

$$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} = \cos 30^\circ$$

$\therefore \quad \theta = 75^\circ$ or 15°

64. I. We have, $AB = BC$

$\Rightarrow$ B is mid point of AC

$\Rightarrow \quad \frac{x+z}{2} = y$ and $\frac{y+z+x+y}{2} = z + x$

$\Rightarrow \quad x + z = 2y$ and $2y = x + z$

$\Rightarrow \quad x, y, z$ are in A. P.

II. The given lines are

$7x + y - 16 = 0$ (1)

$5x - y - 8 = 0$ (2)

and, $x - 5y + 8 = 0$ (3)

Equation of the line passing through the point $A\ (k + 1, 2k)$ and making an angle θ with the positive direction of x-axis is

$$\frac{x-(k+1)}{\cos\theta} = \frac{y-2k}{\sin\theta}$$

$= r_1, r_2, r_3$ (if $AB = r_1, AC = r_2, AD = r_3$)

$\therefore \quad B \equiv [(k + 1) + r_1\cos\theta, 2k + r_1\sin\theta]$

$C \equiv [(k + 1) + r_2\cos\theta, 2k + r_2\sin\theta]$

$D \equiv [(k + 1) + r_3\cos\theta, 2k + r_3\sin\theta]$

Since the points B, C and D lie on the lines (1), (2) and (3), respectively,

$$\therefore \quad r_1 = \frac{9(1-k)}{7\cos\theta + \sin\theta}, r_2 = \frac{3(1-k)}{5\cos\theta - \sin\theta}$$

and, $$r_3 = \frac{9(1-k)}{5\sin\theta - \cos\theta}$$

$$\therefore \quad \frac{1}{r_2} + \frac{1}{r_3} = \frac{5\cos\theta - \sin\theta}{3(1-k)} + \frac{5\sin\theta - \cos\theta}{9(1-k)}$$

$$= \frac{15\cos\theta - 3\sin\theta + 5\sin\theta - \cos\theta}{9(1-k)}$$

$$= \frac{14\cos\theta + 2\sin\theta}{9(1-k)} = \frac{2}{r_1}.$$

Hence, r_2, r_1, r_3 are in H. P.

III. The given line is $x\cos\theta + y\sin\theta + \frac{\sin^2\theta}{\cos\theta} = 0$ (1)

Let p_1, p_2, p_3 be the lengths of the perpendiculars from the points $A\ (m^2, 2m)$, $B\ (mn, m + n)$ and $C\ (n^2, 2n)$ on line (1), then

$$p_1 = \frac{m^2\cos\theta + 2m\sin\theta + \frac{\sin^2\theta}{\cos\theta}}{\sqrt{\sin^2\theta + \cos^2\theta}} = \frac{(m\cos\theta + \sin\theta)^2}{\cos\theta}$$

$$p_2 = mn\cos\theta + (m+n)\sin\theta + \frac{\sin^2\theta}{\cos\theta}$$

$= \frac{mn\cos^2\theta + m\sin\theta\cos\theta + n\sin\theta\cos\theta + \sin^2\theta}{\cos\theta}$

$= \frac{1}{\cos\theta}[m\cos\theta(n\cos\theta + \sin\theta) + \sin\theta\,(n\cos\theta + \sin\theta)]$

$= \frac{(m\cos\theta + \sin\theta)\,(n\cos\theta + \sin\theta)}{\cos\theta}$ and

$p_3 = n^2\cos\theta + 2\,n\sin\theta + \frac{\sin^2\theta}{\cos\theta} = \frac{(n\cos\theta + \sin\theta)^2}{\cos\theta}$

Now, $p_1 p_3 = \frac{(m\cos\theta + \sin\theta)^2\,(n\cos\theta + \sin\theta)^2}{\cos^2\theta} = p_2^2$

Hence, p_1, p_2, p_3 are in G. P.

Previous Year's Questions

65. Let $A(4, 0)$, $B(-1, -1)$ and $C(3, 5)$ be the vertices of a ΔABC.

Then $AB = \sqrt{(-1-4)^2 + (-1-0)^2}$

$= \sqrt{25+1} = \sqrt{26}$

$BC = \sqrt{(3+1)^2 + (5+1)^2} = \sqrt{4^2 + 6^2}$

$= \sqrt{16+36} = \sqrt{52}$

and $CA = \sqrt{(4-3)^2 + (0-5)^2}$

$= \sqrt{1+25} = \sqrt{26}$

$CA^2 + AB^2 = \left(\sqrt{26}\right)^2 + \left(\sqrt{26}\right)^2$

$\therefore = 26 + 26 = 52$

$= BC^2$

$\Rightarrow \quad CA^2 + AB^2 = BC^2$

Thus, the triangle is isosceles and right angled triangle.

66. The equation of parabola is

$y^2 + 4y + 4x + 2 = 0$

$\Rightarrow \quad y^2 + 4y + 4 = -4x - 2 + 4$

$\Rightarrow \quad (y+2)^2 = -4\left(x - \frac{1}{2}\right)$

Transformation $y + 2 = Y$ and $x - \frac{1}{2} = X$ gives

$Y^2 = -4X$

Here $a = 1$

$\therefore$ Equation of directrix is $X = 1$

$\Rightarrow \quad x - \frac{1}{2} = 1$

$\Rightarrow \quad x = \frac{3}{2}$

67. Key Idea: If the triangle is equilateral, then the incentre coincides with the centroid of the triangle.

Let $A(1, \sqrt{3})$, $B(0, 0)$, $C(2, 0)$ be the vertices of a triangle ABC.

$\therefore \quad a = BC\sqrt{(2-0)^2 + (0-0)^2} = 2$

$b = AC\sqrt{(2-1)^2 + (0-\sqrt{3})^2} = 2$

and $c = AB\sqrt{(0-1)^2 + (0-\sqrt{3})^2} = 2$

$b = AC = -J\,(2 - 1f + (0 - V3)^2 = 2$

and $c = AB = V(0 - 1)^2 + (0 - 41) = 2$

$\therefore$ The triangle is an equilateral triangle.

$\therefore$ Incentre is same as centroid of the triangle.

$\Rightarrow$ Co-ordinates of incentre are

$\left(\frac{1+0+2}{3}, \frac{\sqrt{3}+0+0}{3}\right)$ i.e., $\left(1, \frac{1}{\sqrt{3}}\right)$

68. Key Idea: Equations of angle bisectors of lines

$a_1x + b_1y + c_1 = 0$, and

$a_2x + b_2y + c_2 = 0$

are $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

For the two lines $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$, the angle bisectors are given by

$\frac{24x + 7y - 20}{25} = \pm\frac{4x - 3y - 2}{5}$

Taking positive sign, we get

$2x + 11y - 5 = 0$

Therefore, the given three lines are concurrent with one line bisecting the angle between the other two.

69. $\sqrt{3}x + y = 0$ makes an angle 120° with OX and $\sqrt{3}x + y = 0$ makes an angle of 60° with OX. So, the required equation of line is $y - 2 = 0$.

70. Let $p\,(x, y)$ be the point equidistant to the given points, then

$(x - a_1)^2 + (y - b_1)^2 = (x - a_2)^2 + (y - b_2)^2$

$\Rightarrow \quad (a_1 - a_2)\,x + (b_1 - b_2)\,y + \frac{1}{2}(b_2^2 - b_1^2 + a_2^2 - a_1^2) = 0$

Hence, (A) is the correct answer.

71. Given parametric equations

$x = \frac{a\cos t + b\sin t + 1}{3}$, $y = \frac{a\sin t - b\cos t + 1}{3}$

This implies that $\left(x - \frac{1}{3}\right)^2 + y^2 = \frac{a^2 + b^2}{9}$.

Hence, (B) is the correct answer.

72. If C be (h, k) then centroid is $\left(\frac{h}{3}, \frac{(k-2)}{3}\right)$ it lies on $2x + 3y = 1$.

Therefore the locus is $2x + 3y = 9$.

73. Point (4, 3) lies on $\frac{x}{a} + \frac{y}{b} = 1$ with $a + b = -1$ then $\frac{4}{a} + \frac{3}{b} = 1$

$\Rightarrow \quad a = 2, b = -3$ or $a = -2, b = 1$.

Hence $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{-2}+\frac{y}{1}=1$.

74. $m_1+m_2=-\frac{2c}{7}$ and, $m_1m_2=-\frac{1}{7}$

$\Rightarrow\ m_1+m_2=4m_1m_2$ (given)

$\Rightarrow\ c=2.$

75. $m_1+m_2=\frac{1}{4c}, m_1m_2=\frac{6}{4c}$ and $m_1=-\frac{3}{4}$.

Hence $c=-3$.

76. $P=(1,0)$

Let $Q=(h,k)$ then $k^2=8h$

Let (α,β) be the midpoint of PQ, then

$\alpha=\frac{h+1}{2},\ \beta=\frac{k+0}{2}$

$\Rightarrow\ 2\alpha-1=h,\ 2\beta=k.$

$\Rightarrow\ (2\beta)^2=8(2\alpha-1)\ \Rightarrow\ \beta^2=4\alpha-2$

$\Rightarrow\ y^2-4x+2=0.$

77. Required equation is of the from

$ax+2by+3b+\lambda(bx-2ay-3a)=0$

$\Rightarrow\ (a+b\lambda)x+(2b-2a\lambda)y+3b-3\lambda a=0$

$a+b\lambda=0\Rightarrow\lambda=-\frac{a}{b}$

$\Rightarrow\ ax+2by+3b-\frac{a}{b}(bx-2ay-3a)=0$

$\Rightarrow\ ax+2by+3b-ax+\frac{2a^2}{b}y+\frac{3a^2}{b}=0$

$\Rightarrow\ y\left(2b+\frac{2a^2}{b}\right)+3b+\frac{3a^2}{b}=0$

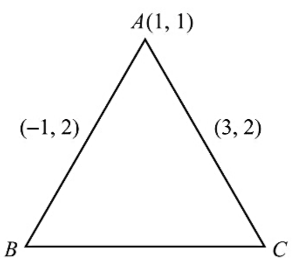
$\Rightarrow\ y\left(\frac{2b^2+2a^2}{b}\right)=-\left(\frac{3b^2+3a^2}{b}\right)$

$\Rightarrow\ y=\frac{-3(a^2+b^2)}{2(b^2+a^2)}=\frac{-3}{2}$

$\Rightarrow\ y=-\frac{3}{2}$ so it is $\frac{3}{2}$ units below x-axis.

78. Vertex of triangle is (1, 1) and midpoint of sides through this vertex is (–1, 2) and (3, 2) $\Rightarrow$ vertices B and C come out to be (–3, 3) and (5, 3)

$\therefore$ centroid is $\left(\frac{1-3+5}{3},\frac{1+3+3}{3}\right)=\left(1,\frac{7}{3}\right)$



79. The required line intersects the axes at points (0, 8) and (6, 0) Hence, the equation of the line is

$y=\frac{-4}{3}(x-6)$

$\Rightarrow\ 4x+3y=24$

80. Given parabola: $y=\frac{a^2x^2}{3}+\frac{a^2x}{2}-2a$

Vertex: (α,β) implies

$$\alpha=\frac{\frac{-a^2}{2}}{\frac{2a^3}{3}}=-\frac{3}{4a},\ \beta=\frac{-\left(\frac{a^4}{4}+4\cdot\frac{a^3}{3}\cdot 2a\right)}{4\frac{a^3}{3}}=-\frac{-\left(\frac{1}{4}+\frac{8}{3}\right)a^4}{\frac{4}{3}a^3}$$

$$=-\frac{35}{12}\frac{a}{4}\times 3=-\frac{35}{16}a$$

$\therefore\ \alpha\beta=-\frac{3}{4a}\left(-\frac{35}{16}\right)a=\frac{105}{64}$

81. We must have

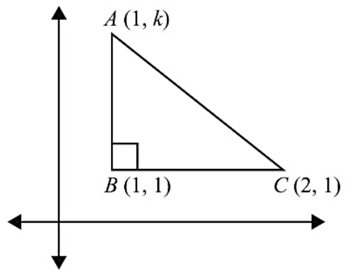
$a^2-3a<0$ and $a^2-\frac{a}{2}>0$

$\Rightarrow\ \frac{1}{2}<a<3$

82. We have $\frac{1}{2}(k-1)=\pm 1$

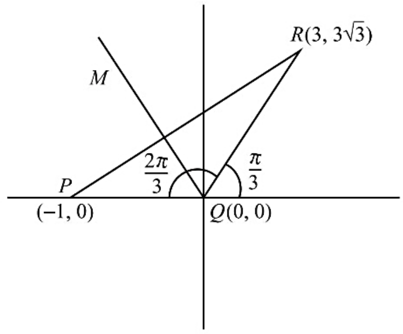
$\Rightarrow\ k-1=\pm 2$

$\Rightarrow\ k=3$ or $k=-1$



83. Slope of the line QM is tan $\frac{2\pi}{3}=-\sqrt{3}$

Hence, the equation of line QM is $y=-\sqrt{3}x$.



84. Equation of bisectors of line $x=0$ and $y=0$ are $y=\pm x$.

Put $y=\pm x$ in $my^2+(1-m^2)xy-mx^2=0$, we get $(1-m^2)x^2=0$

$\Rightarrow\ m=\pm 1.$

85. Slope of the bisector = $k - 1$

Mid-point of $PQ = \left(\frac{k+1}{2}, \frac{7}{2}\right)$

Equation of bisector is

$$y - \frac{7}{2} = (k-1)\left(x - \frac{(k+1)}{2}\right)$$

Put $x = 0$ and $y = -4$.

$\Rightarrow \quad k = \pm 4.$

86. Slope of line $L = -\frac{b}{5}$

Slope of line $K = -\frac{3}{c}$

Link L is parallel to link k.

$\Rightarrow \quad \frac{b}{5} = \frac{3}{c} \quad \Rightarrow \quad bc = 15$

(13, 32) is a point on L

$\Rightarrow \quad \frac{13}{5} + \frac{32}{b} = 1 \quad \Rightarrow \quad \frac{32}{b} = -\frac{8}{5}$

$\Rightarrow \quad b = -20 \quad \Rightarrow \quad c = -\frac{3}{4}$

Equation of K: $y - 4x = 3$

Distance between L and $K = \frac{|52 - 32 + 3|}{\sqrt{17}} = \frac{23}{\sqrt{17}}$

87. $P(-2, -2)$; $Q = (1, -2)$

Equation of angular bisector OR is

$(\sqrt{5} + 2\sqrt{2})x = (\sqrt{5} - \sqrt{2})y$

$\therefore \quad PR : RQ = 2\sqrt{2} : \sqrt{5}$

88. $b^2 = a^2(1 - e^2) = a^2\left(1 - \frac{2}{5}\right) = a^2\frac{3}{5} = \frac{3a^2}{5}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{9}{a^2} + \frac{5}{3a^2} = 1$

$a^2 = \frac{32}{3}$

$b^2 = \frac{32}{5}$

$\therefore$ Required equation of ellipse $3x^2 + 5y^2 - 32 = 0$.

89. Point $p = \left(\frac{6+2}{5}, \frac{12+2}{5}\right)$

$p = \left(\frac{8}{5}, \frac{14}{5}\right)$

$p = \left(\frac{8}{5}, \frac{14}{5}\right)$ lies on $2x + y = k$

$\Rightarrow \quad \frac{16}{5} + \frac{14}{5} = k$

$\Rightarrow \quad k = \frac{30}{5} = 6$

90. Equation of line passing through (1, 2) with slope m is $y - 2 = m(x - 1)$

Area of $\Delta OPQ = \frac{(m-2)^2}{2|m|}$

$\Delta = \frac{m^2 + 4 - 4m}{2m}$

$\Delta = \frac{m}{2} + \frac{2}{m} - 2$

Δ is least if $\frac{m}{2} = \frac{2}{m}$

$\Rightarrow \quad m^2 = 4$

$\Rightarrow \quad m = \pm 2$

$\Rightarrow \quad m = -2$

91. Slope of the incident ray $= -\frac{1}{\sqrt{3}}$.

So, the slope of the reflected ray must be $\frac{1}{\sqrt{3}}$.

Now, the point of incidence is $(\sqrt{3}, 0)$ And so, the equation.

of reflected ray is $y = \frac{1}{\sqrt{3}}(x - \sqrt{3})$.

92. Abscissa $= \frac{ax_1 + bx_2 + cx_3}{a+b+c}$

$= \frac{2\times 2 + 2\sqrt{2}\times 0 + 2\times 0}{2 + 2 + 2\sqrt{2}}$

$= \frac{4}{4 + 2\sqrt{2}} = \frac{2}{2+\sqrt{2}} = 2 - \sqrt{2}.$

Alternate Solution:

Abscissa $= r = (s - a)\tan\frac{A}{2}$

$= \left(\frac{4 + 2\sqrt{2}}{2} - 2\sqrt{2}\right)\tan\frac{\pi}{4} = 2 - \sqrt{2}$

93. Let point of intersection is $(h, -h)$

$\Rightarrow \quad \begin{cases} 4ah - 2ah + c = 0 \\ 5bh - 2bh + d = 0 \end{cases}$

So, $\frac{c}{2a} = -\frac{d}{3b}$

$3bc - 2ad = 0$

94. $S\left(\frac{13}{2}, 1\right), P(2, 2)$

Slope $= -\frac{2}{9}$

Equation will be $\frac{y+1}{x-1} = -\frac{2}{9}$

$9y + 9 + 2x - 2 = 0$

$2x + 9y + 7 = 0$

95. $x + y < 41, x > 0, y > 0$ is bounded region.

Now, number of positive integral solutions of the equation $x + y + k = 41$ will be number of integral co-ordinates in the bounded region.

$= {}^{41-1}C_{3-1} = {}^{40}C_2 = 780.$

96. Let M be mid-point of BB' and AM is $\perp$ bisector of BB' (where A is the point of intersection of the given lines)

$(x - 2)(x - 1) + (y - 2)(y - 3) = 0$

$\Rightarrow \left(\frac{h+2}{2} - 2\right)\left(\frac{h+2}{2} - 1\right) + \left(\frac{k+3}{2} - 2\right)\left(\frac{k+3}{2} - 3\right) = 0$

$\Rightarrow (h - 2)(h) + (k - 1)(k - 3) = 0$

$\Rightarrow x^2 - 2x + y^2 - 4y + 3 = 0$

$\Rightarrow (x - 1)^2 + (y - 2)^2 = 2.$

97. Equation of angle bisector of the lines $x - y + 1 = 0$ and $7x - y - 5 = 0$ is given by

$\frac{x-y+1}{\sqrt{2}} = \pm\frac{7x-y-5}{5\sqrt{2}}$

$\Rightarrow 5(x - y + 1) = 7x - y - 5$

and

$5(x - y + 1) = -7x + y + 5$

$\therefore \; 2x + 4y - 10 = 0 \Rightarrow x + 2y - 5 = 0$ and

$12x - 6y = 0 \Rightarrow 2x - y = 0$

Now equation of diagonals are

$(x + 1) + 2(y + 2) = 0 \Rightarrow x + 2y + 5 = 0$ (1)

and

$2(x + 1) - (y + 2) = 0 \Rightarrow 2x - y = 0$ (2)

Cleary $\left(\frac{1}{3}, \frac{8}{3}\right)$ lies on (1)