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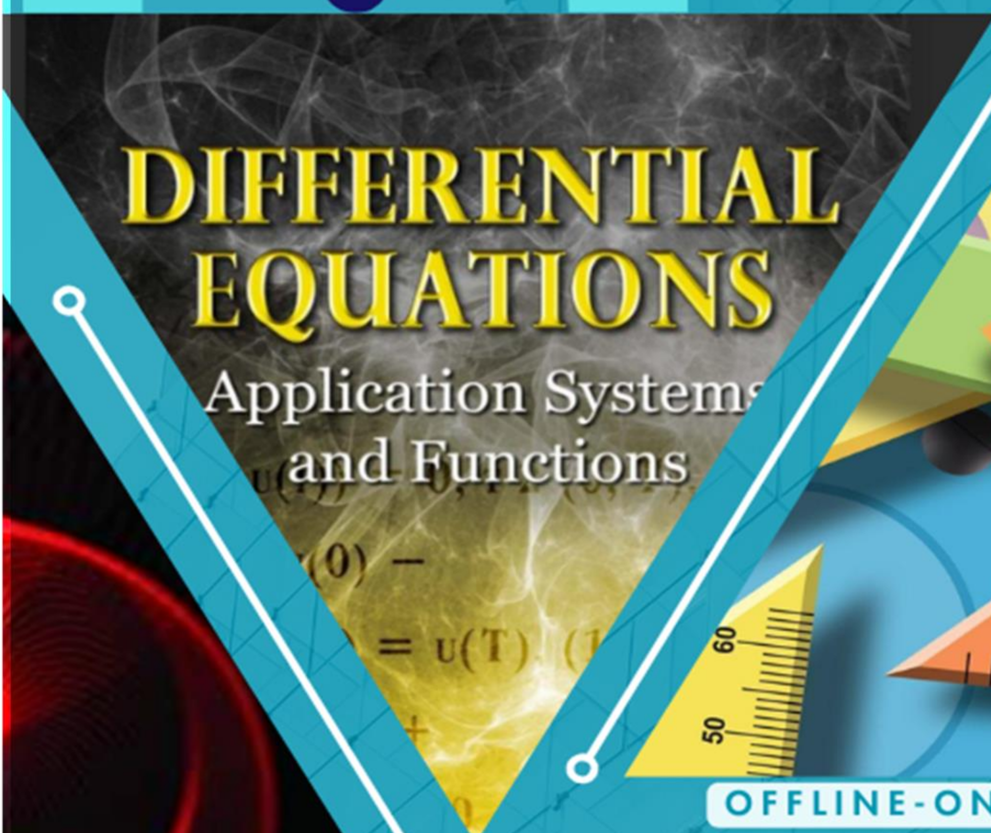

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MATHEMATICS IIT-JEE: DIFFERENTIAL EQUATIONS

An equation involving the derivatives of a dependent variable with respect to one or more independent variables is called a differential equation. *The order* of a *differential equation* is the order of the highest differential coefficient involved. When an equation is a polynomial in all the differential coefficients, the power to which the highest differential coefficient is raised is known as *the degree of the equation*.

Illustration 1

Any differential equation of order 1 is of the form $f(x, y, y') = 0$ e. g.
$y' = y$, $xy' = 2y + x$ are 1 order differential equations. The order of the equation
$y'' + 5y = 0$ is 2.
The order of $x^2 y''' y' + e^x y'' = (x^2 + 3) y^3$ is order 3. The degree of the equations

$$\frac{dy}{dx} + y^2 = 0,\ \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$$

and $\left(\dfrac{d^2y}{dx^2}\right)^2 + \dfrac{dy}{dx} + y = x^2$ are

1, 1 and 2 respectively.

FORMATION OF A DIFFERENTIAL EQUATION

To obtain the differential equation whose solution is the equation $f(x, y, c_1, c_2, \cdots, c_n) = 0$, where x and y are variables and $c_1, c_2, \cdots, c_n$ are arbitrary constants we differentiate the above equation n times successively, so that $n + 1$ equations are obtained. From these $n + 1$ equations, eliminate the constants $c_1, c_2, \cdots c_n$.

Illustration 2

Find the differential equation of family of circles of radius 5cm having centre on x-axis.
The equation of such a circle is

$$(x - a)^2 + y^2 = 25 \qquad (1)$$

Differentiating this equation with respect to x, we get

$$2(x - a) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow x - a = -y\frac{dy}{dx}$$

Substituting this value in (1), we get

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = 25$$

which is the required differential equation.

Illustration 3

Find the differential equation corresponding to the equation $y = ae^x + be^{2x} + ce^{-3x}$ (i)
where a, b, c are arbitrary constants.

$$y' = ae^x + 2be^{2x} - 3ce^{-3x} \qquad (ii)$$
$$y'' = ae^x + 4be^{2x} + 9ce^{-3x} \qquad (iii)$$
$$y''' = ae^x + 8be^{2x} - 27ce^{-3x} \qquad (iv)$$
$$y' - y = be^{2x} - 4ce^{-3x} \qquad (v)$$
$$y'' - y' = 2be^{2x} + 12ce^{-3x} \qquad (vi)$$
$$y''' - y'' = 4be^{2x} - 36ce^{-3x} \qquad (vii)$$

Putting (v) in (vi)

$$y'' - y' = 2(y' - y) + 8ce^{-3x} + 12ce^{-3x}$$
$$\Rightarrow y'' - 3y' + 2y = 20ce^{-3x}$$

Putting this value in (v),

$$be^{2x} = (y' - y) + 4ce^{-3x}$$

$$= (y' - y) + \frac{1}{5}(y'' - 3y' + 2y)$$

$$= \frac{1}{5}y'' + \frac{2}{5}y' - \frac{3}{5}y$$

Substituting all these values in (vii), we have

$$y''' - y'' = \frac{4}{5}(y'' + 2y' - 3y) - \frac{36}{20}(y'' - 3y' + 2y)$$

$$y''' + \left(-1 - \frac{4}{5} + \frac{9}{5}\right)y'' + \left(-\frac{8}{5} - \frac{27}{5}\right)y' + \left(\frac{12}{5} + \frac{18}{5}\right)y = 0$$

$$y''' - 7y' + 6y = 0$$

METHODS OF SOLVING DIFFERENTIAL EQUATIONS SOLUTION OR INTEGRAL OF A DIFFERENTIAL EQUATION

It is a relation between the variables, not involving the differential coefficients such that this relation and the derivatives obtained from it satisfy the given differential equation. The solution of a differential equation is also called its *primitive*.

A solution which involves a number of essentially distinct arbitrary constants equal to the order of the equation is known as the *general solution.*

A solution of a differential equation obtained from a general solution by giving particular values to one or more arbitary constants is called a *particular solution.*

A solution which cannot be obtained from any general solution by any choice of the arbitrary constants is called a *singular solution.*

EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation is said to be linear if the dependent variable and its differential coefficients occur in it in the first degree only and are not multiplied together.

Equations in which the Variables are Separable

These are the equations which can be so expressed that the coefficients of dx is only a function of x and that of dy is only a function of y. The general form of such an equation is $f(x)$ dx = $g(y)$dy. Integrating both the sides, we get the solution.

Illustration 4

Solve $9yy' + 4x = 0$.

Separating variables, we have

$$9y\,dy = -4x\,dx$$

Integrating both sides, we obtain

$$9\frac{y^2}{2} = -2x^2 + \text{Const}$$

$$\Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{4} = \text{Const}$$

Linear Equations

A linear equation of first order and first degree is either of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{(i)}$$

where P and Q are functions of x, or

$$\frac{dx}{dy} + P_1x = Q_1 \qquad \text{(ii)}$$

where P_1 and Q_1 are functions of y. In order to solve equation (i), we multiply both sides by the integrating factor is $e^{\int P dx}$. After the multiplication, left hand side becomes the differential coefficient of $ye^{\int P dx}$ and now integrating both the sides, we have

$$ye^{\int P dx} = \int Qe^{\int P dx}\,dx + C$$

as the solution. Similarly we solve equation (ii)

Equations reducible to linear form. If the given equation is of the form $dy/dx + Py = Qy^n$ where P, Q are functions of x, we can reduce it to a linear equation by dividing both the sides by y^n and then substituting $y^{-n+1} = z$. The given equation will be linear in z.

Illustration 5

Solve $y' - y = e^{2x}$

This is a linear equation with $P = -1$, $Q = e^{2x}$.

The I.F. $= e^{-\int 1 dx} = e^{-x}$. Multiplying with I.F., we have

$$\frac{d}{dx}(ye^{-x}) = e^{-x}y' - ye^{-x} = e^x$$

Integrating both sides

$$ye^{-x} = \int e^x\,dx + C$$
$$= e^x + C$$
$$\Rightarrow y = e^{2x} + ce^x$$

Illustration 6

$y' - Ay = -By^2$, A, B are constants. This is a differential equation reducible to linear equation with $P = -A$, $Q = -B$, $n = 2$. Dividing by y^2, we have

$$\frac{1}{y^2}y' - A\frac{1}{y} = -B$$

$$\text{Put } -\frac{1}{y} = Z \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dZ}{dx}$$

$$\frac{dZ}{dx} + AZ = -B$$

This is a linear equation with $P = A$, $Q = -B$

I.F. $= e^{Ax}$, so

$$\frac{d}{dx}(Ze^{Ax}) = -Be^{Ax}$$

$$\Rightarrow Ze^{Ax} = -\frac{B}{A}e^{Ax} + C$$

$$\Rightarrow -\frac{1}{y} = -\frac{B}{A} + Ce^{-Ax}$$

Homogeneous Equations

It is a differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$, where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x and y of the same degree. A function $f(x, y)$ is said to be homogeneous of degree n if it can be written as $x^n f_1(y/x)$. Such an equation can be solved by putting $y = Vx$. After substituting $y = Vx$, the given equation will have variables separable in V and x.

The equations of the form $dy/dx = (ax + by + c)/(Ax + By + C)$ can be reduced to a homogeneous equation by changing $x = X + h$ and $y = Y + k$, where h and k are the constants to be chosen so that it makes the given equation homogeneous provided $aB - bA \neq 0$.

If $aB - bA = 0$ then the given equation can be written as

$$\frac{dy}{dx} = \frac{b\left(\frac{a}{b}x + y\right) + c}{B\left(\frac{A}{B}x + y\right) + C}$$

Now put $(a/b)x + y = z$ so that the given equation reduces to an equation whose variables are seperable in z and x.

Illustration 7

Solve $y' = \dfrac{x+y}{x-y}$

$y' = f(x, y)$, $f(x, y)$ is a homogeneous function of degree 1. Putting $y = Vx$, we have $\dfrac{dy}{dx} = V + x\dfrac{dv}{dx}$

The given equation reduces to

$$V + x\frac{dV}{dx} = \frac{x(1+V)}{x(1-V)} = \frac{1+V}{1-V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1+V}{1-V} - V = \frac{1+V-V+V^2}{1-V} = \frac{1+V^2}{1-V}$$

$$\Rightarrow \frac{1-V}{1+V^2}dV = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1+V^2} - \frac{1}{2}\frac{2V}{1+V^2}\right)dV = \frac{dx}{x}$$

$$\Rightarrow \tan^{-1}V - \frac{1}{2}\log(1 + V^2) = \log x + C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \log x\,(1 + V^2)^{1/2} + C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \log\,(x^2 + y^2)^{1/2} + C$$

ORTHOGONAL TRAJECTORY

Any curve which cuts every member of a given family of curve at right angle is called an orthogonal trajectory of the family. For example, each straight line $y = mx$ passing through the origin, is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$.

Procedure for finding the orthogonal trajectory

(i) Let $f(x, y, c) = 0$ be the equation, where c is an arbitrary parameter.

(ii) Differentiate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in the equation obtained in (ii)

(iv) Solve the differential equation in (iii).

DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

The equations which are of first order but not of the first degree, the following types of equations are discussed.

(i) Equations solvable for $p = dy/dx$
(ii) Equations solvable for y
(iii) Equations solvable for x
(iv) Clairut's Equations

(i) If there is a quadratic (or third degree) equation in p, solve the equation for p to get two (or three) first order and first degree equations in x, y and p. These equations can be solved as above.

(ii) Suppose that the given differential equation on solving for y, gives

$$y = f(x, p) \qquad (1)$$

Differentiating w.r.t. x, we obtain $p = \dfrac{dy}{dx} = Q\left(x, p, \dfrac{dp}{dx}\right)$,

so that we obtain a new differential equation with variables x and p. Suppose that it is possible to solve this equation. Let the solution be

$$f(x, p, c) = 0 \qquad (2)$$

where, c is the arbitrary constant.

We may either eliminate p between (1) and (2) or we may solve (1) and (2) for x, y.

(iii) In this case differentiate w.r.t. y and proceed as in case (ii).

(iv) The equation of the form $y = px + f(p)$ is known as Clairut's equation.

Differentiating w.r.t. x, we get $p = p + x\dfrac{dp}{dx} + f'(p)\dfrac{dp}{dx}$

$$[x + f'(p)]\frac{dp}{dx} = 0 \quad \Rightarrow \quad \frac{dp}{dx} = 0,\ x + f'(p) = 0$$

If $\dfrac{dp}{dx} = 0$, we have p = constant = c (say). Eliminating p we have $y = cx + f(c)$ as a solution. If $x + f'(p) = 0$, then by eliminating p, we will obtain another solution. This solution is called singular solution.

Remark If $u = rf(\theta)$, then $du = rf'(\theta)d\theta + f(\theta)\,dr$

SECOND ORDER BUT OF DEGREE ONE

If there is a linear second order equations with constant coefficients say $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$ then we solve the quadratic equation $m^2 + a_1 m + a_2 = 0$. There will be three cases

Case I Discriminant of the quadratic equation above is positive say roots are α_1 and α_2 then $y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x}$ is a solution.

Case II If $\alpha_1 = \alpha_2 = \alpha$(say) then a solution is $y = (C_1 x + C_2)e^{\alpha x}$.

Case III $\alpha_1 = \alpha + i\beta$ (complex roots) then a solution is $y = C_1 \cos \alpha x + C_2 \sin \beta x$.

Illustration 8

Solve $y'' + y' - 2y = 0$
Consider the equation $m^2 + m - 2 = 0$
$\Rightarrow (m + 2)(m - 1) = 0$
$\Rightarrow m = -2, 1$
Hence the required solution $y = C_1e^{-2x} + C_2e^x$

Illustration 9

Solve $y'' + 2y' + y = 0$
Consider the equation $m^2 + 2m + 1 = 0$
$(m + 1)^2 = 0$, so
$m = -1$ are coincident roots. Hence required solution $y = (C_1 + C_2x)e^{-x}$.

Illustration 10

Solve $y'' + y = 0$
Consider the equation $m^2 + 1 = 0 \Rightarrow m = \pm i$ are two complex roots with $\alpha = 0$, $\beta = 1$.
Hence required solution is
$y = C_1 \cos 0.x + C_2 \sin x = C_1 + C_2 \sin x$.

Equation of the form $\frac{d^2y}{dx^2} = f(x)$

If the given differential equation is $\frac{d^2y}{dx^2} = f(x)$ then integrating both the sides, we get $\frac{dy}{dx} = F(x) + C_1$, where $F(x) = \int f(x)\,dx$ and C_1 is a parameter.

Integrating again, we have

$$y = G(x) + C_1 x + C_2$$

where $G(x) = \int F(x)\,dx$ and C_1, C_2 are parameter.

Illustration 11

$$\frac{d^2y}{dx^2} = \sin^2 x$$

Integrating, we get

$$\frac{dy}{dx} = \int \sin^2 x\,dx + C_1 = \frac{1}{2}\int (1 - \cos 2x)\,dx + C_1$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C_1$$

Integrating again, we have

$$y = \frac{x^2}{4} + \frac{1}{8}\cos 2x + C_1 x + C_2.$$

SOLVED EXAMPLES
Concept-based
Straight Objective Type Questions

Example 1: The solution of the equation $y' = \frac{y+1}{x}$ is given by

(a) $y = Cx + 1$ (b) $y = Cx - 1$
(c) $y = Cx^2 + 1$ (d) $y = Cx - 2$

Ans. (b)

Solution: The equation is with separable variables, so can be written as

$$\frac{dy}{y+1} = \frac{dx}{x}$$

Integrating, we have

$\log(y + 1) = \log x + \text{Conts}$

$\Rightarrow y + 1 = Cx \Rightarrow y = Cx - 1.$

Example 2: The solution of the equation

$$\frac{xy' - y}{x} = \tan\frac{y}{x} \text{ is given by}$$

(a) $\sin \frac{y}{x} = Cx$ (b) $\cos \frac{y}{x} = Cx$

(c) $\tan \frac{y}{x} = Cx^2$ (d) $\tan \frac{y}{x} = Cx^2 + x$

Ans. (a)

Solution: The given equation can be written as $y' = \tan \frac{y}{x} + \frac{y}{x}$

This is a homogeneous equation of degree 1 so put $y = Vx$.

$$V + x\frac{dV}{dx} = \tan V + V$$

$$x\frac{dV}{dx} = \tan V \Rightarrow \cot V\, dV = \frac{dx}{x}$$

$\Rightarrow$ $\log \sin V = \log x + \text{Const} \Rightarrow \sin y/x = Cx.$

Example 3: The solution of $y' \frac{2xy}{1-x^2} = 1 + x$

$y|_{x=0} = 1$ is given by

(a) $y = \sqrt{\frac{1+x}{1-x}} + \sin^{-1} x$

(b) $y = \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\left[2 + x\sqrt{1-x^2}\right]$

(c) $y = \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\left[2 + x\sqrt{1-x^2} + \sin^{-1} x\right]$

(d) $y = \frac{1}{1-x^2}\left(1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right)$

Ans. (d)

Solution: The equation is a linear equation with integrating factor $e^{-\int \frac{dx}{1-x^2}} = e^{-\int \frac{2x}{1-x^2}dx} = e^{\log(1-x^2)} = (1-x^2)$

Multiplying with I.F., we have

$$\frac{d}{dx}(y(1-x^2)) = (1-x^2)(1+x)$$

$$y(1-x^2) = \int(1 - x^2 + x - x^3)dx + C$$

$$= x - \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4} + C$$

$$y = \frac{1}{1-x^2}\left(x - \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4}\right) + \frac{C}{1-x^2}$$

since $y(0) = 1$ so $C = 1$. Hence

$$y = \frac{1}{(1-x^2)}\left(1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right)$$

Example 4: The solution of $y' + y = -\frac{x}{y}$ is given by

(a) $\frac{1}{y} = -x + e^x$

(b) $y^2 = -x^2 + \frac{1}{2}(1 + e^{-x})$

(c) $y^2 = -x + \frac{1}{2}(1 + e^{-2x})$

(d) $\frac{1}{y} = -\frac{1}{2}x + \frac{3}{2}e^{-x}$

Ans. (c)

Solution: The given equation is reducible to linear equation $yy' + y^2 = -x$

Put $y^2 = Z \Rightarrow 2y\frac{dy}{dx} = \frac{dZ}{dx}$. Thus

$$\frac{1}{2}\frac{dZ}{dx} + Z = -x$$

$$\Rightarrow \frac{dZ}{dx} + 2Z = -2x$$

which is a linear equation with I.F. $= e^{2x}$, so

$$\frac{d}{dx}(Ze^{2x}) = -2xe^{2x}$$

$$\Rightarrow Ze^{2x} = -2\int xe^{2x}dx + C = -2\left[\frac{xe^{2x}}{2} - \int\frac{e^{2x}}{2}dx\right] + C$$

$$= -xe^{2x} + \frac{e^{2x}}{2} + C$$

$$\Rightarrow y^2 = -x + \frac{1}{2} + C\,e^{-2x}$$

For $x = 0, y(0) = 1$ so $1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$.

Thus $y^2 = -x + \frac{1}{2}(1 + e^{-2x})$

Example 5: The solution of $y' = \frac{2x - y + 1}{x - 2y + 1}$ is given by

(a) $x^2 - xy + y^2 + x + y = C$
(b) $x^2 - xy + y^2 + x - y = C$
(c) $x^2 - xy + y^2 - x + y = C$
(d) $-x^2 + 2xy - y^2 - x + 2y = C$

Ans. (b)

Solution: The given equation is reducible to homogeneous equation. Put $y = Y + k, x = X + h$

$$\frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX}$$

Select h, k such that $2h - k + 1 = 0$, $h - 2k + 1 = 0$ so $h = -\frac{1}{3}$ and $k = \frac{1}{3}$. Thus

$$\frac{dY}{dX} = \frac{2X - Y}{X - 2Y} \text{ which is a homogeneous equation}$$

Putting $Y = vX$, we have

$$v + X\frac{dv}{dX} = \frac{2 - v}{1 - 2v}$$

$$\Rightarrow \quad X\frac{dv}{dx} = \frac{2 - v - v + 2v^2}{1 - 2v} = -\frac{2(1 - v + v^2)}{2v - 1}$$

$$\Rightarrow \frac{2v - 1}{v^2 - v + 1}dV = -2\frac{dX}{X}$$

$$\Rightarrow \quad \log(v^2 - v + 1) + \log X^2 = \text{Const}$$

$$\Rightarrow \quad \left(\frac{y^2}{x^2} - \frac{y}{x} + 1\right)X^2 = \text{Const}$$

$$\Rightarrow \quad Y^2 - YX + X^2 = \text{Const}$$

$$\Rightarrow \quad \left(y - \frac{1}{3}\right)^2 - \left(y - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) + \left(x + \frac{1}{3}\right)^2 = \text{Const}$$

$$\Rightarrow \quad y^2 - \frac{2}{3}y + \frac{1}{9} - \left(xy - \frac{1}{3}x + \frac{1}{3}y - \frac{1}{9}\right) + x^2 + \frac{2}{3}x + \frac{1}{9} = \text{Const}$$

$$\Rightarrow \quad y^2 + x^2 - xy - y + x = C$$

Example 6: A particular solution of

$\log\frac{dy}{dx} = 4x + 5y$, $\quad y(0) = 0$ is given by

(a) $5e^{4x} + 4e^{-5y} = 9$ (b) $5e^{4x} + 4e^{5y} = 9$
(c) $4e^{4x} + 5e^{-4x} = 20$ (d) $4e^{4x} + 5e^{-4y} = 9$

Ans. (a)

Solution: $\frac{dy}{dx} = e^{4x + 5y} = e^{4x}. e^{5y}$

$$\Rightarrow \quad e^{-5y}dy = e^{4x}\,dx$$

Integrating, we have

$$\frac{e^{-5y}}{-5} = \frac{e^{4x}}{4} + C$$

$$\Rightarrow \quad C = \frac{e^{4x}}{4} + \frac{e^{-5y}}{5}$$

Putting $x = 0$ and $y(0) = 0$, $C = \frac{1}{4} + \frac{1}{5} = \frac{9}{20}$

Thus $9 = 5e^{4x} + 4e^{-5y}$

Example 7: The motion of a particle is given by $\frac{dx}{dt} = -3a\cos^2 t \sin t$, $\frac{dy}{dt} = 3a\sin^2 t\cos t$ and passes through at $t = 0$, then the path is given by

(a) $(x + a)^{2/3} + (y - a)^{2/3} = 1$
(b) $(x - a)^{2/3} + (y + a)^{2/3} = 2a^{2/3}$
(c) $x^{2/3} + y^{2/3} = a^{2/3}$
(d) $(x + a)^{2/3} + y^{2/3} = a^{2/3}$

Ans. (d)

Solution: $x = -3a\int\cos^2 t\sin t\,dt + C = a\cos^3 t + C$

For $t = 0$, $C = -a$

$y = a\int\sin^2 t\cos t\,dt = a\sin^3 t + G$, at $t = 0$, $y = 0$ so $C_1 = 0$

Thus $x = a(\cos^3 t - 1)$, $y = a\sin^3 t$

$$\Rightarrow \left(\frac{x}{a} + 1\right)^{1/3} = \cos t, \left(\frac{y}{a}\right)^{1/3} = \sin t$$

$$\Rightarrow \left(\frac{x}{a} + 1\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1. \Rightarrow (x + a)^{2/3} + y^{2/3} = a^{2/3}$$

Example 8: The solution of initial value problem $xy' + y = 0$, $y(2) = -2$ is given by

(a) $xy + 4 = 0$ (b) $(y + 2) = 0$
(c) $x - 2 = y + 2$ (d) $(x - 2)(y + 2)^2 = 4$

Ans. (a)

Solution: $x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$

$$\Rightarrow \quad \log y + \log x = \text{Const}$$

$$\Rightarrow \quad xy = \text{Const}$$

Since $y(2) = -2$ so Const $= 2(-2) = -4$

Thus $xy + 4 = 0$ is the required solution

Example 9: The solution of $y' + xy = xy^{-1}$, $y(0) = 2$ is given by

(a) $y^2 = 2 + 2e^{-x^2}$ (b) $y^2 = 3 + e^{x^2}$
(c) $y^2 = 3 + e^{-x^2}$ (d) $y^2 = 2 + 2e^{e^{-x}}$

Ans. (c)

Solution: $y' + xy = xy^{-1} \Rightarrow yy' + xy^2 = x$

This is reducible to linear equation, so put $y^2 = z \Rightarrow y\frac{dy}{dx} = \frac{1}{2}\frac{dz}{dx}$

$$\frac{1}{2}\frac{dz}{dx} + xz = x$$

$$\Rightarrow \quad \frac{dz}{dx} + 2xz = 2x$$

I.F. $= e^{2\int x dx} = e^{x^2}$. Multiplying with I.F., we have

$$\frac{d}{dx}(ze^{x^2}) = 2x\,e^{x^2}$$

$$\Rightarrow \quad z\,e^{x^2} = \int 2x\,e^{x^2}dx + C = e^{x^2} + C$$

$$\Rightarrow \quad y^2 e^{x^2} = e^{x^2} + C$$

Since $y(0) = 1$ so $C = 3$

$$\Rightarrow \quad y^2 = 3 + e^{-x^2}$$

Example 10: The solution of $y' + \cos(x + y) = \cos(x - y)$ and $y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$ is given by

(a) $\sin y = 2 \sin x \cos^2 y$ (b) $\cos y = 2 \sin x \sin^2 y$
(c) $\cos y = 2\cos x \sin^2 y$ (d) $\sin x = 2 \cos y \sin^2 y$

Ans. (b)

Solution: $\frac{dy}{dx} = \cos(x - y) - \cos(x + y) = 2\sin x \sin y$

$\Rightarrow \quad \frac{dy}{\sin y} = 2 \sin x\, dx$

$\Rightarrow \quad -\text{cosec } y \cot y = -2 \cos x + C$

Since $y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$ so $C = 0$. Thus

$\text{cosec } y \cot y = 2 \cos x$

$\Rightarrow \quad \cos y = \cos x \sin^2 y$

LEVEL 1

Straight Objective Type Questions

Example 11: The degree of the differential equation

$$\frac{d^3y}{dx^2} + 5\left(\frac{d^2y}{dx^2}\right)^2 = x^3 \log \frac{d^2y}{dx^2} \text{ is}$$

(a) 1 (b) 2
(c) 3 (d) none of these

Ans. (d)

Solution: Since the equation is not a polynomial in all the differential coefficients so the degree is not defined.

Example 12: The degree of the differential equation $y_2^{3/2} - y_1^{1/2} - 4 = 0$ is

(a) 6 (b) 3
(c) 2 (d) 4

Ans. (a)

Solution: $y_2^{3/2} = y_1^{1/2} + 4$. Squaring both sides, we have $y_2^3 = y_1 + 16 + 8y_1^{1/2}$
$\Rightarrow (y_2^3 - y_1 - 16)^2 = 64y_1 \Rightarrow y_2^6 - 32y_2^3 - 2y_2^3 y_1 + y_1^2 - 32y_1 + 256 = 0$. Hence the degree of the given equation is 6.

Example 13: Which of the following equations is a linear equation of order 3 ?

(a) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\frac{dy}{dx} + y = x$

(b) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + y^2 = x^2$

(c) $x\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = e^x$

(d) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \log x$

Ans. (c)

Solution: The equations in (a), (b) and (c) are of order 3 and in (d) is of order 2. The equations in (c) and (d) are linear, see theory for definition.

Example 14: The order and degree of the differential equation $x^2 = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$ are (respectively)

(a) 2, 1 (b) 2, 3
(c) 2, 2 (d) 2, 6

Ans. (c)

Solution: The given equation can be written as

$$x^4\left(\frac{d^2y}{dx^2}\right)^2 = \left(1+\left(\frac{dy}{dx}\right)^2\right)^3$$

which is clearly of order 2 and degree 2.

Example 15: An equation of the curve in which sub-normal varies as the square of the ordinate is (k is constant of proportionality)

(a) $y = Ae^{kx}$ (b) $y = e^{kx}$
(c) $y^2/2 + kx = A$ (d) $y^2 + kx^2 = A$

Ans. (a)

Solution: According to the given condition $y\frac{dy}{dx} = ky^2$

$\Rightarrow \quad \frac{dy}{y} = k\,dx$ (variables separable equation)

$\Rightarrow \log |y| = kx + C \Rightarrow |y| = Be^{kx} \Rightarrow y = Ae^{kx}$ where $A = \pm B$ and k is the constant of proportionality.

Example 16: A solution of the differential equations $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is

(a) $y = 2$ (b) $y = 2x$
(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

Ans. (c)

Solution: Let $p = \frac{dy}{dx}$ so that the given equation can be written as $p^2 - xp + y = 0$. Therefore $y = xp - p^2$ which is a

Clairut's equation see theory. Differentiating with respect to x, we have

$$p = \frac{dy}{dx} = p + x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$(x - 2p)\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{dx} = 0 \text{ so } p = \text{const} = C \text{ (say)}$$

So $y = Cx - C^2$ is a solution, in particular $C = 2$.

Example 17: The solution of $\frac{dy}{dx} = \frac{1}{2x - y^2}$ is given by

(a) $y = Ce^{-2x} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}$

(b) $x = Ce^{-y} + \frac{1}{4}y^2 + \frac{1}{4}y + \frac{1}{2}$

(c) $x = Ce^{y} + \frac{1}{4}y^2 + y + \frac{1}{2}$

(d) $x = Ce^{2y} + \frac{1}{2}y^2 + \frac{1}{2}y + \frac{1}{4}$

Ans. (d)

Solution: $\frac{dx}{dy} = 2x - y^2$ (This is a linear equation in x).

The integrating factor is $e^{-\int 2dy} = e^{-2y}$.

So $\frac{d}{dy}(xe^{-2y}) = -y^2 e^{-2y}$

Integrating, we have

$$xe^{-2y} = \frac{-y^2e^{-2y}}{-2} - \int ye^{-2y}\,dy + \text{Const}$$

$$= \frac{y^2}{2}e^{-2y} + \frac{ye^{-2y}}{2} - \frac{1}{2}\int e^{-2y}\,dy + \text{Const}$$

$$= \frac{y^2}{2}e^{-2y} + \frac{y}{2}e^{-2y} + \frac{e^{-2y}}{4} + C_1$$

$$\therefore \quad x = C_1 e^{2y} + \frac{y^2}{2} + \frac{y}{2} + \frac{1}{4}$$

Example 18: The curves whose subtangents are proportional to the abscissas of the point of tangency (the proportionality factor is equal to k) is

(a) $y^k = Cx$ (b) $y^k = Cx^2$

(c) $y^{k/2} = Cx^3$ (d) none of these

Ans. (a)

Solution: Subtangent $= y\frac{dx}{dy}$ so according to the given condition $y\frac{dx}{dy} = kx$

$$\Rightarrow \frac{dx}{x} = k\frac{dy}{y} \Rightarrow \log A + \log|x| = k\log|y|$$

$$\Rightarrow A|x| = |y|^k \Rightarrow y^k = \pm Ax = Cx$$

Example 19: The degree of the differential equation of all curves having normal of constant length C is

(a) 1 (b) 3

(c) 4 (d) none of these

Ans. (d)

Solution: According to the given condition

$$y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = C \Rightarrow y^2 + y^2\left(\frac{dx}{dy}\right)^2 = C^2$$

The degree of this equation is 2.

Example 20: Which of the following transformation reduce the differential equation $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$ into the form

$$\frac{du}{dx} + uP(x) = Q(x)$$

(a) $u = \log z$ (b) $u = e^z$

(c) $u = (\log z)^{-1}$ (d) $u = (\log z)^2$

Ans. (c)

Solution: Dividing the given equation by $z(\log z)^2$, we have

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{\log z}\frac{1}{x} = \frac{1}{x^2} \qquad \text{(i)}$$

Putting $\frac{1}{\log z} = u$, we have $\frac{du}{dx} = -(\log z)^{-2}\frac{1}{z}\frac{dz}{dx}$

So (i) can be written as

$$-\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^2} \Rightarrow \frac{du}{dx} - \frac{u}{x} = \frac{-1}{x^2}$$

which is in the required form with $P(x) = -1/x$ and $Q(x) = -1/x^2$.

Example 21: A particular solution of $\log\frac{dy}{dx} = 3x + 4y$, $y(0) = 0$ is

(a) $e^{3x} + 3e^{-4y} = 4$ (b) $4e^{3x} - e^{-4y} = 3$

(c) $3e^{3x} + 4e^{4y} = 7$ (d) $4e^{3x} + 3e^{-4y} = 7$

Ans. (d)

Solution: $\frac{dy}{dx} = e^{3x+4y} = e^{3x}e^{4y} \Rightarrow e^{-4y}\,dy = e^{3x}\,dx$

Thus $\frac{e^{-4y}}{-4} - \frac{e^{3x}}{3} = \text{Const}$. But $y(0) = 0$, so, $-\frac{1}{4} - \frac{1}{3} = C$.

Hence $(e^{-4y}/(-4)) - (e^{3x}/3) = -7/12 \Rightarrow 3e^{-4y} + 4e^{3x} = 7$.

Example 22: The solution of $\frac{dy}{dx} = \frac{ax+b}{cy+d}$ represents a parabola if

(a) $a = 0, c = 0$ (b) $a = 1, b = 2$
(c) $a = 0, c \neq 0$ (d) $a = 1, c = 1$

Ans. (c)

Solution: The given equation has separable variables so $(cy + d)dy = (ax + b)\,dx$. Integrating we have $\frac{cy^2}{2} + dy + K = \frac{ax^2}{2} + bx$, K being the constant of integration. The last equation represents a parabola if $c = 0, a \neq 0$ or $a = 0, c \neq 0$.

Example 23: The differential equation corresponding to the family of curves $y = e^x (a \cos x + b \sin x)$, a and b being arbitrary constants is

(a) $2y_2 + y_1 - 2y = 0$ (b) $y_2 - 2y_1 + 2y = 0$
(c) $2y_2 - y_1 + 2y = 0$ (d) none of these

Ans. (b)

Solution: $y_1 = e^x(-a \sin x + b \cos x) + e^x(a \cos x + b \sin x)$

$\Rightarrow\quad y_1 = e^x(-a \sin x + b \cos x) + y$

$\Rightarrow\quad y_2 = y_1 + e^x(-a \cos x - b \sin x) + e^x(-a \sin x + b \cos x)$

$\Rightarrow\quad y_2 - 2y_1 + 2y = 0.$

Example 24: The solution of $y^5 x + y - x\frac{dy}{dx} = 0$ is

(a) $x^4/4 + (1/5)(x/y)^5 = C$
(b) $x^5/5 + (1/4)(x/y)^4 = C$
(c) $(x/y)^5 + x^4/4 = C$
(d) $(x y)^4 + x^5/5 = C$

Ans. (b)

Solution: The given differential equation can be written as $y^5 x\,dx + y\,dx - x\,dy = 0$. Multiplying by x^3/y^5, we have

$$x^4\,dx + \frac{x^3}{y^3}\left(\frac{y\,dx - x\,dy}{y^2}\right) = 0$$

Integrating, we get $x^5/5 + (1/4)(x/y)^4 = C$,

since $\frac{x^3}{y^3}\left(\frac{y dx - x dy}{y^2}\right) = u^3\,du$ where $u = \frac{x}{y}$.

Example 25: The equation of the curve passing through (3, 9) which satisfies $dy/dx = x + 1/x^2$ is

(a) $6xy = 3x^2 - 6x + 29$
(b) $6xy = 3x^2 - 29x + 6$
(c) $6xy = 3x^3 + 29x - 6$
(d) none of these

Ans. (c)

Solution: The given differential equation has variable separable so integrating, we have $y = x^2/2 - 1/x + C$. This will pass through (3, 9) if $9 = 9/2 - 1/3 + C \Rightarrow C = 29/6$. Hence the required equation is $6xy = 3x^3 + 29x - 6$.

Example 26: The solution of $\frac{x\,dy}{x^2+y^2} = \left(\frac{y}{x^2+y^2} - 1\right)dx$ is

(a) $y = x \cot(c - x)$ (b) $\cos^{-1}(y/x) = -x + c$
(c) $y = x \tan(c - x)$ (d) $y^2/x^2 = x \tan(c - x)$

Ans. (c)

Solution: The given equation can be written as

$$\frac{x\,dy - y\,dx}{x^2+y^2} = -\,dx \;\Rightarrow\; \frac{x\,dy - y\,dx}{x^2} \times \frac{1}{1 + y^2/x^2} = -\,dx$$

$\Rightarrow\quad \frac{1}{1+y^2/x^2}\frac{d}{dx}\left(\frac{y}{x}\right) = -\,dx$. Integrating we have

$\tan^{-1}(y/x) = -x + c$

$\Rightarrow\quad y = x \tan(c - x).$

Example 27: The solution of the equation $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$ is

(a) $\log|2x + y - 1| = C + x + y$
(b) $\log(4x + 2y - 1) = C + 2x + y$
(c) $\log(2x + y + 1) + x + 2y = C$
(d) $\log|2x + y - 1| + x + 2y = C$

Ans. (d)

Solution: Put $2x + y = X \;\Rightarrow\; 2 + \frac{dy}{dx} = \frac{dX}{dx}$. Therefore, the given equation is reduced to (see theory the case when $aB - bA = 0$)

$$\frac{dX}{dx} - 2 = -\frac{X+1}{2X-1} \;\Rightarrow\; \frac{dX}{dx} = \frac{3(X-1)}{2X-1}$$

$$\Rightarrow\quad \frac{2X-1}{3(X-1)}\,dX = dx \;\Rightarrow\; \frac{1}{3}\left[2 + \frac{1}{X-1}\right]dX = dx$$

$$\Rightarrow\quad \frac{1}{3}[2X + \log|X-1|] = x + \text{Const}$$

$$\Rightarrow\quad 2(2x + y) + \log|2x + y - 1| = 3x + \text{Const}$$

$$\Rightarrow\quad x + 2y + \log|2x + y - 1| = C.$$

Example 28: Solution of the differential equation $2y \sin x \frac{dy}{dx} = 2 \sin x \cos x - y^2 \cos x$ satisfying $y(\pi/2) = 1$ is given by

(a) $y^2 = \sin x$ (b) $y = \sin^2 x$
(c) $y^2 = \cos x + 1$ (d) $y^2 \sin x = 4 \cos^2 x$

Ans. (a)

Solution: The given equation can be written as

$$2y \sin x \frac{dy}{dx} + y^2 \cos x = \sin 2x$$

$\Rightarrow \quad \frac{d}{dx}(y^2 \sin x) = \sin 2x$

$\Rightarrow \quad y^2 \sin x = (-1/2)\cos 2x + C.$

so $\quad (y(\pi/2))^2 \sin(\pi/2) = (-1/2)\cos(2\pi/2) + C$

$\Rightarrow \quad C = 1/2.$

Hence $\quad y^2 \sin x = (1/2)(1 - \cos 2x) = \sin^2 x$

$\Rightarrow \quad y^2 = \sin x.$

Example 29: Solution of the differential equation $x dy - y\, dx - \sqrt{x^2 + y^2}\, dx = 0$ is

(a) $y - \sqrt{x^2+y^2} = Cx^2$ (b) $y + \sqrt{x^2+y^2} = Cx^2$

(c) $x + \sqrt{x^2+y^2} = Cy^2$ (d) $x - \sqrt{x^2+y^2} = Cy^2$

Ans. (b)

Solution: Writing the given equation as $\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$ and putting $y = vx$, we have $v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \quad \Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrating, we have $\log(v + \sqrt{1+v^2}) = \log x + \text{Const}.$

$\Rightarrow \quad y/x + \sqrt{1 + y^2/x^2} = Cx \quad \Rightarrow \quad y + \sqrt{x^2+y^2} = Cx^2.$

Example 30: An equation of the curve satisfying $x dy - y\, dx = \sqrt{x^2 - y^2}\, dx$ and $y(1) = 0$ is

(a) $y = x^2 \log|\sin x|$ (b) $y = x \sin(\log|x|)$

(c) $y^2 = x(x-1)^2$ (d) $y = 2x^2(x-1)$

Ans. (b)

Solution: The equation can be written as

$$x^2 \frac{x\,dy - y\,dx}{x^2} = x\sqrt{1 - (y/x)^2}\, dx$$

$\Rightarrow \quad \frac{d(y/x)}{\sqrt{1-(y/x)^2}} = \frac{dx}{x} \quad \Rightarrow \quad \sin^{-1} y/x = \log|x| + \text{Const}$

Since $y(1) = 0$ so Const = 0. Hence $y = x \sin(\log|x|)$.
Note that one can also solve the given equation as a homogenous equation also, i.e. by putting $y = vx$

Example 31: A solution of the equation $x\frac{dy}{dx} = y(\log y - \log x + 1)$ is

(a) $y = xe^{cx}$ (b) $y^2/x = cx$

(c) $y^2 = cx \log x$ (d) $\log y = cx$

Ans. (a)

Solution: Putting $y = vx$ in the given equation, we have

$$v + x\frac{dv}{dx} = v(\log v + 1) \quad \Rightarrow \quad x\frac{dv}{dx} = v \log v$$

$\Rightarrow \quad \frac{dv}{v \log v} = \frac{dx}{x} \quad \Rightarrow \quad \log|\log v| = \log|x| + \text{Const}$

$\Rightarrow \quad \log v = \pm Ax = cx \quad \Rightarrow \quad y = xe^{cx}.$

Example 32: The solution of $x^3 \frac{dy}{dx} + 4x^2 \tan y = e^x \sec y$ satisfying $y(1) = 0$ is

(a) $\tan y = (x-2)e^x \log x$

(b) $\sin y = e^x (x-1)x^{-4}$

(c) $\tan y = (x-1)e^x x^{-3}$

(d) $\sin y = e^x (x-1)x^{-3}$

Ans. (b)

Solution: Rewriting the given equation in the form $x^4 \cos y \frac{dy}{dx} + 4x^3 \sin y = xe^x \quad \Rightarrow \quad \frac{d}{dx}(x^4 \sin y) = xe^x$

$\Rightarrow \quad x^4 \sin y = \int xe^x\, dx$

$\quad = (x-1)e^x + C$

Since $y(1) = 0$, so $C = 0$.
Thus $\sin y = x^{-4}(x-1)e^x$.

Example 33: If for the differential equation $y' = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ the general solution is $y = \frac{x}{\log|Cx|}$ then $\phi(x/y)$ is given by

(a) $-x^2/y^2$ (b) y^2/x^2

(c) x^2/y^2 (d) $-y^2/x^2$

Ans. (d)

Solution: Putting $v = y/x$ so that $\frac{x\,dv}{dx} + v = \frac{dy}{dx}$, we have

$\frac{x\,dv}{dx} + v = v + \phi(1/v) \quad \Rightarrow \quad \frac{dv}{\phi(1/v)} = \frac{dx}{x}$

$\Rightarrow \quad \log|Cx| = \int \frac{dv}{\phi(1/v)}$ (C being constant of integration.)

But $y = \frac{x}{\log|(Cx)|}$ is the general solution so

$$\frac{x}{y} = \frac{1}{v} = \int \frac{dv}{\phi(1/v)} \quad \Rightarrow \phi(1/v) = -v^2$$

$\Rightarrow \quad \phi(x/y) = -y^2/x^2.$

Example 34: The solution $y(x)$ of the differential equation $\frac{d^2y}{dx^2} = \sin 3x + e^x + x^2$ when $y_1(0) = 1$ and $y(0) = 0$ is

(a) $-\frac{\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$

(b) $-\dfrac{\sin 3x}{9}+e^x+\dfrac{x^4}{12}+\dfrac{1}{3}x$

(c) $-\dfrac{\cos 3x}{3}+e^x+\dfrac{x^4}{12}+\dfrac{1}{3}x+1$

(d) none of these

Ans. (a)

◎ **Solution:** Integrating the given differential equation, we have

$$\frac{dy}{dx}=-\frac{\cos 3x}{3}+e^x+\frac{x^3}{3}+C_1$$

but $y_1(0)=1$ so $1=-\dfrac{1}{3}+1+C_1 \Rightarrow C_1=\dfrac{1}{3}$

Again integrating, we get

$$y=-\frac{\sin 3x}{9}+e^x+\frac{x^4}{12}+\frac{1}{3}x+C_2$$

but $y(0)=0$ so $0=1+C_2 \Rightarrow C_2=-1$. Thus

$$y=-\frac{\sin 3x}{9}+e^x+\frac{x^4}{12}+\frac{1}{3}x-1.$$

Example 35: The differential equation $\dfrac{dy}{dx}=\dfrac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(a) variable radii and a fixed centre (0, 1)
(b) variable radii and a fixed centre (0, –1)
(c) fixed radius 1 and a variable centres along the x-axis
(d) fixed radius 1 and variable centres along the y-axis

Ans. (c)

◎ **Solution:** $dx=\dfrac{y}{\sqrt{1-y^2}}dy$

$$\Rightarrow \quad c+x=-\frac{1}{2}\int\frac{2y}{\sqrt{1-y^2}}dy=-\sqrt{1-y^2}$$

$$\Rightarrow \quad (x+c)^2+y^2=1$$

which represents a family of circles of fixed radius 1 and variable centre on the x-axis.

Example 36: Let $f(x)$ be a differentiable on the interval $(0,\infty)$ such that $f(1)=1$, and $\lim\limits_{t\to x}\dfrac{t^2f(x)-x^2f(t)}{t-x}=1$ for each $x>0$. Then $f(x)$ is

(a) $\dfrac{1}{3x}+\dfrac{2x^2}{3}$ (b) $-\dfrac{1}{3x}+\dfrac{4x^2}{3}$

(c) $-\dfrac{1}{x}+\dfrac{2}{x^2}$ (d) $\dfrac{1}{x}$

Ans. (a)

◎ **Solution:** $1=\lim\limits_{t\to x}\dfrac{t^2f(x)-t^2f(t)+t^2f(t)-x^2f(t)}{t-x}$

$$=\lim_{t\to x}\frac{t^2(f(x)-f(t))}{t-x}+\lim_{t\to x}(t+x)f(t)$$

$$=-x^2f'(x)+2xf(x)$$

Thus $f(x)$ is a solution of the differential equation

$$x^2\frac{dy}{dx}-2xy=-1 \Rightarrow \frac{dy}{dx}-\frac{2}{x}y=-\frac{1}{x^2} \qquad (1)$$

This is a linear equation with I.F. $=e^{-\int\frac{2}{x}dx}=\dfrac{1}{x^2}$.

Multiplying (1) with $\left(1/x^2\right)$ we get

$$\frac{d}{dx}\left(\frac{y}{x^2}\right)=-\frac{1}{x^4} \Rightarrow \frac{y}{x^2}=\frac{1}{3x^3}+C$$

Since $f(1)=1$ so $C=\dfrac{2}{3}$ $\therefore$ $y=\dfrac{2}{3}x^2+\dfrac{1}{3x}$.

Example 37: Suppose $y=y(x)$ satisfies the differential equation $ydx+y^2dy=xdy$. If $y(x)>0\ \forall\ x\in \mathbf{R}$ and $y(1)=1$ then $y(-3)$ equals

(a) 1 (b) 2
(c) 3 (d) 5

Ans. (c)

◎ **Solution:** $\dfrac{ydx-xdy}{y^2}=-dy$

$$\Rightarrow \quad d\left(\frac{x}{y}\right)=-dy \Rightarrow x/y=-y+c$$

since $y(1)=1$ so $c=2$ Thus $x/y=-y+2$

when $x=-3$, $-\dfrac{3}{y}=-y+2$

$\Rightarrow$ $-3=-y^2+2y \Rightarrow (y-1)^2=4$

$\Rightarrow$ $y=1\pm 2=3$ or -1

As $y(x)>0$ for all $x\in \mathbf{R}$ so $y(-3)=3$.

Example 38: The equation of the curve whose tangent at any point (x, y) makes an angle $\tan^{-1}(2x+3y)$ with x-axis and which passes through (1, 2) is:

(a) $6x+9y+2=26e^{3(x-1)}$
(b) $6x-9y+2=26e^{3(x-1)}$
(c) $6x+9y-2=26e^{3(x-1)}$
(d) $6x-9y-2=26e^{3(x-1)}$

Ans. (a)

◎ **Solution:** $\dfrac{dy}{dx}=\tan[\tan^{-1}(2x+3y)]=2x+3y$

$$\Rightarrow \quad \frac{dy}{dx}-3y=2x$$

I.F. $=e^{-3x}$ Multilplying (1) by e^{-3x}, we get

$$e^{-3x}\frac{dy}{dx}-3e^{-3x}y=2xe^{-3x} \Rightarrow \frac{d}{dx}[ye^{-3x}]=2xe^{-3x}$$

$$\Rightarrow \quad ye^{-3x} = \int 2xe^{-3x}\,dx = -\frac{2}{3}xe^{-3x} + \frac{2}{3}\int (1)e^{-3x}\,dx$$

$$= -\frac{2}{3}xe^{-3x} - \frac{2}{9}e^{-3x} + C$$

As this curve passes through (1, 2), we get

$$2 = -\frac{2}{9}(3+1) + Ce^{3} \quad \Rightarrow C = \frac{26}{9}e^{-3}$$

Thus, required curve is

$$y = -\frac{2}{9}(3x+1) + \frac{26}{9}e^{3(x-1)} \quad \Rightarrow 6x + 9y + 2 = 26e^{3(x-1)}$$

Example 39: The degree and order respectively of the differential equation of all parabolas whose axis is x-axis, are:

(a) 2, 1 (b) 1, 2
(c) 2, 2 (d) 1, 1

Ans. (b)

Solution: Equation of any parabola whose axis is x-axis is

$$y^2 = 4a(x+b)$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

which is a differential equation of order 2 and degree 1.

Example 40: The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ satisfying the condition $y(1) = 1$ is

(a) $y = \log x + x$ (b) $y = x\log x + x^2$
(c) $y = xe^{(x-1)}$ (d) $y = x\log x + x$

Ans. (d)

Solution: $xdy = xdx + ydx$

$$\frac{xdy - ydx}{x^2} = \frac{dx}{x}$$

$$\Rightarrow \quad d(y/x) = d(\log x) \quad \Rightarrow \quad y = x(\log x + C)$$

Putting $x = 1$, we get $C = 1$, so

$$y = x(\log x + 1)$$

Example 41: The differential equation which represents the family of curves $y = c_1e^{c_2x}$, where c_1 and c_2 are arbitrary constants, is

(a) $yy'' = y'$ (b) $yy'' = y'^2$
(c) $y'' = y^2$ (d) $y'' = y'y$

Ans. (b)

Solution: $y = c_1e^{c_2x} \quad \Rightarrow \quad y' = c_1c_2e^{c_2x}$

$$\Rightarrow \quad y'' = c_1c_2^2e^{c_2x}$$

$$\Rightarrow \quad yy'' = c_1^2c_2^2e^{c_2x}e^{c_2x} = \left(c_1c_2e^{c_2x}\right)^2 = y'^2$$

Example 42: Solution of the differential equation $\cos x\, dy = y(\sin x - y)dx$, $0 < x < \pi/2$ is

(a) $y\tan x = \sec x + c$ (b) $\tan x = (\sec x + c)y$
(c) $\sec x = (\tan x + c)y$ (d) $y\sec x = \tan x + c$

Ans. (c)

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{y(\sin x - y)}{\cos x} \quad \Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

Put $-\frac{1}{y} = z$, we have

$$\frac{dz}{dx} (\tan x)z = -\sec x \qquad (1)$$

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

(1) reduces to $\frac{d}{dx}(z\sec x) = -\sec^2 x$

$$\Rightarrow \quad z\sec x = -\tan x + c_1$$

$$\Rightarrow \quad \sec x = (c + \tan x)y,\ c = -c_1.$$

Example 43: If $\frac{dy}{dx} = y + 3 > 0$ and $y(0)$ and $y(0) = 2$, $y(\log 2)$ is equal to

(a) – 2 (b) 7
(c) 5 (d) 13

Ans. (b)

Solution: $\frac{dy}{y+3} = dx \Rightarrow \log(y+3) = x + c$ when $x = 0$, $y = 2$, therefore $\log 5 = C$

Thus $\log(y+3) = x + \log 5$

When $x = \log 2$, we have

$$\log(y+3) = \log 2 + \log 5 = \log 10$$

$$\Rightarrow \quad y + 3 = 10 \Rightarrow y = 7.$$

Example 44: Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T-t)$, where k > 0 is a constant and T is the total life in years of the equipment. Then the scrap value $V(T)$ of the equipment is:

(a) e^{-kT} (b) $T^2 - \frac{I}{k}$
(c) $I - \frac{kT^2}{2}$ (d) I – K(T)

Ans. (c)

Solution: $\frac{dv}{dt} = -K(T-t),\ K > 0$

$$\Rightarrow \quad dV = -K(T-t)\,dt$$

Integrating, we have $V(t) = \frac{k(T-t)^2}{2} + C$

We have $V(0) = I$, therefore

$$I = \frac{KT^2}{2} + C \quad \Rightarrow \quad C = I - \frac{k}{2}T^2$$

Scrap value $V(T) = C = I - \frac{K}{2}T^2$.

Example 45: The population $p(t)$ at time t of a certain mouse species follows the differential equation $\frac{dp(t)}{dt} = 0.5$ p(t) – 450. If p(0) = 850, then the time at which the population becomes zero is

(a) log 9 (b) $\frac{1}{2}\log 18$
(c) log 18 (d) 2 log 18

Ans. (d)

Solution: The given differential equation in a linear equation with I.F. = $e^{-\int(0.5)dt} = e^{-t/2}$. Multiplying with I.F. we have

$$\frac{d}{dt}\left(e^{-t/2}p\right) = -450\,e^{-t/2} \quad \Rightarrow \quad e^{-t/2}\,p(t) = 900\,e^{-t/2} + C$$

When $t = 0$, $p = 850$,

$$850 = 900 + C \quad \Rightarrow \quad C = -50$$

Thus $p(t) = 900 - 50\,e^{t/2}$

When $p(t) = 0$, we get $900 - 50e^{t/2} = 0$

$\Rightarrow$ $e^{t/2} = 18 \quad \Rightarrow \quad t/2 = \log 18$ i.e. $t = 2\log 18$

Example 46: If the differential equation representing the family of all circles touching y-axis at the origin is $x^2 - y^2 = f(x)y\frac{dx}{dy}$ then $f(x)$ is equal to

(a) $2x$ (b) x
(c) x^2 (d) $3x$

Ans. (a)

Solution: Equation of circles touching y-axis at origin is

$$x^2 + (y-a)^2 = a^2 \qquad \text{(i)}$$

Differentiating, we get

$$2x + 2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y - a = -\frac{x}{\frac{dy}{dx}} \Rightarrow a = y + x\frac{dx}{dy}$$

Substituting this value in (i), we have

$$x^2 + x^2\left(\frac{dx}{dy}\right)^2 = \left(y + x\frac{dx}{dy}\right)^2$$

$$\Rightarrow x^2\left(1 + \left(\frac{dx}{dy}\right)^2\right) = y^2 + x^2\left(\frac{dx}{dy}\right)^2 + 2xy\frac{dx}{dy}$$

$$\Rightarrow \quad x^2 - y^2 = 2xy\frac{dx}{dy}. \text{ So } f(x) = 2x$$

Example 47: The general solution of the differential equation $\cos 2x\left(\frac{dy}{dx} - \sqrt{\frac{1+\tan x}{1-\tan x}}\right) - y = 0$ is given by

(a) $y\sqrt{\tan\left(x + \frac{\pi}{4}\right)} = x + C$

(b) $y\sqrt{\cot\left(x + \frac{\pi}{4}\right)} = x + C$

(c) $y\sqrt{\tan\left(x + \frac{\pi}{4}\right)} = \tan x + C$

(d) $y\sqrt{\cot\left(x + \frac{\pi}{4}\right)} = \cot x + C$

Ans. (b)

Solution: The given equation can be written as

$$\frac{dy}{dx} - \sec 2x\, y = \sqrt{\tan\left(x + \frac{\pi}{4}\right)}$$

This is a linear equation whose I.F. $= e^{-\int \sec 2x\, dx}$

$$= e^{-\frac{1}{2}\log\tan\left(x+\frac{\pi}{4}\right)}$$

$$= \left(\tan\left(x + \frac{\pi}{4}\right)\right)^{-1/2}$$

Multiplying with I.F., we have

$$\frac{d}{dx}\left(y\frac{1}{\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}\right) = 1$$

$$\Rightarrow y\sqrt{\cot\left(x + \frac{\pi}{4}\right)} = x + C$$

Example 48: The curve $y = f(x)$ ($f(x) \ge 0$, $f(0) = 0$) bounding a curvilinear trapezoid with the base $[0, x]$, whose area is proportional to the $(n+1)$th power of $f(x)$ $f(1) = 1$ is given by

(a) $x = y^n$ (b) $y = x^n$
(c) $y + 1 = (x+1)^n$ (d) $x = y^{n+1}$

Ans. (a)

Solution: According to the given condition

$\int_0^x f(x)\,dx = k(f(x))^{n+1}$, K being the constant of proportionality. Differentiating we have

$$f(x) = k(n+1)\,(f(x))^n f'(x)$$

$$y = k(n+1)\,y^n\frac{dy}{dx}$$

$$dx = k(n+1)y^{n-1}\,dy$$

$$\Rightarrow \quad x + C = \frac{k(n+1)y^n}{n}$$

since $y(0) = 0$ so $C = 0$. Also $y(1) = 1$, so $k = \frac{n}{n+1}$. Thus $x = y^n$.

Assertion-Reason Type Questions

Example 49: Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{2y\log y + y - x}.$$

Statement-1: $xy = y^2 \log y + C$ is a solution of the given differential equation

Statement-2: The differential equation is a linear equation in y and x.

Ans. (a)

Solution: $\frac{dx}{dy} = \frac{2y\log y + y - x}{y} = 2\log y + 1 - \frac{x}{y}$

$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y} = 2\log y + 1$ which is linear in x and y.

$$\text{I.F.} = e^{\int \frac{1}{y} dy} = y.$$

So $\quad \frac{d}{dy}(xy) = 2y\log y + y$

$$\Rightarrow \quad xy = 2\int y\log y\, dy + \frac{y^2}{2} + C$$

$$= 2\left[\frac{y^2}{2}\log y - \frac{y^2}{4}\right] + \frac{y^2}{2} + C$$

$$= y^2 \log y + C.$$

Example 50: **Statement-1:** The differential equation of all circles in a plane must be of order 3.

Statement-2: There is only one circle passing through three non-collinear points.

Ans. (a)

Solution: The equation of a circle contains three independent constants if it passes through three non-collinear points.

Example 51: **Statement-1:** Curve satisfying the differential equation $y' = y/2x$ passing through (2, 1) is a parabola with focus (1/4, 0)

Statement-2: The differential equation $y' = y/2x$ is of variable separable.

Ans. (d)

Solution: $\frac{dy}{dx} = \frac{y}{2x} \quad \Rightarrow \quad \frac{2dy}{y} = \frac{dx}{x}$

$\Rightarrow \quad \log y^2 = \log x + \text{const} \quad \Rightarrow \quad y^2 = Cx$, this passes through (2, 1) if $C = 1/2$. Thus $y^2 = 1/2x$ which represents a parabola with focus (1/8, 0).

Example 52: Let $y' + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$

Statement-1: A solution satisfying $y(0) = \pi$ is a periodic function with period 4π.

Statement-1: y can be explicitly represented in terms of x.

Ans. (b)

Solution: $y' = \sin\frac{x-y}{2} - \sin\frac{x+y}{2} = -2\sin\frac{y}{2}\cos\frac{x}{2}.$

$\Rightarrow \quad \text{cosec}\, y/2\; dy = -2\cos(x/2)\, dx$

$\Rightarrow \quad 2\log|\tan(y/4)| = -4\sin(x/2) + \text{Const}$

$\Rightarrow \quad \log|\tan(y/4)| = -2\sin(x/2) + \text{Const}$

$\Rightarrow \quad |\tan(y/4)| = \text{Const}\; e^{-2\sin(x/2)}$

Since $y(0) = \pi$ so Const = 1. Thus $|\tan(y/4)| = e^{-2\sin x/2}$

so $y = 4\tan^{-1}(\pm e^{-2\sin x/2})$ which is periodic with period 4π.

Example 53: Let $(xy^2 + x)dx + (y - x^2y)dy = 0$ satisfy $y(0) = 0$.

Statement-1: The curve represented by the solution of the given differential equation is a circle.

Statement-2: It is circle with radius 1 and centre (0, 0).

Ans. (c)

Solution: $x(1 + y^2)dx + y(1 - x^2)dx = 0$

$$\Rightarrow \quad \frac{x}{1-x^2}dx + \frac{y}{1+y^2}dy = 0$$

$\Rightarrow \quad -\log(1 - x^2) + \log(1 + y^2) = \text{Const}$

$\Rightarrow \quad 1 + y^2 = C(1 - x^2)$

Since $y(0) = 0$ so $C = 1$

$\Rightarrow \quad x^2 + y^2 = 0$ which is a point circle.

Example 54: Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2-1}\; dy - y\sqrt{y^2-1}\; dx = 0$ satisfy $y(2) = 2\sqrt{3}$

Statement 1: $y(x) = \sec(\sec^{-1} x - \pi/6)$

Statement 2: $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

Ans. (c)

Solution: The given equation can be written as

$$\frac{dy}{y\sqrt{y^2-1}} - \frac{dx}{x\sqrt{x^2-1}} = 0$$

Integrating $\sec^{-1} y - \sec^{-1} x = \text{Const}$

Putting $x = 2$, Const $= \sec^{-1} 2/\sqrt{3} - \sec^{-1}2 = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}$

so $y = \sec(\sec^{-1}x - \pi/6)$.

Example 55: **Statement 1:** The differential equation of all circles passing through one fixed point is of order 2.
Statement 2: General equation of a circle involves three arbitrary constants.
Ans. (a)

Solution: General equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

where g, f, c are arbitrary constant. Since the circle passes through a fixed point so the arbitrary constants are two. Hence required differential equation is of order 2.

Example 56: **Statement 1:** The solution of $xy' + y = y^2 \log x$ is given by $y(1 + \log x + Cx) = 1$.
Statement 2: The given differential equation is a linear equation whose I.F. is x.
Ans. (c)

Solution: $xy' + y = y^2 \log x \Rightarrow \frac{1}{y^2} y' + \frac{1}{x}\frac{1}{y} = \frac{\log x}{x}$

This is reducible to linear equation but not a linear equation. Put $\frac{1}{y} = Z \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dZ}{dx}$, so

$$-\frac{dZ}{dx} + \frac{1}{x}Z = \frac{\log x}{x} \Rightarrow \frac{dZ}{dx} - \frac{1}{x}Z = -\frac{\log x}{x}$$

The I.F. is $e^{-\int \frac{1}{x}dx} = \frac{1}{x}$. Multiplying with I.F.

$$\frac{d}{dx}\left(Z \cdot \frac{1}{x}\right) = -\frac{\log x}{x^2}$$

Integrating, we have

$$Z\frac{1}{x} = -\int \frac{\log x}{x^2}dx + C$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{1}{x} = -\left[-\frac{1}{x}\log x + \int \frac{1}{x^2}dx\right] + C$$

$$= -\left[-\frac{1}{x}\log x - \frac{1}{x}\right] + C$$

$$\Rightarrow \quad \frac{1}{y} = [1 + \log x] + Cx$$

$$\Rightarrow \quad 1 = y(1 + \log x + Cx)$$

LEVEL 2

Straight Objective Type Questions

Example 57: The degree of the differential equation satisfying

$$\sqrt{1+x^2} + \sqrt{1+y^2} = A\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right) \text{ is}$$

(a) 2 (b) 3
(c) 4 (d) none of these

Ans. (d)

Solution: Put $x = \tan\theta$ and $y = \tan\phi$. Then $\sqrt{1+x^2} = \sec\theta$, $\sqrt{1+y^2} = \sec\phi$, and the equation becomes

$$\sec\theta + \sec\phi = A(\tan\theta\sec\phi - \tan\phi\sec\theta)$$

$$\Rightarrow \quad \frac{\cos\phi + \cos\theta}{\cos\theta\cos\phi} = A\left(\frac{\sin\theta - \sin\phi}{\cos\theta\cos\phi}\right)$$

$$\Rightarrow \quad \cos\phi + \cos\theta = A(\sin\theta - \sin\phi)$$

$$\Rightarrow \quad 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = 2A\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}$$

$$\Rightarrow \quad \cot\frac{\theta-\phi}{2} = A \quad \Rightarrow \quad \theta - \phi = 2\cot^{-1}A$$

$$\Rightarrow \quad \tan^{-1}x - \tan^{-1}y = 2\cot^{-1}A.$$

Differentiating this, we get $\frac{1}{1+x^2} - \left(\frac{1}{1+y^2}\right)\frac{dy}{dx} = 0$, which is a differential equation of degree 1.

Example 58: The solution of differential equation $\frac{dy}{dx} = \frac{y}{x} + 2\frac{\phi(y/x)}{\phi'(y/x)}$ is

(a) $x^2\phi(y/x) = k$ (b) $y^2\phi(y/x) = k$
(c) $\phi(y/x) = kx^2$ (d) $\phi(y/x) = ky^2$

Ans. (c)

Solution: Putting $\frac{y}{x} = u$ we have $\frac{dy}{dx} = u + x\frac{du}{dx}$. The given differential equation can be written as $u + x\frac{du}{dx} = u$

$+ 2 \dfrac{\phi(u)}{\phi'(u)}$

$$\Rightarrow \quad x\frac{du}{dx} = 2\frac{\phi(u)}{\phi'(u)} \quad \Rightarrow \quad \frac{\phi'(u)}{\phi(u)}\,du = 2\frac{dx}{x}$$

Integrating, we get $\log \phi(u) = \log x^2 + \log k$, so $\phi(u) = kx^2$ i.e. $\phi(y/x) = kx^2$, k being an arbitrary constant.

Example 59: The orthogonal trajectories of the family of curves $a^{n-1}\, y = x^n$ are given by

(a) $x^n + n^2 y = \text{const}$ (b) $ny^2 + x^2 = \text{const}$
(c) $n^2 x + y^n = \text{const}$ (d) $n^2 x - y^n = \text{const}$

Ans. (b)

Solution: Differentiating, we have (see theory)

$$a^{n-1}\frac{dy}{dx} = nx^{n-1} \quad \Rightarrow \quad a^{n-1} = n\,x^{n-1}\frac{dx}{dy}$$

Putting this value in the given equation, we have

$$nx^{n-1}\frac{dx}{dy}\,y = x^n$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, we have $ny = -x\dfrac{dx}{dy}$

$\Rightarrow \quad ny\,dy + x\,dx = 0 \quad \Rightarrow \quad ny^2 + x^2 = \text{const.}$

which is the required family of orthogonal trajectories.

Example 60: The solution of the differential equation $\dfrac{d^2x}{dt^2} + x = 0;\ x(0) = 1,\ x'(0) = 0$

(a) approaches infinity as $t \to \infty$
(b) is a periodic function
(c) is always greater than or equal to unity
(d) does not exist

Ans. (b)

Solution: Consider the equation $m^2 + 1 = 0$, $m = \pm i$. Hence $x = C_1 \cos t + C_2 \sin t$ (see theory)

Now $x(0) = 1 \Rightarrow C_1 = 1$ and $x' = -C_1 \sin t + C_2 \cos t$ so $C_2 = 0$. Hence $x = \cos t$ is the required solution which is a periodic function.

Alternate Solution

We have $2\dfrac{dx}{dt}\dfrac{d^2x}{dt^2} = -2x\dfrac{dx}{dt} \quad \Rightarrow \quad \left(\dfrac{dx}{dt}\right)^2 = -x^2 + C$

Since $x'(0) = 0$ and $x(0) = 1$ so $C = 1$. Hence $\dfrac{dx}{dt} = \pm\sqrt{1-x^2}$

$$\Rightarrow \quad \frac{dx}{\sqrt{1-x^2}} = \pm t$$

$\Rightarrow \quad \sin^{-1} x = t + C_1$ or $\cos^{-1} x = t + C_2$

$\Rightarrow \quad x = \sin(t + C_1)$ or $x = \cos(t + C_2)$

When $t = 0, x = 1$ so $C_1 = \pi/2$ and $C_2 = 0$

Hence $x = \cos t.$

Example 61: The solution of the equation $\dfrac{dy}{dx} = \cos(x - y)$ is

(a) $y + \cot\left(\dfrac{x-y}{2}\right) = c$ (b) $x + \cot\left(\dfrac{x-y}{2}\right) = c$
(c) $x + \tan\left(\dfrac{x-y}{2}\right) = c$ (d) none of these

Ans. (b)

Solution: Putting $u = x - y$, we have $\dfrac{du}{dx} = 1 - \dfrac{dy}{dx}$. The given equation can be written as $1 - \dfrac{du}{dx} = \cos u$

$$\Rightarrow \quad \int \frac{du}{1-\cos u} = dx$$

$$\Rightarrow \quad \frac{1}{2}\int \operatorname{cosec}^2 \frac{u}{2}\,du = \int dx + \text{Const}$$

$$\Rightarrow \quad x + \cot\frac{u}{2} = \text{Const, i.e. } x + \cot\frac{x-y}{2} = c.$$

Example 62: A solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{xy[x^2 \sin y^2 + 1]} \text{ is}$$

(C is an arbitrary constant)

(a) $x^2(\cos y^2 - \sin y^2 - 2C\,e^{-y^2}) = 2$
(b) $y^2(\cos x^2 - (\sin y^2 - 2C\,e^{-y^2}) = 2$
(c) $x^2(\cos y^2 - \sin y^2 - e^{-y^2}) = 4$
(d) none of these

Ans. (a)

Solution: The given differential equation can be written as $\dfrac{dx}{dy} = xy\,[x^2 \sin y^2 + 1]$

$\Rightarrow \quad \dfrac{1}{x^3}\dfrac{dx}{dy} - \dfrac{1}{x^2}\,y = y \sin y^2$. This equation is reducible to linear equation, so putting $-1/x^2 = u$, the last equation can be written as

$$\frac{du}{dy} + 2uy = 2y \sin y^2$$

The integrating factor of this equation is e^{y^2}. So required solution is

$$ue^{y^2} = \int 2y \sin y^2 . e^{y^2}\,dy + C$$
$$= \int (\sin t).e^t\,dt + C \qquad (t = y^2)$$
$$= (1/2)\,e^{y^2}(\sin y^2 - \cos y^2) + C$$

$$\Rightarrow \quad 2u = (\sin y^2 - \cos y^2) + Ce^{-y^2}$$

$$\Rightarrow \quad 2 = x^2[\cos y^2 - \sin y^2 - 2\,Ce^{-y^2}].$$

Example 63: The solution of $(y(1 + x^{-1}) + \sin y)\,dx + (x + \log x + x\cos y)\,dy = 0$ is

(a) $(1 + y^{-1}\sin y) + x^{-1}\log x = C$
(b) $(y + \sin y) + xy\log x = C$
(c) $xy + y\log x + x\sin y = C$
(d) none of these

Ans. (c)

Solution: The given equation can be written as

$$y(1 + x^{-1})dx + (x + \log x)dy + \sin y\,dx + x\cos y\,dy = 0$$
$$\Rightarrow \quad d(y(x + \log x)) + d(x\sin y) = 0$$
$$\Rightarrow \quad y(x + \log x) + x\sin y = C$$

Example 64: If $\phi(x)$ is a differentiable function then the solution of $dy + (y\,\phi'(x) - \phi(x)\,\phi'(x))\,dx = 0$ is

(a) $y = (\phi(x) - 1) + C\,e^{-\phi(x)}$
(b) $y\phi(x) = (\phi(x))^2 + C$
(c) $ye^{\phi(x)} = \phi(x)\,e^{\phi(x)} + C$
(d) $(y - \phi(x)) = (\phi(x))\,e^{-\phi(x)}$.

Ans. (a)

Solution: The given equation can be written in the linear form as follows:

$$\frac{dy}{dx} + y\phi'(x) = \phi(x)\,\phi'(x)$$

The integrating factor of this equation is $e^{\int \phi'(x)\,dx} = e^{\phi(x)}$.

Hence $\quad \dfrac{d}{dx}(ye^{\phi(x)}) = \phi(x)\,\phi'(x)\,e^{\phi(x)}$

Integrating, we have $ye^{\phi(x)} = \int te^t\,dt + C$, (where $t = \phi(x)$)

$= te^t - e^t + C$ Hence $y = (\phi(x) - 1) + C\,e^{-\phi(x)}$.

Example 65: The solution of $y_2 - 2y_1 + y = 0$ is

(a) $y = x^2 e^x + c_1 xe^x + c_2$
(b) $y = (c_1 + c_2 x)\,e^x$
(c) $y = c_1 x^2 e^x + e^x + c_2$
(d) none of these

Ans. (b)

Solution: The given equation can be written as

$$\left(\frac{d}{dx} - 1\right)\left(\frac{dy}{dx} - y\right) = 0 \qquad \text{(i)}$$

If $\dfrac{dy}{dx} - y = u$ then (i) reduces to $\dfrac{du}{dx} - u = 0 \Rightarrow u = c_1 e^x$. Therefore, we have $\dfrac{dy}{dx} - y = c_1e^x$ which is a linear equation whose I.F. is e^{-x}. So $\dfrac{d}{dx}(ye^{-x}) = c_1 \Rightarrow ye^{-x} = c_1x + c_2 \Rightarrow y = (c_1x + c_2)e^x$. (For a short cut see theory).

Example 66: The solution of $y_2 - 7y_1 + 12y = 0$ is

(a) $y = C_1e^{3x} + C_2e^{4x}$ (b) $y = C_1xe^{3x} + C_2e^{4x}$
(c) $y = C_1e^{3x} + C_2xe^{4x}$ (d) none of these.

Ans. (a)

Solution: The given equation can be written as

$$\left(\frac{d}{dx} - 3\right)\left(\frac{dy}{dx} - 4y\right) = 0 \qquad \text{(i)}$$

If $\dfrac{dy}{dx} - 4y = u$ then (i) reduces to $\dfrac{du}{dx} - 3u = 0$

$\Rightarrow \dfrac{du}{u} = 3dx \Rightarrow u = C_1e^{3x}$. Therefore, we have $\dfrac{dy}{dx} - 4y = C_1e^{3x}$ which is a linear equation whose I.F. is e^{-4x}. So $\dfrac{d}{dx}(ye^{-4x}) = C_1e^{-x} \Rightarrow ye^{-4x} = -C_1e^{-x} + C_2$

$\Rightarrow y = C_1e^{3x} + C_2e^{4x}$ (For a short cut see theory).

Example 67: A solution of $y = 2x\left(\dfrac{dy}{dx}\right) + x^2\left(\dfrac{dy}{dx}\right)^4$ is

(a) $y = 2c^{1/2}x^{1/4} + c$ (b) $y = 2\sqrt{c}\,x^2 + c^2$
(c) $y = 2\sqrt{c}\,(x + 1)$ (d) $y = 2\sqrt{cx} + c^2$

Ans. (d)

Solution: Writing $p = \dfrac{dy}{dx}$ and differentiating w.r.t. x, we have

$$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx}$$

$$\Rightarrow \quad 0 = p(1 + 2xp^3) + 2x\frac{dp}{dx}(1 + 2p^3x)$$

$$\Rightarrow \quad p + 2x\frac{dp}{dx} = 0 \Rightarrow 2\frac{dp}{p} = -\frac{dx}{x}$$

$$\Rightarrow \quad 2\log p + \log x = \text{const} \Rightarrow p^2x = c \text{ or } p = \sqrt{\frac{c}{x}}$$

Substituting this value in the given equation, we get

$$y = 2\sqrt{cx} + c^2.$$

Example 68: The equation of the curve not passing through origin and having the portion of the tangent included between the coordinate axes is bisected at the point of contact is

(a) a parabola
(b) an ellipse or a straight line
(c) a circle or an ellipse
(d) a hyperbola

Ans. (d)

◎ **Solution**: The equation of tangent at any point $P(x, y)$ is $Y - y = \frac{dy}{dx}(X - x)$

This intersects cut the X-axis at $A\left(x - y\frac{dx}{dy}, 0\right)$ and the Y-axis at $B\left(0, y - x\frac{dy}{dx}\right)$. According to the given condition

mid point of $AB = (x, y)$

$\Rightarrow \quad x - y\frac{dx}{dy} = 2x$ and $y - x\frac{dy}{dx} = 2y$

$\Rightarrow \quad x + y\frac{dx}{dy} = 0$ and $y + x\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0$

$\Rightarrow \quad \log(xy) = \log c$

$\Rightarrow \quad xy = c$, which is a hyperbola.

Example 69: An equation of the curve for which the portion of y-axis cut off between the origin and the tangent varies as the cube of the abscissa of the point of contact is

(a) $y = Kx^3/3 + Cx$ (b) $y = -Kx^2/2 + C$
(c) $y = -Kx^3/2 + Cx$ (d) $y = Kx^3/3 + Cx^2/2$
(K is constant of proportionality)

Ans. (c)

◎ **Solution**: The portion of y-axis cut off between the origin and the tangent is $y - x\frac{dy}{dx}$. According to the given condition $y - x\frac{dy}{dx} = Kx^3$ (K is constant of proportionality)

$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = -Kx^2$. This is a linear equation whose I.F. is $1/x$.

Hence $\frac{d}{dx}(y/x) = -Kx \Rightarrow y = -Kx^3/2 + Cx$.

Example 70: Through any point (x, y) of a curve which passes through the origin, lines are drawn parallel to the co-ordiante axes. The curve, given that it divides the rectangle formed by the two lines and the axes into two areas, one of which is twice the other, represents a family of

(a) circles (b) parabolas
(c) hyperbolas (d) straight lines

Ans. (d)

◎ **Solution**: Let $P(x, y)$ be the point on the curve passing through the origin $O(0, 0)$, and let PN and PM be the lines parallel to the x- and y-axes, respectively (Fig. 15.1). If the equation of the curve is $y = y(x)$, the area $POM = \int_0^x y\,dx$ and the area $PON = xy - \int_0^x y\,dx$ Assuming that $2(POM) = PON$, we therefore have

$$2\int_0^x y\,dx = xy - \int_0^x y\,dx \quad \Rightarrow \quad 3\int_0^x y\,dx = xy.$$

Differentiating both sides of this gives

$$3y = x\frac{dy}{dx} + y \quad \Rightarrow \quad 2y = x\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{y} = 2\frac{dx}{x}$$

$\Rightarrow \quad \log|y| = 2\log|x| + C$

$\Rightarrow \quad y = Cx^2$, with C being a constant.

This solution represents a parabola. We will get a similar result if we had started instead with $2(PON) = POM$

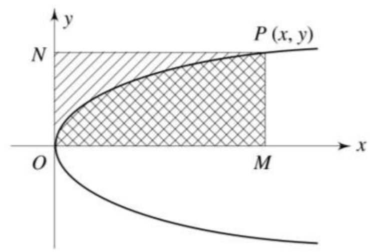

Fig. 15.1

Example 71: The solution of $(y + x + 5)dy = (y - x + 1)dx$ is

(a) $\log((y+3)^2 + (x+2)^2) + \tan^{-1}\frac{y+3}{x+2} = C$

(b) $\log((y+3)^2 + (x-2)^2) + \tan^{-1}\frac{y-3}{x-2} = C$

(c) $\log((y+3)^2 + (x+2)^2) + 2\tan^{-1}\frac{y+3}{x+2} = C$

(d) $\log((y+3)^2 + (x+2)^2) - 2\tan^{-1}\frac{y+3}{x+2} = C$

Ans. (c)

◎ **Solution**: The intersection of $y - x + 1 = 0$ and $y + x + 5 = 0$ is $(-2, -3)$. Put $x = X - 2$, $y = Y - 3$. The given equation reduces to $\frac{dY}{dX} = \frac{Y - X}{Y + X}$. This is a homogeneous equation, so putting $Y = vX$, we get

$$X\frac{dv}{dX} = -\frac{v^2+1}{v+1} \quad \Rightarrow \quad \left(-\frac{v}{v^2+1} - \frac{1}{v^2+1}\right)dv = \frac{dX}{X}$$

$\Rightarrow \quad -\frac{1}{2}\log(v^2+1) - \tan^{-1}v = \log|X| + \text{Const}$

$\Rightarrow \quad \log(Y^2 + X^2) + 2\tan^{-1}\frac{Y}{X} = \text{Const}$

$\Rightarrow \quad \log((y+3)^2 + (x+2)^2) + 2\tan^{-1}\frac{y+3}{x+2} = C.$

Example 72: General solution of the differential equation $y = x\frac{dy}{dx} + \frac{dx}{dy}$ represent

(a) a straight line or a hyperbola
(b) a straight line or a parabola
(c) a parabola or a hyperbola
(d) circles

Ans. (b)

Solution: Putting $p = \frac{dy}{dx}$, the given equation can be written as $y = px + 1/p$. Differentiating w.r.t. x, we have

$$p = \frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \quad (x - (1/p^2))\ \frac{dp}{dx} = 0 \Rightarrow p^2 = \frac{1}{x} \text{ or } \frac{dp}{dx} = 0$$

If $\frac{dp}{dx} = 0$ then p = constant = c putting this value in given equation, we get $y = cx + 1/c$ which represents a straight line. If $p^2 = 1/x$ then $y^2 = (px + 1/p)^2 = p^2x^2 + 1/p^2 + 2x = \frac{1}{x}x^2 + x + 2x = 4x$, which represents a parabola.

Example 73: The curves satisfying the differential equation $(1 - x^2)\, y' + xy = ax$ are

(a) ellipses and hyperbolas
(b) ellipses and parabola
(c) ellipses and straight lines
(d) circles and ellipses

Ans. (a)

Solution: The given equation is linear in y and can be written as

$$\frac{dy}{dx} + \frac{x}{1-x^2}y = \frac{ax}{1-x^2}$$

Its integrating factor is $e^{\int \frac{x}{1-x^2}dx} = e^{-(1/2)\log(1-x^2)}$

$= \frac{1}{\sqrt{1-x^2}}$ if $-1 < x < 1$ and if $x^2 > 1$ then I.F. $= \frac{1}{\sqrt{x^2-1}}$

$$\frac{d}{dx}\left(y\frac{1}{\sqrt{1-x^2}}\right) = \frac{ax}{(1-x^2)^{3/2}} = -\frac{1}{2}a\frac{-2x}{(1-x^2)^{3/2}}$$

$$\Rightarrow \quad y\frac{1}{\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + C \Rightarrow y = a + C\sqrt{1-x^2}$$

$$\Rightarrow \quad (y-a)^2 = C^2(1-x^2)$$

$$\Rightarrow \quad (y-a)^2 + C^2x^2 = C^2$$

Thus if $-1 < x < 1$ the given equation represents an ellipse. If $x^2 > 1$ then the solution is of the form $-(y-a)^2 + C^2x^2 = C^2$ which represents a hyperbola.

Example 74: Consider the differential equation $y^2\,dx + \left(x - \frac{1}{y}\right)dy = 0$. If $y(1) = 1$, then x is given by:

(a) $4 - \frac{2}{y} - \frac{e^{1/y}}{e}$ (b) $3 - \frac{1}{y} + \frac{e^{1/y}}{e}$

(c) $1 + \frac{1}{y} - \frac{e^{1/y}}{e}$ (d) $1 - \frac{1}{y} + \frac{e^{1/y}}{e}$

Ans. (c)

Solution: The given differential equation can be written as $\frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3}$ which is a linear equation with I.F. $= e^{\int(1/y^2)dy} = e^{-1/y}$. Multiplying with the integrating factor

$$\frac{d}{dx}\left(x\,e^{-1/y}\right) = \frac{1}{y^3}e^{-1/y}$$

$$\Rightarrow \quad x\,e^{-1/y} = \int \frac{1}{y^3}e^{-1/y}dy + C$$

$$= \int -u\,e^u du + C \quad (u = -1/y)$$

$$= -u\,e^u + \int e^u\,du + C$$

$$= e^{-1/y}\,(1 + 1/y) + C$$

Thus, $\quad x = 1 + \frac{1}{y} + C\,e^{1/y}$

When $\quad x = 1, y = 1$

$1 = 1 + 1 + C\,e \Rightarrow C = -1/e.$

$$\therefore \quad x = 1 + \frac{1}{y} - \frac{e^{1/y}}{e}.$$

EXERCISE

Concept-based Straight Objective Type Questions

1. The general solution of $y' + xy = 4x$ is given by

(a) $y = e^{(-1/2)x^2} + Cx$ (b) $y = e^{(1/2)x^2} + Cx$

(c) $y = Ce^{(-1/2)x^2} - 4$ (d) $y = e^{(1/2)x^2} + Cx + 4$

2. The solution of $x' + x\tan t = \sec t$, $x(0) = 1$ is

(a) $x = \cot t + \sin t$ (b) $x = \cos t + 2\sin t$

(c) $x = \cos t + \sin^2 t$ (d) $x = \cos^2 t + \sin t$

3. A particular solution of the initial value differential equation

$\log \dfrac{dy}{dx} = 7x + 8y,\ y(0) = 0$ is

(a) $15y = 8e^{-8y} + 7e^{-7y}$
(b) $15x = 7e^{-8y} + 8e^{7x}$
(c) $8e^{-8y} + 7e^{7x} = 15$
(d) $7e^{-8y} + 8e^{7x} = 15$

4. If $x(t)$ is a solution of $(t + 1)dx = (2x + (t + 1)^4)\,dt$, $x(0) = 2$ then $\lim_{t\to1} x(t)$ is

(a) 8 (b) 10
(c) 14 (d) 12

5. The degree of the differential equation $\dfrac{dy}{dx} + x = \left(y + x\dfrac{dy}{dx}\right)^{-4}$ is

(a) 2 (b) 3
(c) 4 (d) 5

6. The solution $y' = \dfrac{1-2y-4x}{1+y+2x}$ is given by

(a) $(y + 2x) + \dfrac{1}{2}(y + 2x)^2 = 2x + C$
(b) $(y + \dfrac{1}{2}(y + 2x)^2) = x + C$
(c) $x + \dfrac{1}{2}(2x + y)^2 = y + C$
(d) $2x + \dfrac{1}{2}(2x + y)^2 = y + C$

7. The solution of $\left(x - y\cos\dfrac{y}{x}\right)dx + x\cos\dfrac{y}{x}\,dy = 0$, $y(1) = 0$ is given by

(a) $\sin\dfrac{y}{x} + \log|x| = 0$ (b) $y\cos\dfrac{y}{x} + \log|x| = 0$
(c) $\tan\dfrac{y}{x} + \log|x| = 0$ (d) $y\sin\dfrac{y}{x} + \log|x| = 0$

8. The solution $y^{n-1}(ay' + y) = x$ is

(a) $(n-1)y^{n-1} = Ce^{-\frac{nx}{a}} + nx + a$
(b) $ny^{n-1} = Ce^{-\frac{nx}{a}} + nx$
(c) $ny^{n} = Ce^{-\frac{nx}{a}} + nx - a$
(d) $ny^{n-1} = Ce^{-\frac{nx}{a}} + nx + a$

9. A curve passing through origin, all the normals to which pass through (1, 2) is

(a) a circle with centre origin
(b) a straight line
(c) a parabola
(d) a circle with centre (1, 2)

10. The solution of $y' = (x + y)^2$, $y(0) = 0$ is given by

(a) $\tan x = x + y$ (b) $\sin^2 y = x + y$
(c) $\tan y = x + y$ (d) $\sin y = x + y$

LEVEL 1

Straight Objective Type Questions

11. $y = ae^{-1/x} + b$ is a solution of $\dfrac{dy}{dx} = \dfrac{y}{x^2}$ when

(a) $a = 1, b = 0$ (b) $a = 3, b = 0$
(c) $a = 1, b = 1$ (d) $a = 2, b = 2$

12. The solution of $y - xy' = 2(x + yy')$, $y(1) = 1$ is

(a) $\tan^{-1}(y/x) + \log(x^2 + y^2) = \log 2$
(b) $\tan^{-1}(y/x) + \log(x^2 + y^2) = \pi/4 + \log 2$
(c) $\tan^{-1}(y/x) + \log(x^2 + y^2) = \log 3$
(d) none of these

13. The solution of $\dfrac{x^3\,dx + yx^2\,dy}{\sqrt{x^2+y^2}} = y\,dx - x\,dy$ is

(a) $\sqrt{x^2+y^2} = Cx$
(b) $\sqrt{x^2+y^2} + y/x = C$
(c) $\sqrt{x^2+y^2} + y/x^2 = C$
(d) $(x^2 + y^2)^2 + xy^2 = C$

14. Let $f(x)$, $g(x)$ be twice differentiable function on [0, 2] satisfying $f''(x) = g''(x)$, $f'(1) = 2g'(1) = 4$ and $f(2) = 3$, $g(2) = 9$, then the value of $f(4) - g(4)$ is

(a) 0 (b) 2
(c) 8 (d) – 2

15. Which of the following differential equation is not degree 1

(a) $x^3 y_2 + (x + x^2)\,y_1^{\,2} + e^x y^3 = \sin x$
(b) $y_2^{1/2} + \sin xy_1 + xy = x$
(c) $\sqrt{y_1 + y} = x + 1$
(d) $y = 2y_1 + \sqrt{y_1 + y}$

16. The general solution of

$$y' = \frac{y}{x} + \frac{x}{y}\frac{\phi(y^2/x^2)}{\phi'(y^2/x^2)}$$ is

(a) $y^2 = K\,\phi\,(y^2/x^2)$ (b) $x^2 = K\,\phi(y^2/x^2)$
(c) $x = K\,\phi(y^2/x^2)$ (d) $y = Kx\,\phi(y^2/x^2)$

17. The solution of $\log \frac{dy}{dx} = 3x + 4y$, $y(0) = 0$ is

(a) $e^{3x} + {}^{-4y} = 4$ (b) $4e^{3x} - e^{-4y} = 3$
(c) $3e^{3x} + 4e^{4y} = 7$ (d) $4e^{3x} + 3e^{-3x} = 7$

18. The solution of the equation $(x + y)\,dy - (x - y)\,dx = 0$ is

(a) $y^2 + 2xy + x^2 = K$ (b) $y^2 + 2xy - x^2 = K$
(c) $x^3 + 2xy - y^2 = K$ (d) $y^2 - 2xy + x^2 = K$

19. The solution of $(1 + y + x^2y)\,dx + (x + x^3)\,dy = 0$ is

(a) $y + \tan^{-1} x = C$ (b) $xy + \tan^{-1} x = C$
(c) $y^2 + \tan^{-1} x = C$ (d) $x^2 + \tan^{-1} y/x = C$

20. The equation of curves which intersect the hyperbola $xy = 4$ at an angle $\pi/2$ is

(a) $y = \frac{x^2}{\sqrt{2}} + C$ (b) $y^2 = \frac{x^3}{3} + C$
(c) $y^2 - x^2 = C$ (d) $xy = x^2 + C$

21. The differential equation of a curve such that the initial ordinate of any tangent at the point of contact is equal to the corresponding subnormal is

(a) a linear equation
(b) not a homogeneous equation
(c) an equation with separable variables
(d) none of these

22. The curve satisfying $y\,dx - x\,dy + \log x\,dx = 0$ $(x > 0)$ passing through $(1, -1)$ is

(a) $y + \log x + 1 = 0$ (b) $-y^2 + \log x + 1 = 0$
(c) $y^3 + (\log x)^2 + 1 = 0$ (d) none of these

23. A differential equation associated with the primitive $y = a + be^{5x} + ce^{-7x}$ is

(a) $y_3 + 2y_2 - y_1 = 0$
(b) $4y_3 + 5y_2 - 20y_1 = 0$
(c) $y_3 + 2y_2 - 35y_1 = 0$
(d) none of these

24. The differential equation of the family of circles passing through the fixed points $(a, 0)$ and $(-a, 0)$ is

(a) $y_1\,(y^2 - x^2) + 2xy + a^2 = 0$
(b) $y_1\,y^2 + xy + a^2x^2 = 0$
(c) $y_1(y^2 - x^2 + a^2) + 2xy = 0$
(d) none of these

25. Solution of the differential equation $(x - y)^2 \frac{dy}{dx} = a^2$ is

(a) $y = \frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right| + c$

(b) $x = \frac{a}{2}\log\left|\frac{x-y+a}{x-y-a}\right| + c$

(c) $y^2 = a\log\left|\frac{x-y+a}{x-y-a}\right| + c$

(d) none of these

26. Solution of the differential equation $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$ is

(a) $\log\left|1 + \tan\left(\frac{x+y}{2}\right)\right| = y + c$

(b) $\log\left|2 + \sec\left(\frac{x+y}{2}\right)\right| = x + c$

(c) $\log|1 + \tan(x + y)| = y + c$
(d) none of these

27. Equation of the curve through the origin satisfying $dy = (\sec x + y \tan x)\,dx$ is

(a) $y \sin x = x$ (b) $y \cos x = x$
(c) $y \tan x = x$ (d) none of these

28. The solution of $(x\,dx + y\,dy)(x^2 + y^2) + (xdy - ydx) = 0$ is

(a) $x^2 + y^2 + \frac{y}{x} = C$

(b) $x^2 + y^2 + 2\tan^{-1}\left(\frac{y}{x}\right) = C$

(c) $\frac{x^2+y}{2} + \tan^{-1}\left(\frac{x}{y}\right) = C$

(d) $\frac{x^2+y^2}{2} + \tan^{-1}\left(\frac{x}{y}\right) = C$

29. General solutions of $y' = e^{y/x} + y/x$ is

(a) $Cx = e^{-y/x}$ (b) $\log|Cx| = -e^{-y/x}$
(c) $\log|Cx| = e^{y/x}$ (d) $\log|Cx| = e^{-y/x}$

30. The general solution of $y' + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$ is

(a) $\log|\tan(y/2)| = C + \sin x/2$
(b) $\log|\tan(y/2)| = C - 2\sin x$
(c) $\log|\tan(y/4)| = C - 2\sin x/2$
(d) none of these

31. The order and the degree (respectively) of the differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, $c > 0$ is a parameter are

(a) 3, 1 (b) 1, 3
(c) 2, 4 (d) 4, 1

32. Let f be a real-valued on **R** such that $f(1) = 1$. If the y-intercept of the tangent at any point $\{(x, y)$ on the curve $y = f(x)$ is equal to cube of the abscissa of P, then the value of $f(-3)$ is equal to
(a) 3 (b) 8
(c) 12 (d) 9

33. The curves given by $2xyy' = y^2 - x^2$ represent
(a) family of circles with centre on y-axis
(b) family of parabolas passing through origin
(c) family of circles with centre on x-axis
(d) family of hyperbola

34. Solution to $9yy' + 4x = 0$ represent
(a) family of circles
(b) family of ellipses
(c) family of straight lines
(d) family of hyperbolas

35. The solution of initial value problem $y' = -2xy$, $y(0) = 1$ represents
(a) a circle with centre on x-axis
(b) an ellipse
(c) bell-shaped curve
(d) a circle with centre on y-axis

Assertion-Reason Type Questions

36. Let $y' = \cos(x - y)$ such that $y(0) = -\pi$ then
Statement-1: y can be expressed explicitly in terms of x
Statement-2: $x + \sec(x - y) = -1$

37. Let $y = f(x)$ be a curve having the following property: The segment of the tangent between the point of tangency and the x-axis is bisected at the point of intersection with the y-axis.
Statement-1: Such curves represent hyperbola
Statement-2: Curves have latus rectum parallel to y-axis.

38. Let $y' = 3x - 2y + 5$
Statement-1: The solution of the above equation is $4y - 6x - 7 = Ce^{-2x}$
Statement-2: The given equation is linear in y and x with I.F. e^{2x}

39. Let a solution $y = y(x)$ of the differential equation $y \sin x + y' \cos x = 1$ satisfy $y(0) = 1$
Statement-1: $y(x) = \sin(x + \pi/4)$
Statement-2: The integrating factor of the given differentiable equation is sec x.

40. Let $xy' + y - e^x = 0$, $y(a) = b$
Statement-1: The solution is given by $yx = e^x + ab - e^a$
Statement-2: The given equation is a linear function with I.F. x.

LEVEL 2

Straight Objective Type Questions

41. A curve such that the area of the trapezoid formed by coordinate axes, ordinate at an arbitrary point and the tangent at this point equals half the square of its abscissa is
(a) $y = Cx^2$ (b) $y = (1/2)x + Cx^2$
(c) $3y = x + Cx^2$ (d) $y = x + Cx^2$

42. A curve passing through origin, all the normals to which pass through (x_0, y_0) is
(a) $yy_0 = x^2 + y^2$ (b) $x^2 + y^2 = 2(xx_0 + yy_0)$
(c) $x^2 + y^2 = x_0^2 + y_0^2$ (d) none of these

43. The trajectories orthogonal to $(2a - x)y^2 = x^3$ is
(a) $x^2 + y^2 = C(y^2 + 2x^2)$
(b) $(x^2 + y^2)^2 = Cxy$
(c) $(x^2 + y^2)^2 = C(y^2 + 2x^2)$
(d) $(x^2 + y^2) = x^3 + xy^2$

44. The general solution of $xy_5 = y_4$ is given by
(a) $y = C_1 x^5 + C_2 x^3 + C_3 x^2 + C_4 x + C_5$
(b) $y = C_1 x^5 + C_2 x^4 + C_3 x^2 + C_4 x + C_5$
(c) $y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4$
(d) none of these
(C_1, C_2, C_3, C_4, C_5 being arbitrary constants)

45. The general solution of $yy'' = (y')^2$ is
(a) $y = C_1 x + C_2$ (b) $y = C_2 e^{C_1 x}$
(c) $y = C_2 + e^{C_1 x}$ (d) $y = e^{C_2 x} + e^{C_1 x}$

46. The curve such that the ratio of the subnormal at any point to the sum of its abscissa and ordinate is equal to the ratio of the ordinate of this point to its abscissa is

(a) $y = \log |Cx|$ (b) $y = x^2 + Cx$
(c) $y = x \log C(x^2 + y^2)$ (d) $y = x \log |Cx|$

47. The curve $y = f(x)$ ($f(x) \geq 0$, $f(0)$, $= 0$, $f(1) = 1$) bounding a curvilinear trapezoid with the base $[0, x]$, whose area is proportional to the $(n + 1)$th power of $f(x)$ is

(a) $y' = x$ (b) $y^{n+1} = x$
(c) $x^n = y$ (d) $x^{n+1} = y$

48. The curve satisfying the equation

$$y_1 = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}$$

passing through $(1, -1)$ is a

(a) straight line (b) circle
(c) ellipse (d) none of these

49. The curve whose subtangent is n times the abscissa of the point of contact and passes through $(2, 3)$ represent

(a) a straight line for $n = 2$
(b) a parabola with vertex $(2, 3)$ for $n = 1$
(c) a circle with centre $(2, 3)$ for $n = 2$
(d) a parabola with vertex origin and axes coincide with x-axis for $n = 2$

50. Solution of $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$ is

(a) $y = 2Cx + C^2$ (b) $x^2 = 2Cy + C^2$
(c) $y^2 = 2Cx + C^2$ (d) $xy = Cx^2 + C^2$

51. If the area of the figure bounded by a curve, the x-axis, and two ordinates, one of which is constant, the other variable is equal to the ratio of the cube of the variable ordinate to the variable abscissa, then the curve is

(a) $(2y^2 - x^2)^3 = Cx^2$ (b) $(y^2 - x^2)^3 = Cx^2$
(c) $(2y - x^2)^2 = Cx^3$ (d) $(2y + x^2)^2 = Cx^3$

52. The curve such that the initial ordinate of any tangent is less than the abscissa of the point of tangency by two unit is

(a) $y = Cx + \log |x| - 2$
(b) $y = Cx - \frac{x^2}{2} \log |x| - 2$
(c) $y = Cx^2 + \log |x| - 2$
(d) $y = Cx - x \log |x| - 2$

53. The solution of $y'(y - x - 4) = x + y - 2$ is given by

(a) $x^2 + 2xy + y^2 - 4x + 16y = C$
(b) $x^2 + 2xy - y^2 - 4x + 8y = C$
(c) $x^2 - 2xy + y^2 - 2x + 4y = C$
(d) $x^2 - 2xy + y^2 - 4x + 8y = C$

54. A solution of $y = xy' - 3y'^3$ is

(a) $y = 2x - 3$ (b) $y = 3x - 9$
(c) $y = x - 3$ (d) $y = 2x - 6$

55. The trajectories orthogonal to $x^2 + y^2 = 2ax$ is

(a) $y = x^2$ (b) $y = C(x^2 + y^2)$
(c) $y = C(x^2 + 2y^2)$ (d) $y^2 = Cx$

56. The solution of $y'' - 4y' + 3y = 0$, $y(0) = 6$ and $y'(0) = 10$ is

(a) $y = 4e^x + 2e^{3x}$ (b) $y = e^x + 6e^{3x}$
(c) $y = (2x + 6)e^x$ (d) $y = 2e^x + 4e^{3x}$

57. The order of the differential equation whose general solution is given by $y = (C_1 + C_2) \cos(x + C_3) - C_4 e^{x + C_5}$, where C_1, C_2, C_3, C_4, C_5 are arbitrary constant is

(a) 5 (b) 4
(c) 3 (d) 2

58. The solution of the equation $\frac{dy}{dx} = \sec(x + y)$ is given by

(a) $y - \tan(x + y) = C$
(b) $y - \frac{1}{2}\tan(x + y) = C$
(c) $y - \tan\frac{1}{2}(x + y) = C$
(d) $y + \frac{1}{2}\tan(x + y) = C$

59. The differential equation corresponding to $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$, where C_i are arbitrary constants and m_1, m_2, m_3 are roots of the equation $m^3 - 7m + 6 = 0$ is

(a) $y_3 - 7y_1 + 6y = 0$
(b) $y_3 - 7y_2 + 6y_1 + y = 0$
(c) $y_3 - 6y_2 + 7y_1 + y = 0$
(d) $y_3 + 7y_2 - 6y = 0$

60. The solution of the differential equation

$\frac{d^2y}{dx^2} = x + e^{3x}$ when $y_1(0) = \frac{1}{3}$ and $y(0) = 0$ is

(a) $y = \frac{x^3}{6} + \frac{e^{3x}}{9} + \frac{1}{3}x - \frac{1}{9}$
(b) $y = \frac{1}{18}(3x^3 + 2e^{3x} - 2)$
(c) $y = \frac{x^3}{6} + \frac{e^{3x}}{9} + \frac{2x}{9} + \frac{1}{3}$
(d) none of these

61. The solution of the differential equation $\frac{d^2y}{dx^2} = e^x \sin x$ when $y_1\left(\frac{\pi}{4}\right) = 0,\ y\left(\frac{\pi}{2}\right) = 0$ is

(a) $y = e^x \sin(x - \pi/4) + x + \frac{\pi}{2}$

(b) $y = (1/2)\, e^x \sin(x + \pi/2)$

(c) $y = (1/\sqrt{2})\, e^x \sin(x - \pi/2)$

(d) $y = (1/2)\, e^x \sin(x - \pi/2)$

Previous Years' AIEEE/JEE Main Questions

1. The solution of the differential equations
$$(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0 \text{ is}$$
(a) $2x\, e^{\tan^{-1} y} = e^{2\tan^{-1} y} + k$
(b) $x\, e^{\tan^{-1} y} = \tan^{-1} y + k$
(c) $x\, e^{2\tan^{-1} y} = e^{\tan^{-1} y} + k$
(d) $(x - 2) = k\, e^{-\tan^{-1} y}$ **[2003]**

2. The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant is
(a) $(x^2 - y^2)y' = 2xy$ (b) $2(x^2 + y^2)y' = xy$
(c) $2(x^2 - y^2)y' = xy$ (d) $(x^2 + y^2)y' = 2xy$
[2004]

3. The solution of the differential equation $y\, dx + (x + x^2y)\, dy = 0$ is
(a) $\frac{1}{xy} + \log y = C$ (b) $-\frac{1}{xy} + \log y = C$
(c) $-\frac{1}{xy} = C$ (d) $\log y = Cx$ **[2004]**

4. If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is
(a) $\log\left(\frac{y}{x}\right) = Cx$ (b) $\log\left(\frac{x}{y}\right) = Cy$
(c) $y\log\left(\frac{x}{y}\right) = Cx$ (d) $x\log\left(\frac{y}{x}\right) = Cy$ **[2005]**

5. The differential equation whose solution is $Ax^2 + By^2 = 1$, where A and B are arbitrary constants is of
(a) second order and first degree
(b) second order and second degree
(c) first order and second degree
(d) first order and first degree **[2006]**

6. The differential equation of all circles passing through the origin and having their centres on the x-axis is
(a) $x^2 = y^2 + xy\frac{dy}{dx}$ (b) $x^2 = y^2 + 3xy\frac{dy}{dx}$
(c) $y^2 = x^2 + 2xy\frac{dy}{dx}$ (d) $y^2 = x^2 - 2xy\frac{dy}{dx}$
[2007]

7. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is
(a) $(x - 2)y'^2 = 25 - (y - 2)^2$
(b) $(y - 2)y'^2 = 25 - (y - 2)^2$
(c) $(y - 2)^2y'^2 = 25 - (y - 2)^2$
(d) $(x - y)^2\, y'^2 = 25 - (y - 2)^2$ **[2008]**

8. The differential equation which represents the family of curves $y = c_1e^{c_2x}$, where c_1 and c_2 are arbitrary constants is
(a) $yy'' = y'$ (b) $yy'' = y'^2$
(c) $y'' = y^2$ (d) $y'' = y'y$ **[2009]**

9. Solution of the differential equation
$\cos x\, dy = y(\sin x - y)dx,\quad 0 < x < \pi/2$ is
(a) $y\tan x = \sec x + C$ (b) $\tan x = (\sec x + C)y$
(c) $\sec x = (\tan x + C)y$ (d) $y\sec x = \tan x + C$
[2010]

10. If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, $y(\log 2)$ is equal to:
(a) –2 (b) 7
(c) 5 (d) 13 **[2011]**

11. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where $k > 0$ is a constant and T is the total life in years of the

equipment. Then the scrap value $V(T)$ of the equipment is:

(a) e^{-kT} (b) $T^2 - \frac{I}{K}$

(c) $I - \frac{kT^2}{2}$ (d) $I - \frac{K(T-t)^2}{2}$

[2011]

12. Consider the differential equation $y^2\,dx + \left(x - \frac{1}{y}\right) dy = 0$. If $y(1) = 1$, then x is given by:

(a) $4 - \frac{2}{y} - \frac{e^{1/y}}{e}$ (b) $3 - \frac{1}{y} + \frac{e^{1/y}}{y}$

(c) $1 + \frac{1}{y} - \frac{e^{1/y}}{e}$ (d) $1 - \frac{1}{y} + \frac{e^{1/y}}{y}$ **[2011]**

13. The population $p(t)$ dt time t of a certain mouse species satisfies the differential equation $\frac{d}{dt}\,p(t) = 0.5\,p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is :

(a) log 9 (b) $\frac{1}{2}\log 18$

(c) log 18 (d) 2 log 18 **[2012]**

14. At present, a firm is manufacturing 2000 items. It is estimated that the change of production P w.r.t. additional number of workers is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is

(a) 3000 (b) 3500

(c) 4500 (d) 2500 **[2013]**

15. Consider the differential equation

$$\frac{dy}{dx} = \frac{y^3}{2(xy^2 - x^2)}$$

Statement 1: The substitution $z = y^2$ transforms the above equation into first order homogeneous differential equation

Statement 2: The solution of this differential equation is $y^2 e^{-y^2/x} = C$. **[2013, online]**

16. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{d\,p(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ equals

(a) $400 - 300\,e^{t/2}$ (b) $300 - 200\,e^{-t/2}$

(c) $600 - 500\,e^{t/2}$ (d) $400 - 300\,e^{-t/2}$ **[2014]**

17. If the differential equation representing the family of all circles touching x-axis at the origin is $(x^2 - y^2)\frac{dy}{dx} = g(x)y$, then $g(x)$ equals

(a) $\frac{1}{2}x$ (b) $2x^2$

(c) $2x$ (d) $\frac{1}{2}x^2$ **[2014, online]**

18. If the general solution of the differential equation $y' = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$, for some function ϕ is given by $y \log |cx| = x$, where C is an arbitrary constant, then $\phi(2)$ is equal to

(a) 4 (b) $\frac{1}{4}$

(c) -4 (d) $-\frac{1}{4}$ **[2014, online]**

19. The general solution of the differential equation $\sin 2x \left(\frac{dy}{dx} - \sqrt{\tan x}\right) - y = 0$ is

(a) $y\sqrt{\tan x} = x + C$

(b) $y\sqrt{\cot x} = \tan x + C$

(c) $y\sqrt{\tan x} = \cot x + C$

(d) $y\sqrt{\cot x} = x + C$ **[2014, online]**

20. If $\frac{dy}{dx} + y\tan x = \sin 2x$ and $y(0) = 1$, then $y(\pi)$ is equal to

(a) 1 (b) -1

(c) -5 (d) 5 **[2014, online]**

21. Let $y(x)$ be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x$ $(x \geq 1)$. Then $y(e)$ is equal to

(a) e (b) 0

(c) 2 (d) $2e$ **[2015]**

22. The solution of the differential equation $y\,dx - (x + 2y^2)dy = 0$ is $x = f(y)$. If $f(-1) = 1$, then $f(1)$ is equal to

(a) 4 (b) 3

(c) 2 (d) 1 **[2015, online]**

23. If $y(x)$ is the solution of the differential equation $(x + 2)\frac{dy}{dx} = x^2 + 4x - 9$, $x \neq -2$ and $y(0) = 0$, then $y(-4)$ is equal to

(a) 0 (b) 1

(c) -1 (d) 2 **[2015, online]**

24. For $x \in \mathbf{R}$, $x \neq 0$ if $y(x)$ is a differentiable function such that

$x\int_1^x y(t)\,dt = (x+1)\int_1^x t\,y(t)\,dt$, then $y(x)$

equals: (where C is a constant)

(a) $C x^3 e^{1/x}$ (b) $\frac{C}{x^2}e^{-1/x}$

(c) $\frac{C}{x}e^{-1/x}$ (d) $\frac{C}{x^3}e^{-1/x}$ **[2016, online]**

25. The solution of the differential equation

$\frac{dy}{dx} + \frac{y}{2}\sec x = \frac{\tan x}{2y}$, where $0 \le x \le \frac{\pi}{2}$, and $y(0) = 0$ is given by

(a) $y^2 = 1 + \frac{x}{\sec x + \tan x}$

(b) $y = 1 + \frac{x}{\sec x + \tan x}$

(c) $y = 1 - \frac{x}{\sec x + \tan x}$

(d) $y^2 = 1 - \frac{x}{\sec x + \tan x}$

[2016, online]

26. If a curve $y = f(x)$ passes through the point $(1, -1)$ and differential equation, $y(1 + xy)dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to

(a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$

(c) $\frac{2}{5}$ (d) $\frac{4}{5}$ **[2016]**

Previous Years' B-Architecture Entrance Examination Questions

1. A particular solution of the initial value differential equation

$\log \frac{dy}{dx} = 3x + 4y,\ y(0) = 0$

(a) $16y = 3(4x - 3 + 3e^{4x})$
(b) $3e^{-4y} - 4e^{3x} = 1$
(c) $4e^{3x} + 3e^{-4y} = 7$
(d) $16y = -3(4x + 3 - 3e^{4x})$ **[2006]**

2. The equation of motion of a particle are given by $\frac{dx}{dt} = t(t+1)$, $\frac{dy}{dt} = \frac{1}{t+1}$

where the particle is at $(x(t))$, $y(t)$ at time t. If the particle is at the origin at $t = 0$ then
(a) $6x = (e^y + 1)(e^y - 1)^2$
(b) $6x = (2e^y - 1)(e^y + 1)^2$
(c) $6x = (e^y - 1)(e^y + 1)^2$
(d) $6x = (2e^y + 1)(e^y - 1)^2$ **[2007]**

3. The degree of the differential equation which has a solution $y^2 = 4a(x + a^2)$ where a is arbitrary constant
(a) 2 (b) 3
(c) 4 (d) 1 **[2009]**

4. Let $u(t)$ and $v(t)$ be two solutions of the differential equation $\frac{dy}{dt} = e^{t^2}y(t) + \sin t$ with $u(2) < v(2)$

Statement 1: $u(t) < v(t)$ for all t

Statement 2: $u - v$ is proportional to a positive function of t **[2010]**

5. If $y(x)$ is a solution of the differential equation $\frac{dy}{dx} + 3y = 2$, then $\lim_{x\to\infty} y(x)$ is equal to

(a) 1 (b) 0

(c) $\frac{3}{2}$ (d) $\frac{2}{3}$ **[2011]**

6. The differential equation $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$ determines a family of circles with
(a) variable radius and fixed centre
(b) variable radius and variable centre
(c) fixed radius and variable centre on x-axis
(d) fixed radius and variable centre on y-axis **[2012]**

7. The general solution of the differential equation $\frac{dy}{dx} + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$ is:

(a) $\log\tan\frac{y}{2} + 2\sin x = C$

(b) $\log\tan\frac{y}{4} + 2\sin\frac{x}{2} = C$

(c) $\log \cot \frac{y}{2} + 2\sin x = C$

(d) $\log \cot \frac{y}{4} + 2\sin \frac{x}{2} = C$ **[2013]**

8. Consider the differential equation, $ydx - (x + y^2)\,dy = 0$. If for $y = 1$, x takes value 1, then the value of x when $y = 4$ is
(a) 9 (b) 16
(c) 36 (d) 64 **[2014]**

9. The general solution of the differential equation $y\,dy + \sqrt{1+y^2}\,dx = 0$ represent a family of
(a) circles
(b) ellipses other than circles
(c) hyperbolas
(d) parabolas **[2015]**

10. The solution of differential equation $\frac{ydx + xdy}{ydx - xdy} = \frac{x^2 e^{xy}}{y^4}$, satisfying $y(0) = 1$, is
(a) $x^3 = 3y^3(1 - e^{-xy})$ (b) $x^3 = 3y^3(-1 + e^{xy})$
(c) $x^3 = 3y^3(1 - e^{xy})$ (d) $x^3 = 3y^3(-1 + e^{-xy})$
[2016]

Answers

Concept-based

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)
5. (d)	**6.** (b)	**7.** (a)	**8.** (c)
9. (d)	**10.** (a)		

Level 1

11. (a)	**12.** (b)	**13.** (b)	**14.** (d)
15. (c)	**16.** (b)	**17.** (c)	**18.** (c)
19. (b)	**20.** (c)	**21.** (a)	**22.** (a)
23. (c)	**24.** (c)	**25.** (a)	**26.** (d)
27. (b)	**28.** (d)	**29.** (b)	**30.** (c)
31. (b)	**32.** (d)	**33.** (c)	**34.** (b)
35. (c)	**36.** (c)	**37.** (d)	**38.** (a)
39. (d)	**40.** (a)		

Level 2

41. (b)	**42.** (b)	**43.** (c)	**44.** (a)
45. (b)	**46.** (d)	**47.** (a)	**48.** (a)
49. (d)	**50.** (c)	**51.** (a)	**52.** (d)
53. (b)	**54.** (c)	**55.** (b)	**56.** (a)
57. (c)	**58.** (c)	**59.** (a)	**60.** (b)
61. (d)			

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (a)	**3.** (b)	**4.** (a)
5. (a)	**6.** (c)	**7.** (c)	**8.** (b)
9. (c)	**10.** (b)	**11.** (c)	**12.** (c)
13. (d)	**14.** (b)	**15.** (b)	**16.** (a)
17. (c)	**18.** (d)	**19.** (d)	**20.** (c)
21. (c)	**22.** (b)	**23.** (a)	**24.** (d)
25. (d)	**26.** (d)		

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (d)	**3.** (b)	**4.** (a)
5. (d)	**6.** (c)	**7.** (a)	**8.** (b)
9. (c)	**10.** (a)		

Hints and Solutions

Concept-based

1. The given equation is a linear equation with I.F. = $e^{-\int x dx} = e^{-\frac{1}{2}x^2}$. so

$$\frac{d}{dx}\left(y\,e^{-\frac{1}{2}x^2}\right) = 4x\,e^{-\frac{1}{2}x^2}$$

$$\Rightarrow y\,e^{-\frac{1}{2}x^2} = 4\int x\,e^{-\frac{1}{2}x^2}\,dx + C = -4\,e^{-\frac{1}{2}x^2} + C$$

so $y = -4 + Ce^{\frac{1}{2}x^2}$

2. I.F. = $e^{\int \tan t\,dt} = e^{\log \sec t} = \sec t$. Multiplying with I. F. we have $\frac{d}{dt}(x \sec t) = \sec^2 t$

$\Rightarrow x \sec t = \tan t + C$

Since $x(0) = 1$ so $C = 1$. Thus $x \sec t \neq \tan^{-1} t + 1$

$\Rightarrow x = \sin t + \cos t$

3. $\frac{dy}{dx} = e^{7x+8y} = e^{7x}\,e^{8y} \Rightarrow e^{-8y}\,dy = e^{7x}\,dx$

so $\frac{e^{-8y}}{-8} = \frac{e^{7x}}{7} + C$. Since $y(0) = 0$ so $-\frac{1}{8} = \frac{1}{7} + C$

$\Rightarrow C = -\frac{1}{8} - \frac{1}{7} = -\frac{15}{56}$. Thus $\frac{e^{-8y}}{-8} = \frac{e^{7x}}{7} - \frac{15}{56}$

$\Rightarrow -7e^{-8y} = 8e^{7x} - 15 \Rightarrow 7e^{-8y} + 8e^{7x} = 15$.

4. $\frac{dx}{dt} - \frac{2}{t+1}x = (t+1)^3$. The I. F. is $e^{-2\int \frac{dt}{t+1}} = \frac{1}{(t+1)^2}$

The equation redues to $\frac{d}{dt}\left(x\frac{1}{(t+1)^2}\right) = (t+1)$

$\Rightarrow \frac{x}{(t+1)^2} = \frac{(t+1)^2}{2} + C$. Since $x(0)=h\,2$,

So, $C = \frac{3}{2}$. Thus $\frac{1}{(t+1)^2} = \frac{(t+1)^2}{2} + \frac{3}{2}$

$\Rightarrow x = \frac{(t+1)^4}{2} + \frac{3}{2}(t+1)^2$. $\lim_{t\to 1} x(t) = 8 + 6 = 14$.

5. $\left(y + x\frac{dy}{dx}\right)^4 \left(\frac{dy}{dx} + x\right) = 1$

$\Rightarrow$ the degree is 5.

6. Put $u = y + 2x \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$, so the given equation reduces to

$\frac{du}{dx} - 2 = \frac{1-2u}{1+u} \Rightarrow \frac{du}{dx} = \frac{1-2u+2+2u}{1+u}$

$\Rightarrow \frac{1+u}{3} du = dx$

$\Rightarrow \left(u + \frac{u^2}{2}\right) = 3x + C$

i. e. $(y + 2x + \frac{(y+2x)^2}{2}) = 3x + C$

$y + \frac{1}{2}(y+2x)^2 = x + C$

7. Dividing by x, we get

$\left(1 - \frac{y}{x}\cos\frac{y}{x}\right)dx + \cos\frac{y}{x}dy = 0$

Put $y = Vx \Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$

$\cos V\left(V + x\frac{dV}{dx}\right) = (V\cos V - 1)$

$\Rightarrow x\cos V\frac{dv}{dx} = -1$

$\Rightarrow \cos V\,dV + \frac{dx}{x} = 0$

$\Rightarrow \sin V + \log|x| = C$

i. e. $\sin\frac{y}{x} + \log|x| = C$. Since $y(1) = 0$ so $C = 0$

Thus $\sin\frac{y}{x} + \log|x| = 0$

8. $a\,y^{n-1}\,y' + y^n = x$, Put $y^n = u$

$\Rightarrow n\,y^{n-1}\frac{dy}{dx} = \frac{du}{dx}$. The given equation reduces to

$\frac{a}{n}\frac{du}{dx} + u = x \Rightarrow \frac{du}{dx} + \frac{n}{a}u = \frac{n}{a}x$

I. F. $= e^{\frac{n}{a}x}$. Multiplying with $e^{\frac{n}{a}x}$, we have

$\frac{d}{dx}\left(ue^{\frac{n}{a}x}\right) = \frac{n}{a}\left[xe^{\frac{n}{a}x}\right]$

$u\,e^{\frac{n}{a}x} = \frac{n}{a}\left[\frac{x\,e^{\frac{n}{a}x}}{n/a} - \frac{1}{(n/a)^2}e^{\frac{n}{a}x}\right] + C$

So $u = x - \frac{a}{n} + Ce^{-\frac{n}{a}x}$

$\Rightarrow n\,y^n = nx - a + Ce^{-\frac{n}{a}x}$

9. Let the curve be $y = f(x)$ so $f(0) = 0$.
Equation of normal at any point (x, y) is

$Y - y = -\frac{1}{f'(x)}(X - x)$

This passes through (1, 2) if

$2 - y = -\frac{1}{f'(x)}(1 - x)$

$2 - y = -\frac{dx}{dy}(1 - x)$

$\Rightarrow (2 - y)dy = -dx(1 - x)$

$2y - \frac{y^2}{2} = -x + \frac{x^2}{2} + C$

Since $y(0) = 0$, so $C = 0$. Hence

$\frac{x^2}{2} + \frac{y^2}{2} - (x + y) = 0$

$x^2 + y^2 - 2x - 4y = 0$
which is a circle with centre (1, 2).

10. Put $x + y = u \Rightarrow 1 + y' = \frac{du}{dx}$. The equation reduces to $\frac{du}{dx} - 1 = u^2 \Rightarrow \frac{du}{1+u^2} = x$

$\Rightarrow \tan^{-1}(x + y) = x + C$
But $y(0) = 0$ so $C = 0$
$\Rightarrow x + y = \tan x$.

Level 1

11. $\frac{dy}{dx} = \frac{1}{x^2}ae^{-1/x} = \frac{1}{x^2}(y - b)$, so $b = 0$ and $a = 1$

12. $\frac{dy}{dx} = \frac{y-2x}{2y+x}$. Putting $y = vx$, we get

$v + x\frac{dv}{dx} = \frac{v-2}{2v+1} \Rightarrow \frac{2v+1}{1+v^2}dv = -2\frac{dx}{x}$

$\Rightarrow \log(1 + v^2) + \tan^{-1} v = -\log x^2 + C$

$\Rightarrow \log(x^2+y^2) + \tan^{-1}\dfrac{y}{x} = C$

Putting $y(1) = 1$, we obtain $C = \dfrac{\pi}{4} + \log 2$.

13. The given equation can be written as

$$x^2 \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = -(x\,dy - y\,dx)$$

$$\Rightarrow \frac{1}{2}\frac{d(x^2+y^2)}{\sqrt{x^2+y^2}} = -\frac{(x\,dy - y\,dx)}{x^2} = -d\left(\frac{y}{x}\right)$$

Integrating we have, $\sqrt{x^2+y^2} = -\dfrac{y}{x} + C$

14. $f''(x) - g''(x) = 0 \Rightarrow f'(x) - g'(x) = C$. Putting $x = 1$, we obtain $c = 2$. Thus $f(x) - g(x) = 2x + C_1$. Putting $x = 2$, we get $C_1 = -10$. Hence $f(4) - g(4) = 8 - 10 = -2$.

15. The degree of the equation in (*a*), (*b*), (*c*) is clearly 1. The equation in (a) can be written as $(y - 2y_1)^2 = 9(1 - y_1)^2$ which is of degree 2.

16. Putting $\dfrac{y}{x} = u$, we have $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$. The given differential equation can be written as $u + x\dfrac{du}{dx} = u + \dfrac{1}{u}\dfrac{\phi(u^2)}{\phi(u^2)}$

$$\Rightarrow \frac{u\,\phi'(u^2)}{\phi(u^2)}du = \frac{dx}{x}$$

$$\Rightarrow \log\phi(u^2) = \log x^2 + \text{const} \Rightarrow x^2 = k\,\phi\left(\frac{y^2}{x^2}\right).$$

17. $\dfrac{dy}{dx} = e^{3x+4y} = e^{3x}\cdot e^{4y} \Rightarrow e^{-4y}\,dy = e^{3x}\,dx$

Thus $\dfrac{e^{-4y}}{-4} - \dfrac{e^{3x}}{3} = \text{Const}$. But $y(0) = 0$, so

$-\dfrac{1}{4} - \dfrac{1}{3} = \text{C}$. Hence $3e^{-4y} + 4e^{3x} = 7$.

18. The given equation can be written as

$x\,dy + y\,dx + y\,dy - x\,dx = 0$

$\Rightarrow d(xy + y^2/2 - x^2/2) = 0$

$\Rightarrow 2xy + y^2 - x^2 = \text{const.}$

19. Write the given equation as

$dx + (1 + x^2)(y\,dx + x\,dy) = 0$

$$\Rightarrow \frac{dx}{1+x^2} + d(xy) = 0 \Rightarrow \tan^{-1}x + xy = c.$$

20. $y + x\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$. Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, we have $\dfrac{dx}{dy} = \dfrac{y}{x} \Rightarrow y^2 - x^2 = \text{Const}$.

21. Equation of tangent at (x, y) to curve $y = f(x)$ is $Y - y = f'(x)(X - x)$. Putting $X = 0$, the initial ordinate of the tangent is $y - x f'(x)$. The subnormal at this point is given by $y\dfrac{dy}{dx}$, so we have

$$y\frac{dy}{dx} = y - x\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{x+y}$$

$\Rightarrow \dfrac{dx}{dy} - \dfrac{x}{y} = 1$ which is a linear equation.

22. Multiplying the given equation by $1/x^2$, we get

$$\frac{y\,dx}{x^2} - \frac{x\,dy}{x^2} + \frac{\log x}{x^2}dx = 0$$

$$-y\,d\left(\frac{1}{x}\right) - \left(\frac{1}{x}\right)dy - \log x\,d\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow d\left(\frac{y}{x}\right) + d\left(\frac{\log x}{x}\right) - \frac{dx}{x^2} = 0$$

$$\Rightarrow \frac{y}{x} + \frac{\log x}{x} + \frac{1}{x} + C = 0.$$

The curve passes through $(1, -1)$, so $C = 0$
Thus the required curve is $y + \log x + 1 = 0$.

23. $y_1 = 5\,b\,e^{5x} - 7c\,e^{-7x}$ (i)

$y_2 = 25\,b\,e^{5x} + 49\,c\,e^{-7x}$ (ii)

and $y_3 = 125\,b\,e^{5x} - 343\,c\,e^{-7x}$ (iii)

Multiplying (i) by 7 and the result to (ii) gives

$y_2 + 7y_1 = 60\,\text{b}\,e^{5x}$ (iv)

Multiplying (i) by 5 and substracting the result from (ii), we get

$y_2 - 5y_1 = 84\,C\,e^{-7x}$ (v)

use (iv) and (v) to replace to e^{5x} and $C\,e^{-7x}$ in (iii)

$$y_3 = 125\left(\frac{y_2 + 7y_1}{60}\right) - 343\left(\frac{y_2 - 5y_1}{84}\right)$$

$\Rightarrow y_3 + 2y_2 - 35y_1 = 0.$

24. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + C = 0$, it passes through $(a, 0)$ and $(-a, 0)$ of $a^2 + 2ga + C = 0$ and $a^2 - 2ga + C = 0$. Substracting we obtain $g = 0$ and so $C = -a^2$. Hence the equation of circle reduces to $x^2 + y^2 + 2fy - a^2 = 0$. Differentiating, we have $x + yy_1 + fy_1 = 0$

$\Rightarrow f = -\dfrac{x + yy_1}{y_1}$. Thus, we have

$$x^2 + y^2 - a^2 - \frac{2x(x + yy_1)}{y_1} = 0$$

$$\Rightarrow x^2 - y^2 - a^2 - \frac{2xy}{y_1} = 0$$

$\Rightarrow y_1(y_2 - x_2 + a_2) + 2xy = 0.$

25. Put $x - y = u$. Then $\dfrac{du}{dx} = 1 - \dfrac{du}{dx}$, and the given equation becomes $u^2\left(1 - \dfrac{du}{dx}\right) = a^2$

$$\Rightarrow \quad dx = \left(\frac{u^2}{u^2 - a^2}\right) du = \left(1 + \frac{a^2}{u^2 - a^2}\right) du$$

Integrating, we obtain

$$x = u + \frac{a^2}{2a} \log \left|\frac{u-a}{u+a}\right| + C$$

$$= x - y + \frac{a}{2} \log\left|\frac{x-y-a}{x-y+a}\right| + C$$

$$\Rightarrow \quad y = \frac{a}{2} \log\left|\frac{x-y-a}{x-y+a}\right| + C.$$

26. Putting $x + y = u$, the given equation becomes

$$\frac{du}{1 + \sin u + \cos u} = dx$$

$$\Rightarrow \quad \int \frac{du}{1 + \sin u + \cos u} = x + C$$

Substituting $\tan(u/2) = t$, to calculate the integral on the L.H.S, we obtain its value as $\log | 1 + t |$ so that the required solution is

$$\log\left|1 + \tan\frac{x+y}{2}\right| = x + C.$$

27. The given equation can be written as

$$\frac{dy}{dx} - y \tan x = \sec x$$

which is linear equation with integrating factor

$$e^{-\int \tan x\, dx} = e^{-\log \sec x} = \cos x.$$

Multiplying with the integrating factor and integrating we obtain $y \cos x = x + C$. Since it passes through (0, 0) so we have $C = 0$.

28. The given equation can be written as

$$d\left(\frac{1}{2}(x^2 + y^2)\right) + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$$

$$d\left(\frac{1}{2}(x^2 + y^2)\right) + \frac{d(y/x)}{1 + y^2/x^2} = 0$$

Integrating, we obtain

$$\frac{1}{2}(x^2 + y^2) + \tan^{-1} y/x = C$$

29. Put $y/x = u$, so that $\frac{dy}{dx} = x\frac{du}{dx} + u$,

From the given equation, we have

$$x\frac{du}{dx} + u = e^u + u \Rightarrow e^{-u}\, du = \frac{dx}{x}$$

$$\Rightarrow -e^{-u} = \log x + \text{Const}$$

$$\Rightarrow \log |Cx| = -e^{-y/x}$$

30. $y' = \sin\frac{x-y}{2} - \sin\frac{x+y}{2} = -2\cos\frac{x}{2}\sin\frac{y}{2}$

$$\Rightarrow \frac{dy}{\sin y/2} = -2\cos x/2\, dx.$$

Integrating, we have

$$2\log\left|\tan\frac{y}{4}\right| = \text{const} - 4\sin x/2$$

$$\Rightarrow \log\left|\tan\frac{y}{4}\right| = C - 2\sin x/2.$$

31. Differentiating the given equation $y^2 = 2c\,(x + \sqrt{c})$ (1)

$$2y\frac{dy}{dx} = 2c \quad \Rightarrow \quad c = y\frac{dy}{dx}$$

Putting this value in (1)

$$y^2 = 2y\frac{dy}{dx}\left(x + \sqrt{y\frac{dy}{dx}}\right)$$

$$\Rightarrow y - 2x\frac{dy}{dx} = 2\left(y\frac{dy}{dx}\right)^{3/2}$$

$$\Rightarrow \left(y - 2x\frac{dy}{dx}\right)^2 = 4y^3\left(\frac{dy}{dx}\right)^3$$

Thus the order of the differential equation is 1 and the degree is 3.

32. Equation of tangent a $P(x, y)$ to the curve $y = f(x)$ is

$$Y - y = f'(x)\,(X - x)$$

Intercept on Y-axis is $y - xf'(x)$

We are given, $y - xf'(x) = x^3$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = -x^2$$

This is a linear equation whose I.F. is $\frac{1}{x}$.

Multiplying with I.F., the above equation reduces to

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = -x \quad \Rightarrow \quad f(x) = y = -\frac{1}{2}x^3 + Cx$$

As $f(1) = 1$ so $C = 3/2$. Thus $f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x$, so $f(3) = 9$.

33. Putting $y = Vx$, the given equation reduces

$$\frac{dx}{dx} = -\frac{V^2 + 1}{2V} \quad \Rightarrow \quad \frac{2V dV}{1 + V^2} = -\frac{dx}{x}$$

$\Rightarrow (1 + V^2)x = \text{Const}$

$\Rightarrow x^2 + y^2 = Cx \quad \Rightarrow \quad (x - C/2)^2 + y^2 = C^2/4$

which represents circle with centre $(C/2, 0)$ i.e., on x-axis.

34. $9y dy = -4x dx \quad \Rightarrow \quad \frac{9}{2}y^2 = -2x^2 + \text{const}$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = \text{Const}$, so represents a family of ellipses.

35. $\frac{dy}{y} = -2xdx \Rightarrow \log y = -x^2 + C$

$\Rightarrow \quad y = Ce^{-x^2}$

For $x = 0, y = 1$ so $C = 1$

Thus $y = e^{-x^2}$ which is bell shaped.

36. Put $x - y = u \Rightarrow 1 - \frac{dy}{dx} = \frac{du}{dx}$. The given equation reduces to $1 - \frac{du}{dx} = \cos u$

$\Rightarrow \frac{du}{1-\cos u} = dx \Rightarrow \frac{1}{2}\int \text{cosec}^2 (u/2) du = x + C$

$\Rightarrow -\cot \frac{u}{2} = x + C$ i.e. $-\cot \frac{x-y}{2} = x + C$

Putting $y(0) = -\pi$, we have $C = 0$
$y = x - 2\cot^{-1}(-x)$.

37. Let (x, y) be the point of tangency. Equation of tangent is $Y - y = f'(x)(X - x)$. This intersects x-axis at $\left(x - \frac{1}{f'(x)}y, 0\right)$. The mid point is $\left(x - \frac{1}{2f'(x)}y, \frac{y}{2}\right)$ is on Y-axis so

$$x - \frac{1}{2f'(x)}y = 0$$

$\Rightarrow \quad 2\frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad y^2 = Cx$

which represents parabola with latus ractum parallel to Y-axis.

38. $\frac{dy}{dx} + 2y = 3x + 5 \Rightarrow ye^{2x} = \int (3x + 5)\, e^{2x} + C$

$$= (3x+5)\frac{e^{2x}}{2} - \frac{3}{4}e^{2x} + C$$

$\Rightarrow \quad y = \left(\frac{3x+5}{2} - \frac{3}{4}\right) + Ce^{-2x}$

$\Rightarrow \quad 4y - 6x - 7 = Ce^{-2x}$

39. It is linear equation with I.F. $e^{\int \tan x\, dx} = \sec x$, Required solution is $y = \sin x + \cos x$.

40. I.F. $= e^{\int \frac{1}{x} dx} = x$ so $\frac{d}{dx}(xy) = e^x$

$\Rightarrow \quad xy = e^x + C$ Put $y(a) = b$
$ab - e^a = C$, so $xy = e^x + ab - e^a$.

Level 2

41. The tangent at any point (x, y) meets the Y-axis at $\left(0, y - x\frac{dy}{dx}\right)$. According to the given condition

$\frac{1}{2}x^2$ = Area of the trapezoid

$$= \left(y + y - x\frac{dy}{dx}\right)x$$

$\Rightarrow \frac{dy}{dx} - 2\frac{y}{x} = -\frac{1}{2}$ (which is a linear equation)

I. F. $= e^{-2\int \frac{1}{x} dx} = x^{-2}$

Multiplying with I.F. and integrating, we have

$yx^{-2} = -\frac{1}{2}\int x^{-2}\, dx + C$

$$= \frac{1}{2}x^{-1} + C$$

$\Rightarrow y = \frac{1}{2}x + Cx^2$.

42. Equation of normal at any point (x, y)

$Y - y = -\frac{1}{f'(x)}(X - x)$

This passes through (x_0, y_0) if

$y_0 - y = \frac{1}{f'(x)}(x_0 - x)$

$(y_0 - y)\, dy + (x_0 - x)\, dx = 0$

$\frac{(y_0 - y)^2}{2} + \frac{(x_0 - x)^2}{2} = C$

$(y - y_0)^2 + (x - x_0)^2 = \text{Const}$

Since $y(0) = 0$ so Const $= y_0^2 + x_0^2$

Thus $y^2 - 2yy_0 + y_0^2 + x^2 + x_0^2 - 2xx_0 = x_0^2 + y_0^2$
$x^2 + y^2 = 2(xx_0 + yy_0)$.

43. $(2a - x)\, 2y\frac{dy}{dx} - y^2 = 3x^2$ (Differentiating given equation)

$2\,(2a - x)y\frac{dy}{dx} = 3x^2 + y^2$

Eliminating a between the last equation and given equation i.e. $(2a - x)\, y^2 = x^2$, we have

$y\,(3x^2 + y^2)\frac{dx}{dy} = 2x^3$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have

$2x^3 + y(3x^2 + y^2)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x^3}{(3x^2 + y^2)\, y}$, which is a homogeneous equation.

Putting $y = Vx$

$$V + x\frac{dV}{dx} = -\frac{2}{(3+V^2)V}$$

$$\Rightarrow x\frac{dV}{dx} = -\left[\frac{2+3V^2+V^4}{(3+V^2)V}\right]$$

$$\frac{dx}{x} = -\frac{(3+V^2)V}{(1+V^2)(2+V^2)} = -\left[\frac{2V}{1+V^2} - \frac{V}{2+V^2}\right]$$

$$\Rightarrow \log x^2 = -\log\frac{(1+V^2)^2}{2+V^2} + \text{Const}$$

$$\Rightarrow \frac{x^2(1+y^2/x^2)^2}{2+y^2/x^2} = \text{Const}$$

$$\Rightarrow (x^2+y^2)^2 = C(y^2+2x^2).$$

44. $xy_5 = y_4 \Rightarrow x\frac{d}{dx}y_4 = y_4$

$$\Rightarrow \frac{dy_4}{y_4} = \frac{dx}{x} \Rightarrow y_4 = (\text{Const})\ x$$

So $y_3 = \text{const}\ x^2 + \text{const}$

$$\Rightarrow y_2 = C_1'x^3 + C_2'x + C_3'$$

$$\Rightarrow y_1 = C_1''x^4 + C_2''x^2 + C_3''x + C_4''$$

$$\Rightarrow y = C_1x^5 + C_2x^3 + C_3x^2 + C_4x + C_5.$$

45. $yy'' = y'^2 \Rightarrow \frac{y''}{y'} = \frac{y'}{y}$ Integrating, we get

$\log y' = \log y + \text{const} \Rightarrow y' = Cy$

$$\Rightarrow \frac{dy}{y} = C\,dx \Rightarrow \log y = Cx + C'$$

$$y = C_2e^{C_1x}.$$

46. According to the given condition

$$\frac{y\frac{dy}{dx}}{x+y} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Putting $y = Vx$, $V + \frac{dV}{dx} = 1 + V$

$$\Rightarrow dV = \frac{dx}{x}$$

$$\Rightarrow V = \log x + \text{const}$$

$$\Rightarrow y = x\log Cx.$$

47. $\int_0^x f(t)\,dt = K\,y^{n+1}$

Differentiating, we have

$$y = f(x) = K(n+1)y^n\frac{dy}{dx}$$

$$\Rightarrow dx = K(n+1)y^{n-1}dy$$

$$\Rightarrow x + C = K\frac{(n+1)}{n}y^n$$

Since $f(0) = 0$, $C = 0$

$$f(1) = 1,\ 1 = \frac{K(n+1)}{n} \Rightarrow K = \frac{n}{n+1}$$

Thus $x = y^n$.

48. Putting $y = Vx$

$$V + x\frac{dV}{dx} = \frac{V^2-2V-1}{V^2+2V-1}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{V^2-2V-1-V^3-2V^2+V}{V^2+2V-1}$$

$$\frac{dx}{x} = -\frac{V^2+2V-1}{V^3+V^2+V+1}dV$$

$$= -\left(-\frac{1}{V+1} + \frac{2V}{V^2+1}\right)dV$$

Integrating both sides, we get

$\log x - \log(V+1) + \log(V^2+1) + \log C = O$

$$\Rightarrow C\left(\frac{x(V^2+1)}{V+1}\right) = 1 \Rightarrow C(y^2+x^2) = y + x$$

If this curve passes through $(-1, 1)$, we get $2C = 0$ i.e.

$C = 0$. Hence the required curve is $y + x = 0$ which is a straight line.

49. According to the given condition

$$\frac{y}{f'(x)} = nx$$

$$y\frac{dx}{dy} = nx \Rightarrow \frac{dy}{y} = \frac{1}{n}\frac{dx}{x}$$

Integrating $y^n = Cx$. This passes through

$(2, 3)$ if $3^n/2 = C$

So $y^n = \frac{3^nx}{2}$

$\Rightarrow \left(\frac{y}{3}\right)^n = \frac{x}{2}$. For $n = 2$. This represents a parabola whose axes coincide with x-axis.

50. $x = \left(\frac{1}{2p} - \frac{p}{2}\right)y$. Differentiating both sides

w·r·t y $\left(p = \frac{dy}{dx}\right)$

$$\frac{1}{p} = \frac{1}{2p} - \frac{p}{2} + y\left(-\frac{1}{2p^2} - \frac{1}{2}\right)\frac{dp}{dy}$$

$$\frac{1}{2p} + \frac{p}{2} = -y\left(\frac{1+p^2}{2p^2}\right)\frac{dp}{dy}$$

$$\Rightarrow \frac{1+p^2}{2p} = -y\left(\frac{1+p^2}{2p^2}\right)\frac{dp}{dy}$$

$$-\frac{dy}{y} = \frac{dp}{p} \Rightarrow py = C$$

i.e. $y\,dy = c\,dx \Rightarrow y^2 = 2Cx + C^2$.

51. According to the given condition

$$\int_0^x f(t)\,dt = \frac{y^3}{x}$$

Differentiating, we get

$$y = \frac{3y^2}{x}\frac{dy}{dx} - \frac{y^3}{x^2}$$

$$\frac{x^2y + y^3}{x^2} = \frac{3y^2}{x}\frac{dy}{dx}$$

$$\Rightarrow \frac{x^2y + y^3}{3y^2x} = \frac{dy}{dx}$$

Putting $y = Vx$

$$\frac{V + V^3}{3V^2} = V + x\frac{dV}{dx}$$

$$\Rightarrow \frac{1+V^2}{3V} - V = x\frac{dV}{dx}$$

$$\Rightarrow \frac{1-2V^2}{3V} = x\frac{dV}{dx}$$

$$\Rightarrow \frac{3V}{1-2V^2}dV = \frac{dx}{x}$$

Integrating, we have

$$\Rightarrow \frac{-3}{4}\log(1 - 2V^2) = \log x + \text{Const}$$

$$\Rightarrow x^4(1 - 2V^2)^3 = \text{Const}$$

$$\Rightarrow (2y^2 - x^2)^3 = Cx^2.$$

52. The initial ordinate of tangent is equal to

$y - x\frac{dy}{dx}$, so

$$y - x\frac{dy}{dx} = x - 2$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = -1 + \frac{2}{x}$$

This is a linear equation with I.F. $\frac{1}{x}$

$$\frac{d}{dx}\left(y.\frac{1}{x}\right) = -\frac{1}{x} + \frac{2}{x^2}$$

$$y\cdot\frac{1}{x} = -\log|x| - 2^{x-1} + C$$

$$y = Cx - x\log|x| - 2.$$

53. $y' = \frac{x+y-2}{y-x-4}$. Put $x = X + h$, $y = Y + K$

$$\frac{dY}{dX} = \frac{X + Y + (h + K - 2)}{Y - X + (-h + K - 4)}$$

Select h, K such that $h + K - 2 = 0$, $-h + K - 4 = 0$

$\Rightarrow K = 3$, $h = -1$

$$\frac{dY}{dX} = \frac{X+Y}{Y-X}$$

Putting $Y = vX$, we have

$$v + X\frac{dV}{dX} = \frac{1+V}{V-1}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+V}{V-1} - V = \frac{1+2V-V^2}{V-1}$$

$$\frac{dX}{X} = \frac{-1}{2}\frac{-2V+2}{1+2V-V^2}dV$$

Integrating, we have $X^2(1 + 2V - V^2) = \text{Const}$

$X^2 - Y^2 + 2XY = \text{Const}$

$(x + 1)^2 - (y - 3)^2 + 2(x + 1)(y - 3) = \text{Const}$

$x^2 + 2xy - y^2 - 4x + 8y = \text{Const}.$

54. $y = xp - 3p^3$, $p = \frac{dy}{dx}$. Differentiating w·r·t·x,

we have $p = p + x\frac{dp}{dx} - 9p^2\frac{dp}{dx}$

$$\Rightarrow (x - 9p^2)\frac{dp}{dx} = 0$$

$\frac{dp}{dx} = 0 \Rightarrow p = \text{Const}$ so $y = Cx - 3C^3$

$y = x - 3$ is solution with $C = 1$.

55. $2x + 2y\frac{dy}{dx} = 2a \Rightarrow a = x + y\frac{dy}{dx}$

Eliminating a, we have

$$x^2 + y^2 = 2\left(x + y\frac{dy}{dx}\right)x$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$x^2 + y^2 = 2\left(x^2 - xy\frac{dx}{dy}\right)$$

$$\Rightarrow \frac{y^2 - x^2}{-2xy} = \frac{dx}{dy}$$

$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$. Putting $y = Vx$, we have

$$V + x\frac{dV}{dx} = \frac{2V}{1 - V^2}$$

$$\frac{1 - V^2}{V(1 + V^2)} dV = \frac{dx}{x}$$

$$\left[\frac{1}{V} - \frac{2V}{1 + V^2}\right] dV = \frac{dx}{x}$$

$$\Rightarrow \frac{V}{x(1 + V^2)} = \text{const}$$

$$\Rightarrow y = C(x^2 + y^2).$$

56. $m^2 - 4m + 3 = 0 \Rightarrow (m - 1)(m - 3) = 0$

so $m = 1, 3$

Hence $y = C_1 e^x + C_2 e^{3x}$

$6 = y(0) = C_1 + C_2$, $10 = y'(0) = C_1 + 3C_2$

$C_2 = 2$, $C_1 = 4$. Thus $y = 4e^x + 2e^{3x}$.

57. Since $C_1 + C_2$ is one constant $C_4\, e^{x + C_5}$
$= C_1 e^{C_5} e^x$

Thus there are three arbitrary constants. So the order of the differential equation is 3.

58. Put $x + y = u \Rightarrow 1 + \frac{dy}{dx} = \frac{du}{dx}$. The given equation reduces to

$$\frac{du}{dx} - 1 = \sec u$$

$$\Rightarrow \frac{du}{1 + \sec u} = dx$$

$$\Rightarrow \left(1 - \frac{1}{1 + \cos u}\right) du = dx$$

$$\Rightarrow u - \tan u/2 = x + C$$

$$\Rightarrow x + y - \tan\frac{x + y}{2} = x + C$$

$$\Rightarrow y - \tan\frac{x + y}{2} = C.$$

59. Clearly $y_3 - 7y_1 + 6y = 0$ is the required equation

60. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = x + e^{3x}$. Integrating, we have

$$\frac{dy}{dx} = \frac{x^2}{2} + \frac{e^{3x}}{3} + C_1$$

$$\Rightarrow y = \frac{x^3}{6} + \frac{e^{3x}}{9} + C_1 x + C_2$$

Since $y(0) = 0$ so $C_2 = -\frac{1}{9}$ and $\frac{1}{3} = y_1(0)$

$$= \frac{1}{3} + C_1 \Rightarrow C_1 = 0$$

Hence $y = \frac{x^3}{6} + \frac{e^{3x}}{9} - \frac{1}{9} = \frac{1}{18}(3x^3 + 2e^{3x} - 2)$.

61. $\frac{dy}{dx} = \int e^x \sin x\, dx = \frac{e^x \sin(x - \pi/4)}{\sqrt{2}} + C_1$

$$\Rightarrow y = \frac{1}{\sqrt{2}}\frac{e^x \sin(x - \pi/2)}{\sqrt{2}} + C_1 x + C_2$$

$$0 = y\left(\frac{\pi}{2}\right) = 0 + C_1\frac{\pi}{2} + C_2$$

$$0 = y_1\left(\frac{\pi}{4}\right) = \frac{1}{2}e^{\pi/4}\left[-\sin\frac{\pi}{4} + \cos\pi/4\right] + C_1$$

$$\Rightarrow C_1 = 0 \text{ So } C_2 = 0$$

Hence $y = \frac{1}{2}e^x \sin\left(x - \frac{\pi}{2}\right)$.

Previous Years' AIEEE/JEE Main Questions

1. $(1 + y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$

$$\Rightarrow (1 + y^2)\frac{dx}{dy} + x = e^{\tan^{-1} y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2}x = \frac{1}{1 + y^2}e^{\tan^{-1} y}$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$$\frac{d}{dy}\left(xe^{\tan^{-1} y}\right) = \frac{e^{2\tan^{-1} y}}{1 + y^2}$$

$$\Rightarrow xe^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1 + y^2} dy + \text{const}$$

$$= \frac{1}{2}e^{2\tan^{-1} y} + \text{const}$$

$$\Rightarrow 2xe^{\tan^{-1} y} = e^{\tan^{-1} y} + k$$

2. $x^2 + y^2 - 2ay = 0 \Rightarrow 2a = \dfrac{x^2 + y^2}{y}$

Differentiating, we get

$$\frac{(2x+2yy')y-(x^2+y^2)y'}{y^2} = 0$$

$\Rightarrow (x^2 - y^2)y' = 2xy$

3. We can write the given differential equation as

$$(ydx + xdy) + \frac{1}{y}(xy)^2\, dy = 0$$

$$\Rightarrow \frac{d(xy)}{(xy)^2}+\frac{1}{y}dy = 0$$

$$\Rightarrow -\frac{1}{xy}+\log y = C.$$

4. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{x}\left[\log\frac{y}{x}+1\right] \qquad (1)$$

Put $y/x = v$, so that

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting this in (1), we get

$$v+x\frac{dv}{dx} = v(\log v + 1)$$

$$\Rightarrow \frac{dv}{v\log v} = \frac{dx}{x}$$

$$\Rightarrow \int\frac{dv}{v\log v} = \int\frac{dx}{x}$$

$\Rightarrow \log \log v = \log x + \log c$

$\Rightarrow \log v = cx$ or $\log\left(\dfrac{y}{x}\right) = cx$

5. Differentiating w.r.t. x, we get

$$2Ax + 2By\frac{dy}{dx} = 0$$

$$\Rightarrow -Ax = By\frac{dy}{dx} \qquad (1)$$

Again differentiating w.r.t. x, we get

$$-A = B\left(\frac{dy}{dx}\right)^2 + By\left(\frac{d^2y}{dx^2}\right)$$

$$\Rightarrow -Ax = Bx\left(\frac{dy}{dx}\right)^2 + Bxy\left(\frac{d^2y}{dx^2}\right)$$

$$\Rightarrow By\frac{dy}{dx} = Bx\left(\frac{dy}{dx}\right)+Bxy\left(\frac{d^2y}{dx^2}\right) \quad \text{[from (1)]}$$

$$\Rightarrow xy\frac{d^2y}{dx^2}+(x-y)\frac{dy}{dx} = 0$$

which is of second order and first degree.

6. Equation of any circle through origin and having centre on the x-axis is

$$(x - a)^2 + y^2 = a^2 \qquad (1)$$

Differentiating w.r.t. x, we get

$2(x - a) + 2yy' = 0$

$\Rightarrow a = x + yy'$

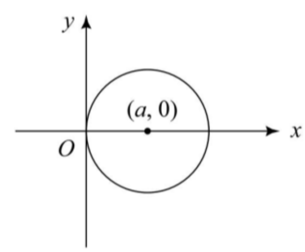

Fig. 15.2

Putting this in (1), we get

$(-yy')2 + y^2 = (x + yy')^2$

$\Rightarrow y^2 = x^2 + 2xy\dfrac{dy}{dx}$

7. Let the centre be at $(a, 2)$. Equation of circle of radius 5 is

$$(x - a)^2 + (y - 2)^2 = 25 \qquad (1)$$

Differentiating, we have

$$2(x - a) + 2(y - 2)\frac{dy}{dx} = 0$$

$$\Rightarrow x - a = -(y - 2)\frac{dy}{dx}$$

Putting this value in (1), we have

$$(y - 2)^2\left(\frac{dy}{dx}\right)^2 + (y - 2)^2 = 25$$

$(y - 2)^2\, y'^2 = 25 - (y - 2)^2.$

8. $y = c_1 e^{c_2 x}$

$\Rightarrow y' = c_1 c_2 e^{c_2 x}$

$\Rightarrow y'' = c_1 c_2^2 e^{c_2 x}$

$\therefore yy'' = c_1^2 c_2^2 e^{c_2 x} e^{c_2 x}$

$= \left(c_1 c_2 e^{c_2 x}\right)^2 = (y')^2$

9. We can write the given differential equation as

$$\frac{dy}{dx} = \frac{y(\sin x - y)}{\cos x} \Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = -\sec x$$

Put $-\frac{1}{y} = z$, so that the above equation becomes

$$\frac{dz}{dx} + (\tan x)z = -\sec x \quad (1)$$

I.F. $= e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

Multiplying (1) by $\sec x$, we get

$\Rightarrow \frac{d}{dx}[(\sec x)z] = -\sec^2 x \Rightarrow (\sec x)z = c_1 - \tan x$

$\Rightarrow \frac{-\sec x}{y} = c_1 - \tan x \Rightarrow \sec x = (c + \tan x)\, y$

Where $c = -c_1$.

10. $\frac{dy}{dx} = y + 3$

$\Rightarrow \int \frac{dy}{y+3} = \int dx \Rightarrow \ln(y + 3) = x + c$

When $x = 0$, $y = 2$, therefore, $\ln 5 = c$

Thus, $\ln(y + 3) = x + \ln 5$

When $x = \ln 2$, we get

$\ln(y + 3) = \ln 2 + \ln 5 \Rightarrow y + 3 = 10$ or $y = 7$

11. $\frac{dV}{dt} = -k(T - t),\ k > 0$

$\Rightarrow dV = -k(T - t)dt$

$\Rightarrow \int dV = -k\int(T - t)dt$

$\Rightarrow V(t) = \frac{k}{2}(T - t)^2 + C$

We have $V(0) = I$, therefore

$I = \frac{k}{2}T^2 + C \Rightarrow C = I - \frac{k}{2}T^2$

Also, scrap value $= V(T) = C$

$= I - \frac{k}{2}T^2$

12. $y^2 dx + \left(x - \frac{1}{y}\right)dy = 0$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3} \quad (1)$$

I.F. $= e^{\int (1/y^2)dy} = e^{-1/y}$

Multiplying (1) by $e^{-1/y}$, we get

$\frac{d}{dx}\left[xe^{-1/y}\right] = \frac{1}{y^3}e^{-1/y}$

$\Rightarrow xe^{-1/y} = \int \frac{1}{y^3}e^{-1/y}dy = I$ (say)

To evaluate I, put $-1/y = t$, so that

$I = \int(-t)e^t\, dt = -te^t + \int e^t\, dt$

$= e^t(1 - t) = e^{-1/y}(1 + 1/y)$.

Thus,

$xe^{-1/y} = e^{-1/y}(1 + 1/y) + c$

$\Rightarrow x = 1 + \frac{1}{y} + ce^{1/y}$

When $x = 1$, $y = 1$.

Hence,

$1 = 1 + 1 + ce \Rightarrow c = -1/e$

$\therefore x = 1 + \frac{1}{y} - \frac{e^{1/y}}{e}$

13. $\frac{d}{dt}p(t) - 0.5p(t) = -450$. I.F. $= e^{-0.5t}$

$\Rightarrow \frac{d}{dt}(p(t)\, e^{-0.5t}) = -450\, e^{-0.5t}$

$\Rightarrow p(t)e^{-0.5t} = 450 \times 2.\ e^{-0.5t} + C$

Putting $t = 0$, $850 = 900 + C$

$\Rightarrow C = -50$

$\Rightarrow p(t) = 900\, e^{-0.5t} - 50$

$p(t) = 0 \Rightarrow e^{-0.5t} = \dfrac{1}{18} \Rightarrow -0.5t = -\log 18$

$\Rightarrow t = 2\log 18.$

14. $\dfrac{dP}{dx} = 100 - 12\sqrt{x}$

$\Rightarrow dP = \left(100 - 12\sqrt{x}\right)dx$

$\Rightarrow P = \int\left(100 - 12\sqrt{x}\right)dx$

$P(x) = 100\,x - 8x^{3/2} + c$

We have $P(0) = 2000$,

Therefore, $c = 2000$.

Thus, $P(25) = 100\,(25) - 8(25)^{3/2} + 2000 = 3500$

15. $z = y^2 \Rightarrow \dfrac{dz}{dx} = 2y\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{y^3}{2(xy^2 - x^2)} \Rightarrow 2y\dfrac{dy}{dx} = \dfrac{2y^4}{2(xy^2 - x^2)}$

$\Rightarrow \dfrac{dz}{dx} = \dfrac{z^2}{\left(xz - x^2\right)}$

which a first order homogeneous differential equation. Put $z = ux \Rightarrow \dfrac{dz}{dx} = u + x\dfrac{du}{dx}$

$u + x\dfrac{du}{dx} = \dfrac{u^2x^2}{\left(x^2u - x^2\right)} = \dfrac{u^2}{u-1}$

$\Rightarrow x\dfrac{du}{dx} = \dfrac{u^2 - u^2 + u}{u-1} = \dfrac{u}{u-1}$

$\Rightarrow \dfrac{u-1}{u}du = \dfrac{dx}{x}$

$\Rightarrow u - \log x = \log x + C$

$\Rightarrow u = \log ux + C$

$\Rightarrow ux = ce^{u}$

$\Rightarrow ze^{-z/x} = c$

$\Rightarrow y^2e^{-y^2/x} = c$

16. $\dfrac{d}{dt}p(t) - \dfrac{1}{2}p(t) = -200.$

I.F. is $e^{-\frac{1}{2}t}$, so

$\dfrac{d}{dt}\left(e^{-\frac{1}{2}t}p(t)\right) = -200e^{-\frac{1}{2}t}$

$\Rightarrow e^{-\frac{1}{2}t}p(t) = 400e^{-\frac{1}{2}t} + C$

Putting $t = 0$

$100 = p(0) = 400 + C$

$\Rightarrow C = -300$

Thus $p(t) = 400 - 300\,e^{\frac{1}{2}t}$

17. An equation of circle touching the x-axis at the origin is

$x^2 + (y - a)^2 = a^2$ (1)

where a is parameter.

Differentiating (1) w.r.t. x,

we get

$2x + 2(y - a)y' = 0$

$\Rightarrow a = y + \dfrac{x}{y'}$

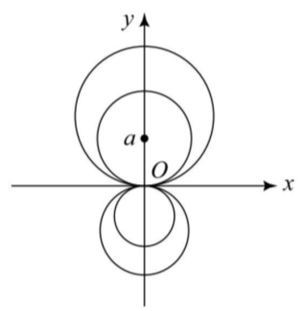

Fig. 15.3

Putting this in (1), we get

$x^2 + \left(\dfrac{x}{y'}\right)^2 = \left(y + \dfrac{x}{y'}\right)^2$

$\Rightarrow x^2 = y^2 + (2xy)\,\dfrac{1}{y'}$

$\Rightarrow (x^2 - y^2)\,\dfrac{dy}{dx} = (2x)y$

$\therefore\ g(x) = 2x$

18. $y'\ \ln|cx| + y\left(\dfrac{1}{x}\right) = 1$

$$\Rightarrow y'\left(\frac{x}{y}\right)+\frac{y}{x} = 1$$

$$\Rightarrow y' = \frac{y}{x}-\left(\frac{y}{x}\right)^2 = \frac{y}{x}+\phi\left(\frac{y}{x}\right)$$

$$\Rightarrow \phi(u) = -\frac{1}{u^2}$$

$$\therefore \phi(2) = -\frac{1}{4}$$

19. Given differential equation is

$$\frac{dy}{dx} - \text{cosec }(2x)y = \sqrt{\tan x} \qquad (1)$$

$$\text{I.F.} = e^{-\int \text{cosec}(2x)dx}$$

$$= e^{-(1/2)\log(\tan x)} = \frac{1}{\sqrt{\tan x}}$$

Multiplying (1) $\frac{1}{\sqrt{\tan x}}$, we get

$$\frac{d}{dx}\left(\frac{y}{\sqrt{\tan x}}\right) = 1$$

$$\Rightarrow y\sqrt{\cot x} = x + c$$

20. I.F. $= e^{\int(\tan x)dx} = e^{\log \sec x} = \sec x$

Multiplying the given differential equation by $\sec x$, we get

$$\frac{d}{dx}((\sec x)y) = 2 \sin x$$

$$\Rightarrow (\sec x)\, y = -2 \cos x + C$$

Putting $x = 0$, $y = 1$, we get

$$1 = -2 + C \Rightarrow C = 3$$

When $x = \pi$, we get

$$(-1)y = -2 \cos \pi + 3 \Rightarrow y = -5$$

21. We can write the given differential equation as

$$\frac{dy}{dx}+\frac{1}{x\log x}y = 2 \qquad (1)$$

$$\text{I.F.} = e^{\int dx/x \log x} = e^{\log (\log x)} = \log x$$

Multiplying (1) by $\log x$, we get

$$(\log x)\frac{dy}{dx}+\frac{1}{x}y = 2 \log x$$

$$\Rightarrow \frac{d}{dx}[(\log x)y] = 2 \log x$$

$$\Rightarrow (\log x)y = 2(x \log x - x) + C$$

When $x = 1$, we get

$$0 = 2(0 - 1) + C \Rightarrow C = 2$$

When $x = e$, we get

$$(\log e)\, y(e) = 2(e \log e - e) + 2$$

$$\Rightarrow y(e) = 2$$

22. Write the differential equation as

$$\frac{dx}{dy}+\left(-\frac{1}{y}\right)x = 2y$$

$$\text{I.F.} = e^{\int(-1/y)dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying (1) by $1/y$, we get

$$\frac{1}{y}\frac{dx}{dy}+\left(-\frac{1}{y^2}\right)x = 2$$

$$\Rightarrow \frac{dx}{dy}\left(\frac{x}{y}\right) = 2$$

$$\Rightarrow \frac{x}{y} = 2y + c$$

$$\Rightarrow x = 2y^2 + cy$$

When $y = -1$, $x = 1$,

Therefore, $1 = 2 - c \Rightarrow c = 1$

Thus, $f(y) = 2y^2 + y \Rightarrow f(1) = 3$.

23. We can write differential equation as

$$\frac{dy}{dx} = \frac{x^2+4x-9}{x+2} = x + 2 - \frac{13}{x+2}$$

$$\Rightarrow y = \frac{1}{2}(x + 2)^2 - 13 \log |x + 2| + C$$

As $y(0) = 0$, we get

$$0 = 2 - 13 \log |2| + C \text{ or } C = 13 \log 2 - 2$$

Also, $y(-4) = 2 - 13 \log 2 + 13 \log 2 - 2$

$= 0$.

24. $x\int_1^x y(t)\, dt = (x + 1) \int_1^x ty(t)\, dt$

Differentiating both the sides, w.r.t. x, we get

$$\int_1^x y(t)dt + xy(x) = \int_1^x ty(t) + (x+1)xy(x)$$

$$\Rightarrow \int_1^x y(t)dt = \int_1^x ty(t)dt + x^2 y(x)$$

Differentiating both the sides w.r.t. x, we get

$$y(x) = xy(x) + 2xy(x) + x^2y'(x)$$

$$\Rightarrow (1 - 3x)\, y(x) = x^2y'(x)$$

$$\Rightarrow \frac{y'(x)}{y(x)} = \frac{1}{x^2} - \frac{3}{x}$$

Integrating we get

$$\Rightarrow \ln|y(x)| = \frac{1}{x} - 3 \ln |x| + A$$

$$\Rightarrow \ln |x^3y(x)| = -\frac{1}{x} + A$$

$$x^3y(x) = \pm\, e^A e^{-1/x} = Ce^{-1/x}$$

where C is a constant

$$\Rightarrow y(x) = \frac{c}{x^3}\, e^{-1/x}$$

25. $\frac{dy}{dx} + \frac{1}{2} y \sec x = \frac{\tan x}{2y}$

$$\Rightarrow \frac{d}{dx}(y^2) + (\sec x)y^2 = \tan x \quad (1)$$

I.F. $= e^{\int \sec x\, dx} = e^{\ln(\sec x + \tan x)}$

$= \sec x + \tan x$

Multiplying (1) by $\sec x + \tan x$, we get

$$\frac{d}{dx}[y^2 (\sec x + \tan x)] = \tan x (\sec x + \tan x)$$

$$\Rightarrow y^2 (\sec x + \tan x) = \sec x + \tan x - x + C$$

When $x = 0$, $y = 1$, so that

$1 = 1 + C \Rightarrow C = 0$

Thus, $y^2 = 1 - \frac{x}{\sec x + \tan x}$

26. We can write the differential equation as

$$y^{-2}\frac{dy}{dx} - \frac{1}{x} y^{-1} = 1 \quad (1)$$

Put $-y^{-1} = t \Rightarrow y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$

We can write (1) as

$$\frac{dt}{dx} + \frac{1}{x}t = 1$$

$$\Rightarrow x\frac{dt}{dx} + t = x$$

$$\Rightarrow \frac{d}{dx}(tx) = x$$

$$\Rightarrow tx = \frac{1}{2}x^2 + C$$

$$\Rightarrow -\frac{1}{y}x = \frac{1}{2}x^2 + C$$

As it passes through (1, –1), $1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

$\therefore -\frac{1}{y}x = \frac{1}{2}x^2 + \frac{1}{2}$. When $x = -\frac{1}{2}$,

$$-\frac{1}{y}\left(-\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

$$\Rightarrow y = \frac{4}{5}.$$

Previous Years' B-Architecture Entrance Examination Questions

1. $\log\frac{dy}{dx} = 3x + 4y$

$$\Rightarrow \frac{dy}{dx} = e^{3x + 4y} = e^{3x}\, e^{4y}$$

$$\Rightarrow e^{-4y}\, dy = e^{3x}\, dx$$

$$\Rightarrow -\frac{e^{-4y}}{4} = \frac{e^{3x}}{3} + \text{const}$$

$y(0) = 0 \Rightarrow -\frac{1}{4} = \frac{1}{3} + \text{const} \Rightarrow \text{const} = -\frac{7}{12}$

so $-3e^{-4y} = 4e^{3x} - 7$

$$\Rightarrow 4e^{3x} + 3e^{-4y} = 7$$

2. $\frac{dx}{dt} = t(t + 1) \Rightarrow x = \frac{t^3}{3} + \frac{t^2}{2} + C_1$

But at $t = 0$, $x = 0$ so $C_1 = 0$

$$\text{Thus } x = \frac{t^3}{3} + \frac{t^2}{2} \quad (1)$$

$\frac{dy}{dx} = \frac{1}{t+1} \Rightarrow y = \log (t + 1) + C_2$

But at $t = 0$, $y = 0$ so $C_2 = 0$

Thus $y = \log (t + 1) \Rightarrow t = e^y - 1.$

Putting in (1), we have

$6x = t^2 (2t + 3)$

$= (e^y - 1)^2 (Ze^y + 1).$

3. Differentiating $y^2 = 4a(x + a^2)$ (1)

We have, $2y\frac{dy}{dx} = 4a \Rightarrow a = \frac{yy'}{2}$

Putting in (1)

$y^2 = 4\frac{yy'}{2}\left(x+\frac{1}{4}y^2y'^2\right)$

$= 2xy\, y' + y^3\, y'^3$

So the degree is 3.

4. $u'(t) = e^{t^2}\ u(t) + \sin t,\ v'(t) = e^{t^2}\ v(t) + \sin t$

Subtracting, we have

$u'(t) - v'(t) = e^{t^2}(u(t) - v(t))$

$\Rightarrow \log (u - v) = \int e^{t^2} dt + c$

$u(t) - v(t) = C\ e^{\int et^2 dt}$

so $u - v$ is proportional to a positive function of t. Since $u(2) < v(2)$ so $C < 0$.

Hence $u(t) < v(t)$ for all t.

5. $\frac{dy}{dx} + 3y = 2 \Rightarrow$ I.F. $= e^{3x}$. So

$\frac{d}{dx}\left(ye^{3x}\right) = 2e^{3x}$

$\Rightarrow ye^{3x} = \frac{2}{3}e^{3x} + C \Rightarrow y = \frac{2}{3}+Ce^{-3x}$

$\lim\limits_{x\to\infty} y(x) = \frac{2}{3}.$

6. $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y} \Rightarrow -\frac{1}{2}\frac{-2y}{\sqrt{1-y^2}}dy = x$

$\Rightarrow -\sqrt{1-y^2} = x + c$

$\Rightarrow 1 - y^2 = (x + c)^2$

$\Rightarrow (x + c)^2 + y^2 = 1,$

which represent a circle with radius 1 and centre on x-axis so fixed radius and variable centre on x-axis.

7. $\frac{dy}{dx} = \sin\frac{x-y}{2} - \sin\frac{x+y}{2}$

$= -2\cos\frac{x}{2}\sin\frac{y}{2}$

$\Rightarrow \frac{dy}{\sin y/2} = -2\cos\frac{x}{2}dx$

$\Rightarrow 2\log\tan y/4 = -4\sin\frac{x}{2}+c$

$\Rightarrow \log\ \tan\frac{y}{4}+2\sin\frac{x}{2} = C$

8. $y\,dx - (x + y^2)dy = 0$

$\Rightarrow \frac{ydx - xdy}{y^2} = dy$

$\Rightarrow d\left(\frac{x}{y}\right) = dy$

$\Rightarrow \frac{x}{y} = y + c$

Since $y(1) = 1$ so $c = 0$

$\Rightarrow x = y^2$

$x = 16$, when $y = 4$.

9. $y\,dy + \sqrt{1+y^2}dx = 0$

$\Rightarrow dx + \frac{y}{\sqrt{1+y^2}}dy = 0$

$\Rightarrow x + \sqrt{1+y^2} = c$

$\Rightarrow (c - x)^2 = 1 + y^2$

$\Rightarrow (x - c)^2 - y^2 = 1,$

which represents a hyperbola.

10. Write the given differential equation as

$e^{-xy}\ d(xy) = \left(\frac{x}{y}\right)^2 d\left(\frac{x}{y}\right)$

Integrating, we get

$-\ e^{-xy} = \frac{1}{3}\left(\frac{x}{y}\right)^3 + C$

When $y(0) = 1$, we get

$-\ e^0 = \frac{1}{3}(0) + C \Rightarrow C = -1$

$\therefore\ 1 - e^{-xy} = \frac{1}{3}\left(\frac{x}{y}\right)^3$

$\Rightarrow 3y^3(1 - e^{-xy}) = x^3$