AEP STUDY CIRCLE

Since 2001...

IIT-JEE | NEET | CBSE

ACCENTS EDUCATIONAL PROMOTERS

OFFLINE-ONLINE LEARNING ACADEMY

MATHEMATICS

DEFINITE-INTEGRATION

IIT-JEE

CBSE

DEFINITE-INTEGRATION

The Success Destination...

NEET

Call Us:
+91 99395 86130, +91 7739650505

Visit Us:
www.aepstudycircle.com

2ND FLOOR, SATKOUDI COMPLEX, THANA CHOWK, RAMGARH, JHARKAHAND - 829122

THE NEWTON-LEIBNITZ FORMULA

If $F(x)$ is one of the antiderivatives of a continuous function $f(x)$ on $[a, b]$ i.e., $F'(x) = f(x)$ $(a < x < b)$, then we have the following formula due to Newton and Leibnitz: (FUNDAMENTAL THEOREM OF CALCULUS)

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. Then $\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty,\,h\to 0\\ nh=b-a}} h\sum_{r=1}^{n} f(a+rh)$. In particular,

$$\lim_{n\to\infty}\sum_{r=1}^{r=n}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

Illustration 1

$$\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{r}{n} = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

Illustration 2

$$\lim_{n\to\infty}\left[\frac{1^2}{n^3}+\frac{2^2}{n^3}+\cdots+\frac{(n-1)^2}{n^3}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}\left(\frac{r}{n}\right)^2 = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(u)\,du = \int_a^b f(t)\,dt$

4. If $f(x) \ge 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \ge 0$. If $f(x) > 0$ for all points of $[a, b]$. Then $\int_a^b f(x)\,dx > 0$.

5. If $f(x) \le g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b f(x)\,dx$.

6. $\int_a^b (f_1(x) + f_2(x))\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx$ and $\int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx$

7. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

8. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

9. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

10. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

11. $\int_0^a f(x)\,dx = \int_0^{a/2} f(x) + \int_0^{a/2} f(a-x)\,dx$. In particular,

$$\int_0^a f(x)\,dx = \begin{cases} 0 & \text{if } f(a-x) = -f(x) \\ 2\int_0^{a/2} f(x)\,dx & \text{if } f(a-x) = f(x) \end{cases}$$

12. If $f(-x) = f(x)$ (i.e., f is an even function), then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

13. If $f(-x) = -f(x)$ (i.e., f is an odd function), then $\int_{-a}^{a} f(x)dx = 0$.

14. If f is continuous on $[a, b]$, then the integral function g defined by $g(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$ is derivable on $[a, b]$, and $g'(x) = f(x)$ for all $x \in [a, b]$.

15. If m and M are the smallest and greatest values of a function $f(x)$ on an interval $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

16. If $f(x)$ is continuous on $[a, b]$, then there exists a point $c \in (a, b)$ such that $\int_a^b f(x)dx = f(c)(b-a)$. The number $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ is called the mean value of the function $f(x)$ on the interval $[a, b]$. The above result is called the first mean value theorem for integrals.

17. If $f(x)$ is periodic with period T then $\int_a^b f(x)dx = \int_{a+nT}^{b+nT} f(x)\,dx$, where n is an integer. In particular, $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)dx.$

18. If the function $\varphi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at every point $x \in (a, b)$, and $f(t)$ is continuous for $\varphi(a) \le t \le \varphi(b)$, then

$$\frac{d}{dx}\left(\int_{\varphi(x)}^{\psi(x)} f(t)\,dt\right) = f(\psi(x))\frac{d\psi}{dx} - f(\varphi(x))\frac{d\varphi}{dx}.$$

19. *Change of variables* If the function $f(x)$ is continuous on $[a, b]$ and the function $x = \varphi(t)$ is continuously differentiable on the interval $[t_1, t_2]$ and $a = \varphi(t_1)$, $b = \varphi(t_2)$, then

$$\int_a^b f(x)dx = \int_{t_1}^{t_2} f(\varphi(t))\,\varphi'(t)dt$$

20. Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. For any $\alpha \in [c, d]$, if $I(\alpha) = \int_a^b f(x, \alpha)dx$, then $I'(\alpha) = \int_a^b f'(x, \alpha)dx$, where $I'(\alpha)$ is the derivative of $I(\alpha)$ w.r.t. α, and $f'(x, \alpha)$ is the derivative of $f(x, \alpha)$ w.r.t. α, keeping x constant.

21. If $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$, then

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \left(\int_a^b f^2(x)\,dx\right)^{1/2}\left(\int_a^b g^2(x)\,dx\right)^{1/2}.$$

22. Let f be continuous and non-negative at all points of $[a, b]$ and $f(x_0) > 0$ at least at one point x_0 within this interval then $\int_a^b f(x)\,dx > 0$.

Illustration 3

Evaluate $\int_0^2\left(\frac{x^2}{4} - 7x + 5\right)dx$

The given integral is equal to

$$\frac{1}{4}\int_0^2 x^2dx - 7\int_0^2 x\,dx + 5\int_0^2 dx$$

$$= \frac{1}{4}\frac{x^3}{3}\Big|_0^2 - 7\frac{x^2}{2}\Big|_0^2 + 5x\Big|_0^2$$

$$= \frac{1}{4}\left[\frac{8}{3} - 0\right] - \frac{7}{4}[4-0] + 5[2-0]$$

$$= \frac{2}{3} - 14 + 10 = -\frac{10}{3}$$

Illustration 4

Find the average value of $f(x) = 4 - x^2$ on $[0, 3]$.

$$\text{average }(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{1}{3-0}\int_0^3 (4-x^2)dx$$

$$= \frac{1}{3}4x - \frac{x^3}{3}\Big|_0^3 = \frac{1}{3}\left[12 - \frac{27}{3}\right] = 1$$

Illustration 5

The value of $\int_0^1 \sin x^2 dx$ cannot be 2

For $x \in [0, 1]$, $0 \le \sin x^2 \le 1$, so

$$0 \le \int_0^1 \sin x^2 dx \le \int_0^1 1.dx = 1.$$

INTEGRALS WITH INFINITE LIMITS

If a function $f(x)$ is continuous for $a \le x < \infty$, then by definition,

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx \quad (1)$$

If there exists a finite limit on the right-hand side of (1), then the improper integral is said to be convergent; otherwise it is divergent.

Geometrically, the improper integral (1) for $f(x) > 0$, is the area of the figure bounded by the graph of the function $y = f(x)$, the straight line $x = a$ and the x-axis. Similarly, we can define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

and $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

REDUCTION FORMULAE FOR $\int_0^{\pi/2}\sin^n x\,dx$ and $\int_0^{\pi/2}\sin^p x\cos^q x\,dx$

$$\int_0^{\pi/2}\cos^n x\,dx = \int_0^{\pi/2}\sin^n x\,dx$$

$$= \begin{cases} \dfrac{n-1}{n}\times\dfrac{n-3}{n-2}\times\cdots\times\dfrac{1}{2}\cdot\dfrac{\pi}{2} & \text{if } n \text{ is even} \\ \dfrac{n-1}{n}\times\dfrac{n-3}{n-2}\times\cdots\dfrac{2}{3} & \text{if } n \text{ is odd} \end{cases}$$

$$I_{p,q} = \int_0^{\pi/2}\sin^p x\cos^q x\,dx$$

$$= \begin{cases} \dfrac{p-1}{p+q}\times\dfrac{p-3}{p+q-2}\times\cdots\times\dfrac{2}{3+q}\cdot I_{1,q} & \text{if } p \text{ is odd} \\ \dfrac{q-1}{p+q}\times\dfrac{p-3}{p+q-2}\times\cdots\dfrac{1}{2+q}I_{0,q} & \text{if } p \text{ is even} \end{cases}$$

where $I_{1,q} = \dfrac{1}{q+1}$ and

$$I_{0,q} = \begin{cases} \dfrac{q-1}{q}\dfrac{q-3}{q-2}\cdots\dfrac{2}{3} & \text{if } q \text{ is odd} \\ \dfrac{q-1}{q}\cdot\dfrac{q-3}{q-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2} & \text{if } q \text{ is even} \end{cases}$$

Note that if p is odd then it does not matter whether q is even or odd.

SOLVED EXAMPLES
Concept-based
Straight Objective Type Questions

Example 1: The value of $\int_0^1 \dfrac{x\,dx}{(x^2+1)^2}$ is

(a) $\dfrac{1}{2}$ (b) 1

(c) $\dfrac{1}{3}$ (d) $\dfrac{1}{4}$

Ans. (d)

Solution: Substituting $x^2 + 1 = t$. When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$$\int_0^1 \frac{x\,dx}{(x^2+1)^2} = \frac{1}{2}\int_1^2 \frac{dt}{t^2} = -\frac{1}{2}\frac{1}{t}\bigg|_1^2 = -\frac{1}{2}\left[\frac{1}{2}-1\right] = \frac{1}{4}$$

Example 2: The value of $\int_{1/\pi}^{2/\pi}\dfrac{\sin 1/x}{x^2}dx$ is equal

(a) 1 (b) $\dfrac{1}{2}$

(c) –1 (d) $\dfrac{1}{4}$

Ans. (a)

Solution: Substitute $\dfrac{1}{x} = t \Rightarrow -\dfrac{1}{x^2}\,dx = dt$

When $x = \dfrac{1}{\pi}$, $t = \pi$ and when $x = \dfrac{2}{\pi}$, $t = \dfrac{\pi}{2}$. Thus

$$\int_{1/\pi}^{2/\pi}\frac{\sin 1/x}{x^2}dx = -\int_{\pi}^{\pi/2}\sin t\,dt = \cos t\bigg|_{\pi}^{\pi/2} = -(-1) = 1$$

Example 3: If $\int_0^1 \dfrac{\sqrt{e^x}}{\sqrt{e^x+e^{-x}}}dx = \log\dfrac{e+A}{1+\sqrt{2}}$ then A is equal to

(a) e^2+1 (b) $\sqrt{e^2+1}$

(c) e^2 (d) e^2-1

Ans. (b)

Solution: $\int_0^1 \dfrac{\sqrt{e^x}}{\sqrt{e^x+e^{-x}}}dx = \int_0^1 \dfrac{e^x}{\sqrt{e^{2x}+1}}dx$. Putting $e^x = t$ the last integral is equal to

$$\int_1^e \frac{dt}{\sqrt{t^2+1}} = \log\left|t+\sqrt{t^2+1}\right|\bigg|_1^e$$

$$= \log(e+\sqrt{e^2+1}) - \log(1+\sqrt{2})$$

$$= \log\frac{e+\sqrt{e^2+1}}{1+\sqrt{2}}$$

So $A = \sqrt{e^2+1}$

Example 4: Value of x satisfying $\int_{\log 2}^{x}\frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$ is

(a) log 2 (b) 3 log 2
(c) 2 log 2 (d) 2

Ans. (c)

Solution: Substituting $E^x - 1 = t^2$, $e^x\,dx = 2t\,dt$

$$\Rightarrow \quad dx = \frac{2t\,dt}{t^2+1}, \text{ so}$$

$$\frac{\pi}{6} = \int_{\log 2}^{x}\frac{dx}{\sqrt{e^x-1}} = \int_1^{\sqrt{e^x-1}}\frac{2t\,dt}{(t^2+1)t}$$

$$= 2\int_1^{\sqrt{e^x-1}}\frac{dt}{t^2+1} = 2\tan^{-1}t\Big|_1^{\sqrt{e^x-1}}$$

$$= 2\tan^{-1}\sqrt{e^x-1} - \frac{\pi}{2}$$

$$\Rightarrow \quad 2\tan^{-1}\sqrt{e^x-1} = \frac{2\pi}{3}$$

$$\Rightarrow \quad \sqrt{e^x-1} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\Rightarrow \quad e^x - 1 = 3 \quad \text{i.e. } e^x = 4 \Rightarrow x = \log 4.$$

Example 5: The value of $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} x^{10} \sin^9 x\,dx$ is equal to

(a) 0 (b) 1
(c) $\frac{\pi}{4}$ (d) $\left(\frac{\pi}{8}\right)^{10}$

Ans. (a)

Solution: Let $f(x) = x^{10}\sin^9 x$, $f(-x) = (-x)^{10}\sin^9(-x)$
$= -x^{10}\sin^9 x = -f(x)$

Hence f is an odd function. Now using Prop. 13 Page 13.2, we have $\int_{-a}^{a} f(x)dx = 0$. Thus

$$\int_{-\pi/8}^{\pi/8} x^{10}\sin^9 x\,dx = 0$$

Example 6: $\int_0^{\pi/2}\max(\sin x, \cos x)\,dx$ is equal to

(a) 1 (b) 2
(c) $2-\sqrt{2}$ (d) $\sqrt{2}$

Ans. (d)

Solution: $\max(\sin x, \cos x) = \begin{cases}\cos x & 0 \le x \le \frac{\pi}{4} \\ \sin x & \frac{\pi}{4} \le x \le \frac{\pi}{4}\end{cases}$
$x \in [0, \pi/2]$

$$\int_0^{\pi/2}\max(\sin x, \cos x)\,dx = \int_0^{\pi/4}\cos x\,dx + \int_{\pi/4}^{\pi/2}\sin x\,dx$$

$$= \sin x\Big|_0^{\pi/4} - \cos x\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$

Example 7: If $\int_0^{\pi/2}\frac{\cos x\,dx}{6-5\sin x+\sin^2 x} = \log K$ then K is equal to

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$
(c) $\frac{1}{3}$ (d) $\frac{5}{3}$

Ans. (b)

Solution: Putting $\sin x = t$, $\cos x\,dx = dt$. When $x = 0$, $\sin x = 0$ and when $x = \pi/2$, $\sin x = 1$. So the given integral reduces to

$$\int_0^1\frac{dt}{6-5t+t^2} = \int_0^1\frac{dt}{\left(t-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\frac{1}{2}}\log\left|\frac{t-\frac{5}{2}-\frac{1}{2}}{t-\frac{5}{2}+\frac{1}{2}}\right|\Bigg|_0^1$$

$$= \log\left|\frac{t-3}{t-2}\right|\Bigg|_0^1 = \log\frac{4}{3}$$

Example 8: $\int_{-1}^{1}|x|\,dx$ is equal to

(a) 1 (b) 0
(c) 2 (d) $\frac{1}{2}$

Ans. (a)

Solution: The integrand $f(x) = |x|$ is an even function so

$$\int_{-1}^{1}|x|\,dx = 2\int_0^1|x|\,dx = 2\int_0^1 x\,dx = 2\frac{x^2}{2}\Big|_0^1 = 1$$

Example 9: The value of $\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right]$ is equal to

(a) π (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

Ans. (b)

Solution:

$$=\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}$$

$$=\frac{1}{n}\left[\frac{1}{\sqrt{4-\frac{1}{n^2}}}+\frac{1}{\sqrt{4-\frac{2^2}{n^2}}}+\cdots+\frac{1}{\sqrt{4-\left(\frac{n}{n}\right)^2}}\right]$$

So, required limit is equal to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{4-\left(\frac{r}{n}\right)^2}}=\int_0^1\frac{dx}{\sqrt{4-x^2}}$$

$$=\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}.$$

Example 10: $\int_0^{10\pi}\sqrt{1-\sin^2 x}\,dx$ is equal to

(a) 0 (b) 10
(c) 10π (d) 20

Ans. (d)

Solution:

$$\int_0^{10\pi}\sqrt{1-\sin^2 x}\,dx=\int_0^{10\pi}|\cos x|\,dx$$

$$=10\int_0^{\pi}|\cos x|\,dx \quad \text{(Prop. 16)}$$

$$=20\int_0^{\pi/2}|\cos x|\,dx \quad \text{(Prop. 10)}$$

$$=20\int_0^{\pi/2}\cos x\,dx=20\sin x\Big|_0^{\pi/2}=20.$$

LEVEL 1

Straight Objective Type Questions

Example 11: The value $I=\int_0^{\pi/2}\frac{1}{1+\cot x}dx$ is

(a) π (b) $\pi/2$
(c) $\pi/4$ (d) $\pi/3$

Ans. (c)

Solution: Let $f(x)=\frac{1}{1+\cot x}=\frac{\sin x}{\sin x+\cos x}$, so

$$f(\pi/2-x)=\frac{\sin(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}$$

$$=\frac{\cos x}{\cos x+\sin x}$$

Considering $2I=\int_0^{\pi/2}f(x)\,dx+\int_0^{\pi/2}f(\pi/2-x)\,dx$

$$\left(\text{using}\int_0^a f(x)dx=\int_0^a f(a-x)\,dx\right)$$

$$=\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}dx+\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}dx$$

$$=\int_0^{\pi/2}\frac{\sin x+\cos x}{\cos x+\sin x}dx=\int_0^{\pi/2}1\cdot dx=\frac{\pi}{2}$$

So $I=\pi/4$

Example 12: The value of $I=\int_0^{\pi}\frac{x\,dx}{4\cos^2 x+9\sin^2 x}$ is

(a) $\pi^2/12$ (b) $\pi^2/4$
(c) $\pi^2/6$ (d) $\pi^2/3$

Ans. (a)

Solution: Using Property 9, we have

$$I=\int_0^{\pi}\frac{(\pi-x)dx}{4\cos^2(\pi-x)+9\sin^2(\pi-x)}$$

$$=\pi\int_0^{\pi}\frac{dx}{4\cos^2 x+9\sin^2 x}-\int_0^{\pi}\frac{x\,dx}{4\cos^2 x+9\sin^2 x}$$

$$2I=\pi\int_0^{\pi}\frac{dx}{4\cos^2 x+9\sin^2 x}$$

$$=2\pi\int_0^{\pi/2}\frac{dx}{4\cos^2 x+9\sin^2 x}$$

(using Property 9)

$$= 2\pi \int_0^{\pi/2} \frac{\sec^2 x\, dx}{4 + 9\tan^2 x}$$

$$= 2\pi \int_0^{\infty} \frac{dt}{4 + 9t^2} \qquad (t = \tan x)$$

$$= \frac{2\pi}{9}\int_0^{\infty} \frac{dt}{t^2 + 4/9} = \frac{2\pi}{9}\cdot\frac{3}{2}\tan^{-1}\frac{3}{2}t\Big|_0^{\infty} = \frac{\pi^2}{6}$$

Hence $I = \pi^2/12$.

Example 13: If $\int_e^x tf(t)\,dt = \sin x - x\cos x - \frac{x^2}{2}$ for all $x \in \mathbf{R} \sim \{0\}$ then the value of $f(\pi/6)$ is

(a) 0 (b) 1
(c) – 1/2 (d) 3/2

Ans. (c)

Solution: Differentiating both the sides and using Property 17, we have

$xf(x) = \cos x - \cos x + x\sin x - x$, so $f(x) = \sin x - 1$.

Hence $f(\pi/6) = \sin \pi/6 - 1 = -1/2$.

Example 14: The value of $\int_0^2 x^{[x^2+1]}\,dx$, where $[x]$ is the greatest integer less than or equal to x is

(a) 2 (b) 8/3
(c) 4 (d) none of these

Ans. (d)

Solution: For $x \in [0, 2]$, $x^2 + 1 \in [1, 5]$, we must break $[0, 2] = [0, 1] \cup [1, \sqrt{2}] \cup [\sqrt{2}, \sqrt{3}] \cup [\sqrt{3}, 2]$.

Hence $\int_0^2 x^{[x^2+1]}\,dx$

$$= \int_0^1 x^{[x^2+1]}\,dx + \int_1^{\sqrt{2}} x^{[x^2+1]}\,dx + \int_{\sqrt{2}}^{\sqrt{3}} x^{[x^2+1]}\,dx + \int_{\sqrt{3}}^{2} x^{[x^2+1]}\,dx$$

$$= \int_0^1 x\,dx + \int_1^{\sqrt{2}} x^2\,dx + \int_{\sqrt{2}}^{\sqrt{3}} x^3\,dx + \int_{\sqrt{3}}^{2} x^4\,dx$$

$$= \frac{1}{2} + \frac{1}{3}\left[2^{3/2} - 1\right] + \frac{1}{4}[9 - 4] + \frac{1}{5}\left[32 - 3^{5/2}\right]$$

$$= \frac{469}{60} + \frac{1}{3}2^{3/2} - \frac{1}{5}3^{5/2}.$$

Example 15: Let $f(x) = \int_1^x \sqrt{2 - t^2}\,dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(a) ± 1 (b) $\pm 1/\sqrt{2}$
(c) $\pm 1/2$ (d) 0 and 1

Ans. (a)

Solution: Using Property 18, we have $f'(x) = \sqrt{2 - x^2}$ and thus the given equation reduces to $x^2 - \sqrt{2 - x^2} = 0$ $\Rightarrow (x^2 + 2)(x^2 - 1) = 0$. Thus the real roots are given by $x = \pm 1$.

Example 16: The integral $\int_{-1/2}^{1/2} \left([x] + 2\log\frac{1+x}{1-x}\right) dx$ equals

(a) – 1/2 (b) 0
(c) 1 (d) 2 log (1/2)

Ans. (a)

Solution: $\int_{-1/2}^{1/2} \left([x] + 2\log\frac{1+x}{1-x}\right) dx = \int_{-1/2}^{1/2} [x]\,dx$

(since $\log\frac{1+x}{1-x}$ is an odd function)

$$= \int_{-1/2}^{0} [x]\,dx + \int_{0}^{1/2} [x]\,dx = \int_{-1/2}^{0} (-1)\,dx = -\frac{1}{2}.$$

Example 17: The value

$$I = \int_{-1}^{1} \left(\cos^{-1} x + \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x}\right) dx \text{ is}$$

(a) $\pi/2$ (b) 0
(c) 2π (d) π

Ans. (d)

Solution: Let $f(x) = \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x}$ then $f(-x) = -f(x)$ so $\int_{-1}^{1} f(x)\,dx = 0$ (Property 12)

Let $g(x) = \cos^{-1} x$ then $I_1 = \int_{-1}^{1} \cos^{-1} x\,dx$

$$= \int_{-1}^{1} \cos^{-1}(-x)\,dx \qquad \text{(Property 8)}$$

$$= \int_{-1}^{1} (\pi - \cos^{-1} x)\,dx = 2\pi - I_1$$

Thus $I = I_1 = \pi$.

Example 18: The value of $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\,dx$ is

(a) 0 (b) 2/3
(c) 4/3 (d) 1/3

Ans. (c)

Solution: $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\, |\sin x|\, dx$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\, |\sin x|\, dx$$

(the integand is an even function)

$$= 2\int_{0}^{\pi/2} \sqrt{\cos x}\, \sin x\, dx$$

$$= -\frac{4}{3}(\cos x)^{3/2}\Big|_0^{\pi/2} = \frac{4}{3}.$$

Example 19: If $n \in \mathbf{N}$, the value of $\int_0^n [x]\, dx$ (where $[x]$ is the greatest integer function) is

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n-1)}{2}$

(c) $n(n-1)$ (d) none of these

Ans. (b)

Solution: $\int_0^n [x]\, dx = \sum_{i=1}^{n}\int_{i-1}^{i} [x]\, dx = \sum_{i=1}^{n}\int_{i-1}^{i} (i-1)\, dx$

$$= \sum_{i=1}^{n} (i-1) = \frac{n(n-1)}{2}.$$

Example 20: If $F(x) = \int_3^x \left(2 + \frac{d}{dt}\cos t\right) dt$ then $F'\left(\frac{\pi}{6}\right)$ is equal to

(a) 1/2 (b) 2

(c) 3/4 (d) 3/2

Ans. (d)

Solution: $F(x) = \int_3^x (2 - \sin t)dt$ so $F'(x) = 2 - \sin x$.

Thus $F'(\pi/6) = 2 - 1/2 = 3/2$.

Example 21: $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$ equals

(a) $1 + \sqrt{5}$ (b) $-1 + \sqrt{5}$

(c) $-1 + \sqrt{2}$ (d) $1 + \sqrt{2}$

Ans. (b)

Solution: Required limit $= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} \frac{r/n}{\sqrt{1+r^2/n^2}}$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}}\, dx = \sqrt{1+x^2}\Big|_0^2 = \sqrt{5} - 1.$$

Example 22: The value of the integral $\int_0^{n\pi+t} (|\cos x| + |\sin x|)\, dx$ is

(a) n (b) $2n + \sin t + \cos t$

(c) $\cos t$ (d) $\sin t - \cos t + 4n + 1$

Ans. (d)

Solution: Since the period of $|\sin x| + |\cos x|$ is $\pi/2$ so

$\int_0^{n\pi+t} (|\sin x| + |\cos x|)\, dx = 2n\int_0^{\pi/2} (|\sin x| + |\cos x|)dx + \int_0^t (|\sin x| + |\cos x|)\, dx$

$$= 2n\int_0^{\pi/2} (\sin x + \cos x)\, dx + \int_0^t (\sin x + \cos x)\, dx$$

$$= (2n)(2) + \sin t - \cos t + 1$$

$$= (4n+1) + \sin t - \cos t.$$

Example 23: If $g(x) = \int_0^x \cos^4 t\, dt$ then $g(x+\pi)$ equals

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$

(c) $g(x)\, g(\pi)$ (d) $g(x)/g(\pi)$

Ans. (a)

Solution: $g(x+\pi) = \int_0^{x+\pi} \cos^4 t\, dt$

$$= \int_0^{\pi} \cos^4 t\, dt + \int_{\pi}^{x+\pi} \cos^4 t\, dt$$

$$= g(\pi) + I_1$$

In I_1, put $t = \pi + u$, so that

$$I_1 = \int_0^x \cos^4(\pi+u)\, du = \int_0^x \cos^4 u\, du = g(x).$$

Example 24: The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is

(a) 0 (b) 1

(c) $\pi/4$ (d) $\pi/2$

Ans. (c)

Solution: $I = \int_0^{\pi/2} \frac{dx}{1+\tan^3 x} = \int_0^{\pi/2} \frac{dx}{1+\tan^3(\pi/2-x)}$

$$= \int_0^{\pi/2} \frac{dx}{1+\cot^3 x}$$

$$= \int_0^{\pi/2} \frac{\tan^3 x}{1+\tan^3 x}\, dx = \int_0^{\pi/2}\left(1 - \frac{1}{1+\tan^3 x}\right) dx = \frac{\pi}{2} - I.$$

Thus $I = \pi/4$.

Example 25: The value of $\int_0^4 3^{\sqrt{2x+1}}\, dx$ is

(a) $\frac{6}{\log 3}\left(13 - \frac{4}{\log 3}\right)$ (b) $\frac{66}{\log 3}$

(c) $\frac{6}{\log 3}\left(13 - \frac{5}{\log 3}\right)$ (d) none of these

Ans. (d)

Solution: Putting $2x + 1 = t^2$, we have $dx = t\,dt$, so

$$\Rightarrow \int_0^4 3^{\sqrt{2x+1}}\,dx = \int_1^3 3^t\, t\,dt = \left.\frac{t\cdot 3^t}{\log 3}\right|_0^3 - \frac{1}{\log 3}\int_0^3 3^t\,dt$$

$$= \frac{(3)\cdot 3^3 - 3}{\log 3} - \frac{1}{(\log 3)^2}[3^3 - 3^1] = \frac{78}{\log 3} - \frac{24}{(\log 3)^2}$$

Example 26: The value of $\int_0^1 x(1-x)^{99}\,dx$ is

(a) $\frac{1}{10100}$ (b) $\frac{11}{10100}$

(c) $\frac{1}{10010}$ (d) none of these

Ans. (a)

Solution: $\int_0^1 x(1-x)^{99}\,dx = \int_0^1 (1-x)(1-(1-x))^{99}\,dx$ (Property 9)

$$= \int_0^1 (x^{99} - x^{100})\,dx = \left.\frac{x^{100}}{100} - \frac{x^{101}}{101}\right|_0^1 = \frac{1}{10100}.$$

Example 27: The value of $\sum_{n=1}^{1000}\int_{n-1}^{n} e^{x-[x]}\,dx$ is ([x] is the greatest integer function)

(a) $\frac{e^{1000}-1}{1000}$ (b) $\frac{e^{1000}-1}{e-1}$

(c) $1000(e-1)$ (d) $\frac{e-1}{1000}$

Ans. (c)

Solution: Since the period of the function $x - [x]$ is 1 so

$$\sum_{n=1}^{1000}\int_{n-1}^{n} e^{x-[x]}\,dx = \int_0^{1000} e^{x-[x]}\,dx$$

$$= 1000\int_0^1 e^{x-[x]}\,dx = 1000\int_0^1 e^x\,dx = 1000\,(e-1).$$

Example 28: If $f: \mathbf{R} \to \mathbf{R}$, $g: \mathbf{R} \to \mathbf{R}$ are continuous functions then the value of the integral $\int_{-\pi/2}^{\pi/2} [(f(x) + f(-x))(g(x) - g(-x))]\,dx$ is

(a) 1 (b) 0

(c) – 1 (d) π

Ans. (b)

Solution: Let $F(x) = (f(x) + f(-x))(g(x) - g(-x))$ so

$$F(-x) = (f(-x) + f(x))(g(-x) - g(x)) = -F(x).$$

Hence $\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$ (using Property 11)

Example 29: If $f(x) = (1 + \tan x)(1 + \tan(\pi/4 - x))$ and $g(x)$ is a function with domain **R**, then $\int_0^1 x^3\, g\,o\,f(x)\,dx$ is

(a) $\frac{1}{2}\,g(\pi/4)$ (b) $\frac{1}{4}\,g(2)$

(c) $\frac{1}{4}\,g(1)$ (d) none of these

Ans. (b)

Solution: $f(x) = (1+\tan x)\left(1 + \frac{1-\tan x}{1+\tan x}\right)$

$$= (1+\tan x)\left(\frac{2}{1+\tan x}\right) = 2.$$

So $\int_0^1 x^3\, g\,o\,f(x)\,dx = \int_0^1 x^3\, g(2)\,dx$

$$= \left.\frac{x^4 g(2)}{4}\right|_0^1 = \frac{1}{4}\,g(2).$$

Example 30: The $\lim_{n\to\infty} S_n$ if

$$S_n = \frac{1}{2n} + \frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-4}} + \ldots + \frac{1}{\sqrt{3n^2+2n-1}} \text{ is}$$

(a) $\pi/2$ (b) 2

(c) 1 (d) $\pi/6$

Ans. (d)

Solution: $\lim_{n\to\infty} S_n$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\sqrt{4-0}} + \frac{1}{\sqrt{4-1/n^2}} + \frac{1}{\sqrt{4-4/n^2}} + \ldots + \frac{1}{\sqrt{4-\left(\frac{n-1}{n}\right)^2}}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{4-(r/n)^2}} = \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left.\sin^{-1}\frac{x}{2}\right|_0^1 = \frac{\pi}{6}.$$

Example 31: The value of $\int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)\,d\theta$ is

(a) 1 (b) 0

(c) 2 (d) none of these

Ans. (b)

Solution: $I = \int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)\,d\theta$

$$= \int_{-\pi}^{3\pi} \log(\sec(2\pi-\theta) - \tan(2\pi-\theta))\,d\theta$$

$$= \int_{-\pi}^{3\pi} \log(\sec\theta + \tan\theta)\,d\theta.$$

Thus $2I = \int_{-\pi}^{3\pi} [\log(\sec\theta - \tan\theta) + \log(\sec\theta + \tan\theta)]\, d\theta$

$$= \int_{-\pi}^{3\pi} \log(\sec^2\theta - \tan^2\theta)d\theta$$

$$= \int_{-\pi}^{3\pi} \log 1\, d\theta = 0.$$

Example 32: If $\int_0^\infty \frac{dx}{(x^2+4)(x^2+9)} = k\pi$ then the value of k is

(a) 1/60 (b) 1/80
(c) 1/40 (d) 1/20

Ans. (a)

Solution:

$$\int_0^\infty \frac{dx}{(x^2+4)(x^2+9)} = \frac{1}{5}\int_0^\infty \left[\frac{1}{x^2+4} - \frac{1}{x^2+9}\right]dx$$

$$= \frac{1}{5}\left[\frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^\infty$$

$$= \frac{1}{5}\left[\frac{\pi}{4} - \frac{\pi}{6}\right] = \frac{\pi}{60}, \text{ so } k = \frac{1}{60}.$$

Example 33: If $\int_{-1}^{3/2} |x \sin \pi x|\, dx = k/\pi^2$, then the value of k is

(a) $3\pi + 1$ (b) $2\pi + 1$
(c) 1 (d) 4

Ans. (a)

Solution:

$$I = \int_{-1}^{1} |x\sin\pi x dx| + \int_1^{3/2} |x\sin\pi x| dx$$

$$= 2\int_0^1 x\sin\pi x\, dx - \int_1^{3/2} x\sin\pi x\, dx$$

$$= 2\left[\frac{-x\cos\pi x}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_0^1 - \left[\frac{-x\cos\pi x}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

$$= 2\left[\frac{1}{\pi}\right] - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{3\pi+1}{\pi^2}, \text{ so } k = 3\pi + 1.$$

Example 34: If $f(0) = 2$, $f'(x) = f(x)$, $\phi(x) = x + f(x)$ then $\int_0^1 f(x)\,\phi(x)\,dx$ is

(a) e^2 (b) $2e^2$
(c) $2e$ (d) $2e^{-3/2}$

Ans. (b)

Solution: Since $f'(x) = f(x)$ and $f(0) = 2$ so $f(x) = 2e^x$, thus $\phi(x) = x + 2e^x$. Hence

$$\int_0^1 f(x)\,\phi(x)\,dx = 2\int_0^1 (xe^x + 2e^{2x})dx$$

$$= 2\,[xe^x - e^x|_0^1 + e^{2x}|^1{}_0]$$

$$= 2\,[([e - e] + 1) + e^2 - 1] = 2e^2.$$

Example 35: The value of $\lim_{x\to 0} \frac{\int_0^x \cos t^2 dt}{x}$ is

(a) 1 (b) 0
(c) – 1 (d 2

Ans. (a)

Solution: Let $f(x) = \int_0^x \cos t^2\, dt$ and $g(x) = x$. Then $f(0) = g(0) = 0$.

$$\therefore \quad \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\int_0^x \cos t^2 dt}{x} = \lim_{x\to 0}\frac{\cos x^2\cdot 1 - \cos 0\cdot 0}{1}$$

$$= \lim_{x\to 0}\frac{\cos x^2}{1} = \cos 0 = 1.$$

Example 36: If $f(x) = \int_{1/x^2}^{x^2} \cos\sqrt{t}\, dt$, then $f'(1)$ is equal to

(a) cos 1 (b) 2 cos 1
(c) 4 cos 1 (d) none of these

Ans. (c)

Solution: $f'(x) = \cos\sqrt{x^2}\,\frac{d}{dx}(x^2) - \cos\sqrt{1/x^2}\,\frac{d}{dx}\frac{1}{x^2}$

(Using Property 18)

$$= 2x\cos x + \frac{2}{x^3}\cos(1/x). \text{ Hence } f'(1) = 4\cos 1.$$

Example 37: The least value of the function

$$F(x) = \int_0^x (3\sin u + 4\cos u)\,du$$

on the interval $(5\pi/4, 4\pi/3]$ is

(a) $3/2 - \sqrt{3}/2$ (b) $\frac{5-4\sqrt{3}}{2}$
(c) $\frac{7-4\sqrt{3}}{2}$ (d) $\frac{9-4\sqrt{3}}{2}$

Ans. (d)

Solution: We have $F'(x) = 3\sin x + 4\cos x$. Since $\sin x$ and $\cos x$ assume negative values in the third quadrant, we have $F'(x) < 0$ for all $x \in (5\pi/4, 4\pi/3)$ so $F(x)$ assumes the least value at the point $x = 4\pi/3$. Thus the least value is

$$F(4\pi/3) = \int_0^{4\pi/3} (3\sin u + 4\cos u)du$$

$$= (-3\cos u + 4\sin u)\Big|_0^{4\pi/3}$$

$$= -3\cos\frac{4\pi}{3} + 4\sin\frac{4\pi}{3} - (-3)$$

$$= \frac{9}{2} - \frac{4\sqrt{3}}{2} = \frac{9-4\sqrt{3}}{2}.$$

Example 38: The integral $\int_0^{1/a} \frac{\log(1+ax)}{1+a^2x^2}\,dx$ $(a > 0)$ is equal to

(a) $a(\log 2)\frac{\pi}{8}$ (b) $\frac{1}{a}(\log 2)\frac{\pi}{4}$

(c) $\frac{1}{a}(\log 2)\frac{\pi}{8}$ (d) $\frac{1}{a^2}(\log 2)$

Ans. (c)

Solution: Put $ax = t$, the given integral reduces to

$$\frac{1}{a}\int_0^1 \frac{\log(1+t)}{1+t^2}dt = \frac{1}{a}\int_0^{\pi/4} \frac{\log(1+\tan u)}{1+\tan^2 u}\sec^2 u\,du \ (t - \tan u)$$

$$= \frac{1}{a}\int_0^{\pi/4} \log(1+\tan u)\,du$$

$$= \frac{1}{a}\int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)du$$

$$= \frac{1}{a}\int_0^{\pi/4} \log\left(1+\frac{1-\tan u}{1+\tan u}\right)du$$

$$= \frac{1}{a}\int_0^{\pi/4} [\log 2 - \log(1+\tan u)]\,du$$

$$\Rightarrow \quad \frac{2}{a}\int_0^{\pi/4} \log(1+\tan u)\,du = \frac{1}{a}(\log 2)\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{1}{a}\int_0^{\pi/4} \log(1+\tan u)\,du = \frac{1}{a}(\log 2)\frac{\pi}{8}$$

Example 39: For $m > 0$, $n > 0$, let $I_{m,n} = \int_0^1 x^m (\log x)^n\,dx$, then $I_{5,5}$ is given by

(a) $-\frac{5!}{6^5}$ (b) $-\frac{5!}{5^5}$

(c) $-\frac{5!}{6^6}$ (d) $\frac{5!}{6^6}$

Ans. (c)

Solution: Integrating by parts, we obtain

$$I_{m,n} = \frac{x^{m+1}(\log x)^n}{m+1}\Big|_0^1 - \frac{n}{m+1}\int_0^1 x^m(\log x)^{n-1}dx$$

Since $\lim_{x\to 0+} x^m(\log x)^n = \lim_{x\to 0+} (x^{m/n}\log x)^n = 0$

So, $$I_{m,n} = -\frac{n}{m+1}\,I_{m,n-1}$$

Hence $$I_{5,5} = -\frac{5}{6}I_{5,4}$$

$$= -\frac{5}{6}\left(-\frac{4}{6}I_{5,3}\right)$$

$$= (-1)^3\frac{5.4.3}{6^3}I_{5,2} = (-1)^4\frac{5.4.3.2}{6^4}I_{5,1}$$

$$I_{5,1} = \int_0^1 x^5\log x\,dx = \frac{x^6\log x}{6}\Big|_0^1 - \frac{1}{6}\int_0^1 x^6\frac{1}{x}dx$$

$$= -\frac{1}{6^2}$$

So $$I_{5,5} = (-1)^5\frac{5!}{6^6} = -\frac{5!}{6^6}$$

Example 40: If $I_n = \int_0^1 (\cos^{-1}x)^n\,dx$ then $I_6 - 360\,I_2$ is given by

(a) $6\left(\frac{\pi}{2}\right)^5 - 24\left(\frac{\pi}{2}\right)^3$ (b) $6\left(\frac{\pi}{2}\right)^5 - 120\left(\frac{\pi}{2}\right)^3$

(c) $6\left(\frac{\pi}{2}\right)^5$ (d) $6\left(\frac{\pi}{2}\right)^5 - 4\left(\frac{\pi}{2}\right)^3$

Ans. (b)

Solution: Integrating by parts, we obtain

$$I_n = \int_0^1 (\cos^{-1}x)^n\,dx = x\cos^{-1}x\ \Big|_0^1 + \int_0^1 n\frac{(\cos^{-1}x)^{n-1}x}{\sqrt{1-x^2}}dx$$

$$= n\int_0^1 \frac{x}{\sqrt{1-x^2}}(\cos^{-1}x)^{n-1}dx$$

$$= n\left[-\sqrt{1-x^2}\,(\cos^{-1}x)^{n-1}\Big|_0^1 - \int_0^1 (n-1)(\cos^{-1}x)^{n-2}dx\right]$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

$$I_6 = 6.\left(\frac{\pi}{2}\right)^5 - 6.5\,I_4$$

$$= 6.\left(\frac{\pi}{2}\right)^5 - 30\left[4\left(\frac{\pi}{2}\right)^3 - 12I_2\right]$$

$$I_6 - 360\,I_2 = 6.\left(\frac{\pi}{2}\right)^5 - 120\left(\frac{\pi}{2}\right)^3$$

Example 41: Let $I_n = \int_0^{\pi/2} x^n \cos x \, dx$ then $I_8 + 56I_6$ is equal to

(a) $\left(\frac{\pi}{2}\right)^6$ (b) $\left(\frac{\pi}{2}\right)^8$

(c) $\left(\frac{\pi}{2}\right)^6 - 1$ (d) $\left(\frac{\pi}{2}\right)^8 - 1$

Ans. (b)

Solution:

$$I_n = \int_0^{\pi/2} x^n \cos x \, dx$$

$$= x^n \sin x\Big|_0^{\pi/2} - n\int_0^{\pi/2} x^{n-1}\sin x \, dx$$

$$= \left(\frac{\pi}{2}\right)^n - n\left[-x^{n-1}\cos x\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} x^{n-2}\cos x \, dx\right]$$

$$= \left(\frac{\pi}{2}\right)^n - n(n-1)\, I_{n-2}$$

$$I_8 + 8.7\, I_6 = \left(\frac{\pi}{2}\right)^8$$

Example 42: If a function $f : [0, 27] \to \mathbf{R}$ is differentiable then for some $0 < \alpha < \beta < 3$, $\int_0^{27} f(x)dx$ is equal to

(a) $3[\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3)]$

(b) $3[\alpha^2 f(\alpha) + \beta^2 f(\beta)]$

(c) $3[\alpha^2 f(\alpha^3) + \frac{1}{2}\beta^2 f(\beta^3)]$

(d) $3[\alpha^2 f(\alpha) + \frac{1}{2}\beta^2 f(\beta)]$

Ans. (c)

Solution: Let $g(x) = \int_0^{x^3} f(t)dt$ so $g(0) = 0$

$$\Rightarrow \quad g'(x) = 3x^2 f(x^3)$$

$$\int_0^{27} f(t)dt = g(3) = \frac{g(3) - g(1)}{3-1} + \frac{1}{2}\,\frac{g(1) - g(0)}{1-0}$$

$$= g'(\alpha) + \frac{1}{2}g'(\beta)$$

for some $\alpha \in (1, 3)$, $\beta \in (0, 1) \subseteq (0, 3)$. but $g'(\alpha) = 3\alpha^2 f(\alpha^3)$ and $g'(\beta) = 3\beta^2 f(\beta)^3$

Therefore, $\int_0^{27} f(t)dt = 3[\alpha^2 f(\alpha^3) + \frac{1}{2}\beta^2 f(\beta^3)]$

Example 43: The difference between the greatest and the least value of the function $F(x) = \int_0^x (t+1)\, dt$ on [2, 3] is

(a) 3 (b) 2
(c) 7/2 (d) 3/2

Ans. (c)

Solution: Differentiating the given function, we get

$$F'(x) = t + 1]_{t=x}\frac{dx}{dx} - [t+1]_{t=0}\frac{d0}{dx} = x + 1.$$

This is positive for all $x \in [2, 3]$, so F is an increasing function in this interval. Therefore its greatest value is $F(3) = \int_0^3 (t+1)\, dt$ and its least value is $F(2) = \int_0^2 (t+1)\, dt$, so that the required difference between these values is $\int_0^3 (t+1)dt - \int_0^2 (t+1)dt = \int_2^3 (t+1)dt = \frac{7}{2}$.

Example 44: The value of the integral $\int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x}\, dx$ is

(a) log 2 (b) log 3
(c) (1/4) log 3 (d) (1/8) log 3

Ans. (c)

Solution: The integral can be written

$$-\int_0^{\pi/4} \frac{\sin x + \cos x}{(\sin x - \cos x)^2 - 4}\, dx.$$

Now put $t = \sin x - \cos x$. Then $dt = (\cos x + \sin x)dx$, and the integral becomes

$$-\int_{-1}^{0} \frac{dt}{t^2 - 4} = -\frac{1}{4}\left[\log\left|\frac{t-2}{t+2}\right|\right]_{-1}^{0}$$

$$= -\frac{1}{4}(\log 1 - \log 3) = \frac{1}{4}\log 3.$$

Example 45: The inflection points on the graph of function $y = \int_0^x (t-1)(t-2)^2\, dt$ are

(a) $x = -1$ (b) $x = 3/2$
(c) $x = 4/3$ (d) $x = 1$

Ans. (c)

Solution: $\frac{dy}{dx} = (x-1)(x-2)^2$ so $\frac{d^2y}{dx^2} = (x-2)(3x - 4)$. The points of inflection are given by $\frac{d^2y}{dx^2} = 0$ so $x = 2$, $x = 4/3$ are points of inflection.

Example 46: The value of the integral $\int_0^{\pi/2} \frac{dx}{1 + \frac{1}{6}\sin^2 x}$ is

(a) $\frac{\pi}{2}\sqrt{\frac{6}{7}}$ (b) $\frac{\pi}{3}\sqrt{\frac{2}{3}}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

Ans. (a)

Solution: $I = \int_0^{\pi/2} \frac{dx}{1+\frac{1}{6}\sin^2 x} = \int_0^{\pi/2} \frac{\sec^2 x\,dx}{\sec^2 x + \frac{1}{6}\tan^2 x}$

$$= \frac{6}{7}\int_0^{\infty} \frac{dt}{6/7 + t^2} \quad (t = \tan x)$$

$$= \frac{6}{7} \times \sqrt{\frac{7}{6}} \tan^{-1} t\sqrt{\frac{7}{6}}\Bigg|_0^{\infty} = \sqrt{\frac{6}{7}} \cdot \frac{\pi}{2}.$$

Example 47: The value of $\int_{e^{-1}}^{e^2} \left|\frac{\log x}{x}\right| dx$ is

(a) 3/2 (b) 5/2
(c) 3 (d) 5

Ans. (b)

Solution: $\int_{e^{-1}}^{e^2} \left|\frac{\log x}{x}\right| dx = -\int_{e^{-1}}^{1} \frac{\log x}{x} dx + \int_1^{e^2} \frac{\log x}{x} dx$

(since $\log x < 0$ for $x \in [e^{-1}, 1]$ and $\log x > 0$ for $x \in (1, e^2)$)

$$= -\int_{-1}^{0} t\,dt + -\int_0^2 t\,dt = -\frac{t^2}{2}\Bigg|_{-1}^{0} + \frac{t^2}{2}\Bigg|_0^2 = \frac{1}{2} + 2 = \frac{5}{2}$$

Example 48: If $f(x) = \begin{cases} e^{\cos x}\sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise,} \end{cases}$ then $\int_{-2}^{3} f(x)\,dx =$

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (c)

Solution: $\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_2^3 f(x)\,dx$

$$= \int_{-2}^{2} e^{\cos x}\sin x\,dx + \int_2^3 2\,dx$$

$$= 0 + 2 = 2 \left(\text{since } \int_{-a}^{a} g(x)dx = 0 \text{ if } g(-x) = -g(x)\right)$$

Example 49: $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to

(a) 2 (b) – 2
(c) 1/2 (d) – 1/2

Ans. (a)

Solution: $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x} = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(\pi - x)}$ (Property 10)

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x}$$

$\therefore$ $$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\right) dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{2}{1-\cos^2 x} dx$$

$$= -2\cot x\Big|_{\pi/4}^{3\pi/4} = 4. \text{ Hence } I = 2$$

Example 50: The value of $\int_0^{\pi^2/4} \sin\sqrt{x}\,dx$ is

(a) 0 (b) 1
(c) 2 (d) none of these

Ans. (c)

Solution: Putting $\sqrt{x} = t$ so that $dx = 2t\,dt$

$$\int_0^{\pi^2/4} \sin\sqrt{x}\,dx = 2\int_0^{\pi/2} t\sin t\,dt$$

$$= 2\left[-t\cos t]_0^{\pi/2} + \int_0^{\pi/2} \cos t\,dt\right]$$

$$= 2[0 + \sin t|_0^{\pi/2}] = 2$$

Example 51: The value of the integral $\int_0^{100\pi} \sqrt{1-\cos 2x}\,dx$ is

(a) $100\sqrt{2}$ (b) $200\sqrt{2}$
(c) 0 (d) 100π

Ans. (b)

Solution: We have $\sqrt{1-\cos 2x} = \sqrt{2}\,|\sin x|$. Since the period of $|\sin x|$ is π, so $\int_0^{100\pi} \sqrt{1-\cos 2x}\,dx =$ $\sqrt{2}\int_0^{100\pi} |\sin x|\,dx = 100\sqrt{2}\int_0^{\pi} \sin x\,dx = 200\sqrt{2}$.

Example 52: Whenever $a < b$, the value of $\int_a^b \frac{|x|}{x} dx$ is

(a) $b - a$ (b) $a - b$
(c) $|b| - |a|$ (d) $|b| + |a|$

Ans. (c)

Solution: If $0 \le a < b$, then $f(x) = \frac{|x|}{x} = 1$, therefore, $\int_a^b f(x)\,dx = b - a$. If $a < b \le 0$ then $f(x) = -1$ and so $\int_a^b f(x)\,dx = a - b$. Finally if $a < 0 < b$ then $\int_a^b f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^b f(x)\,dx$

$$= -(0 - a) + (b - 0) = b - (-a)$$

The above three cases can be represented by

$$\int_a^b \frac{|x|}{x} dx = |b| - |a|.$$

Example 53: Let f be an odd function then $\int_{-1}^{1}(|x|+f(x)\cos x)\,dx$ is equal to

(a) 0 (b) 1
(c) 2 (d) none of these

Ans. (b)

Solution: The function $g(x)=|x|$ is an even function and $h(x)=f(x)\cos x$ is an odd function so $\int_{-1}^{1}(|x|+f(x)\cos x)\,dx$ $=2\int_0^1|x|\,dx=2\int_0^1 x\,dx=1.$

Example 54: Let f be a periodic continuous function with period $T>0$. If $I=\int_0^T f(x)dx$ Then the value of $I_1=\int_4^{4+4T}f(3x)dx$ is

(a) I (b) $2I$
(c) $3I$ (d) $4I$

Ans. (d)

Solution: Put $3x=y$ in I_1

$$I_1=\frac{1}{3}\int_{12}^{12+12T}f(y)dy$$

$$=\frac{1}{3}\left[\int_{12}^{T}f(y)dy+\sum_{K=1}^{11}\int_{KT}^{(K+1)T}f(y)dy+\int_{12T}^{12+12T}f(y)dy\right]$$

But $\int_{KT}^{(K+1)T}f(y)dy=\int_0^T f(u)du$,

(Put $KT+u=y$ or use Property 17)

and $\int_{12T}^{12+12T}f(y)dy=\int_0^{12}f(u)du$

Hence $I_1=\frac{1}{3}\left[\int_{12}^{T}f(y)dy+11I+\int_0^{12}f(u)du\right]$

$$=\frac{1}{3}\left[\int_0^T f(y)dy+11I\right]=\frac{1}{3}\times 12I=4I.$$

Example 55: Let $f(x)=\int_0^x\sqrt{6-u^2}\,du$. Then the real roots of the equation $x^2-f'(x)=0$ are

(a) $x=\pm\sqrt{6}$ (b) $x=\pm\sqrt{3}$
(c) $x=\pm\sqrt{2}$ (d) $x=\pm 1$

Ans. (c)

Solution: Since $f'(x)=\sqrt{6-x^2}$, the equation $x^2-f'(x)=0$ becomes

$$x^2-\sqrt{6-x^2}=0\ \Rightarrow\ x^4+x^2-6=0$$

$$\Rightarrow\ (x^2-2)(x^2+3)=0\ \Rightarrow\ x^2-2=0\ (x^2+3\neq 0)$$

$$\Rightarrow\ x=\pm\sqrt{2}.$$

Example 56: If $f(x)=\int_{x^2}^{x^2+4}e^{-t^2}dt$, then the function $f(x)$ increases in

(a) $(-\infty, 0)$ (b) $(0,\infty)$
(c) $(-1, 2)$ (d) $(-2,\infty)$

Ans. (a)

Solution: $f'(x)=2xe^{-(x^2+4)^2}-2xe^{-x^4}$

$$=2xe^{-(x^2+4)^2}\left[1-e^{16+8x^2}\right]$$

Since $1-e^{16+8x^2}<0$ for all x so f increases for $x<0$.

Example 57: If $\int_{\sin x}^{1}t^2 f(t)dt=1-\sin x$, then $f\left(1/\sqrt{3}\right)$ is equal to

(a) $1/3$ (b) $1/\sqrt{3}$
(c) $\sqrt{3}$ (d) 3

Ans. (d)

Solution: Differentiating both the sides, we obtain

$-\sin^2 x\, f(\sin x)\cos x=-\cos x$

$\Rightarrow\ f(\sin x)=\frac{1}{\sin^2 x}$, if $\cos x\neq 0$

If $\sin x=1/\sqrt{3}$ then $\cos x\neq 0$ so $f\left(1/\sqrt{3}\right)=3.$

Example 58: The integral $I=\int_{-3/2}^{3/2}\left([x]+x^3+\log_a\left(x+\sqrt{x^2+1}\right)\right)dx$ is equal

(a) 0 (b) $-3/2$
(c) 1 (d) $3/2$

Ans. (b)

Solution: Let $f(x)=x^3+\log_a\left(x+\sqrt{x^2+1}\right)$

$$f(-x)=-x^3+\log_a\left(-x+\sqrt{x^2+1}\right)$$

$$=-x^3+\log_a\frac{1}{x+\sqrt{x^2+1}}$$

$$=-x^3+\log_a\left(x+\sqrt{x^2+1}\right)=-f(x)$$

$\therefore\ \int_{-3/2}^{3/2}f(x)dx=0$

Thus $I=\int_{-3/2}^{3/2}[x]dx$

$$=\int_{-3/2}^{-1}(-2)dx+\int_{-1}^{0}-1\,dx+\int_0^1 0\,dx+\int_1^{3/2}1\,dx$$

$=(-2)[-1+3/2]+(-1)[0+1]+0+1[3/2-1]$
$=-1-1+1/2=-3/2.$

Example 59: $\int_0^{\pi/4} \log\left(1+\tan^2\theta+2\tan\theta\right)d\theta =$

(a) $\pi \log 2$ (b) $(\pi \log 2)/2$
(c) $(\pi \log 2)/4$ (d) $\log 2$

Ans. (c)

Solution: Let $I = \int_0^{\pi/4} \log(1+\tan\theta)d\theta$

$$= \int_0^{\pi/4} \log(1+\tan(\pi/4-\theta))d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} [\log 2 - \log(1+\tan\theta)]d\theta$$

$\Rightarrow$ $2I = \pi/4 \log 2.$

Required integral $= 2I = \frac{\pi}{4}\log 2$

Example 60: $\int_a^b \sqrt{(x-a)(b-x)}\,dx\,(b>a)$ is equal to

(a) $\pi(b-a)^2/8$ (b) $\pi(b+a)^2/8$
(c) $\pi(b-a)^2$ (d) $\pi(b+a)^2$

Ans (a)

Solution: Let $t = \frac{x-a+x-b}{2} = x - \frac{1}{2}(a+b)$

and $\alpha = \frac{b-a}{2}$, so that

$$\int_a^b \sqrt{(x-a)(b-x)}dx = \int_\alpha^\alpha \sqrt{(t+\alpha)(\alpha-t)}dt$$

$$= 2\int_0^\alpha \sqrt{\alpha^2-t^2}\,dt$$

$$= t\sqrt{\alpha^2-t^2}+\alpha^2\sin^{-1}\left(\frac{t}{\alpha}\right)\Bigg]_0^\alpha$$

$$= \frac{\pi\alpha^2}{2} = \frac{\pi(b-a)^2}{8}.$$

Example 61: $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right) =$

(a) $\log 2$ (b) $\log 3$
(c) $\log 5$ (d) 0

Ans. (b)

Solution: $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{n+2n}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{2n}\frac{1}{n+k} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n}\frac{1}{1+k/n}$$

$$= \int_0^2 \frac{dx}{1+x} = \log(1+x)\Big|_0^2 = \log 3.$$

Example 62: Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}dx$.

Then which one of the following is true

(a) $I > \frac{2}{3}$ and $J > 2$ (b) $I < \frac{2}{3}$ and $J < 2$

(c) $I < \frac{2}{3}$ and $J > 2$ (d) $I > \frac{2}{3}$ and $J < 2$

Ans. (b)

Solution: Since $\sin x < x$ for $x > 0$, we get

$$\frac{\sin x}{\sqrt{x}} < \sqrt{x} \quad \text{for} \quad 0<x<1$$

$\therefore$ $I = \int_0^1 \frac{\sin x}{\sqrt{x}}dx < \int_0^1 \sqrt{x}\,dx = \frac{2}{3}$

$$J = \int_0^1 \frac{\cos x}{\sqrt{x}}dx < \int_0^1 \frac{1}{\sqrt{x}}dx = 2$$

Example 63: $I = \int_0^\pi [\cot x]dx$, where $[\cdot]$ denotes the greatest integer function, is equal to

(a) -1 (b) $-\pi/2$
(c) $\pi/2$ (d) 1

Ans. (b)

Solution: $I = \int_0^\pi [\cot x]dx = \int_0^{\pi/2} ([\cot x]+[\cot(\pi-x)])dx$

$$= \int_0^{\pi/2} ([\cot x]+[-\cot x])dx$$

Put $\cot x = t$, so that

$$I = \int_0^\infty [t]+[-t]\frac{dt}{1+t^2}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\int_{k-1}^{k}([t]+[-t])\frac{dt}{1+t^2}$$

But $[t] + [-t] = -1$ for $k-1 < t < k$, therefore

$$\int_{k-1}^{k}([t]+[-t])\frac{dt}{1+t^2}$$

$$= \int_{k-1}^{k}(-1)\frac{dt}{1+t^2} = \left[\tan^{-1}k - \tan^{-1}(k-1)\right]$$

$\therefore$ $I = -\lim_{n\to\infty}\sum_{k=1}^{n}\left(\tan^{-1}k-\tan^{-1}(k-1)\right)$

$$= -\lim_{n\to\infty}\left[\tan^{-1} n - \tan^{-1} 0\right] = -\frac{\pi}{2}.$$

Example 64: Let $p(x)$ be a function defined on **R** such that $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$. Then $\int_0^1 p(x)dx$ equals

(a) 41 (b) 42

(c) $\sqrt{41}$ (d) 21

Ans. (d)

Solution:

$$\int_0^1 p(x)dx = \int_0^1 1 \,.\, p(x)dx = [xp(x)]_0^1 - I_1 = p(1) - I_1$$

where

$$I_1 = \int_0^1 xp'(x)dx = \int_0^1 (1-x)p'(1-x)dx$$

$$= \int_0^1 (1-x)p'(x)dx = \int_0^1 p'(x)dx - I_1$$

$$\Rightarrow \quad 2I_1 = [p(x)]_0^1 = p(1) - p(0)$$

Thus,

$$\int_0^1 p(x)dx = p(1) - \frac{1}{2}(p(1) - p(0))$$

$$= \frac{1}{2}(p(1) + p(0)) = \frac{1}{2}(41+1) = 21$$

Example 65: The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is

(a) log 2 (b) $\pi \log^2$

(c) $\frac{\pi}{8}\log 2$ (d) $\frac{\pi}{2}\log 2$

Ans. (b)

Solution: Let $I = 8\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

$$= 8\int_0^{\pi/4} \log\frac{(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta \; (x = \tan\theta)$$

So $\quad I = 8\int_0^{\pi} \log(1+\tan\theta)d\theta$

$$= 8\int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$$

$$\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$= 8\int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= 8\int_0^{\pi/4} \log\frac{2}{1+\tan\theta}d\theta = (8\log 2)\frac{\pi}{4} - I$$

$$\Rightarrow \quad 2I = (8\log 2)\frac{\pi}{4} \quad \Rightarrow \quad I = \pi\log 2$$

Example 66: For $x \in (0, 5\pi/2)$, define $f(x) = \int_0^x \sqrt{t}\sin t\,dt$ Then f has

(a) local maximum at π and local minimum at 2π.

(b) local maximum at π and 2π

(c) local minimum at π and 2π

(d) local minimum at π and local maximum at 2π

Ans. (a)

Solution: $f'(x) = \sqrt{x}\sin x$, so $f'(x) = 0$ $\Rightarrow x = \pi, 2\pi$ but $f''(x) = \sqrt{x}\cos x - \frac{1}{2\sqrt{x}}\sin x$, so $f''(\pi) = -\sqrt{\pi} < 0$ and $f''(2\pi) = \sqrt{2\pi} > 0$. Hence f has local maximum at $x = \pi$ and local minimum at $x = 2\pi$.

Example 67: Let [.] denote the greatest integer function then the value of $\int_0^{1\cdot 5} x\left[x^2\right]dx$ is

(a) 0 (b) 3/2

(c) 3/4 (d) 5/4

Ans. (c)

Solution: Put $x^2 = t$, so that $x\,dx = \frac{1}{2}dt$

$$I = \frac{1}{2}\int_0^{2\cdot 25}[t]dt = \frac{1}{2}\int_0^1 [t]dt + \int_1^2 [t]dt + \int_1^{2\cdot 25}[t]dt$$

$$= \frac{1}{2}\left[0 + \int_1^2 dt + 2\int_2^{2\cdot 25} dt\right]$$

$$= \frac{1}{2}[1 + 2\times .25] = \frac{3}{4}$$

Example 68: If $g(x) = \int_0^x \cos 4t\,dt$, , then $g(x+\pi)$ equals

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$

(c) $g(x)\,g(\pi)$ (d) $g(x)\,g(\pi)$

Ans. (a), (b)

Solution: $g(x+\pi) = \int_0^{x+\pi} \cos 4t\,dt$

$$= \int_0^{\pi} \cos 4t\,dt + \int_{\pi}^{x+\pi} \cos 4t\,dt$$

$$= g(\pi) + I$$

where $\quad I = \int_{\pi}^{x+\pi} \cos 4t\,dt$, Put $t = \pi + \theta$, so that

$$I = \int_0^x \cos(4\pi + 4\theta)d\theta = \int_0^x \cos 4\theta d\theta = g(x)$$

So $\quad g(x+\pi) = g(\pi) + g(x)$ but

$$g(\pi) = \int_0^{\pi} \cos 4t\,dt = \frac{1}{4}(\sin 4t\Big|_0^{\pi} = 0.$$

$\therefore \quad g(x+\pi) = g(x) - g(\pi)$ also.

Example 69: Let $f : [-1, 2] \to [0, \infty]$ be a continuous function such that $f(x) = f(1 - x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} x\, f(x)dx$, , and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$ and $x = 2$ and the x-axis. Then

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$

(c) $2R_1 = R_2$ (d) $3R_1 = R_2$

Ans. (c)

Solution:

$$R_1 = \int_{-1}^{2} x\, f(x)dx = \int_{-1}^{2} (2+(-1)-x)\, f(2+(-1)-x)dx$$

$$= \int_{-1}^{2} (1-x)\, f(1-x)dx = \int_{-1}^{2} (1-x)\, f(x)dx$$

$$\Rightarrow 2R_1 = \int_{-1}^{2} f(x)dx = R_2.$$

Example 70: The value of

$$I = \int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin\left(\log 6 - x^2\right)} dx \text{ is}$$

(a) $\frac{1}{4}\log\frac{3}{2}$ (b) $\frac{1}{2}\log\frac{3}{2}$

(c) $\log\frac{3}{2}$ (d) $\frac{1}{6}\log\frac{3}{2}$

Ans. (a)

Solution: Put $x^2 = t$, so that

$$I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t}{\sin t + \sin(\log 6 - t)} dt$$

$$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 2 + \log 3 - t)}{\sin(\log 2 + \log 3 - t) + \sin t} dt$$

$$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin t + \sin(\log 6 - t)} dt$$

$$2I = I + I$$

$$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t + \sin(\log 6 - t)}{\sin t + \sin(\log 6 - t)} dt$$

$$= \frac{1}{2}[\log 3 - \log 2]$$

$$\Rightarrow \quad I = \frac{1}{4}\log\frac{3}{2}.$$

Example 71: Let f be a continuous function satisfying $\int_{-\pi}^{t^2} (f(x) + x^2)dx = \pi + \frac{4}{3}t^3$ for all t, then $f(\pi^2/4)$ is equal to

(a) $\pi - \frac{\pi^4}{8}$ (b) $\frac{\pi}{2} - \left(\frac{\pi}{4}\right)^4$

(c) $\frac{\pi}{2} - \left(\frac{\pi}{4}\right)^2$ (d) $\pi - \frac{\pi^4}{16}$

Ans. (d)

Solution: Differentiating both sides, we have

$$(f(t^2) + t^4)\, 2t = 4t^2$$

$$f(t^2) + t^4 = 2t$$

$$\Rightarrow \quad f(t^2) = 2t - t^4$$

$$\text{so} \quad f\left(\frac{\pi^2}{4}\right) = f\left(\left(\frac{\pi}{2}\right)^2\right) = 2\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^4$$

$$= \pi - \frac{\pi^4}{16}.$$

Example 72: For a continuous function f, the value $\int_0^{\infty} f(x^n + x^{-n})\frac{\log x}{x} + \frac{1}{1+x^2} dx$ is

(a) $\frac{\pi}{2}$ (b) 0

(c) $-\pi$ (d) 2π

Ans. (a)

Solution: Putting $\frac{1}{x} = t \Rightarrow dx = -\frac{1}{t^2} dt$, so

$$I = \int_0^{\infty} f(x^n + x^{-n}) \log x \frac{dx}{x}$$

$$= -\int_{\infty}^{0} f(t^{-n} + t^n)\left(\log\frac{1}{t}\right) t \frac{dt}{t^2}$$

$$= -\int_0^{\infty} f(t^{-n} + t^n) \log t \frac{dt}{t} = -I$$

$$\Rightarrow \quad 2I = 0 \quad \Rightarrow \quad I = 0$$

The given integral is equal to

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \tan^{-1} x\Big|_0^{\infty} = \frac{\pi}{2}$$

Assertion-Reason Type Questions

Example 73: If $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ then

Statement-1: $\int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx = \sqrt{\pi}$

Statement-2: $\lim_{x\to\infty} e^{-x^2} = 0$

Ans. (b)

Solution: Put $x = t^2$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx = 2\int_0^\infty \frac{e^{-t^2}}{t} t\, dt = 2\int_0^\infty e^{-t^2} dt$$

Example 74: If $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ then

Statement-1: $\int_0^\infty \frac{\sin ax \cos bx}{x} dx = \pi/2 \ (a > b > 0)$

Statement-2: $\lim_{x\to 0} \frac{\sin ax \cos bx}{x} = a$

Ans. (b)

Solution: $\sin ax \cos bx = \frac{1}{2}\,[\sin(a+b)x + \sin(a-b)x]$

$$\int_0^\infty \frac{\sin ax \cos bx}{x} dx$$

$$= \frac{1}{2}\left[\int_0^\infty \frac{\sin(a+b)x}{x} dx + \int_0^\infty \frac{\sin(a-b)x}{x} dx\right]$$

$$= \frac{1}{2}\left[\int_0^\infty \frac{\sin t}{t} dt + \int_0^\infty \frac{\sin u}{u} du\right]$$

$$= \pi/2$$

Example 75: **Statement-1:** $\int_{-\pi/3}^{\pi/3} x^{10} \sin^9 x\, dx = 0$

Statement-2: $f(x) = x^{2n}$ is an even function and $g(x) = \sin^{2m+1}x$ is an odd function, m and n are integers.

Ans. (a)

Solution: $x^{10}\sin^9 x$ is an odd function so

$$\int_{-a}^{a} x^{10}\sin^9 x\, dx = 0.$$

Example 76: Suppose that f is an odd function and $F(x) = \int_a^x f(t)\, dt$.

Statement-1: F is an even function

Statement-2: $\int_{-a}^{a} f(t)\, dt = 0$

Ans. (a)

Solution: $F(x) = \int_a^x f(t)\, dt = -\int_{-a}^{-x} f(u)\, du \quad (u = -t)$

$$= \int_{-a}^{-x} f(u)\, du = \int_{-a}^{a} f(u)\, du + \int_a^{-x} f(u)\, du$$

$$= \int_a^{-x} f(u)\, du = F(-x)$$

Example 77: Let $I = \int_{10}^{18} \frac{\cos x}{\sqrt{1+x^4}} dx$

Statement-1: $|I| < 0.1$

Statement-2: $\left|\frac{\cos x}{\sqrt{1+x^4}}\right| < 0.1$

Ans. (b)

Solution: Since $\cos x \le 1$, the condition $x > 10$ yields

$$\left|\frac{\cos x}{\sqrt{1+x^4}}\right| < 10^{-2} < 10^{-1}$$

$$|I| < 8.10^{-2} < 10^{-1}.$$

Example 78: **Statement-1:** $\int_0^{2\pi} \cos^{2m+1} x\ dx = 0\ (m > 0)$

Statement-2: $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ if $f(2a-x) = f(x)$

Ans. (a)

Solution: $\int_0^{2\pi} \cos^{2m+1} x\, dx = 2\int_0^{\pi} \cos^{2m+1} x\, dx$

$= 0$ (since $\cos(\pi - x) = -\cos x$)

Example 79:

Statement-1: $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2$

Statement-2: $\int_0^{\pi/2} \log \sin\theta\, d\theta = -\pi \log 2$

Ans. (c)

Solution:

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$2I = \frac{\pi}{4}\log 2 \quad\Rightarrow\quad I = \frac{\pi}{8}\log 2$$

$$I_1 = \int_0^{\pi/2} \log\sin\theta\, d\theta = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$= \int_0^{\pi/2} \log\cos\theta\ d\theta$$

$$2I_1 = \int_0^{\pi/2} \log(\sin\theta\cos\theta)\ d\theta = \int_0^{\pi/2} \log\left(\log\frac{\sin 2\theta}{2}\right)d\theta$$

$$= \int_0^{\pi/2} \log\sin 2\theta\ d\theta - \frac{\pi}{2}\log 2 = I_2 - \frac{\pi}{2}\log 2$$

but $I_2 = \int_0^{\pi/2} \log\sin 2\theta\ d\theta = \frac{1}{2}\int_0^{\pi} \log\sin t\, dt$

$$= \frac{1}{2}.2\int_0^{\pi/2} \log\sin t\, dt = I_1$$

Hence, $I_1 = -\frac{\pi}{2}\log 2$.

LEVEL 2

Straight Objective Type Questions

Example 80: Let f: $(0, \infty) \to \mathbf{R}$ and $F(x) = \int_0^x f(t)\ dt$.

If $F(x^2) = x^2(1 + x)$ then $f(4)$ equals

(a) 5/4 (b) 7

(c) 4 (d) 2

Ans. (c)

Solution: $F(x^2) = \int_0^{x^2} f(t)\ dt$, therefore, $x^2(1 + x) = \int_0^{x^2} f(t)\ dt$. Differentiating both sides w.r.t. x using Property 17, we have

$$2x + 3x^2 = f(x^2)\cdot 2x \Rightarrow f(x^2) = 1 + (3/2)x$$

Putting $x = 2$, we have $f(4) = 1 + 3 = 4$.

Example 81: If $I_1 = \int_{1/e}^{\tan x} \frac{t}{1+t^2}\ dt$ and $I_2 = \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$

then the value of $I_1 + I_2$ is

(a) 1/2 (b) 1

(c) $e/2$ (d) $(1/2)(e + 1/e)$

Ans. (b)

Solution: Putting $t = 1/u$ in I_2 we have

$$I_2 = -\int_e^{\tan x} \frac{u\ du}{1+u^2} = -\int_{1/e}^{\tan x} \frac{u\ du}{1+u^2} + \int_{1/e}^{e} \frac{u\ du}{1+u^2}$$

$$= -I_1 + \frac{1}{2}\int_{1/e}^{e} \frac{2u\ du}{1+u^2}$$

So $I_1 + I_2 = \frac{1}{2}\ \log(u^2+1)]_{1/e}^{e}$

$$= \frac{1}{2}\left[\log\left(e^2+1\right) - \log\left(\frac{e^2+1}{e^2}\right)\right]$$

$$= \frac{1}{2}\times 2 = 1.$$

Example 82: If $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\ dt$ and $g(x) = \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\ dt$ then the value of $f(x) + g(x)$ is

(a) π (b) $\pi/4$

(c) $\pi/2$ (d) $\sin^2 x + \sin x + x$

Ans. (b)

Solution: $f'(x) + g'(x) = \sin^{-1}(\sin x)\ 2\sin x\cos x - \cos^{-1}(\cos x)\ 2\sin x\cos x$ (Property 17)

$$= x\sin 2x - x\sin 2x = 0$$

for all $x \in \mathbf{R}$. Hence $f(x) + g(x) = \text{constant} = C$ (say)

Putting $x = \pi/4$, $C = \int_0^{1/2} \sin^{-1}\sqrt{t}\ dt + \int_0^{1/2} \cos^{-1}\sqrt{t}\ dt$

$$= \int_0^{1/2} \frac{\pi}{2}\ dt = \frac{\pi}{4}$$

Hence $f(x) + g(x) = \pi/4$.

Example 83: Solution of the equation $\int_{\log 2}^{x} \frac{dx}{\sqrt{e^x - 1}} = \frac{\pi}{6}$ are

(a) $x = \log 6$ (b) $x = 2 \log 2$
(c) $x = 3$ (d) $x = 1/2$

Ans. (b)

Solution: Putting $e^x - 1 = t^2$, we have $e^x\, dx = 2t\, dt$

so $$\int_{\log 2}^{x} \frac{dx}{\sqrt{e^x - 1}} = \int_{1}^{\sqrt{e^x-1}} \frac{2t\, dt}{t(t^2+1)} = 2\int_{1}^{\sqrt{e^x-1}} \frac{dt}{t^2+1}$$

$$= 2\left(\tan^{-1}\sqrt{e^x - 1} - \frac{\pi}{4}\right)$$

Thus the given equation reduces to

$$\tan^{-1}\sqrt{e^x - 1} - \frac{\pi}{4} = \frac{\pi}{12} \Rightarrow \sqrt{e^x - 1} = \tan\frac{\pi}{3} = \sqrt{3}$$

so $$e^x = 4 \Rightarrow x = 2 \log 2$$

Example 84: The value of $\lim_{m\to\infty} \frac{\int_0^{\pi/2} \sin^{2m} x\, dx}{\int_0^{\pi/2} \sin^{2m+1} x\, dx}$

(a) 0 (b) 1/2
(c) 2 (d) 1

Ans. (d)

Solution: We know that $I_{2n} = \int_0^{\pi/2} \sin^{2n} x\, dx$

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \ldots \times \frac{1}{2} \times \frac{\pi}{2},$$

$$I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} x\, dx$$

$$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \ldots \times \frac{2}{3} \text{ and}$$

Also, $$I_{2m+1} = \frac{2m}{2m+1} I_{2m-1}.$$

For all $x \in (0, \pi/2)$, $\sin^{2m-1} x > \sin^{2m} x > \sin^{2m+1} x$
Integrating from 0 to $\pi/2$, we get $I_{2m-1} \geq I_{2m} \geq I_{2m+1}$

hence $$\frac{I_{2m-1}}{I_{2m+1}} \geq \frac{I_{2m}}{I_{2m+1}} \geq 1 \quad \text{(i)}$$

Also $$\frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}.$$

Hence $$\lim_{m\to\infty} \frac{I_{2m-1}}{I_{2m+1}} = \lim_{m\to\infty} \frac{2m+1}{2m} = 1.$$

From (i) and using sandwich theorem we have

$$\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1.$$

Example 85: Let f be a positive function and

$$I_1 = \int_{1-k}^{k} x f(x(1-x))\, dx,\ I_2 = \int_{1-k}^{k} f(x(1-x))\, dx$$

where $2k - 1 > 0$ then I_1/I_2 is

(a) 2 (b) k
(c) 1/2 (d) 1

Ans. (c)

Solution: Since $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$, we have

$$I_1 = \int_{1-k}^{k} (k+1-k-x) f((k+1-k-x)(1-(k+1-k-x))\, dx.$$

$$= \int_{1-k}^{k} (1-x) f((1-x)x)\, dx$$

$$= \int_{1-k}^{k} f((1-x)x)\, dx - \int_{1-k}^{k} x f((1-x)x)\, dx$$

$$= I_2 - I_1.$$

So $2I_1 = I_2$ and $I_1/I_2 = 1/2$.

Example 86: If the equality

$$\int_0^x \frac{bt\cos 4t - a\sin 4t}{t^2}\, dt = \frac{a\sin 4x}{x} - 1$$

holds for all x such that $0 < x < \pi/4$ then a and b are given by

(a) $a = 1/4, b = 1$ (b) $a = 2, b = 2$
(c) $a = -1, b = 4$ (d) $a = 2, b = 4$

Ans. (a)

Solution: $$\int_0^x \frac{bt\cos 4t - a\sin 4t}{t^2}\, dt$$

$$= b\int_0^x \frac{\cos 4t}{t}\, dt - a\int_0^x \frac{\sin 4t}{t^2}\, dt$$

$$= b\int_0^x \frac{\cos 4t}{t}\, dt - a\left[-\frac{\sin 4t}{t}\Big|_0^x + 4\int_0^x \frac{\cos 4t}{t}\, dt\right]$$

$$= (b - 4a)\int_0^x \frac{\cos 4t}{t}\, dt + \frac{a\sin 4x}{x} - 4a$$

Thus $$(b-4a)\int_0^x \frac{\cos 4t}{t}\, dt + \frac{a\sin 4x}{x} - 4a$$

$$= \frac{a\sin 4x}{x} - 1$$

i.e. $$(b-4a)\int_0^x \frac{\cos 4t}{t}\, dt = 4a - 1.$$

Since R.H.S. is independent of x, so we must have $b - 4a = 0$ and $4a - 1 = 0$ i.e., $a = 1/4$, $b = 1$.

Example 87: The value of $\int_0^{\pi/2} \frac{1 + 2\cos x}{(2+\cos x)^2}\, dx$ is

(a) −1/2 (b) 2
(c) 1/2 (d) none of these

Ans. (c)

Solution: $\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}dx = \int_0^{\pi/2} \frac{2(\cos x+2)-3}{(2+\cos x)^2}dx$

$= 2\int_0^{\pi/2} \frac{dx}{(2+\cos x)} - 3\int_0^{\pi/2} \frac{dx}{(2+\cos x)^2}$

$= 4\int_0^1 \frac{dt}{3+t^2} - 6\int_0^1 \frac{1+t^2}{(3+t^2)^2}dt \qquad (t = \tan x/2)$

$= -2\int_0^1 \frac{dt}{3+t^2} + 12\int_0^1 \frac{dt}{(3+t^2)^2}$

$= -2\int_0^1 \frac{dt}{3+t^2} + 12\left[\frac{1}{2.3}\frac{t}{(t^2+3)}\Big|_0^1 + \frac{1}{2.3}\int_0^1 \frac{dt}{3+t^2}\right]$

$= \frac{1}{2}$.

Example 88: The value of $\int_{-1}^{3} \{|x-2| + [x]\}\, dx$, where $[x]$ denotes the greatest integer less than or equal to x is

(a) 5 (b) 7
(c) 4 (d) 3

Ans. (b)

Solution: $\int_{-1}^{3} \{|x-2|+[x]\}dx = \int_{-1}^{0} \{|x-2|+[x]\}dx + \int_0^1 \{|x-2|+[x]\}dx + \int_1^2 \{|x-2|+[x]\}dx + \int_2^3 \{|x-2|+[x]\}dx$

$= \int_{-1}^{0} (2-x-1)\,dx + \int_0^1 (2-x+0)dx + \int_1^2 (2-x+1)dx + \int_2^3 (x-2+2)\,dx$

$= x - \frac{x^2}{2}\Big|_{-1}^{0} + 2x - \frac{x^2}{2}\Big|_0^1 + 3x - \frac{x^2}{2}\Big|_1^2 + \frac{x^2}{2}\Big|_2^3$

$= -\left(-1-\frac{1}{2}\right) + \left(2-\frac{1}{2}\right) + (6-2) - \left(3-\frac{1}{2}\right) + \frac{9}{2} - 2 = 7.$

Example 89: If $f(x) = \begin{vmatrix} \sin x+\sin 2x+\sin 3x & \sin 2x & \sin 3x \\ 3+4\sin x & 3 & 4\sin x \\ 1+\sin x & \sin x & 1 \end{vmatrix}$

then the value of $\int_0^{\pi/2} f(x)\,dx$ is

(a) 3 (b) 0
(c) 2/3 (d) 1/3

Ans. (d)

Solution: By applying the operation $C_1 \to C_1 - C_2 - C_3$, $f(x)$ can be written as

$$f(x) = \begin{vmatrix} \sin x & \sin 2x & \sin 3x \\ 0 & 3 & 4\sin x \\ 0 & \sin x & 1 \end{vmatrix}$$

$= \sin x\,(3 - 4\sin^2 x) = \sin 3x$

So $\int_0^{\pi/2} f(x)\,dx = -\frac{\cos 3x}{3}\Big|_0^{\pi/2} = \frac{1}{3}$.

Example 90: If a is a positive integer, then the number of values of a satisfying

$$\int_0^{\pi/2} \left\{a^2\left(\frac{\cos 3x}{4} + \frac{3}{4}\cos x\right) + a\sin x - 20\cos x\right\} dx \le \frac{a^2}{3}$$ are

(a) one (b) two
(c) three (d) four

Ans. (d)

Solution: The L.H.S. of the above inequality is equal to $a^2\left(\frac{\sin 3x}{12} + \frac{3}{4}\sin x\right) - a\cos x - 20\sin x\Big|_0^{\pi/2} =$

$a^2\left(-\frac{1}{12}+\frac{3}{4}\right) - a\,(0-1) - 20 = \frac{2a^2}{3} + a - 20.$

Thus the given inequality is $(2a^2/3) + a - 20 \le -a^2/3$

i.e., $a^2 + a - 20 \le 0 \Leftrightarrow -5 \le a \le 4$

Since a is a positive integer so $a = 1, 2, 3, 4$.

Example 91: The area bounded by the curve $y = f(x) = x^4 - 2x^3 + x^2 + 3$, x-axis and ordinates corresponding to minimum of the function $f(x)$ is

(a) 1 (b) 91/30
(c) 30/9 (d) 4

Ans. (b)

Solution: $f'(x) = 4x^3 - 6x^2 + 2x = 2x(2x^2 - 3x + 1) = 2x(2x-1)(x-1)$.

Since f is a differentiable function, so extremum points of $f(x)$, we must have $f'(x) = 0$ so $x = 0, 1/2, 1$. Now $f''(x) = 12x^2 - 12x + 2$, $f''(0) = 2$, $f''(1) = 2$ and $f''(1/2) = 3 - 6 + 2 = -1$. Thus the function has minimum at $x = 0$ and $x = 1$. Therefore, the required area $= \int_0^1 (x^4 - 2x^3 + x^2 + 3)\,dx$ (See Chapter 14 for area as definite integral)

$$= \left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} + 3x\right)\Big|_0^1 = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} + 3 = \frac{91}{30}.$$

Example 92: The value of $\int_0^{\pi} \sin^n x \cos^{2m+1} x\, dx$ is

(a) $\dfrac{(2m+1)!}{(n!)^2}$ (b) $\dfrac{(2m+1)!}{n!}$

(c) $\int_0^{\pi} \cos^{2m-1} x\, dx$ (d) none of these

Ans. (c)

Solution: $I = \int_0^{\pi} \sin^n x \cos^{2m+1} x\, dx$

$= \int_0^{\pi} \sin^n (\pi - x) \cos^{2m+1} (\pi - x)\, dx$

$= -\int_0^{\pi} \sin^n x \cos^{2m+1} x\, dx = -I.$

So $2I = 0$ which implies that $I = 0$ Also, $\int_0^{\pi} \cos^{2m-1} x\, dx = 0$ by a similar reasoning.

Example 93: The value of the integral

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \frac{2x}{1+x^2}\, dx \text{ is}$$

(a) 0 (b) $\dfrac{\pi}{\sqrt{3}} + \log \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

(c) $\dfrac{1}{\sqrt{3}} + \dfrac{\pi}{2} \log \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ (d) none of these

Ans. (d)

Solution: $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \frac{2x}{1+x^2}\, dx$

$$= \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \left(\frac{\pi}{2} - \sin^{-1} \frac{2x}{1+x^2}\right) dx$$

$$= \frac{\pi}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\, dx$$

($\sin^{-1}(-x) = -\sin^{-1} x$, so the last integral is zero)

$$= \pi \int_0^{1/\sqrt{3}} \left(-1 + \frac{1}{1-x^4}\right) dx$$

$$= \pi \int_0^{1/\sqrt{3}} \left(-1 + \frac{1}{2}\left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right)\right) dx$$

$$= \left[-\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\left[\frac{1}{2} \log \frac{1+x}{1-x} + \tan^{-1} x\right]\Bigg|_0^{1/\sqrt{3}}\right]$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\left[\frac{1}{2} \log \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\pi}{6}\right].$$

Example 94: The value of $\int_0^{\pi/2} \sqrt{\sin 2\theta}\, \sin \theta\, d\theta$ is

(a) 1 (b) 0
(c) $\pi/2$ (d) $\pi/4$

Ans. (d)

Solution: $I = \int_0^{\pi/2} \sqrt{\sin 2\theta}\, \sin \theta\, d\theta$

$$= \int_0^{\pi/2} \sqrt{\sin (\pi - 2\theta)}\, \sin (\pi/2 - \theta) d\theta$$

$$= \int_0^{\pi/2} \sqrt{\sin 2\theta}\, \cos \theta\, d\theta$$

So $$2I = \int_0^{\pi/2} \sqrt{\sin 2\theta}\, (\sin \theta + \cos \theta)\, d\theta$$

In $2I$, put $\sin \theta - \cos \theta = t$, so that $t^2 = 1 - \sin 2\theta \Rightarrow 1 - t^2 = \sin 2\theta$. Also $dt = (\cos \theta + \sin \theta)\, d\theta$.

$$\therefore \quad 2I = \int_{-1}^{1} \sqrt{1-t^2}\, dt = 2\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right]_0^1$$

$$= 2\frac{\pi}{4} = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}.$$

Example 95: Given $I_m = \int_1^e (\log x)^m\, dx$. If $\dfrac{I_m}{K} + \dfrac{I_{m-2}}{L} = e$ then values of K and L are

(a) $1-m, \dfrac{1}{m}$ (b) $\dfrac{1}{1-m}, m$

(c) $\dfrac{1}{1-m}, \dfrac{m(m-2)}{m-1}$ (d) $\dfrac{m}{m-1}, m-2$

Ans. (a)

Solution: $I_m = \int_1^e (\log x)^m\, dx = x (\log x)^m \Big|_1^e - m \int_1^e (\log x)^{m-1}\, dx$

$$= e - m\left[x (\log x)^{m-1}\Big|_1^e - (m-1)\int_1^e (\log x)^{m-2}\, dx\right]$$

$$= e - me + m(m-1) I_{m-2} = (1-m)e + m(m-1) I_{m-2}$$

So $\dfrac{I_m}{1-m} + m I_{m-2} = e$. Thus $K = 1 - m$ and $L = \dfrac{1}{m}$.

Example 96: The numbers P, Q and R for which the function $f(x) = Pe^{2x} + Qe^x + Rx$ satisfies $f(0) = -1$, $f'(\log 2) = 31$ and $\int_0^{\log 4} [f(x) - Rx]\, dx = 39/2$ are given by

(a) $P = 2, Q = -3, R = 4$
(b) $P = -5, Q = 2, R = 3$
(c) $P = 5, Q = -2, R = 3$
(d) $P = 5, Q = -6, R = 3$

Ans. (d)

◎ **Solution:** We have $f'(x) = 2Pe^{2x} + Qe^{x} + R$, so that $31 = f'(\log 2) = 8P + 2Q + R$. Also, $-1 = f(0) = P + Q$. Further,

$$\frac{39}{2} = \int_0^{\log 4} [f(x) - Rx]\,dx = \int_0^{\log 4} [Pe^{2x} + Qe^{x}]\,dx$$

$$= \frac{P}{2}e^{2x} + Qe^{x}\Big|_0^{\log 4} = \frac{P}{2} \times 16 + 4Q - \frac{P}{2} - Q$$

$$= \frac{15P}{2} + 3Q$$

Solving the above equations, we get $P = 5$, $Q = -6$ and $R = 3$.

Example 97: For $0 < \alpha < \pi$. The value of the definite integral $\int_0^1 dx/(x^2 + 2x\cos\alpha + 1)$ is equal to

(a) $\alpha/(2\sin\alpha)$ (b) $\tan^{-1}(\sin\alpha)$
(c) $\alpha\sin\alpha$ (d) $(\alpha/2)(\sin\alpha)$

Ans. (a)

◎ **Solution:** $\displaystyle\int_0^1 \frac{dx}{x^2 + 2x\cos\alpha + 1}$

$$= \int_0^1 \frac{dx}{(x+\cos\alpha)^2 + 1 - \cos^2\alpha}$$

$$= \int_0^1 \frac{dx}{(x+\cos\alpha)^2 + \sin^2\alpha}$$

$$= \frac{1}{\sin\alpha}\tan^{-1}\frac{x+\cos\alpha}{\sin\alpha}\Big|_0^1$$

$$= \frac{1}{\sin\alpha}\left(\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\frac{\cos\alpha}{\sin\alpha}\right)$$

$$= \frac{1}{\sin\alpha}\left(\tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}\cot\alpha\right)$$

$$= \frac{1}{\sin\alpha}\left(\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2} + \alpha\right) = \frac{\alpha}{2}\sin\alpha.$$

Example 98: The value of the integral $\displaystyle\int_0^\infty \frac{x\log x}{(1+x^2)^2}\,dx$ is

(a) 7 (b) 0
(c) 5 log 13 (d) 2 log 5

Ans. (b)

◎ **Solution:** $\displaystyle\int_0^\infty \frac{x\log x}{(1+x^2)^2}\,dx$

$$= \int_0^1 \frac{x\log x}{(1+x^2)^2}\,dx + \int_1^\infty \frac{x\log x}{(1+x^2)^2}\,dx$$

Put $x = 1/y$ in the second integral, so that $dx = (-1/y^2)dy$. If $x \to \infty$, then $y \to 0$, and if $x = 1$, then $y = 1$.

$$\therefore \int_1^\infty \frac{x\log x}{(1+x^2)^2}\,dx = \int_1^0 \frac{1\cdot\log y^{-1}}{y\left(1+\frac{1}{y^2}\right)^2}\left(-\frac{1}{y^2}\right)dy$$

$$= -\int_0^1 \frac{y\log y}{(1+y^2)^2}\,dy$$

$$\Rightarrow \int_0^\infty \frac{x\log x}{(1+x^2)^2}\,dx = \int_0^1 \frac{x\log x}{(1+x^2)^2}\,dx - \int_0^1 \frac{x\log x}{(1+x^2)^2}\,dx = 0.$$

Example 99: If I is the greatest of the definite integrals

$$I_1 = \int_0^1 e^{-x}\cos^2 x\,dx, \qquad I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx,$$

$$I_3 = \int_0^1 e^{-x^2}\,dx, \qquad I_4 = \int_0^1 e^{-x^2/2}\,dx$$

then

(a) $I = I_1$ (b) $I = I_2$
(c) $I = I_3$ (d) $I = I_4$

Ans. (d)

◎ **Solution:** For $0 < x < 1$, we have $(1/2)x^2 < x^2 < x$ i.e., $-x^2 > -x$, so that $e^{-x^2} > e^{-x}$. Hence $\int_0^1 e^{-x^2}\cos^2 x\,dx > \int_0^1 e^{-x}\cos^2 x\,dx$. Also $\cos^2 x \le 1$, therefore

$$\int_0^1 e^{-x^2}\cos^2 x\,dx \le \int_0^1 e^{-x^2}\,dx < \int_0^1 e^{-x^2/2}\,dx = I_4.$$

Hence I_4 is the greatest integral.

Example 100: The value of the integral $\displaystyle\int_0^1 \frac{x^c - 1}{\log x}\,dx$, $c > 0$ is

(a) $\log c$ (b) $2\log(c+1)$
(c) $3\log c$ (d) $\log(c+1)$

Ans. (d)

◎ **Solution:** Note that the integral is not an elementary function.

Let $\displaystyle I(c) = \int_0^1 \frac{x^c - 1}{\log x}\,dx$ so

$$I'(c) = \int_0^1 \frac{x^c\log x\,dx}{\log x} = \int_0^1 x^c\,dx = \frac{1}{c+1}.$$

Hence $\displaystyle I(c) = \int \frac{1}{c+1}\,dc + C = \log|c+1| + C.$

But $I(0) = 0$, hence $C = 0$ so $I(c) = \log(c+1)$.

Example 101: The value of the integral

$$\int_{-1}^1 \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx \text{ is}$$

(a) $\pi/2$ (b) $\pi/4$
(c) $-\pi/2$ (d) none of these

Ans. (c)

◎ **Solution:** $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = \frac{-1}{1+x^2}$

$$\Rightarrow \quad I = \int_{-1}^{1}\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx = \int_{-1}^{1}\frac{-dx}{1+x^2} = -\tan^{-1}x\Big|_{-1}^{1}$$

$$= -\frac{\pi}{4}+\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}$$

Note that $I = \tan^{-1}(1/x)|_{-1}^{1} = \frac{\pi}{4}-\left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$ is incorrect, since the function $\tan^{-1}(1/x)$ is not an antiderivative of $(d/dx)[\tan^{-1}(1/x)]$ on the interval $[-1, 1]$.

◉ **Example 102:** The value of $\int_0^{\pi}\frac{\sin(n+1/2)x}{\sin(x/2)}dx\,(n\in \mathbf{N})$ is

(a) π (b) 2π
(c) 3π (d) none of these

Ans. (a)

◎ **Solution:** We have,

$$2\sin\frac{x}{2}\left(\frac{1}{2}+\cos x+\cos 2x+\ldots+\cos nx\right)$$

$$= \sin\frac{x}{2}+2\sin\frac{x}{2}\cos x+2\sin\frac{x}{2}\cos 2x+\ldots+2\sin\frac{x}{2}\cos nx$$

$$= \sin\frac{x}{2}+\sin\frac{3x}{2}-\sin\frac{x}{2}+\sin\frac{5x}{2}-\sin\frac{3x}{2}+\ldots$$

$$+\sin\left(n+\frac{1}{2}\right)x-\sin\left(n-\frac{1}{2}\right)x = \sin\left(n+\frac{1}{2}\right)x$$

$$\therefore \quad \frac{1}{2}+\cos x+\cos 2x+\ldots+\cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin(x/2)}$$

$$\Rightarrow \quad \int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin(x/2)}dx$$

$$= 2\left(\int_0^{\pi}\frac{1}{2}dx+\int_0^{\pi}\cos x\,dx+\ldots+\int_0^{\pi}\cos nx\,dx\right)$$

$$= 2\left(\frac{\pi}{2}+\sin x\Big|_0^{\pi}+\ldots+\frac{\sin nx}{n}\Big|_0^{\pi}\right) = \pi$$

◉ **Example 103:** The equation of the tangent to the curve

$$y = \int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^2}}$$

at $x = 1$ is

(a) $\sqrt{2}\,y-1 = x$ (b) $\sqrt{3}\,x-1 = y$
(c) $\sqrt{3}\,x+1 = y$ (d) $\sqrt{2}y+1 = x$

Ans. (d)

◎ **Solution:** Differentiating the given equation, we get

$$\frac{dy}{dx} = \left[\frac{1}{\sqrt{1+t^2}}\right]_{t=x^3}\cdot\frac{d}{dx}(x^3)-\left[\frac{1}{\sqrt{1+t^2}}\right]_{t=x^2}\cdot\frac{d}{dx}(x^2)$$

(using Property 17)

$$= \frac{3x^2}{\sqrt{1+x^6}}-\frac{2x}{\sqrt{1+x^4}}$$

Therefore the slope of the required tangent is

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{3}{\sqrt{2}}-\frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

This tangent passes through the point $x = 1$ and $y(1) = \int_1^1\frac{dt}{\sqrt{1+t^2}} = 0$, so that its equation is $y-0 = \frac{1}{\sqrt{2}}(x-1)$

$$\Rightarrow \quad \sqrt{2}\,y+1 = x.$$

◉ **Example 104:** The mean value of the function $f(x) = \frac{1}{x^2+x}$ on the interval [1, 3/2] is

(a) log (6/5) (b) 2 log (6/5)
(c) 4 (d) log 3/5

Ans. (b)

◎ **Solution:** Mean value $= \frac{1}{b-a}\int_a^b f(x)\,dx$ (Property 14)

$$= \frac{1}{3/2-1}\int_1^{3/2}\frac{1}{x^2+x}dx = 2\int_1^{3/2}\left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$= 2(\log x-\log(x+1))\Big|_1^{3/2}$$

$$= 2[\log(3/2)-\log(5/2)-(\log 1-\log 2)]$$

$$= 2\log(6/5).$$

◉ **Example 105:** The value of $\int_1^{16}\tan^{-1}\sqrt{\sqrt{x}-1}\,dx$ is

(a) $\frac{16\pi}{3}+2\sqrt{3}$ (b) $\frac{4}{3}\pi-2\sqrt{3}$
(c) $\frac{4}{3}\pi+2\sqrt{3}$ (d) $\frac{16}{3}\pi-2\sqrt{3}$

Ans. (d)

◎ **Solution:** Integrating by parts, the given integral is equal to

$$x\tan^{-1}\sqrt{\sqrt{x}-1}\Big|_1^{16}-\int_1^{16}\frac{x}{\sqrt{x}}\frac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}}dx$$

$= \frac{16}{3}\pi - \frac{1}{4}\int_1^{16} \frac{dx}{\sqrt{\sqrt{x}-1}}$

$= \frac{16}{3}\pi - \frac{1}{4}\int_0^{\sqrt{3}} \frac{4t(1+t^2)}{t} dt \;(\sqrt{x} = 1 + t^2)$

$= \frac{16}{3}\pi - (\sqrt{3}+\sqrt{3}) = \frac{16}{3} - 2\sqrt{3}$.

Example 106: Let $g(x) = \int_0^x f(t)\, dt$, where f is such that $1/2 \le f(t) \le 1$ for $0 \le t \le 1$ and $0 \le f(t) \le 1/2$ for $1 \le t \le 2$. Then $g(2)$ satisfies the inequality

(a) $-3/2 \le g(2) < 1/2$ (b) $0 \le g(2) < 2$
(c) $3/2 < g(2) \le 5/2$ (d) $2 < g(2) < 4$

Ans. (b)

Solution: $g(2) = \int_0^2 f(t)\, dt = \int_0^1 f(t)\, dt + \int_1^2 f(t)\, dt$

Since $1/2 \le f(t) \le 1$ for $t \in [0, 1]$ so $1/2 \le \int_0^1 f(t)\, dt \le 1$ by Property 13. Also $0 \le f(t) \le 1/2$ for $t \in [1, 2]$ so $0 \le \int_1^2 f(t)\, dt \le 1$. Hence $0 + 1/2 \le \int_0^1 f(t)\, dt + \int_1^2 f(t)\, dt \le 1 + 1/2$ i.e. $1/2 \le g(2) \le 3/2 \Rightarrow 0 \le g(2) < 2$.

Example 107: A function f is defined by $f(x) = \frac{1}{2^{r-1}}$, $\frac{1}{2^r} < x \le \frac{1}{2^{r-1}}$, $r = 1, 2, 3, \ldots$ then the value of $\int_0^1 f(x)\, dx$ is equal

(a) 1/3 (b) 1/4
(c) 2/3 (d) 1/3

Ans. (c)

Solution: $\int_0^1 f(x)\, dx = \sum_{r=1}^{\infty} \int_{2^{-r}}^{2^{-(r-1)}} \frac{1}{2^{r-1}} dx$

$= \sum_{r=1}^{\infty} \frac{1}{2^{r-1}} [2^{-(r-1)} - 2^{-r}]$

$= \sum_1^{\infty} 2^{-2(r-1)} - \sum_1^{\infty} 2^{-2r+1}$

$= (2^2 - 2)\sum_1^{\infty} 2^{-2r} = 2 \cdot \frac{1}{4} \cdot \frac{1}{1-1/4} = \frac{2}{3}$

Example 108: If $y = \int_1^{t^3} \sqrt[3]{z} \log z\, dz$ and $x = \int_{\sqrt{t}}^{3} z^2 \log z\, dz$ then $\frac{dy}{dx}$ is

(a) $-4t^{5/2}$ (b) $32t^{5/2}$
(c) $-8t^{3/2}$ (d) $-36t^{5/2}$

Ans. (d)

Solution: As is known, we find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

$\frac{dy}{dt} = t \log t^3 \times 3t^2 = 9t^3 \log t$ (Property 17)

and $\frac{dx}{dt} = 0 - t \log \sqrt{t} \times \frac{1}{2\sqrt{t}} = -\frac{1}{4}\sqrt{t} \log t$.

So $\frac{dy}{dx} = \frac{9t^3 \log t}{-\frac{1}{4}\sqrt{t} \log t} = -36t^2\sqrt{t}$.

Example 109: Let $I_1 = \int_{-4}^{-5} e^{(x-5)^2} dx$ and $I_2 = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$ then the value of $I_1 + I_2$ is

(a) 0 (b) 4/3
(c) 7/4 (d) 5/4

Ans. (a)

Solution: Put $x = -t - 4$ in I_1 so

$I_1 = \int_{-4}^{-5} e^{(x+5)^2} dx = -\int_0^1 e^{(-t+1)^2} dt = -\int_0^1 e^{(t-1)^2} dt$

Applying the substituting $x = \frac{t}{3} + \frac{1}{3}$ in I_2, we have

$I_2 = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx = \int_0^1 e^{(t-1)^2} dt$. Thus $I_1 + I_2 = 0$.

Example 110: A line tangent to the graph of the function $y = f(x)$ at the point $x = a$ forms an angle $\pi/3$ with the axis of abscissas and an angle $\pi/4$ at the point $x = b$ then $\int_a^b f''(x)\, dx$ is (f'' is assumed to be a continuous function)

(a) 0 (b) $1 - \sqrt{3}$
(c) $\sqrt{2} - 1$ (d) $\sqrt{3} - 1$

Ans. (b)

Solution: $\int_a^b f''(x)\, dx = f'(b) - f'(a)$. According to the given conditions $f'(b) = \tan \pi/4 = 1$ and $f'(a) = \tan \pi/3 = \sqrt{3}$. Hence the required integral equals to $1 - \sqrt{3}$.

Example 111: The value of $\int_0^{\pi} [2 \sin x]\, dx$, where [,] represents the greatest integer function is

(a) π (b) 2π
(c) $-\pi$ (d) $2\pi/3$

Ans. (d)

Solution: $\int_0^{\pi} [2\sin x]\, dx$

$= \int_0^{\pi/6} [2\sin x]\, dx + \int_{\pi/6}^{\pi/2} [2\sin x]\, dx + \int_{\pi/2}^{5\pi/6} [2\sin x]\, dx + \int_{5\pi/6}^{\pi} [2\sin x]\, dx$

$= \int_0^{\pi/6} 0\, dx + \int_{\pi/6}^{\pi/2} 1\, dx + \int_{\pi/2}^{5\pi/6} 1\, dx + \int_{5\pi/6}^{\pi} 0\, dx$

$$= \frac{\pi}{2} - \frac{\pi}{6} + \frac{5\pi}{6} - \frac{\pi}{2} = \frac{2\pi}{3}.$$

Example 112: For any $t \in \mathbf{R}$ and f a continuous function, let

$$I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f(x(2-x))\, dx$$

and $$I_2 = \int_{\sin^2 x}^{1+\cos^2 t} f(x(2-x))\, dx$$ then I_1/I_2 is equal to

(a) 2 (b) 1
(c) 4 (d) none of these

Ans. (b)

Solution: $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} (2-x) f((2-x)(2-(2-x)))\, dx$

$$= \int_{\sin^2 t}^{1+\cos^2 t} (2-x) f(x(2-x))\, dx$$

$$= 2\int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\, dx - \int_{\sin^2 t}^{1+\cos^2 t} x f(x(2-x))\, dx = 2I_2 - I_1$$

Therefore, $2I_1 = 2I_2$ and so $I_1/I_2 = 1$.

EXERCISE
Concept-based
Straight Objective Type Questions

1. The value of $\int_{\pi/4}^{\pi/2} \cot\theta \ \text{cosec}^2\theta \, d\theta$ is

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$
(c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

2. The value of $\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right)$ is

(a) π (b) $-\frac{\pi}{2}$
(c) $\frac{1}{\pi}$ (d) $\frac{2}{\pi}$

3. The value of $\int_{-\pi/2}^{\pi/2} \frac{x\sin x}{e^x + 1} dx$ is equal to

(a) 2π (b) 1
(c) 2 (d) $\frac{\pi}{2}$

4. $\int_{-\pi/2}^{\pi/2} (x^{14}\sin^{11} x + \sin^2 x)\, dx$ is equal to

(a) 0 (b) π
(c) $\frac{\pi}{2}$ (d) $-\pi$

5. $\int_0^{\pi/2} \frac{dx}{2\cos x + 3}$ is equal to

(a) $\frac{2}{\sqrt{5}}\tan^{-1}\frac{1}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{5}}\tan^{-1}\frac{1}{\sqrt{5}}$
(c) 1 (d) $\frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{5}}$

6. If $\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x} dx = \frac{\pi}{4} + A$, then A is equal to

(a) 0 (b) $\frac{1}{2}$
(c) 1 (d) $-\frac{1}{2}$

7. If $\int_0^2 \frac{dx}{\sqrt{x+1} + \sqrt{(x+1)^3}}$ is equal to $k\pi$ then k is equal to

(a) $\frac{1}{2}$ (b) 2
(c) $\frac{1}{6}$ (d) $\frac{1}{3}$

8. If f is continuously differentiable function then $\int_0^{2.5} [x^2]\, f'(x)\, dx$ is equal to

(a) $6f(6.25) + \sum_{i\in\{1,\sqrt{2},\sqrt{3},2,\sqrt{5}\}} f(i)$

(b) $6f(6.25) - \sum_{i\in\{1,\sqrt{2},\sqrt{3},2,\sqrt{5},\sqrt{6}\}} f(i)$

(c) $6f(2.5) - \sum_{i\in\{1,\sqrt{2},\sqrt{3},2,\sqrt{5},\sqrt{6}\}} f(i)$

(d) $6f(2.5) - \sum_{i\in\{1,\sqrt{2},\sqrt{3},\sqrt{5}\}} f(i)$

9. The value of $\int_0^1 x^3(1-x)^{11}dx$ is equal to

(a) $\frac{1}{11}+\frac{3}{12}+\frac{3}{13}+\frac{1}{14}$

(b) $-\frac{1}{11}+\frac{3}{12}-\frac{3}{13}+\frac{1}{14}$

(c) $\frac{8}{13}$

(d) $\frac{1}{11}-\frac{3}{12}+\frac{3}{14}-\frac{1}{14}$

10. The value of $\int_0^3 |2-x|dx$ is

(a) 1 (b) 2
(c) 4 (d) 3

LEVEL 1

Straight Objective Type Questions

11. The value of $\int_{-2}^{2} |1-x|\, dx$ is

(a) 2 (b) 0
(c) 4 (d) 5

12. If $f(x) = \int_0^x \sin^8 t\, dt$, then $f(x+\pi)$ equals

(a) $\frac{f(x)}{f(\pi)}$ (b) $f(x)f(\pi)$
(c) $f(x)+f(\pi)$ (d) $f(x)-f(\pi)$

13. Suppose that the graph of $y = f(x)$, contains the points (0, 4) and (2, 7). If f' is continuous then $\int_0^2 f'(x)\,dx$ is equal to

(a) 2 (b) – 2
(c) 3 (d) none of these

14. A polynomial P is positive for $x > 0$, and the area of the region bounded by $P(x)$, the x-axis, and the vertical lines $x = 0$ and $x = K$ is $K^2(K+3)/3$. The polynomial $P(x)$ is

(a) x^2+2x (b) x^2+x+1
(c) x^2+2x+1 (d) x^3+1

15. The value of $\lim_{x\to 0}\frac{\int_0^x \sin t^2\,dt}{\sin x^2}$ is

(a) 1 (b) 0
(c) 2 (d) none of these

16. Let $f(x) = \int_1^x e^{-t^2/2}(1-t^2)\,dt$ then

(a) f has maximum at $x = 0$
(b) f has minimum at $x = -1$
(c) f has maximum at $x = -1$
(d) f has no critical point

17. A line tangent to the graph of the function $y = f(x)$ at the point $x = a$ forms an $\pi/3$ with y-axis and at $x = b$ an angle $\pi/4$ with x-axis then $\int_a^b f''(x)\,dx$ is

(a) $1-\frac{1}{\sqrt{3}}$ (b) $-\frac{\pi}{12}$
(c) $\frac{\pi}{12}$ (d) $\sqrt{3}-1$

18. The function $F(x) = \int_0^x \log\frac{1-t}{1+t}\,dt$ is

(a) an even function (b) an odd function
(c) a periodic function (d) none of these

19. If $f(n) = \frac{1}{n}[(n+1)(n+2)\ldots(n+n)]^{1/n}$, then $\lim_{n\to\infty} f(n)$ equals

(a) e (b) $1/e$
(c) $2/e$ (d) $4/e$

20. The value of the integral $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ for $\beta > \alpha$, is

(a) $\sin^{-1}\alpha/\beta$ (b) $\pi/2$
(c) $\sin^{-1}\beta/2\alpha$ (d) π.

21. The value of the integral $\int_{-\pi/3}^{\pi/3} \frac{x \sin x}{\cos^2 x} dx$ is

(a) $\left(\frac{\pi}{3} - \log \tan \frac{3\pi}{2}\right)$

(b) $2\left(\frac{2\pi}{3} - \log\left(2+\sqrt{3}\right)\right)$

(c) $3\left(\frac{\pi}{2} - \log \tan \frac{\pi}{12}\right)$

(d) none of these.

22. The value of $\int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$ is

(a) 1/2 (b) 1
(c) $\pi/4$ (d) none of these

23. The equation of the tangent to the curve

$$y = \int_{x^4}^{x^6} \frac{dt}{\sqrt{1+t^2}} \quad \text{at } x = 1 \text{ is}$$

(a) $\sqrt{2}\,y + 1 = x$ (b) $\sqrt{3}\,x + 1 = y$

(c) $\sqrt{3}\,x + 1 + \sqrt{3} = y$ (d) none of these

24. The value of $\int_{-1}^{1} x|x|\, dx$ is

(a) 2 (b) 1
(c) 0 (d) none of these

25. The difference between the greatest and least values of the function

$F(x) = \int_0^x (t+1)\, dt$ on [1, 3] is

(a) 8 (b) 2
(c) 6 (d) 11/2

26. The value of $\lim_{x\to\infty} \frac{\left(\int_0^x e^x\, dx\right)^2}{\int_0^x e^{2x^2}\, dx}$ is

(a) 1 (b) 2
(c) 3 (d) 0

27. The absolute value of $\int_{10}^{19} \frac{\sin x}{1+x^8} dx$ is

(a) less than 10^{-7} (b) more than 10^{-7}
(c) less than 10^{-9} (d) none of these

28. The value of the integral $\int_0^3 \frac{dx}{\sqrt{x+1}+\sqrt{5x+1}}$ is

(a) 11/15 (b) 14/15
(c) 2/5 (d) none of these

29. Let $f(x) = \{x\}$, the fractional part of x then $\int_{-1}^{1} f(x)\, dx$ is equal to

(a) 1 (b) 2
(c) 0 (d) 1/2

30. The value of $\int_{-\pi/2}^{\pi/2} \cos t \sin(2t - \pi/4)\, dt$ is

(a) – 1/3 (b) 1/3
(c) $\sqrt{2}/3$ (d) $-\sqrt{2}/3$

31. The value of $\int_0^\infty \frac{x \log x}{\left(1+x^2\right)^2} dx$ is

(a) 0 (b) 1
(c) $\sqrt{3}/2$ (d) $\sqrt{5}/2$

32. $\int_0^1 |\sin 2\pi x|dx$ is equal to

(a) 0 (b) $-1/\pi$
(c) $1/\pi$ (d) $2/\pi$

33. The value of $\lim_{x\to\infty} \frac{\int_0^x \left(\tan^{-1} x\right)^2 dx}{\sqrt{x^2+1}}$ is

(a) $\pi/4$ (b) $\pi^2/2$
(c) $\pi^2/4$ (d) π

34. If $A(t) = \int_{-1}^{t} e^{-|x|} dx$, then $\lim_{t\to\infty} A(t)$ is equal to

(a) $2 - e^{-1}$ (b) $3 - e^{-1}$
(c) 4 (d) 0

35. If the value of $\int_{-2}^{2} |x \cos \pi x|\, dx = k/\pi$ then the value of k is

(a) 4 (b) 8
(c) 12 (d) none of these

36. The value of $\int_0^\alpha \frac{dx}{1-\cos\alpha \cos x}$ $(0 < \alpha < \pi/2)$ is

(a) $\frac{\pi}{\sin\alpha}$ (b) $\frac{\pi}{2\cos\alpha}$

(c) $\frac{\pi}{\cos\alpha}$ (d) $\frac{\pi}{2\sin\alpha}$

37. The value of

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(2n-1)^2}\right) \text{ is}$$

(a) 1 (b) 1/3
(c) 1/2 (d) 3/2

38. The value of

$$\lim_{n\to\infty}\left(\frac{1}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \cdots + \frac{n^2}{n^3+n^3}\right) \text{ is}$$

(a) $\frac{1}{3}$ (b) $\frac{1}{3}\log 2$

(c) $\frac{1}{2}\log 3$ (d) $\frac{1}{3}\log 3$

39. The value of $\int_0^{\pi} \frac{\sin 2nx}{\sin x}\, dx$ is

(a) 0 (b) 1
(c) 1/2 (d) $\pi/3$

40. If $f(x) = A \sin \frac{\pi x}{2} + B$, $f'(1/2) = \sqrt{2}$ and $\int_0^1 f(x)\, dx = \frac{2A}{\pi}$ then the constants A and B are respectively

(a) $\pi/2, \pi/2$ (b) $2/\pi, 3/\pi$
(c) $0, -4/\pi$ (d) $4/\pi, 0$

41. If $f'(x) = g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\, g(x)\, dx$ is equal to

(a) $g(b) - g(a)$ (b) $f(b) - f(a)$
(c) $\frac{(f(b))^2 - (f(a))^2}{2}$ (d) $\frac{(g(b))^2 - (g(a))^2}{2}$

42. Let $I_1 = \int_a^b \left(\int_a^x f(t)\, dt\right) dx$ and $I_2 = \int_a^b (b - x)\, f(x)\, dx$ then

(a) $I_1 = I_2$
(b) $I_1 = (b - a) + I_2$
(c) $I_1 = (b - a)\, I_2$
(d) none of these

43. The least value of the function $F(x) = \int_x^2 \log_{1/3} t\, dt$, $x \in [1/10, 4]$ is at $x =$

(a) 1/10 (b) 4
(c) 1 (d) none of these

44. The greatest value of the function $F(x) = \int_4^x (t^2 - 8t + 16)\, dt$ on [0, 5] is

(a) 5 (b) 1/3
(c) 4 (d) 2/3

45. If for a continuous function f, $\int_{-a}^{a} f(x)\, dx = K \int_0^a (f(x) + f(-x))\, dx$ then the value of K is

(a) 1 (b) 1/2
(c) 2 (d) a

46. If $J_m = \int_1^e \log^m x\, dx$ then $J_8 + 8J_7$ is equal to

(a) 2 (b) e^8
(c) $2e$ (d) e

47. Let $f_1(x) = \int_0^x f(t)\, dt$, $f_2(x) = \int_0^x f_1(t)\, dt$ and $f_3(x) = \int_0^x f_2(t)\, dt$ if $f_3(x) = A \int_0^x f(t)\, (x - t)^2\, dt$ then the value of A is

(a) 1 (b) 1/2
(c) 2 (d) none of these

48. If the variables x and y are related by the equation $x = \int_0^y \frac{dx}{\sqrt{1+9x^2}}$ and $\frac{d^2y}{dx^2}$ is proportional to y then the constant of proportionality is

(a) 4 (b) 6
(c) 5 (d) 9

49. If $y = f(x)$ be an invertible function with inverse g and $h(x) = x\, f(x)$, then $\int_{f(a)}^{f(b)} g(x)dx + \int_a^b f(x)dx$ is equal to

(a) $h(a) - h(b)$ (b) $h(b) - h(a)$
(c) $h(a) + h(b)$ (d) $b\, h(b) - a\, h(a)$

50. The value of $\int_0^{[x]} \frac{2^t}{2^{[t]}}\, dt$

(a) $[x] \log 2$ (b) $\frac{[x]}{\log 2}$
(c) $\frac{1}{2}\frac{[x]}{\log 2}$ (d) $\frac{1}{4}\frac{[x]}{\log 2}$

51. If $a_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx$ then $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$ are in

(a) A.P.
(b) G.P.
(c) H.P.
(d) Arith – Geo progression

52. $\int_0^{21} [x]^4\, dx$ is equal

(a) 44100 (b) 2100
(c) 48400 (d) 42400

53. If $x^2 f(x) + f(1/x) = 0$ for all $x \in \mathbf{R} \sim \{0\}$ then $\int_{\cos\theta}^{\sec\theta} f(x)dx =$

(a) $\sin 2\theta$ (b) 1
(c) $\sec\theta - \cos\theta$ (d) 0

Assertion-Reason Type Questions

54. Let $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Statement-1: $I = \pi$

Statement-2: The integrand is even and $\lim\limits_{b \to \infty} F(b) = \pi$, $F(b) = \int_0^b \frac{dx}{1+x^2}$

55. Let $I = \int_0^{\pi/2} \frac{dx}{5+3\cos^2 x}$

Statement-1: $\frac{\pi}{16} \le I \le \frac{\pi}{10}$

Statement-2: $\frac{1}{8} \le \frac{1}{5+3\cos^2 x} \le \frac{1}{5}$.

56. Let $I = \int_0^{\pi} \frac{\cos x}{\sqrt{1-\sin^2 x}}\, dx$

Statement-1: $I = \pi$

Statement-2: The integrand can be expressed as $X_{[0,\ \pi/2]}\ X_{[\pi/2,\ \pi]}$; X_A being the characteristic function of A.

57. Let $I = \int_0^{\pi} x^2 \cos x\, dx$ and $J = \int_0^{\pi} x \sin x\, dx$

Statement-1: $I = -2\pi$

Statement-2: $I = 2J$

58. Let $I = \int_1^2 \frac{10x^2}{(x^3+1)^2}\, dx$ and $J = \int_2^9 \frac{dx}{x^2}$

Statement-1: $I = \frac{35}{27}$

Statement-2: $3I = 10J$

LEVEL 2

Straight Objective Type Questions

59. The value of $\int_{\pi/4}^{\pi/3} \frac{\mathrm{d}x}{\sin x + \tan x}$ is

(a) 3/4 (b) 1/2
(c) $2/3 - \sqrt{2}/2$ (d) none of these

60. If f is integrable on $[a, b]$ then $\int_a^b [f(x) - f_{\text{ave}}]\, \mathrm{d}x$ is equal to

(a) $f(b) - f(a)$ (b) $\frac{1}{2}(f(b) - f(a))$
(c) $f(a) - f(b)$ (d) 0

61. If $m \ne n$, $m, n \in \mathbf{N}$ then the value of $\int_0^{2\pi} \cos mx \cos nx\, \mathrm{d}x$ is

(a) 0 (b) 2π
(c) π (d) dependent on m and n

62. If $Q(x) = \frac{1}{2} a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \ldots + a_n \cos nx + b_n \sin nx$, then the value of $\int_0^{2\pi} Q(x) \sin kx\, \mathrm{d}x$ $(k = 1, 2, \ldots n)$

(a) πa_k (b) πa_0
(c) πb_n (d) πb_k

63. The value of $\int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{\mathrm{d}}{\mathrm{d}x} \left(\tan^{-1} \frac{1}{x} \right) + x^3 \right) \mathrm{d}x$ is

(a) $\pi/2$ (b) $\pi/4$
(c) 1 (d) none of these

64. The value of $\int_{1/2}^{1} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right) \mathrm{d}x$ is

(a) sin 1 (b) cos 1
(c) 2 sin 1 (d) none of these

65. Let f be a continuous function $[a, b]$ such that $f(x) > 0$ for all $x \in [a, b]$. If $F(x) = \int_a^x f(t)\, \mathrm{d}t$ then

(a) F is differentiable but not increasing on $[a, b]$
(b) F is differentiable and increasing on $[a, b]$
(c) F is continuous and decreasing on $[a, b]$
(d) F is neither differentiable nor increasing on $[a, b]$

66. Let f be a continuous function on **R** satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$ with $f(1) = 2$ and g be a function satisfying $f(x) + g(x) = e^x$ then the value of the integral $\int_0^1 f(x)\, g(x)\, dx$ is

(a) $\frac{1}{e} - 4$ (b) $\frac{1}{4}(e - 2)$
(c) 2/3 (d) $\frac{1}{2}(e - 3)$

67. If $f(x) = \text{cosec}(x + \pi/3)\,\text{cosec}(x + \pi/6)$, then the value of $\int_0^{\pi/2} f(x)\, dx$ is

(a) $2 \log 3$ (b) $-2 \log 3$
(c) $\log 3$ (d) 1/4

68. The equal to integral $\int_0^1 \log\left(\sqrt{1-x} + \sqrt{1+x}\right) dx$ is

(a) $\frac{1}{2}\left(\log 2 - \frac{1}{2} + \frac{\pi}{4}\right)$ (b) $\frac{1}{2}\left(\log 2 - 1 + \frac{\pi}{2}\right)$
(c) $\frac{1}{3}\left(\log 4 - 1 + \frac{\pi}{8}\right)$ (d) $\frac{1}{4}\left(\log 3 - 1 + \frac{\pi}{2}\right)$.

69. If $I = \int_0^{1/2} \frac{dx}{\sqrt{1 - x^{2n}}}$ for $n \geq 1$, the value of I is

(a) less than 1 (b) more than 1/2
(c) more than 1 (d) less than 1/2

70. The numbers A, B and C such that a function of the form $f(x) = Ax^2 + Bx + C$ satisfies the conditions $f'(1) = 8$, $f(2) + f''(2) = 33$ and $\int_0^1 f(x)\, dx = 7/3$, are

(a) $A = 1, B = -4, C = 2$
(b) $A = 7, B = -6, C = 3$
(c) $A = 8, B = -6, C = 3$
(d) none of these.

71. The equation of the tangent to the curve

$$y = \int_x^{x^2} \log t\, dt \quad \text{at } x = 2 \text{ is}$$

(a) $y - 6 \log 2 = 7 \log 2(x - 2)$
(b) $y - \log 2e^{1/3} = (\log 2)x$
(c) $y - 8 \log 2e^{-1/3} = 5 \log 2\, x$
(d) $y + 8 \log 2 + 2 = (7 \log 2)x$

72. The value of $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2 - \sin\theta}{2 + \sin\theta}\right) d\theta$ is

(a) 0 (b) 1
(c) 2 (d) none of these

73. The value of the integral $\int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x}\, dx$ is

(a) $\log 2$ (b) $\log 3$
(c) $(1/4) \log 3$ (d) $(1/8) \log 3$.

74. If $I_1 = \int_x^1 \frac{dt}{1+t^2}$ and $I_2 = \int_1^{1/x} \frac{dt}{1+t^2}$ for $x > 0$, then

(a) $I_1 > I_2$ (b) $I_1 = I_2$
(c) $I_2 > I_1$ (d) $I_2 = (\pi/2) - \tan^{-1} x$

75. The solution of the equation

$$\int_{\sqrt{2}}^{x} \frac{dx}{x\sqrt{x^2 - 1}} = \frac{\pi}{12} \text{ is given by}$$

(a) 1 (b) 2
(c) 3 (d) $\sqrt{3}$

76. The mean value of the function $f(x) = \frac{2}{e^x + 1}$ on the interval [0, 2] is

(a) $\log \frac{2}{e^2 + 1}$ (b) $1 + \log \frac{2}{e^2 + 1}$
(c) $2 + \log \frac{2}{e^2 + 1}$ (d) $2 + \log\left(e^2 + 1\right)$

77. The value of $\int_0^{\pi} x \log(\sin x)\, dx$ is (given that $\int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2} \log 2$)

(a) $-\frac{\pi^2}{2} \log 2$ (b) $-\frac{\pi^2}{4} \log 2$
(c) $-\frac{\pi^2}{8} \log 2$ (d) none of these

78. $\int_{-\pi}^{\pi} (\cos px - \sin qx)^2\, dx$, where p and q are integers is equal to

(a) $-\pi$ (b) 0
(c) π (d) 2π

79. Let f, g and h be continuous functions on $[0, a]$ such that $f(x) = f(a - x)$, $g(x) = -g(a - x)$ and $3h(x) - 4h(a - x) = 5$. Then $\int_0^a f(x)\, g(x)\, h(x)\, dx$ is equal to

(a) 5/4 (b) 3/4
(c) 1 (d) none of these

80. If $\int_{\pi/2}^{x} \sqrt{3 - 2\sin^2 z}\, dz + \int_0^y \cos t\, dt = 0$ then $\frac{dy}{dx}$ at $(\pi/2, \pi)$ is

(a) 1 (b) 2
(c) $\sqrt{3}$ (d) $= 0$

81. If $P(x)$ is a polynomial of least degree that has a maximum equal to 6 at $x = 1$, and a minimum equal to 2 at $x = 3$, then $\int_0^1 P(x)\,dx$ equals

(a) 17/4 (b) 13/4
(c) 19/4 (d) 5/4

82. Let $P(x)$ be a polynomial of least degree whose graph has three points of inflextion $(-1, -1)$, $(1,1)$ and a point with abscissa 0 at which the curve is inclined to the axis of abscissas at an angle of 60°. Then $\int_0^1 P(x)\,dx$ equals to

(a) 1 (b) $\sqrt{2}-1$
(c) $2\sqrt{2}-1$ (d) none of these

83. If $S_n = \frac{e^{1/n}}{n^2} + \frac{2}{n^2}e^{2/n} + \frac{3}{n^2}e^{3/n} + \cdots + \frac{1}{n}e$ then $\lim_{n\to\infty} S_n$ is

(a) 1 (b) 0
(c) e (d) none of these

84. The value of $\int_0^1 \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^{k+2}2^k}{k!}\,dx$ is

(a) $e^2 - 1$ (b) 2
(c) $\frac{e^2-1}{2}$ (d) $\frac{e^2-1}{4}$

85. The value of $\int_0^1 \frac{x^{2\alpha}-1}{\log x}\,dx$, if $\alpha = \frac{2n-1}{2}$ is

(a) $\log n$ (b) $2 \log n$
(c) $\log 2n$ (d) $(1/2) \log n$

86. The value of α in $(-\pi, 0)$ satisfying $\sin \alpha + \int_\alpha^{2\alpha} \cos 2x\,dx = 0$ is

(a) $-\pi/2$ (b) $-\pi$
(c) $-\pi/3$ (d) $-\pi/4$

87. Let $g(x) = \begin{cases} 1, & 0 \le x < 1 \\ x^3, & 1 \le x < 4 \\ \sqrt{x}, & 4 \le x < 9 \end{cases}$ then $\int_0^9 g(x)\,dx$ is

(a) 243/12 (b) $9\sqrt{3}$
(c) 929/12 (d) none of these

88. The value of $\lim_{x\to 0} \frac{\int_{-x}^{x} f(t)\,dt}{\int_0^{2x} f(t+4)\,dt}$ is (where f is a continuous function and $f(x) > 0$ for all x)

(a) $f(0)$ (b) 0
(c) $f(4)/f(0)$ (d) $f(0)/f(4)$

89. If $f(x) = \begin{cases} e^{\cos x}\sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise} \end{cases}$ then $\int_{-2}^{3} f(x)\,dx =$

(a) 0 (b) 1
(c) 2 (d) 3

90. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$, $a > 0$, is

(a) $\pi/2$ (b) $a\pi$
(c) π (d) 2π

91. Let $f : (0, \infty) \to \mathbf{R}$ and let

$$F(x) = \int_0^x f(t)\,dt$$

If $F(x^2) = x^2(1 + x)$, then $f(4)$ equals

(a) $\frac{5}{4}$ (b) 7
(c) 4 (d) 2

92. Let $f(x) = \int_1^x \sqrt{2-t^2}\,dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(a) ± 1 (b) $\pm\frac{1}{\sqrt{2}}$
(c) $\pm\frac{1}{2}$ (d) 0 and 1

93. If $l(m, n) = \int_0^1 t^m (1 + t)^n\,dt$, then the expression for $l(m, n)$ in terms of $l(m + 1, n - 1)$ is

(a) $\frac{2^n}{m+1} - \frac{n}{m+1} l(m+1, n-1)$

(b) $\frac{n}{m+1} l(m+1, n-1)$

(c) $\frac{2^n}{m+1} + \frac{n}{m+1} l(m+1, n-1)$

(d) $\frac{m}{n+1} l(m+1, n-1)$

94. The definite integral $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$ is equal to

(a) 1 (b) π
(c) $\frac{\pi}{2} - 1$ (d) $\frac{\pi}{2} + \frac{1}{2}$

95. If, for $t > 0$ the definite integral

$\int_0^{t^2} x\,f(x)\,dx = \frac{2}{5}\,t^5$, then $f\left(\frac{4}{25}\right)$ is equal to

(a) $\frac{2}{5}$ (b) $-\frac{2}{5}$

(c) $\frac{2}{\sqrt{5}}$ (d) $-\frac{2}{\sqrt{5}}$

96. The definite integral

$\int_{-2}^{0} (x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1))\,dx$

equals

(a) -4 (b) 0

(c) 4 (d) 6

Previous Years' AIEEE/JEE Main Questions

1. $\int_0^{\sqrt{2}} [x^2]\,dx$ is

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$

(c) $\sqrt{2}-1$ (d) $\sqrt{2}-2$ **[2002]**

2. $I_n = \int_0^{\pi/4} \tan^n x\,dx$, then $\lim_{n\to\infty} n(I_n + I_{n-2})$ equals

(a) $\frac{1}{2}$ (b) 1

(c) ∞ (d) 0 **[2002]**

3. $\int_{\pi}^{10\pi} |\sin x|\,dx$ is

(a) 20 (b) 8

(c) 10 (d) 18 **[2002]**

4. If $y = f(x)$ makes positive intercept of 2 and 1 unit on x-axis and y-axis and encloses an area of $\frac{3}{4}$ square unit with axes then $\int_0^2 x\,f'(x)\,dx$ is

(a) $\frac{3}{2}$ (b) 1

(c) $\frac{5}{4}$ (d) $-\frac{3}{4}$ **[2002]**

5. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ be a function that satisfies $f(x) + g(x) = x^2$. Then the value of the integral

$\int_0^1 f(x)\,g(x)\,dx$ is

(a) $e + \frac{e^2}{2} - \frac{3}{2}$ (b) $e - \frac{e^2}{2} - \frac{3}{2}$

(c) $e + \frac{e^2}{2} + \frac{5}{2}$ (d) $e - \frac{e^2}{2} - \frac{5}{2}$ **[2003]**

6. If $f(y) = e^y$, $g(y) = y$, $y > 0$ and $F(t) = \int_0^t f(t-y)\,g(y)\,dy$, then

(a) $F(t) = e^t - (1+t)$

(b) $F(t) = te^t$

(c) $F(t) = te^{-t}$

(d) $F(t) = 1 - e^t(1+t)$ **[2003]**

7. If $f(a+b-x) = f(x)$, then $\int_a^b x\,f(x)\,dx$ is equal to

(a) $\frac{a+b}{2}\int_a^b f(x)\,dx$

(b) $\frac{b-a}{2}\int_a^b f(x)\,dx$

(c) $\frac{a+b}{2}\int_a^b f(a+b-x)\,dx$

(d) $\frac{a+b}{2}\int_a^b f(b-x)\,dx$ **[2003]**

8. The value of the integral $I = \int_0^1 x(1-x)^n\,dx$ is

(a) $\frac{1}{n+2}$ (b) $\frac{1}{n+1} - \frac{1}{n+2}$

(c) $\frac{1}{n+1} + \frac{1}{n+2}$ (d) $\frac{1}{n+1}$ **[2003]**

9. The value of $\lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t\,dt}{x \sin x}$ is

(a) 2 (b) 1

(c) 0 (d) 3 **[2003]**

10. The value of $\int_{-2}^{3}|1-x^2|dx$ is

(a) $\frac{7}{3}$ (b) $\frac{14}{3}$

(c) $\frac{28}{3}$ (d) $\frac{1}{3}$ **[2004]**

11. The value of $I = \int_0^{\pi/2} \frac{(\sin x+\cos x)^2}{\sqrt{1+\sin 2x}}dx$ is

(a) 2 (b) 1
(c) 0 (d) 3 **[2004]**

12. If $\int_0^{\pi} x f(\sin x)dx = A\int_0^{\pi/2} f(\sin x)dx$, then A is

(a) $\frac{\pi}{4}$ (b) π

(c) 0 (d) 2π **[2004]**

13. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} x\, g(x(1-x))\, dx$, and $I_2 = \int_{f(-a)}^{f(a)} g(x(1-x))\, dx$, then the value of $\frac{I_2}{I_1}$ is

(a) – 1 (b) – 3
(c) 2 (d) 1 **[2004]**

14. $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\cdots\frac{1}{n^2}\sec^2 1\right]$ equals

(a) tan 1 (b) tan 1

(c) $\frac{1}{2}$ sec 1 (d) $\frac{1}{2}$ cosec 1 **[2005]**

15. Let $f:\mathbf{R}\to\mathbf{R}$ be a differentiable function having $f(2)=6$, $f'(2)=\frac{1}{48}$. Then $\lim_{x\to 0}\int_6^{f(x)}\frac{4t^3}{x-2}dt$ equals

(a) 12 (b) 18
(c) 24 (d) 36 **[2005]**

16. If $I_1 = \int_0^1 2^{x^2}dx$, $I_2 = \int_0^1 2^{x^3}dx$, $I_3 = \int_1^2 2^{x^2}dx$ and $I_4 = \int_1^2 2^{x^3}dx$ then

(a) $I_3 = I_4$ (b) $I_3 > I_4$
(c) $I_2 > I_1$ (d) $I_1 > I_2$ **[2005]**

17. The value of $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}$, $a > 0$ is

(a) $\frac{\pi}{a}$ (b) 2π

(c) $a\pi$ (d) $\frac{\pi}{2}$ **[2005]**

18. The value of the integral, $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}dx$ is

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{3}{2}$ (d) 2 **[2006]**

19. $\int_{-3\pi/2}^{-\pi/2}\left[(x+\pi)^3+\cos^2(x+3\pi)\right]dx$ is equal to

(a) $\left(\frac{\pi}{4}\right)-1$ (b) $\frac{\pi^4}{32}$

(c) $\left(\frac{\pi^4}{32}\right)+\left(\frac{\pi}{2}\right)$ (d) $\frac{\pi}{2}$ **[2006]**

20. $\int_0^{\pi} x f(\sin x)dx$ is equal to

(a) $\pi\int_0^{\pi/2} f(\cos x)dx$ (b) $\pi\int_0^{\pi} f(\cos x)dx$

(c) $\pi\int_0^{\pi} f(\sin x)dx$ (d) $\frac{\pi}{2}\int_0^{\pi/2} f(\sin x)dx$

[2006]

21. The value of $\int_1^a [x] f'(x)dx$, $a > 1$, where $[x]$ denotes the greatest integer not exceeding x is

(a) $af([a]) - \{f(1)+f(2)+\ldots+f(a)]$
(b) $af(a) - \{f(1)+f(2)+\ldots+f([a])\}$
(c) $[a]\,f(a) - \{f(1)+f(2)+\ldots+f([a])\}$
(d) $[a]\,f([a]) - \{f(1)+f(2)+\ldots+f(a)\}$

[2006]

22. Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$ where $f(x) = \int_1^x \frac{\log t}{1+t}dt$. Then $F(e)$ equals

(a) $\frac{1}{2}$ (b) 0

(c) 1 (d) 2 **[2007]**

23. The solution for x of the equation

$$\int_{\sqrt{2}}^{x}\frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2} \text{ is}$$

(a) 2 (b) $-\sqrt{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) $2\sqrt{2}$ **[2007]**

24. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}dx$. Then which one of the following is true?

(a) $I > 2/3$, $J > 2$ (b) $I < 2/3$, $J < 2$
(c) $I < 2/3$, $J > 2$ (d) $I > 2/3$, $J < 2$ **[2008]**

25. $\int_0^{\pi}[\cot x]dx$, where [ ] denotes the greatest integer function, is equal to
(a) -1 (b) $-\pi/2$
(c) $\pi/2$ (d) 1 **[2009]**

26. Let $p(x)$ be a function defined on R such that $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$. Then $\int_0^1 p(x)\,dx$ equals
(a) 41 (b) 42
(c) $\sqrt{41}$ (d) 21 **[2010]**

27. The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is
(a) $\log 2$ (b) $\pi \log 2$
(c) $\frac{\pi}{8}\log 2$ (d) $\frac{\pi}{2}\log 2$ **[2011]**

28. For $x \in (0, 5\pi/2)$, define $F(x) = \int_0^x \sqrt{t}\sin t\,dt$ then f has
(a) local maximum at π and local minimum at 2π.
(b) local maximum at π and 2π
(c) local minimum at π and 2π
(d) local minimum at π and local maximum at 2π **[2011]**

29. Let [.] denote the greatest integer function then the value of $\int_0^{1.5} x[x^2]\,dx$ is
(a) 0 (b) $\frac{3}{2}$
(c) $\frac{3}{4}$ (d) $\frac{5}{4}$ **[2011]**

30. If $g(x) = \int_0^x \cos 4t\,dt$, then $g(x+\pi)$ equals
(a) $g(x)+g(\pi)$ (b) $g(x)-g(\pi)$
(c) $g(x)\,g(\pi)$ (d) $\frac{g(x)}{g(\pi)}$ **[2012]**

31. The intercepts on x-axis by the tangents to the curve, $y = \int_0^x |t|\,dt$, $x \in \mathbf{R}$, which are parallel to the line $y = 2x$, are equal to
(a) ± 2 (b) ± 3
(c) ± 4 (d) ± 1 **[2013]**

32. **Statement-1:** The value of the integral
$$\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{6}$$
Statement-2: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ **[2013]**

33. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x}dx$ is
(a) π (b) $\frac{\pi}{2}$
(c) 4π (d) $\frac{\pi}{4}$ **[2013, online]**

34. If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$, then $\frac{d^2y}{dx^2}$ is equal to
(a) y (b) $\sqrt{1+y^2}$
(c) $\frac{y}{\sqrt{1+y^2}}$ (d) y^2 **[2013, online]**

35. The integral $\int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x}\,dx$ is equal to
(a) $\log 2\sqrt{2}$ (b) $\log 2$
(c) $2\log 2$ (d) $\log \sqrt{2}$ **[2013, online]**

36. The integral $\int_0^{\pi} \sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals
(a) $\pi - 4$ (b) $\frac{2\pi}{3}-4$
(c) $4\sqrt{3}-4$ (d) $4\sqrt{3}-4-\frac{\pi}{3}$ **[2014]**

37. The integral $\int_0^{1/2} \frac{\log(1+2x)}{1+4x^2}dx$ equals
(a) $\frac{\pi}{4}\log 2$ (b) $\frac{\pi}{8}\log 2$
(c) $\frac{\pi}{16}\log 2$ (d) $\frac{\pi}{32}\log 2$ **[2014, online]**

38. If for $n \ge 1$, $P_n = \int_1^e (\log x)^n dx$, then $P_{10} - 90P_8$ is equal to
(a) -9 (b) $10e$
(c) $-9e$ (d) 10 **[2014, online]**

39. If [,] denotes the greatest integer function, then the integral $\int_0^{\pi} [\cos x]\,dx$ is equal to
(a) $\frac{\pi}{2}$ (b) 0
(c) -1 (d) $-\frac{\pi}{2}$ **[2014, online]**

40. If for a continuous function $f(x)$, $\int_{-\pi}^{t}(f(x)+x)\,dx = \pi^2 - t^2$, for all $t \geq -\pi$, then $f\left(-\frac{\pi}{3}\right)$ is equal to:

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$ **[2014, online]**

41. Let function F be defined as $F(x) = \int_1^x \frac{e^t}{t}dt$ $x > 0$ then the value of the integral $\int_1^x \frac{e^t}{t+a}dt$ where $a > 0$ is

(a) $e^a\,[F(x) - F(1 + a)]$
(b) $e^{-a}\,[F(x + a) - F(a)]$
(c) $e^a\,[F(x + a) - F(1 + a)]$
(d) $e^{-a}\,[F(x + a) - F(1 + a)]$ **[2014, online]**

42. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)}dx$ is equal to

(a) 2 (b) 4
(c) 1 (d) 6 **[2015]**

43. Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that $f(2 - x) = f(2 + x)$ and $f(4 - x) = f(4 + x)$ for all $x \in \mathbf{R}$ and $\int_0^2 f(x)\,dx = 5$. Then the value of $\int_{10}^{50} f(x)\,dx$ is:

(a) 80 (b) 100
(c) 125 (d) 200 **[2015, online]**

44. Let f: $(-1, 1) \to \mathbf{R}$ be a continuous function. If $\int_0^{\sin x} f(t)\,dt = \frac{\sqrt{3}}{2}x$, then $f\left(\frac{\sqrt{3}}{2}\right)$ is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{3}$

(c) $\sqrt{\frac{3}{2}}$ (d) $\frac{1}{2}$ **[2015, online]**

45. For $x > 0$, let $f(x) = \int_1^x \frac{\log t}{1+t}dt$. Then $f(x) + f\left(\frac{1}{x}\right)$ is equal to

(a) $\frac{1}{4}(\log x)^2$ (b) $\frac{1}{2}(\log x)^2$

(c) $\log x$ (d) $\frac{1}{4}\log x^2$ **[2015 online]**

46. If $2\int_0^1 \tan^{-1} x\,dx = \int_0^1 \cot^{-1}(1 - x + x^2)dx$, then $\int_0^1 \tan^{-1}(1 - x + x^2)dx$ is equal to

(a) $\frac{\pi}{2} + \log 2$ (b) $\log 2$

(c) $\frac{\pi}{2} - \log 4$ (d) $\log 4$ **[2016, online]**

47. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)\cdots(3n)}{n^{2n}}\right)^{1/n}$ is equal to:

(a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$

(c) $\frac{9}{e^2}$ (d) $3\log 3 - 2$ **[2016]**

48. The value of the integral $\int_4^{10} \frac{[x^2]}{[x^2 - 28x + 196] + [x^2]}dx$, where $[x]$ denotes the greatest integer less than or equal to x, is

(a) $\frac{1}{3}$ (b) 6

(c) 7 (d) 3 **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. $\int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx$ is

(a) 0 (b) –2
(c) 1 (d) 2 **[2006]**

2. If f is continuously differentiable function then $\int_0^{1.5}[x^2]f'(x)dx$ is

(a) $f(1.5) - f(\sqrt{2}) - f(1)$
(b) $f(1.5) + f(\sqrt{2}) + f(1)$
(c) $2f(1.5) + f(\sqrt{2}) + f(1)$
(d) $2f(1.5) - f(\sqrt{2}) - f(1)$ **[2006]**

3. $\int_{-a}^{a}\log(x+\sqrt{x^2+1})\,dx$

(a) $2\log(a^2+1)$ (b) $2\log(\sqrt{a^2+1}-a)$
(c) 0 (d) $2\log(a+\sqrt{a^2+1})$ **[2007]**

4. If $f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $n = 0, 1, 2\ldots$ then $\underset{n\to\infty}{lt}\int_{1/2^n}^{1} f(x)dx$ equals
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) 0 (d) $\frac{1}{2}$ **[2007]**

5. The value of $\int_0^1 x^2(1-x)^9\, dx$ is
(a) $\frac{1}{610}$ (b) $\frac{1}{630}$
(c) $\frac{1}{640}$ (d) $\frac{1}{660}$ **[2008]**

6. The value of $\int_0^1 \max\left(e^x, e^{1-x}\right)\, dx$ equals
(a) $2(e-1)$ (b) $2(e-\sqrt{e})$
(c) $2(e+\sqrt{e})$ (d) $2(e+1)$ **[2010]**

7. $\int_0^{\pi/2} \min(\sin x, \cos x)\, dx$ equals to
(a) $2\sqrt{2}$ (b) $\sqrt{2}$
(c) $2-\sqrt{2}$ (d) $2+\sqrt{2}$ **[2011]**

8. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \int_0^1 \frac{x^2+t^2}{2-t}dt$. Then the curve $y = f(x)$ is
(a) an ellipse (b) a straight line
(c) a parabola (d) a hyperbola **[2011]**

9. The equation of a curve is given by $y = f(x)$, where $f'(x)$ is a continuous function. The tangent at points $(1, f(1))$, $(2, f(2))$ and $(3, f(3))$ make angles $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively with positive x-axis. Then $\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)dx$ is:
(a) 1 (b) $-\frac{1}{\sqrt{3}}$
(c) $\frac{1}{\sqrt{3}}$ (d) 0 **[2011]**

10. Using the fact that $0 \le f(x) \le g(x)$, $c < x < d$ $\Rightarrow \int_c^d f(x)dx \le \int_c^d g(x)dx$, we can conclude that $\int_1^3 \sqrt{3+x^3}$ lies in the interval
(a) $\left(\frac{1}{2}, 3\right)$ (b) $\left(2, \sqrt{30}\right)$
(c) $\left(\frac{3}{2}, 5\right)$ (d) $\left(4, 2\sqrt{30}\right)$ **[2012]**

11. $\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{n^2}{n^2+1^2}+\frac{n^2}{n^2+2^2}+\cdots+\frac{n^2}{n^2+(n-1)^2}\right]$ is equal to
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$ **[2012]**

12. If $f(x) = x|x|$, then for any real number a and b with $a < b$, the value of $\int_a^b f(x)dx$ equals
(a) $\frac{1}{3}(|b|^3-|a|^3)$ (b) $\frac{1}{3}|b^3-a^3|$
(c) $\frac{1}{3}(a^3+b^3)$ (d) $\frac{1}{3}(a^3-b^3)$ **[2013]**

13. The integral $\int_{\sqrt{\log 5}}^{\sqrt{\log 7}} \frac{x\cos x^2}{\cos(\log 35-x^2)+\cos x^2}dx$ is equal to
(a) $\frac{1}{4}\log\frac{5}{7}$ (b) $\frac{1}{2}\log\frac{5}{7}$
(c) $\frac{1}{4}\log\frac{7}{5}$ (d) $\frac{1}{2}\log\frac{7}{5}$ **[2013]**

14. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} x\, g\{x(1-x)\}dx$ and $I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}dx$ where g is not an identity function. Then the value of $\frac{I_2}{I_1}$ is
(a) $\frac{1}{2}$ (b) 2
(c) 1 (d) -1 **[2014]**

15. The integral $\int_0^{1/2} \frac{e^x(2-x^2)}{(1-x)^{3/2}(1+x)^{1/2}}dx$ is equal to

(a) $\sqrt{3e}$ (b) $\sqrt{3e}-1$

(c) $\sqrt{\frac{e}{3}}$ (d) $\sqrt{\frac{e}{3}}-1$ **[2015]**

16. The integral $I = \int_0^2 [x^2]dx$ ([t] denotes the greatest integer less than or equal to t) is equal to:

(a) $5-2\sqrt{3}$ (b) $5-\sqrt{2}-\sqrt{3}$

(c) $6-\sqrt{2}-\sqrt{3}$ (d) $3-\sqrt{2}$

Answers

Concept-based

1. (b)	**2.** (d)	**3.** (b)	**4.** (c)
5. (a)	**6.** (d)	**7.** (c)	**8.** (c)
9. (d)	**10.** (b)		

Level 1

11. (d)	**12.** (c)	**13.** (c)	**14.** (a)
15. (b)	**16.** (b)	**17.** (a)	**18.** (a)
19. (d)	**20.** (d)	**21.** (b)	**22.** (b)
23. (d)	**24.** (c)	**25.** (c)	**26.** (d)
27. (a)	**28.** (d)	**29.** (a)	**30.** (d)
31. (a)	**32.** (d)	**33.** (c)	**34.** (a)
35. (b)	**36.** (a)	**37.** (c)	**39.** (b)
39. (a)	**40.** (d)	**41.** (c)	**42.** (a)
43. (c)	**44.** (b)	**45.** (a)	**46.** (d)
47. (b)	**48.** (d)	**49.** (b)	**50.** (b)
51. (c)	**52.** (a)	**53.** (d)	**54.** (c)
55. (a)	**56.** (d)	**57.** (c)	**58.** (a)

Level 2

59. (d)	**60.** (d)	**61.** (a)	**62.** (d)
63. (d)	**64.** (d)	**65.** (b)	**66.** (c)
67. (b)	**68.** (b)	**69.** (a)	**70.** (b)
71. (d)	**72.** (a)	**73.** (c)	**74.** (a)
75. (b)	**76.** (c)	**77.** (a)	**78.** (d)
79. (d)	**80.** (a)	**81.** (c)	**82.** (d)
83. (a)	**84.** (d)	**85.** (c)	**86.** (c)
87. (c)	**88.** (d)	**89.** (c)	**90.** (a)
91. (c)	**92.** (a)	**93.** (a)	**94.** (c)
95. (a)	**96.** (c)		

Previous Years' AIEEE/JEE Main Questions

1. (c)	**2.** (b)	**3.** (d)	**4.** (d)
5. (b)	**6.** (a)	**7.** (a)	**8.** (b)
9. (b)	**10.** (c)	**11.** (a)	**12.** (b)
13. (c)	**14.** (b)	**15.** (b)	**16.** (d)
17. (d)	**18.** (c)	**19.** (d)	**20.** (a)
21. (c)	**22.** (a)	**23.** (b)	**24.** (b)
25. (b)	**26.** (d)	**27.** (b)	**28.** (a)
29. (c)	**30.** (a)(b)	**31.** (d)	**32.** (c)
33. (b)	**34.** (a)	**35.** (a)	**36.** (d)
37. (c)	**38.** (c)	**39.** (d)	**40.** (a)
41. (d)	**42.** (a)	**43.** (b)	**44.** (b)
45. (b)	**46.** (b)	**47.** (b)	**48.** (d)

Previous Years' B-Architecture Entrance Examination Questions

1. (a)	**2.** (d)	**3.** (c)	**4.** (b)
5. (d)	**6.** (b)	**7.** (c)	**8.** (c)
9. (b)	**10.** (d)	**11.** (c)	**12.** (b)
13. (c)	**14.** (b)	**15.** (b)	**16.** (b)

Hints and Solutions

Concept-based

1. $\int_{\pi/4}^{\pi/2} \cot\theta \operatorname{cosec}^2\theta \, d\theta = -\int_1^0 u\,du \quad (u = \cot\theta)$

$$= \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

2. Reqd. limit $= \lim_{n\to\infty} \frac{1}{n}\sum_{n=1}^{n} \sin\frac{r\pi}{n}$

$$= \int_0^1 \sin\pi x\,dx = -\frac{\cos\pi x}{\pi}\Big|_0^1$$

$$= \frac{-1}{\pi}[-1-1] = \frac{2}{\pi}$$

3. $I = \int_{-\pi/2}^{\pi/2} \frac{x\sin x}{e^x+1}dx = \int_{-\pi/2}^{\pi/2} \frac{(-x)\sin(-x)}{e^{-x}+1}dx$ (Prop. 11)

$$= \int_{-\pi/2}^{\pi/2} \frac{(x\sin x)e^x}{e^x+1}dx$$

$$2I = \int_{-\pi/2}^{\pi/2} \frac{x\sin x}{e^x+1} + \int_{-\pi/2}^{\pi/2} \frac{e^x(x\sin x)}{e^x+1}\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{(e^x+1)(x\sin x)}{e^x+1}\,dx$$

$$= \int_{-\pi/2}^{\pi/2} x\sin x\,dx = 2\int_0^{\pi/2} x\sin x\,dx$$

(Prop. 12)

$$= 2\left[-x\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx\right]$$

$$= 2\sin x\Big|_0^{\pi/2} = 2$$

$\Rightarrow$ $I = 1$.

4. The function $f(x) = x^{14}\sin^{11}x$ is an odd function, so $\int_{-\pi/2}^{\pi/2} x^{14}\sin^{11}x\,dx = 0$. Thus the given integral is equal to $\int_{-\pi/2}^{\pi/2}\sin^2 x\,dx = 2\int_0^{\pi/2}\sin^2 x\,dx = 2.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{2}$

5. Put $\tan x/2 = t$. The given integral reduces to

$$\int_0^1 \frac{2dt}{(1+t^2)\left(2\frac{(1-t^2)}{1+t^2}+3\right)} = 2\int_0^1 \frac{dt}{2-2t^2+3+3t^2}$$

$$= 2\int_0^1 \frac{dt}{t^2+5}$$

$$= \frac{2}{\sqrt5}\tan^{-1}\frac{t}{\sqrt5}\Big|_0^1 = \frac{2}{\sqrt5}\tan^{-1}\frac{1}{\sqrt5}$$

6. $$\int_0^{\pi/4}\frac{x\sin x}{\cos^3 x}dx = \int_0^{\pi/4} x\tan x\sec^2 x\,dx$$

$$= x\frac{\tan^2 x}{2}\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\tan^2 x\,dx$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\int_0^{\pi/4}(\sec^2 x - 1)dx$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}[\tan x - x]^{\pi/4}$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\left[1-\frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{1}{2}$$

7. The given integral in equal to

$$\int_0^2 \frac{dx}{\sqrt{x+1}(2+x)} = \int_1^{\sqrt3}\frac{2t\,dt}{t(t^2+1)}, t^2 = x+1$$

$$= 2\int_1^{\sqrt3}\frac{dt}{t^2+1} = 2\tan^{-1}t\Big|_1^{\sqrt3}$$

$$= 2\left[\frac{\pi}{3}-\frac{\pi}{4}\right] = \frac{\pi}{6}.$$

8. $$\int_0^{2.5}[x^2]f'(x)dx$$

$$= \int_0^1 [x^2]f'(x)dx + \int_1^{\sqrt2}[x^2]f'(x) + \int_{\sqrt2}^{\sqrt3}[x^2]f'(x)dx$$
$$+ \int_{\sqrt3}^{2}[x^2]f'(x)dx + \int_2^{\sqrt5}[x^2]f'(x)\,dx$$
$$+ \int_{\sqrt5}^{\sqrt6}[x^2]f'(x)\,dx + \int_{\sqrt6}^{2.5}[x^2]f'(x)dx$$

$$= 0 + \int_1^{\sqrt2}f'(x)dx + 2\int_{\sqrt2}^{\sqrt3}f'(x)dx + 3\int_{\sqrt3}^{2}f'(x)dx$$
$$+4\int_2^{\sqrt5}f'(x)dx + 5\int_{\sqrt5}^{\sqrt6}f'(x)\,dx + 6\int_{\sqrt6}^{2.5}f'(x)\,dx$$

$$= \left(f(\sqrt2) - f(1)\right) + 2\left(f(\sqrt3) - f(\sqrt2)\right)$$
$$+3\left(f(2) - f(\sqrt3)\right) + 4\left(f(\sqrt5) - f(2)\right)$$
$$+5\left[f(\sqrt6) - f(\sqrt5)\right] + 6\left[f(2.5) - f(\sqrt6)\right]$$

$$= 6f(2.5) - (f(1) + f(\sqrt2) + f(\sqrt3) + f(2)$$
$$+ (f(\sqrt5) + f(\sqrt6))$$

9. Given integral is equal to

$$\int_0^1 (1-x)^3(1-(1-x))^{11}dx = \int_0^1 x^{11}(1-x)^3 dx$$

$$= \int_0^1 x^{11}(1-3x+3x^2-x^3)dx$$

$$= \int_0^1 (x^{11} - 3x^{12} + 3x^{13} - x^{14})dx$$

$$= \frac{1}{11} - \frac{3}{12} + \frac{3}{13} - \frac{1}{14}$$

10. $$|2-x| = \begin{cases} 2-x, & 0 \le x \le 2 \\ x-2, & 2 < x \le 3 \end{cases}$$

$$\int_0^3 |2-x|dx = \int_0^2 (2-x)dx + \int_2^2 (x-2)dx$$

$$= 2x - x^2\Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^3$$

$$= 4 - 4 + \frac{8}{2} - 4 - \left(\frac{4}{2} - 4\right)$$

$$= 2$$

Level 1

11. $|1 - x| = \begin{cases} 1-x & , \quad x \le 1 \\ x-1 & , \quad x > 1 \end{cases}$ so

$$\int_{-2}^{2} |1-x|\,dx = \int_{-2}^{1} |1-x|\,dx + \int_{1}^{2} |1-x|\,dx$$

$$= \int_{-2}^{1} (1-x)\,dx + \int_{1}^{2} (x-1)\,dx$$

$$= 5.$$

12. $f(x + \pi) = \int_0^{x+\pi} \sin^8 t\, dt$

$$= \int_0^{\pi} \sin^8 t\, dt\, t \int_{\pi}^{x+\pi} \sin^8 t\, dt$$

$$= f(x) + \int_0^x \sin^8 (\pi - u)\, du = f(\pi) + f(x).$$

13. $f(0) = 4$, $f(2) = 7$ and $\int_0^2 f'(x)dx = f(2) - f(0) = 7 - 4 = 3.$

14. We have $\int_0^K P(t)dt = \frac{K^2(K+3)}{3}$. Differentiating w.r.t K, we get $P(K) = \frac{K^2}{3} + \frac{2K}{3}(K+3) = K^2 + 2K.$

15. $\lim\limits_{x \to 0} \frac{\int_0^x \sin t^2 dt}{\sin x^2} = \lim\limits_{x \to 0} \frac{\sin x^2}{\cos x^2 . 2x}$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\sin x^2}{x^2} . x . \cos x = 0$$

16. $f'(x) = e^{-x^2/2}(1 - x^2) > 0$ for $-1 < x < 0$ and is negative for $x < -1$. $f(x)$ has minimum at $x = -1$.

17. $f'(a) = \tan\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $f'(b) = \tan \frac{\pi}{4} = 1$, so

$$\int_a^b f''(x)dx = f'(b) - f'(a) = 1 - \frac{1}{\sqrt{3}}.$$

18. $F(-x) = \int_0^{-x} \log\frac{1-t}{1+t}\,dt = -\int_0^x \log\frac{1+u}{1-u}\,du \; (u = -t)$

$$= \int_0^x \log\frac{1-u}{1+u}\,du = F(x)$$

19. $A = \lim\limits_{n \to 0} f(n)$

$$= \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\cdots\left(\frac{n+n}{n}\right)\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$$

$$\log A = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right) = \int_0^1 \log(1+x)\,dx$$

$$= \log\frac{4}{e}$$

$$\Rightarrow \quad A = \frac{4}{e}$$

20. The integrand can be written as

$$\frac{1}{\sqrt{\left(\frac{B-\alpha}{2}\right)^2 - \left(x - \frac{\alpha+\beta}{2}\right)^2}}$$

So the value of the required integral is

$$\sin^{-1}\frac{x - \frac{\alpha+\beta}{2}}{\frac{\beta-\alpha}{2}}\Bigg|_{\alpha}^{\beta} = \sin^{-1}(1) - \sin^{-1}(-1) = \pi$$

21. The integrand is an even function, so the given integral is equal to

$$2\int_0^{\pi/3} \frac{x \sin x}{\cos^2 x}\,dx = 2[x \sec x]_0^{\pi/3} - \int_0^{\pi/3} \sec x\, dx]$$

$$= 2.\frac{\pi}{3}.2 - 2[\log(\sec x + \tan x)]_0^{\pi/3}$$

$$= \frac{4\pi}{3} - 2\log\left(2+\sqrt{3}\right)$$

22. $\frac{1}{t(1+t^2)} = \frac{1}{t} - \frac{t}{1+t^2}$. So the required integral is equal to

$$\int_{1/e}^{\tan x} \frac{t\,dt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t} - \int_{1/e}^{\cot x} \frac{t\,dt}{1+t^2}$$

$$= \frac{1}{2}\log\left(1+t^2\right)\Big|_{1/e}^{\tan x} + \log t\Big|_{1/e}^{\cot x} - 2\log\left(1+t^2\right)\Big|_{1/e}^{\cot x}$$

$= \log |\sec x| + \log |\cot x| + 1 - \log |\operatorname{cosec} x| = 1.$

23. $\frac{dy}{dx} = \frac{1}{\sqrt{1+t^2}}\Bigg|_{t=x^3} \frac{d}{dx}(x^3) - \frac{1}{\sqrt{1+t^2}}\Bigg|_{t=x^2} \frac{d}{dx}(x^2)$

$$= \frac{3x^2}{\sqrt{1+x^6}} - \frac{2x}{\sqrt{1+x^4}}$$

$$\frac{dy}{dx}\Big|_{x=1} = \frac{1}{\sqrt{2}}.$$

Also $y(1) = 0$. So the equation of tangent is $y - 0 = \frac{1}{\sqrt{2}}(x-1)$.

24. The integrand is an odd function, so the given integral is zero.

25. $F'(x) = x + 1 > 0$ for $x \in [1, 3]$. Therefore, the greatest value is $F(3)$ and the least value is $F(1)$. The required difference is

$$\int_0^3 (t+1)dt - \int_0^1 (t+1)dt = \int_1^3 (t+1)dt = 6.$$

26. $$\lim_{x\to\infty} \frac{\left(\int_0^x e^x dx\right)^2}{\int_0^x e^2 x^2 dx} = \lim_{x\to\infty} \frac{(e^x-1)^2}{\int_0^x e^{2x^2} dx}$$

$$= \lim_{x\to\infty} \frac{2(e^x-1)e^x}{e^{2x^2}} = 2\lim_{x\to\infty} \frac{1-e^{-x}}{e^{2x^2-2x}} = 0.$$

27. For $x \ge 10$, we have $|\sin x| \le 1$ and $1 + x^8 \ge 10^8$

$$\Rightarrow \quad \frac{1}{1+x^8} \le 10^{-8}$$

Therefore,

$$1\left|\int_{10}^{19} \frac{\sin x}{1+x^8} dx\right| \le \int_{10}^{19} \frac{|\sin x|}{1+x^8} dx \le \int_{10}^{19} 10^{-8} dx$$

$$= 9(10^{-8}) < 10^{-7}$$

28. The integrand can be written as $\frac{\sqrt{x+1}-\sqrt{5x+1}}{-4x}$.

So the given integral is of the form $I_1 + I_2$

$$I_1 = \frac{-1}{4}\int_0^3 \frac{\sqrt{x+1}}{x} dx = -\frac{1}{2}\int_1^2 \frac{t^2}{t^2-1} dt$$

$$= -\frac{1}{2}\int_1^2 \left(1 + \frac{1}{t^2-1}\right) dt$$

$$= -\frac{1}{2}\left(1 - \frac{1}{2}\lim_{t\to 1} \log\frac{|t-1|}{t+1}\right)$$

$(x + 1 = t^2)$

and $$I_2 = \frac{1}{4}\int_0^3 \frac{\sqrt{5x+1}}{x} dx$$

$$= \frac{1}{4}\int_1^4 \frac{2u^2}{u^2-1} du \; (u^2 = 5x+1)$$

$$= \frac{1}{2}\left(3 + \frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\lim_{u\to 1} \log\frac{|u-1|}{u+1}\right)$$

Thus $I_1 + I_2 = 1 + \frac{1}{4}\log\frac{3}{5}$.

29. $f(x) = x - [x]$ so $\int_{-1}^1 f(x)dx = \int_{-1}^0 (x+1)\,dx + \int_0^1 x\,dx$

$$= 0 - \left(\frac{1}{2} - 1\right) + \frac{1}{2} = 1$$

30. $$\int_{-\pi/2}^{\pi/2} \cos t \sin\left(2t - \frac{\pi}{4}\right) dt$$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \cos t\,(\sin 2t - \cos 2t)\,dt\,.$$

$$= -\frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \cos t \cos 2t\,dt$$

$$= -\frac{\sqrt{2}}{2}\int_0^{\pi/2} (\cos 3t + \cos t)\,dt$$

$$= -\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}} + 1\right) = -\frac{1}{\sqrt{2}}.\frac{2}{3} = -\frac{\sqrt{2}}{3}.$$

31. Putting $u = 1/x$

$$I = \int_0^\infty \frac{x\log x}{(1+x^2)} dx = \int_\infty^0 \frac{u\log u}{(1+u^2)^2} du = -I$$

So $I = 0$.

32. $$\int_0^1 |\sin 2\pi x|\,dx = \int_0^{1/2} |\sin 2\pi x|\,dx + \int_{1/2}^1 |\sin 2\pi x|\,dx$$

$$= \int_0^{1/2} \sin 2\pi x\,dx - \int_{1/2}^1 \sin 2\pi x\,dx$$

$$= -\frac{\cos 2\pi x}{2\pi}\bigg|_0^{1/2} + \frac{\cos 2\pi x}{2\pi}\bigg|_{1/2}^1 = \frac{2}{\pi}.$$

33. $$\lim_{x\to\infty} \frac{\int_0^x (\tan^{-1} x)^2\,dx}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{(\tan^{-1} x)^2\sqrt{x^2+1}}{x}$$

(L' Hôpital Rule)

$$= \lim_{x\to\infty} (\tan^{-1} x)^2 \sqrt{1 + \frac{1}{x^2}} = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}.$$

34. $$A(t) = \int_{-1}^0 e^x\,dx + \int_0^t e^{-x}\,dx = 1 - e^{-1} - (e^{-t} - 1)$$

$$\lim_{t\to\infty} A(t) = 2 - e^{-1}.$$

35. Since the integrand is an even function so required integral is equal to $2\int_0^2 |x\cos\pi x|\,dx$

$$= 2\left[\int_0^{1/2} x\cos\pi x\,dx - \int_{1/2}^{3/2} x\cos\pi x\,dx + \int_{3/2}^2 x\cos\pi x\,dx\right]$$

$$= 2\left[x\frac{\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\bigg|_0^{1/2} - \left(x\frac{\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\right)\bigg|_{1/2}^{3/2} + \left(\frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\right)\bigg|_{3/2}^2\right]$$

$$= 2\left[\frac{1}{2\pi} - \frac{1}{\pi^2} - \left(-\frac{3}{2\pi} - \frac{1}{2\pi}\right) + \frac{1}{\pi^2} - \left(-\frac{3}{2\pi}\right)\right] = \frac{8}{\pi}$$

36. Putting $\tan\frac{x}{2} = t$, the given integral reduces to

$$= 2\int_0^{\tan\alpha/2} \frac{dt}{(1-\cos\alpha) + t^2(1+\cos\alpha)}$$

$$= \frac{1}{\cos^2\frac{\alpha}{2}} \int_0^{\tan\alpha/2} \frac{dt}{t^2 + \tan^2\frac{\alpha}{2}}.$$

$$= \frac{1}{\cos^2\frac{\alpha}{2}} \frac{1}{\tan\frac{\alpha}{2}} \tan^{-1}\frac{t}{\tan\frac{\alpha}{2}}\Bigg|_0^{\tan\alpha/2}$$

$$= \frac{1}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}.\frac{\pi}{4} = \frac{\pi}{2\sin\alpha}$$

37. Required limit $= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{n}{(n+r)^2}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{\left(1+\frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{dx}{(1+x)^2} = -\frac{1}{1+x}\Bigg|_0^1.$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}$$

38. Required limit $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{n^3 + r^3}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{n=1}^{n} \frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\log 2.$$

39. For the integrand $f(\pi - x) = -f(x)$.

40. $f'(x) = A\frac{\pi}{2}\cos\frac{\pi}{2}x \Rightarrow \sqrt{2} = f'\left(\frac{1}{2}\right) = \frac{A\pi}{2\sqrt{2}}$

so $A = \frac{4}{\pi}$. Also

$$\frac{2A}{\pi} = \int_0^1 f(x)\,dx = -A\frac{2}{\pi}\cos\frac{\pi}{2}x + Bx\Big|_0^1$$

$$= B + \frac{2A}{\pi} \quad\Rightarrow\quad B = 0.$$

41. $\int_a^b f(x)\,g(x)\,dx = \int_a^b g'(x)\,g(x)\,dx$

$$= \frac{1}{2}\left((g(b))^2 - (g(a))^2\right)$$

42. Let $F(x) = \int_\alpha^x f(t)\,dt$ then $F'(x) = f(x)$.

$$I_1 = \int_a^b F(x)\,dx = x\,F(x)\Big|_a^b - \int_a^b x\,f(x)\,dx$$

$$= bF(b) - \int_a^b x\,f(x)\,dx$$

$$= \int_a^b (b-x)\,f(x)\,dx = I_2.$$

43. $f'(x) = -\log_{1/3} x = \frac{\log x}{\log 3}$. So $F'(x) < 0$ for $\frac{1}{10} < x < 1$ and $F'(x) > 0$ for $x > 1$. Hence f has least value at $x = 1$.

44. $F'(x) = x^2 - 8x + 16 = (x-4)^2 > 0$ for $x \in [0, 5] \sim \{4\}$
So the greatest value of F is $F(5)$

$$= \int_4^5 (t-4)^2\,dt = \frac{1}{3}.$$

45. $\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx$

$$= -\int_a^0 f(-x)\,dx + \int_0^a f(x)\,dx = \int_0^a (f(-x) + f(x))\,dx.$$

46. $J_m = \int_1^e \log^m x\,dx = x\log^m x\Big|_1^e - \int_1^e m\,x\frac{\log^{m-1}x}{x}\,dx$

$= e - m\,J_{m-1}$

So $J_8 + 8\,J_7 = e$.

47. Note that $\frac{d}{dx} f_3(x) = f_2(x)$, $\frac{d}{dx} f_2(x) = f(x)$ and $\frac{d}{dx} f_1(x) = f(x)$. Integrating by parts

$$f_3(x) = t\,f_2(t)\Big|_0^x - \int_0^x t\,f_1(t)\,dt$$

$$= x\,f_2(x) - \int_0^x t\,f_1(t)\,dt = \int_0^x (x-t)\,f_1(t)\,dt$$

$$= -\frac{(x-t)^2}{2} f_1(t)\Big|_0^x + \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt$$

$$= \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt. \text{ Thus } A = \frac{1}{2}.$$

48. $\frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}} \Rightarrow \frac{dy}{dx} = \sqrt{1+9y^2}$

$\frac{d^2y}{dx^2} = \frac{1}{2}\frac{18y}{\sqrt{1+9y^2}}\frac{dy}{dx} = 9y$. Thus the constant of proportionality is equal to 9.

49. Put $g(x) = t \Rightarrow x = f(t) \Rightarrow dx = f'(t)\,dt$

$$I = \int_a^b t\,f'(t)dt + \int_a^b f(x)dx$$
$$= [t\,f(t)]_a^b - \int_a^b 1.\,f(t)dt + \int_a^b f(x)dx$$
$$= h(t)]_a^b = h(b) - h(a)$$

50. $I = \int_0^{[x]} \frac{2^t}{2^{[t]}}\,dt = \int_0^{[x]} 2^{t-[t]}dt$

The function $g(t) = 2^{t-[t]}$ is periodic with period 1, Therefore

$$I = [x]\int_0^1 2^{t-[t]}dt = [x]\int_0^1 2^t\,dt = \frac{[x]}{\log 2}[2t]_0^1$$
$$= \frac{[x]}{\log 2}.$$

51. $n > 1$, $a_n - a_{n-1} = \int_0^{\pi/2} \frac{\sin^{-2} nx - \sin^2(n-1)x}{\sin x}\,dx$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)x\sin x}{\sin x}\,dx$$
$$= \int_0^{\pi/2} \sin(2n-1)x\,dx = -\left.\frac{\cos(2n-1)x}{2n-1}\right|_0^{\pi/2}$$
$$= \frac{1}{2n-1}$$

which is a Harmonic Progression.

52. $\int_0^{21}[x]^4\,dx = \sum_{r=0}^{20}\int_r^{r+1}[x]^4\,dx$

$$= \sum_{r=0}^{20} r^4 = \left(\frac{20\times 21}{2}\right)^2 = 44100$$

53. $I = \int_{\cos\theta}^{\sec\theta} f(x)dx = -\int_{\cos\theta}^{\sec\theta}\frac{1}{x^2}f\left(\frac{1}{x}\right)dx \left(\text{Put } \frac{1}{x} = t\right)$

$$= \int_{\sec\theta}^{\cos\theta} f(t)dt = -\int_{\cos\theta}^{\sec\theta} f(t)\,dt = -I$$

Hence $2I = 0 \Rightarrow I = 0$

54. The integrand is even, so

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = 2\int_0^{\infty}\frac{dx}{1+x^2}$$

$$\int_0^{\infty}\frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b\frac{dx}{1+x^2} = \lim_{b\to\infty}\left(\tan^{-1}b - 0\right) = \frac{\pi}{2}$$

55. Since $0 \le \cos^2 x \le 1$, so

$$\frac{1}{8} \le \frac{1}{5+3\cos^2 x} \le \frac{1}{5}$$

$$\Rightarrow \frac{\pi}{16} \le \int_0^{\pi/2}\frac{dx}{5+3\cos^2 x} \le \frac{\pi}{10}$$

56. $\frac{\cos x}{\sqrt{1-\sin^2 x}} = \frac{\cos}{|\cos x|} = \begin{cases} 1 & x\in[0,\pi/2] \\ -1 & , x\in(\pi/2,\pi)\end{cases}$

$$\int_0^{\pi}\frac{\cos x}{\sqrt{1-\sin^2 x}} = \int_0^{\pi/2}dx - \int_{\pi/2}^{\pi}dx = 0$$

57. Integrating by parts,

$$\int_0^{\pi} x^2\cos x = x^2\sin x\Big|_0^{\pi} - \int_0^{\pi}2x\sin x\,dx = -2J$$

$$J = \int_0^{\pi} x\sin x\,dx = \int_0^{\pi}\pi\sin x\,dx - \int_0^{\pi}x\sin x\,dx$$

$$\Rightarrow 2J = \pi(-\cos x)|_0^{\pi} = 2\pi \Rightarrow J = \pi.$$

58. Set $x^3 + 1 = u$

$$\Rightarrow \int_1^2\frac{10x^2}{\left(x^3+1\right)^2}dx = \frac{10}{3}\int_2^9\frac{du}{u^2} = \frac{10}{3}\left[-\frac{1}{u}\right]_2^9 = \frac{35}{27}.$$

Level 2

59. $I = \int_{\pi/4}^{\pi/3}\frac{\cos x}{\sin x(\cos x+1)}dx$

$$= \int_{\pi/4}^{\pi/3}\frac{\cos x\sin x}{\left(1-\cos^2 x\right)(1+\cos x)}dx$$

Put $\cos x = t$, so that

$$I = \int_{1/2}^{1/\sqrt{2}}\frac{t\,dt}{(1+t)^2(1-t)}$$

$$= \int_{1/2}^{1/\sqrt{2}}\left[\frac{1}{4}\left(\frac{1}{1+t}+\frac{1}{1-t}\right) - \frac{1}{2(1+t)^2}\right]dt$$

$$= \left[\frac{1}{4}\log\frac{1+t}{1-t} + \frac{1}{2(1+t)}\right]_{1/2}^{1/\sqrt{2}}$$

$$= \frac{1}{4}\log\left[\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\left(\frac{2}{3}\right)\right] + \frac{1}{2+\sqrt{2}} - \frac{1}{3}.$$

60. $\int_a^b[f(x) - fave]\,dx$

$$= \int_a^b f(x)\,dx - (fave)\int_a^b dx$$

$$= \int_a^b f(x)\,dx - \left[\frac{1}{b-a}\int_a^b f(x)\,dx\right](b-a)$$

$= 0.$

61. $\int_0^{2\pi} \cos mx \cos nx\, dx$

$= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]dx$

$= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x\right]_0^{2\pi}$

$= 0.$

62. For $r \neq k,\ r,\ k \in \mathbf{N}$,

$\int_0^{2\pi} \sin k\, x \sin r\, x$

$= \frac{1}{2}\int_0^{2\pi} [\cos(k+r)x + \cos(k-r)x]\, dx = 0$

and $\int_0^{2\pi} \sin kx \sin kx\, dx = \frac{1}{2}\int_0^{2\pi} [1 - \cos(2kx)]\, dx$

$= \frac{1}{2}\ (2\pi) = \pi$

Also, $\int_0^{2\pi} \sin kx \cos(rx)\, dx = 0\ \forall\ r,\ k \in \mathbf{N}$

Thus, $\int_0^{2\pi} Q(x) \sin(kx)\, dx = \pi a_k.$

63. Imitate Example 101.

64. $\int_{1/2}^{1}\left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right]$

$= x^2 \sin\left(\frac{1}{x}\right)\Big]_{1/2}^{1} - \int_{1/2}^{1} x^2\left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right) dx$

$- \int_{1/2}^{1} \cos\left(\frac{1}{x}\right) dx$

$= \sin\ (1) - \frac{1}{4} \sin\ (2).$

65. $F'(x) = f(x) > 0$

$\Rightarrow$ F is differentiable and increasing on $[a, b]$.

66. First show that $f(x) = 2x,\ x \in \mathbf{R}$

Now, $\int_0^1 f(x)\, g(x)\, dx$

$= \int_0^1 2x(e^x - 2x)\, dx$

$= 2xe^x\Big]_0^1 - 2\int_0^1 e^x dx - \frac{4}{3}x^3\Big]_0^1$

$= 2e - 2\ (e - 1) - \frac{4}{3} = \frac{2}{3}.$

67. $I = \int_0^{\pi/2} \operatorname{cosec}(x + \pi/3) \operatorname{cosec}(x + \pi/6)\, dx$

$= \frac{1}{\sin(\pi/6)}\int_0^{\pi/2} \frac{\sin[(x+\pi/3)-(x+\pi/6)]}{\sin(x+\pi/3)\sin(x+\pi/6)}\, dx$

$= 2\int_0^{\pi/2} [\cot(x + \pi/6) - \cot(x + \pi/3)]\, dx$

$= 2 \log\left|\frac{\sin(x+\pi/6)}{\sin(x+\pi/3)}\right|\Bigg]_0^{\pi/2}$

$= 2\left[\log\left(\frac{\sqrt{3}/2}{1/2}\right) - \log\left(\frac{1/2}{\sqrt{3}/3}\right)\right]$

$= 2 \log\ 3.$

68. $I = \int_0^1 \log\left(\sqrt{1-x} + \sqrt{1+x}\right) dx$

$= x \log\ \left(\sqrt{1-x} + \sqrt{1+x}\right)\Big]_0^1$

$- \int_0^1 x\left(\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\right)\left(\frac{-1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}}\right) dx$

$= \log\ (\sqrt{2})$

$+ \frac{1}{2}\int_0^1 \frac{x}{\sqrt{1-x}+\sqrt{1+x}}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1-x^2}}\right)\ dx$

$= \frac{1}{2}\log 2$

$+ \frac{1}{2}\int_0^1 \frac{x}{(1+x)-(1-x)}\ \frac{1+x+1-x-2\sqrt{1-x^2}}{\sqrt{1-x^2}}\, dx$

$= \frac{1}{2}\log 2 + \frac{1}{2}\int_0^1\left(\frac{1}{\sqrt{1-x^2}} - 1\right) dx$

$= \frac{1}{2}\left(\log 2 + \frac{\pi}{2} - 1\right).$

69. For $n \geq 1,\ 0 \leq x \leq 1/2,\ x^{2n} \leq x^2,$

$\Rightarrow 0 \leq 1 - x^2 \leq 1 - x^{2n} \Rightarrow \frac{1}{\sqrt{1-x^{2n}}} \leq \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} \leq \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big]_0^{1/2}$

$= \frac{\pi}{6} < 1.$

70. $8 = f'(1) = 2A + B$

$33 = f\ (2) + f''(2) = 6A + 2B + C$

Also, $\frac{7}{3} = \frac{1}{3}A + \frac{1}{2}B + C$

$\Rightarrow 14 = 2A + 3B + 6C$

Solving we obtain

$A = 7,\ B = -6,\ C = 3$

71. $y = \int_x^{x^2} \log t\, dt = (t \log t - t)\Big]_x^{x^2}$

$= (2x^2 - x) \log x - x^2 + x$

$\Rightarrow y(2) = 6 \log 2 - 2$

Also, $y'(2) = 7 \log 2$

Thus, equation of tangent at $x = 2$ is

$y - (6 \log 2 - 2) = (7 \log 2)(x - 2)$

or $y + 8 \log 2 + 2 = (7 \log 2)(x)$

72. $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin\theta}{2+\sin\theta}\right) d\theta = 0$, as the integrand is an odd function.

73. Put $\sin x - \cos x = t$, so that

$$I = \int_{-1}^{0} \frac{dt}{3+1-t^2}$$

$$= \frac{1}{2(2)} \log\left|\frac{2+t}{2-t}\right|\Bigg]_{-1}^{0} = \frac{1}{4}\log 3.$$

74. In I_2, put $t = 1/u$ to obtain

$$I_2 = \int_1^x \frac{(-1/u^2)\,du}{1+1/u^2} = \int_x^1 \frac{dt}{1+t^2} = I_1$$

75. $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{12}$

$\Rightarrow \sec^{-1} x - \sec^{-1}\sqrt{2} = \pi/12$

$\Rightarrow \sec^{-1} x = \pi/3$

$\Rightarrow x = \sec(\pi/3) = 2.$

76. Average $= \frac{1}{2}\int_0^2 \frac{2}{e^x+1}\,dx = \int_0^2 \frac{e^{-x}}{1+e^{-x}}\,dx$

$= -\log(1+e^{-x})\Big]_0^2$

$= \log 2 - \log(1 + e^{-2})$

$= 2 + \log[2/(1 + e^2)].$

77. $I = \int_0^\pi x \log(\sin x)\,dx$

$$= \int_0^\pi (\pi - x)\log[\sin(\pi - x)]\,dx$$

$$\Rightarrow 2I = \pi\int_0^\pi \log \sin x\,dx$$

$$= \pi\left[\int_0^{\pi/2} \log \sin x\,dx + \int_0^{\pi/2} \log\sin(\pi - x)\,dx\right]$$

$$\Rightarrow I = \pi\int_0^{\pi/2} \log(\sin)\,dx = \frac{\pi^2}{2}\log 2.$$

78. $I = \int_{-\pi}^{\pi}\left[\cos^2 px + \sin^2 qx - 2\cos px \sin qx\right]dx$

$$= \int_0^\pi [2 + \cos 2px - \cos 2qx]\,dx = 2\pi.$$

[$\because$ third term in the integrand is odd]

79. $I = \int_0^a f(x)g(x)h(x)\,dx$

$$= \int_0^a f(a-x)g(a-x)h(a-x)\,dx$$

$$= \int_0^a f(x)\{-g(x)\}\left[\frac{3}{4}h(x) - \frac{5}{4}\right]dx$$

$$\frac{7}{4}I = \frac{5}{4}\int_0^a f(x)g(x)\,dx$$

$$\Rightarrow I = \frac{5}{7}\,I_1$$

where $I_1 = \int_0^a f(x)g(x)\,dx$

$$= \int_0^a f(a-x)g(a-x)\,dx$$

$$= \int_0^a f(x)[-g(x)]\,dx = -I_1$$

$\Rightarrow 2I_1 = 0 \Rightarrow I_1 = 0$

Thus, $I = 0$.

80. Differentiating w·r·t· x, we get

$$\sqrt{3 - 2\sin^2 x} + \frac{dy}{dx}\cos y = 0$$

Putting $x = \pi/2$ and $y = \pi$, we get

$$\left.\frac{dy}{dx}\right]_{(\pi/2,\,\pi)} = 1.$$

81. We have

$P'(x) = a(x - 1)(x - 3)$

$\quad = a[(x - 2)^2 - 1]$

$\Rightarrow P(x) = a\left[\frac{1}{3}(x-2)^3 - x\right] + b$

As $6 = P(1) = -\frac{4}{3}a + b$

$2 = P(3) = -\frac{8}{3}a + b$

$\Rightarrow a = 3,\ b = 10$

Thus

$P(x) = (x - 2)^3 - 3x + 10$

$\Rightarrow \int_0^1 P(x)\,dx = \frac{19}{4}.$

82. We have $P''(x) = a(x + 1)x(x - 1)$

$= a\,(x^3 - x)$

$\Rightarrow P'(x) = a\left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right) + b$

$\Rightarrow P(x) = a\left(\frac{1}{20}x^5 - \frac{1}{6}x^3\right) + bx + c$

As

$-1 = a\left(-\frac{1}{20} + \frac{1}{6}\right) - b + c$

$1 = a\left(\frac{1}{20} - \frac{1}{6}\right) + b + c$

$\Rightarrow c = 0$

and $1 = -\frac{7}{60}a - a + b$

Also, $\sqrt{3} = b$

$\Rightarrow a = 60\left(\sqrt{3}-1\right)/7$

We have

$\int_0^1 P(x)\,dx = -\frac{1}{30}a + \frac{1}{2}b$

$= \frac{2}{7}\left(1-\sqrt{3}\right) + \frac{\sqrt{3}}{2}$.

83. $S_n = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}e^{k/n}$

$\lim_{n\to\infty} S_n = \int_0^1 x\,e^x\,dx = 1.$

84. As $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^{k+2}\,2^k}{k!} = x^2\sum_{k=0}^{\infty}\frac{(2x)^k}{k!} = x^2\,e^{2x}$

Thus, $\int_0^1 \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{x^{k+2}\,2^k}{k!}\right)dx$

$= \int_0^1 x^2\,e^{2x}\,dx = \frac{1}{4}\left(2x^2 - 2x + 1\right)e^{2x}\Big]_0^1$

$= \frac{1}{4}(e^2 - 1).$

85. Let $I = \int_0^1 \frac{x^{2\alpha}-1}{\log x}\,dx$

$\Rightarrow \frac{dI}{d\alpha} = \int_0^1 \frac{2x^{2\alpha}\log x}{\log x}\,dx = \frac{2}{2\alpha+1}\,x^{2\alpha+1}\Big]_0^1$

$= \frac{2}{2\alpha+1}$

$\Rightarrow I = \log\,(2\alpha + 1) + C$

When $\alpha = 0$, $I = 0 \Rightarrow C = 0$

Thus, $I = \log\,(2\alpha + 1) = \log\,(2n)$.

86. $\sin\alpha + \int_{\alpha}^{2\alpha}\cos 2x\,dx = 0$

$\Rightarrow \sin\alpha\ \frac{1}{2}\ (\sin 4\alpha - \sin 2\alpha) = 0$

$\Rightarrow \sin\alpha\,[1 + \cos(3\alpha)] = 0$

As $-\pi < \alpha < 0$, $\sin\alpha \neq 0$, therefore

$\cos(3\alpha) = -1 \Rightarrow \alpha = -\pi/3$.

87. $\int_0^9 g(x)\,dx = \int_0^1 dx + \int_1^4 x^3 dx + \int_4^9 \sqrt{x}\,dx$

$= 1 + \frac{1}{4}(4^4 - 1) + \frac{2}{3}(9^{3/2} - 4^{3/2})$

$= \frac{929}{12}$.

88. $\lim_{x\to 0}\frac{\int_{-x}^{x} f(2t)\,dt}{\int_0^{2x} f(t+4)\,dt}$ $\quad\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\frac{(1)f(2x) - (f(-2x))(-1)}{2f(2x+4)}$

$= \lim_{x\to 0}\frac{f(2x) + f(-2x)}{2f(2x+4)}$

$= \frac{f(0)+f(0)}{2f(4)}$ $\quad$ [$\because f$ is continuous]

$= \frac{f(0)}{f(4)}$

89. $\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} e^{\cos x}\sin x\,dx + \int_2^3 2\,dx$

$= 0 + 2 = 2.$

[integrand of first integral is odd]

90. Let $I = \int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}\,dx$ $\quad$ (1)

Put $x = -\theta$, to obtain

$I = \int_{\pi}^{-\pi}\frac{\cos^2\theta}{1+a^{-\theta}}(-1)d\theta = \int_{-\pi}^{\pi}\frac{a^x\cos^2 x}{1+a^x}\,dx$ $\quad$ (2)

Adding (1) and (2), we get

$2I = \int_{-\pi}^{\pi}\cos^2 x\,dx = 2\int_0^{\pi}\cos^2 x\,dx = \pi$

$\Rightarrow I = \pi/2.$

91. $F(x) = \int_0^{\pi} f(t)\,dt$

$\Rightarrow F'(x) = f(x)$

$\therefore f(4) = F'(2)$

Also, $2xF'(x^2) = 2x\,(1 + x) + x^2$

$\Rightarrow F'(x^2) = 1 + 3x/2$

$\therefore f(4) = F'(2^2) = 1 + 3 = 4.$

92. $f'(x) = \sqrt{2-x^2}$

Now, $x^2 - f'(x) = 0$

$\Rightarrow x^2 = \sqrt{2-x^2}$

$\Rightarrow x = \pm 1$

93. $I\,(m, n) = \left.\frac{1}{m+1} t^{m+1}(1+t)^n\right]_0^1$

$-\frac{n}{m+1} I\,(m + 1, n - 1)$

$\Rightarrow I\,(m, n) = \frac{2^n}{m+1} - \frac{n}{m+1}\; I\,(m + 1, n - 1)$

94. $I = \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx$

$= \left.\left(\sin^{-1} x + \sqrt{1-x^2}\right)\right]_0^1 = \frac{\pi}{2} - 1.$

95. Differentiating both the sides of

$\int_0^{t^2} x\, f(x) = \frac{2}{5} t^5$

we get

$(2t)\,(t^2)\,f\,(t^2) = 2t^4$

$\Rightarrow f\,(t^2) = t$

$\Rightarrow f\left(\frac{4}{25}\right) = \frac{2}{5}.$

96. $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]\,dx$

$= \int_{-2}^{0} \left[(x+1)^3 + 2 + (x+1)\cos(x+1)\right] dx$

Put $x + 1 = t$, therefore,

$I = \int_{-1}^{1} [t^3 + 2 + t\cos t]\,dt$

$= 4$ [$\because$ $t^3 + t \cos t$ is an odd function].

Previous Years' AIEEE/JEE Main Questions

1. $\int_0^{\sqrt{2}} [x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx$

$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx$

$= \sqrt{2} - 1$

2. $I_n = \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)\,dx$

$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx - I_{n-2}$

$\Rightarrow I_n + I_{n-2} = \left.\frac{\tan^{n-1} x}{n-1}\right|_0^{\pi/4} = \frac{1}{n-1}$

$\Rightarrow n(I_n + I_{n-2}) = \frac{n}{n-1} = \frac{1}{1-\frac{1}{n}}$

$\Rightarrow \lim_{n\to\infty} n(I_n + I_{n-2}) = 1.$

3. $\int_{\pi}^{10\pi} |\sin x|\,dx = \int_0^{9\pi} |\sin(\pi+u)|\,du = \int_0^{9\pi} |\sin u|\,du$

$= 9\int_0^{\pi} |\sin u|\,du$ ($|\sin u|$ has period π)

$= 9 \int_0^{\pi} \sin u\,du = -9 \cos u|_0^{\pi}$

$= 18$

4. $\int_0^2 x f'(x)dx = x f(x)|_0^2 - \int_0^2 f(x)dx$

$= 2 f(2) - \int_0^2 f(x)dx$

$= 0 - \frac{3}{4} = -\frac{3}{4}$

5. $f'(x) = f(x) \Rightarrow \frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx$

$\Rightarrow \log y = x + \text{const.}$

$\Rightarrow y = Ce^x$. Since $y(0) = 1$ so $C = 1$

Hence $f(x) = e^x$,

$\int_0^1 f(x)g(x)\,dx = \int_0^1 e^x(x^2 - e^x)\,dx$

$= \int_0^1 x^2 e^x\,dx - \int_0^1 e^{2x}\,dx$

$= \left[x^2 e^x\Big|_0^1 - 2\int_0^1 x e^x\,dx\right] - \left.\frac{e^{2x}}{2}\right|_0^1$

$= \left[e - 2\left[xe^x\Big|_0^1 - \int_0^1 e^x\,dx\right]\right] - \frac{1}{2}\,(e^2 - 1)$

$= [e - 2e + 2e - 2] - \frac{1}{2}\,(e^2 - 1)$

$= e - 2 - \frac{1}{2}e^2 + \frac{1}{2} = e - \frac{1}{2}e^2 - \frac{3}{2}$

6. $F(t) = \int_0^t f(t-y)g(y)\,dy = \int_0^t e^{t-y} y\,dy$

$= e^t \int_0^t y\, e^{-y}\, dy$

$= e^t \left[-y e^{-y}\Big|_0^t + \int_0^t e^{-y}\, dy \right]$

$= e^t\, [-\, t\, e^{-t} - [e^{-t} - 1]$

$= -\, t - 1 + e^t = e^t - (t + 1).$

7. $\int_a^b x\, f(x)\, dx = \int_a^b (a+b-x) f(a+b-x)\, dx$

$= \int_a^b (a+b-x) f(x)\, dx$

$= (a + b) \int_a^b f(x)\, dx - \int_a^b x\, f(x)\, dx$

$\Rightarrow 2 \int_a^b x\, f(x)\, dx = (a + b) \int_a^b f(x)\, dx$

$\Rightarrow \int_a^b x\, f(x)\, dx = \frac{a+b}{2} \int_a^b f(x)\, dx$

8. $I = \int_0^1 x(1-x)^n\, dx = \int_0^1 (1-x)(1-(1-x))^n\, dx$

$= \int_0^1 (1-x) x^n\, dx$

$= \int_0^1 x^n\, dx - \int_0^1 x^{n+1}\, dx$

$= \frac{1}{n+1} - \frac{1}{n+2}.$

9. $\lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t\, dt}{x \sin x}$

$= \lim_{x\to 0} \frac{\tan t\big|_0^{x^2}}{x \sin x} = \lim_{x\to 0} \frac{\tan x^2}{x^2} \cdot \frac{x}{\sin x} = 1$

10. $\int_{-2}^{3} |1 - x^2|\, dx$

$= \int_{-2}^{-1} |1-x^2|\, dx + \int_{-1}^{1} |1-x^2|\, dx + \int_1^3 |1-x^2|\, dx$

$= -\int_{-2}^{-1} (1-x^2)\, dx + \int_{-1}^{1} (1-x^2)\, dx - \int_1^3 (1-x^2)\, dx$

$= -x\big|_{-2}^{-1} + \frac{x^3}{3}\Big|_{-2}^{-1} + x\big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1} - x\big|_1^3 + \frac{x^3}{3}\Big|_1^3$

$= -\, 1 + \frac{7}{3} + 2 - \frac{2}{3} - 2 + \frac{26}{3} = \frac{28}{3}.$

11. We have $1 + \sin 2x = (\sin x + \cos x)^2$. Thus,

$I = \int_0^{\pi/2} (\sin x + \cos x)\, dx$

$= (-\cos x + \sin x)]_0^{\pi/2}$

$= -\cos \frac{\pi}{2} + \sin \frac{\pi}{2} - (-\cos 0 + \sin 0) = 2$

12. Let $I = \int_0^{\pi} x f(\sin x) dx = \int_0^{\pi} (\pi - x) f\{\sin(\pi - x)\} dx$

$= \pi \int_0^{\pi} f(\sin x) dx - I$

$2I = \pi \int_0^{\pi/2} [f(\sin x) + f\{\sin(\pi - x)\}] dx$

$I = \pi \int_0^{\pi/2} f(\sin x)\, dx$

13. $f(a) + f(-a) = \frac{e^a}{1+e^a} + \frac{e^{-a}}{1+e^{-a}} = 1$

Let $f(-a) = b$, so $f(a) = 1 - b$

$I_1 = \int_b^{1-b} x g(x(1-x))\, dx$

$= \int_b^{1-b} (1-b+b-x)\ g((1 - b + b - x)\ (1 - 1 + b - b + x))dx$

$= \int_b^{1-b} (1-x)\ g((1 - x)x)dx$

$= I_2 - I_1$

$\Rightarrow 2I_1 = I_2 \Rightarrow \frac{I_2}{I_1} = 2.$

14. The given limit can be written as

$\lim_{n\to\infty} \frac{1}{n} \sum \frac{k}{n} \sec^2 \left(\frac{k}{n}\right)^2 = \int_0^1 x \sec^2 x^2\, dx$

$= \frac{1}{2} \tan x^2 \Big|_0^1 = \frac{1}{2}\ \tan 1.$

15. $\lim_{x\to 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt$

$= \lim_{x\to 2} \frac{1}{x-2} t^4 \Big]_6^{f(x)}$

$= \lim_{x\to 2} \frac{[f(x)]^4 - 6^4}{x-2}$

$= \lim_{x\to 2} \left(\frac{f(x)-6}{x-2} \right) (f(x) + 6)\ ([f(x)]^2 + 36)$

$= f'(2)\ (f(2) + 6)\ ([f(2)]^2 + 36)$

$= \frac{1}{48}\ (6 + 6)\ [36 + 36) = 18.$

16. For $0 < x < 1$, $x^2 > x^3$

$\Rightarrow 2^{x^2} > 2^{x^3} \Rightarrow \int_0^1 2^{x^2} dx > \int_0^1 2^{x^3} dx$ or $I_1 > I_2$

For $1 < x < 2$, $x^2 < x^3$

$\Rightarrow \int_1^2 2^{x^2} dx < \int_1^2 2^{x^3} dx$ or $I_3 < I_4$

17. Putting $x = -y$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$$

we get

$$I = \int_{\pi}^{-\pi} \frac{[\cos(y)]^2}{1+a^{-y}} (-1) dy = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_{-\pi}^{\pi} \frac{(1+a^x)\cos^2 x}{1+a^x} dx = \int_{-\pi}^{\pi} \cos^2 x\, dx$$

$$= 2 \int_0^{\pi} \cos^2 x\, dx = \int_0^{\pi} (1+\cos 2x)\, dx$$

$$= \left(x + \frac{1}{2}\sin 2x\right)\Big]_0^{\pi} = \pi$$

$\Rightarrow I = \pi/2.$

18. Let $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}} dx \quad (1)$

$$= \int_3^6 \frac{\sqrt{3+6-x}}{\sqrt{9-(3+6-x)}+\sqrt{3+6-x}} dx$$

$$= \int_3^6 \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}} dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_3^6 dx = 6 - 3 = 3$$

$$\Rightarrow I = \frac{3}{2}.$$

19. Put $x + \pi = \theta$, so that

$$I = \int_{-\pi/2}^{\pi/2} [\theta^3 + \cos^2(\theta + 2\pi)]\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

20. Let $I = \int_0^{\pi} x f(\sin x)\, dx$

$$= \int_0^{\pi} (\pi - x) f\{\sin(\pi - x)\}\, dx$$

$$= \pi \int_0^{\pi} f(\sin x)\, dx - I$$

$$\Rightarrow 2I = \pi \int_0^{\pi/2} [f(\sin x) + f(\sin(\pi - x))]\, dx$$

$$= 2\pi \int_0^{\pi/2} f(\sin x) dx$$

$$\Rightarrow I = \pi \int_0^{\pi/2} f\{\sin(\pi/2 - x)\} dx$$

$$= \pi \int_0^{\pi/2} f(\cos x) dx$$

21. Let $m \le a < m + 1$, $m \in \mathbf{N}$, so that $[a] = m$

$$\int_1^a [x] f'(x) dx$$

$$= \int_1^2 f'(x) dx + \int_2^3 2 f'(x) dx + \cdots + \int_{m-1}^{m} (m-1) f'(x) dx$$

$$+ \int_m^a m f'(x) dx$$

$= (f(2) - f(1)) + 2\,(f(3) - f(2)) + \ldots$

$+ (m-1)\,(f(m) - f(m-1)) + m(f(a) - f(m))$

$= mf(a) - [f(1) + f(2) + \ldots + f(m)]$

$= [a]\, f(a) - (f(1) + \ldots + f([a])).$

22. $F(e) = f(e) + f\left(\frac{1}{e}\right)$

$$= \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$$

In the second integral, put $t = 1/u$ to obtain

$$\int_1^{1/e} \frac{\log t}{1+t} dt = \int_1^e \frac{\log(1/u)}{1+1/u}\left(-\frac{1}{u^2}\right) du$$

$$= \int_1^e \frac{\log u}{u(1+u)} du$$

$$\therefore F(e) = \int_1^e \left[\frac{\log t}{1+t} + \frac{\log t}{t(1+t)}\right] dt$$

$$= \int_1^e \frac{\log t}{1+t}\left(1 + \frac{1}{t}\right) dt$$

$$= \int_1^e \frac{\log t}{t} dt = \frac{1}{2}(\log t)^2\Big]_1^e$$

$$= \frac{1}{2}.$$

23. $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$

$$\Rightarrow \sec^{-1} t\Big]_{\sqrt{2}}^{x} = \frac{\pi}{2}$$

$\Rightarrow \sec^{-1} x - \sec^{-1} \sqrt{2} = \frac{\pi}{2}$

$\Rightarrow \sec^{-1} x = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

$\Rightarrow x = \sec\left(\frac{3\pi}{4}\right) = -\sqrt{2}$

24. For $0 < x < 1$, $\sin x < x$, so

$$I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx = \int_0^1 \sqrt{x}\, dx = \frac{2x^{3/2}}{3}\Bigg|_0^1 = \frac{2}{3}$$

$$I = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\Big|_0^1 = 2$$

25. Using $\int_0^{2a} f(x)dx = \int_0^a [f(x) + f(2a-x)]dx$, we can write

$$I = \int_0^{\pi} [\cot x]dx = \int_0^{\pi/2} ([\cot x] + [\cot(\pi - x)])dx$$

$$= \int_0^{\pi/2} ([\cot x] + [-\cot x])dx$$

Put $\cot x = t$, so that

$$I = \int_0^{\infty} ([t] + [-t])\frac{dt}{1+t^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{k-1}^{k} ([t] + [-t])\frac{dt}{1+t^2}$$

But $[t] + [-t] = -1$ for $k - 1 < t < k$, therefore

$$\int_{k-1}^{k} ([t] + [-t])\frac{dt}{1+t^2} = \int_{k-1}^{k} (-1)\frac{dt}{1+t^2} =$$

$- [\tan^{-1} k - \tan^{-1} (k-1)]$

$$\therefore\ I = -\lim_{n\to\infty} \sum_{k=1}^{n} (\tan^{-1} k - \tan^{-1}(k-1))$$

$$= -\lim_{n\to\infty} [\tan^{-1} n - \tan^{-1} 0] = -\frac{\pi}{2}.$$

26. $\int_0^1 p(x)dx = \int_0^1 1.p(x)dx = [xp(x)]_0^1 - I_1 = p(1) - I_1$

where

$$I_1 = \int_0^1 xp'(x)dx = \int_0^1 (1-x)p'(1-x)dx$$

$$= \int_0^1 (1-x)p'(x)dx = \int_0^1 p'(x)dx - I_1$$

$\Rightarrow 2I_1 = [p(x(x)]_0^1 = p(1) - p(0)$

Thus,

$$\int_0^1 p(x)dx = p(1) - \frac{1}{2}(p(1) - p(0))$$

$$= \frac{1}{2}(p(1) + p(0)) = \frac{1}{2}(41 + 1) = 21$$

27. Let $I = 8\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Put $x = \tan\theta$, so that

$$I = 8\int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta\, d\theta$$

$$= 8\int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= 8\int_0^{\pi/4} \log\left[1 + \frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$= 8\int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= 8\left[\int_0^{\pi/4} (\log 2)dx - \int_0^{\pi/4} \log(1+\tan\theta)d\theta\right]$$

$$= 8\left[\frac{\pi}{4}\log 2\right] - I$$

$\Rightarrow 2I = 2\pi \log 2 \Rightarrow I = \pi \log 2.$

28. $F'(x) = \sqrt{x} \sin x$

For local maximum or local minimum, set $f'(x) = 0$

$\Rightarrow x = \pi, 2\pi$ as $x \in (0, 5\pi/2)$

As $\sqrt{x} > 0$ for $0 < x < 5\pi/2$

$f'(x) > 0$ for $0 < x < \pi$

< 0 for $\pi < x < 2\pi$

> 0 for $2\pi < x < 5\pi/2$

By the first derivative test, f has a local maximum at $x = \pi$ and a local minimum at $x = 2\pi$.

29. $\int_0^{1.5} x[x^2]dx = \int_0^1 x[x^2]dx + \int_1^{\sqrt{2}} x[x^2]dx + \int_{\sqrt{2}}^{1.5} x[x^2]dx$

$$= 0 + \int_1^{\sqrt{2}} x\,dx + 2\int_{\sqrt{2}}^{1.5} x\,dx$$

$$= 0 + \frac{x^2}{2}\Bigg|_1^{\sqrt{2}} + 2\frac{x^2}{2}\Bigg|_{\sqrt{2}}^{1.5} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

30. $g(x+\pi) = \int_0^{x+\pi}\cos(4t)dt = I_1 + I_2$

Where $I_1 = \int_0^{\pi}\cos(4t)dt = g(\pi)$ and

$I_2 = \int_{\pi}^{x+\pi}\cos(4t)dt$

In I_2, put $t = \pi + \theta$, so that

$I_2 = \int_0^x \cos(4\pi + 4\theta)d\theta = \int_0^x \cos(4\theta)d\theta$

$= g(x)$.

However, $g(\pi) = \int_0^{\pi}\cos 4t dt = \frac{1}{4}(\sin 4t)\Big]_0^{\pi}$

$= \frac{1}{4}(\sin 4\pi - \sin 0) = 0.$

Therefore, $g(x+\pi) = g(x) + g(\pi)$ and $g(x+\pi) = g(x) - g(\pi)$ both are correct options.

31. $\frac{dy}{dx} = |x|$, so $|x| = 2 \Rightarrow x = \pm 2$. If $x = 2$, then y

$= \int_0^2 |t|\,dt = \int_0^2 t dt = \frac{t^2}{2}\Big|_0^2 = 2$. Hence the equation

tangent is $Y - 2 = 2(X-2)$. The intercept on x-axis is given by $0 - 2 = 2(X-2) \Rightarrow X = 1$.

If $x = -2$, then $y = \int_0^{-2}|t|\,dt = -\int_0^{-2} t dt = -\frac{t^2}{2}\Big|_0^{-2}$

$= -2$

Hence the equation of tangent is

$Y + 2 = 2(X+2)$

The x-axis intercept is given by $0 + 2 = 2(X+2)$

$\Rightarrow X = -1$.

32. $I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}$

$= \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$

$= \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}} = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}dx$

$2I = I + I$

$= \int_{\pi/6}^{\pi/3}\left(\frac{1}{1+\sqrt{\tan x}} + \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\right)dx$

$= \int_{\pi/6}^{\pi/3}dx = \frac{\pi}{6}$

33. $I = \int_{-\pi/2}^{\pi/2}\frac{\sin^2 x}{1+2^x}dx = \int_{-\pi/2}^{\pi/2}\frac{\sin^2 x}{1+2^{-x}}dx$

$= \int_{-\pi/2}^{\pi/2}\frac{2^x\sin^2 x}{1+2^x}dx$

$2I = I + I = \int_{-\pi/2}^{\pi/2}\sin^2 x\,dx = 2\int_0^{\pi/2}\sin^2 x\,dx$

$= \int_0^{\pi/2}(1-\cos 2x)\,dx = \frac{\pi}{2} - \frac{\sin 2x}{2}\Big|_0^{\pi/2}$

$= \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$

34. $\frac{dx}{dy} = \frac{1}{\sqrt{1+y^2}} \Rightarrow \frac{dy}{dx} = \sqrt{1+y^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{1+y^2}}\cdot 2y\frac{dy}{dx} = y.$

35. $\int_{7\pi/4}^{7\pi/3}\sqrt{\tan^2 x}\,dx = \int_{7\pi/4}^{7\pi/3}|\tan x|dx$

$= \int_{7\pi/4}^{2\pi}|\tan x|dx + \int_{2\pi}^{7\pi/3}|\tan x|dx$

$= -\int_{7\pi/4}^{2\pi}\tan x\,dx + \int_{2\pi}^{7\pi/3}\tan x\,dx$

$= -\log|\sec x|\Big|_{7\pi/4}^{2\pi} + \log|\sec x|\Big|_{2\pi}^{7\pi/3}$

$= -\left[\log 1 - \log\sqrt{2}\right] + (\log 2 - \log 1)$

$= \log 2\sqrt{2}$

36. $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\,dx = \int_0^{\pi}\left|2\sin\frac{x}{2} - 1\right|dx$

$= 2\int_0^{\pi/2}|2\sin t - 1|dt$

$= 2\left[\int_0^{\pi/6}|2\sin t - 1|dt + \int_{\pi/6}^{\pi/2}|2\sin t - 1|\,dt\right]$

$= 2\left[-\int_0^{\pi/6}(2\sin t - 1)dt + \int_{\pi/6}^{\pi/2}(2\sin t - 1)dt\right]$

$= 2\left[2\cos t\Big|_0^{\pi/6} + \frac{\pi}{6} + -2\cos t\Big|_{\pi/6}^{\pi/2} - \frac{\pi}{3}\right]$

$= 2\left[\sqrt{3} - 2 + \frac{\pi}{6} + \sqrt{3} - \frac{\pi}{3}\right]$

$$= 2\left[2\sqrt{3}-2-\frac{\pi}{6}\right]$$

$$= 4\sqrt{3}-4-\frac{\pi}{3}$$

37. $\int_0^{1/2}\frac{\log(1+2x)}{1+4x^2}dx = \frac{1}{2}\int_0^{\pi/4}\log(1+\tan t)dt$

$(2x = \tan t)$

$$I = \frac{1}{2}\int_0^{\pi/4}\log(1+\tan t)dt = \frac{1}{2}\int_0^{\pi/4}\log\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/4}\log\left(1+\frac{1-\tan t}{1+\tan t}\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/4}(\log 2-\log(1+\tan t))dt$$

$$= \frac{1}{2}\frac{\pi}{4}\log 2 - I$$

$$I = \frac{\pi}{16}\log 2$$

38. $P_n = \int_1^e(\log x)^n dx = x(\log x)^n \Big|_1^e - n\int_1^e(\log x)^{n-1}dx$

$= e - n\,P_{n-1}$

So $P_n = e - n[e-(n-1)\,P_{n-2}]$

Putting $n = 10$, $P_{10} = e - 10e + 10.9\,P_8$
$\Rightarrow P_{10} - 90\,P_8 = -9e$

39. $\int_0^{\pi}[\cos x]dx = \int_0^{\pi/2}[\cos x]dx + \int_{\pi/2}^{\pi}[\cos x]dx$

$$= 0 + \int_{\pi/2}^{\pi}(-1)dx = -\frac{\pi}{2}$$

40. Differentiating both sides, we have

$f(t) + t = -2t \Rightarrow f(t) = -3t,$

so $f\left(-\frac{\pi}{3}\right) = \pi.$

41. Let $I = \int_1^x \frac{e^t}{t+a}dt$

Put $t + a = u$, then

$$I = \int_{1+a}^{x+a}\frac{e^{u-a}}{u}du$$

$$= e^{-a}\left[\int_1^{x+a}\frac{e^u}{u}du - \int_1^{1+a}\frac{e^u}{u}du\right]$$

$$= e^{-a}[F(x+a) - F(1+a)]$$

42. $I = \int_2^4\frac{\log(x^2)}{\log(x^2)+\log((6-x)^2)}dx$ (1)

$$= \int_2^4\frac{\log((2+4-x)^2)}{\log((2+4-x)^2)+\log((6-(2+4-x))^2)}$$

$= \int_2^4\frac{\log((6-x)^2)}{\log((6-x)^2)+\log(x^2)}dx$ (2)

Adding (1) and (2), we get

$$2I = \int_2^4\frac{\log(x^2)+\log((6-x)^2)}{\log(x^2)+\log((6-x)^2)}dx$$

$2I = 2 \Rightarrow I = 1$

43. $f(4 + x) = f(4 - x)$

$= f(2 + (2 - x))$

$= f(2 - (2 - x))$

$= f(x)\ \forall\ x \in \mathbf{R}$

Thus, f is a periodic function with period 4.

Now,

$$5 = \int_0^2 f(x)dx = \int_0^2 f(2-x)dx = \int_0^2 f(2+x)dx$$

$$= \int_2^4 f(u)du = \int_2^4 f(x)dx$$

$$\therefore\ \int_0^4 f(x)dx = \int_0^2 f(x)dx + \int_2^4 f(x)dx = 10$$

As f is periodic with period 4,

$$\int_{10}^{50} f(x)dx = \int_{10}^{10+4(10)} f(x)dx = 10\int_0^4 f(x)dx$$

$= 10(10) = 100.$

44. Differentiating both the sides, we get

$(\cos x)\,f(\sin x) = \frac{\sqrt{3}}{2}$

To obtain $f\left(\frac{\sqrt{3}}{2}\right)$, put $x = \frac{\pi}{3}$, so that

$$\frac{1}{2}f\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow f\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}.$$

45. $f\left(\frac{1}{x}\right) = \int_1^{1/x}\frac{\log t}{1+t}dt$

Put $t = 1/u$, so that $dt = (-1/u^2)\,du$, then

$$f\left(\frac{1}{x}\right) = \int_1^x \frac{\log(1/u)}{1+1/u}\left(-\frac{1}{u^2}\right)du$$

$$= \int_1^x \frac{\log u}{u(1+u)}du = \int_1^x \frac{\log t}{t(1+t)}dt$$

$$\therefore f(x) + f\left(\frac{1}{x}\right) = \int_1^x \left(1+\frac{1}{t}\right)\frac{\log t}{1+t}dt$$

$$= \int_1^x \frac{\log t}{t}dt$$

$$= \frac{1}{2}(\log t)^2\Big]_1^x = \frac{1}{2}(\log x)^2.$$

46. Let $I = \int_0^1 \tan^{-1}(1-x+x^2)dx$

$$= \int_0^1 \left(\frac{\pi}{2} - \cot^{-1}(1-x+x^2)\right)dx$$

$$= \frac{\pi}{2} - 2\int_0^1 \tan^{-1} x\,dx$$

$$= \frac{\pi}{2} - 2x\tan^{-1} x\Big]_0^1 + \int_0^1 \frac{2x}{1+x^2}dx$$

$$= \frac{\pi}{2} - 2\tan^{-1}(1) + \text{In}(1+x^2)\Big]_0^1$$

$$= \frac{\pi}{2} - 2\left(\frac{\pi}{4}\right) + \log 2$$

$$= \log 2$$

47. Let $y = \lim\limits_{x\to\infty}\left(\frac{(n+1)(n+2)\ldots(n+2n)}{n^{2n}}\right)^{1/n}$

$$\log y = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{2n}\log\left(1+\frac{k}{h}\right)$$

$$= \int_0^2 \log(1+x)dx$$

$$= x\log(1+x)\Big]_0^2 - \int_0^2 \frac{x}{1+x}dx$$

$$= 2\log 3 - \int_0^2 \frac{x+1-1}{x+1}dx$$

$$= 2\log 3 - 2 + \log(x+1)\Big|_0^2$$

$$= 2\log 3 - 2 + \log 3$$

$$= \log 27 - \log e^2 = \log\frac{27}{e^2} \Rightarrow y = \frac{27}{e^2}.$$

48. Let $I = \int_4^{10} \frac{[x^2]dx}{[(14-x)^2]+[x^2]}$ (1)

$$\int_4^{10} \frac{[(10+4-x)^2]}{[(14-(10+4-x))^2]+[(10+4+x)^2]}dx$$

$$I = \int_4^{10} \frac{[(14-x)^2]}{[x^2]+[(14+x)^2]}dx \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_4^{10} dx = 6 \Rightarrow I = 3$$

Previous Years' B-Architecture Entrance Examination Questions

1. $I_1 = \int_{-4}^{-5} e^{(x+5)^2}dx = -\int_0^1 e^{t^2}dt \quad (t = x+5)$

$$I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2}dx = \int_{1/3}^{2/3} e^{(3x-2)^2}dx \quad (3x-2=t)$$

$$= \frac{1}{3}\int_{-1}^0 e^{t^2}dt = -\frac{1}{3}\int_1^0 e^{t^2}dt$$

$$= \frac{1}{3}\int_0^1 e^{t^2}dt$$

$$I_1 + 3I_2 = 0$$

2. $\int_0^{1.5}[x^2]f'(x)dx$

$$= \int_0^1 [x^2]f'(x)dx + \int_1^{\sqrt{2}}[x^2]f'(x)dx + \int_{\sqrt{2}}^{1.5}[x^2]f'(x)dx$$

$$= 0 + \int_1^{\sqrt{2}} f'(x)dx + 2\int_{\sqrt{2}}^{1.5} f'(x)dx$$

$$= f(\sqrt{2}) - f(1) + 2\,[f(1.5) - f(\sqrt{2})]$$

$$= 2f(1.5) - f(\sqrt{2}) - f(1)$$

3. $I = \int_{-a}^a \log(x+\sqrt{x^2+1})dx = \int_{-a}^a \log(-x+\sqrt{x^2+1})dx$

$$\left(\int_{-a}^a f(x)dx = \int_{-a}^a f(-x)dx\right)$$

$$\int_{-a}^a \log\left(\frac{-x^2+x^2+1}{x+\sqrt{x^2+1}}\right)dx = -\int_{-a}^a \log(x+\sqrt{x^2+1})dx$$

$$= -I$$

$$\Rightarrow I = 0$$

4. $\int_{1/2^n}^1 f(x)dx$

$$= \int_{1/2^n}^{1/2^{n-1}} f(x)dx + \int_{1/2^{n-1}}^{1/2^{n-2}} f(x)dx + \cdots + \int_{1/2}^1 f(x)dx$$

$$= \frac{1}{2^{n-1}}\left[\frac{1}{2^{n-1}} - \frac{1}{2^n}\right] + \frac{1}{2^{n-2}}\left[\frac{1}{2^{n-2}} - \frac{1}{2^{n-1}}\right] + \cdots + 1\left[1-\frac{1}{2}\right]$$

$$= \frac{1}{2^{2n-1}} + \frac{1}{2^{2n-3}} + \cdots + \frac{1}{2} = \frac{1}{2}\left[\frac{1-(1/2^2)^n}{1-1/2^2}\right] = \frac{2}{3}\left[1-\left(\frac{1}{4}\right)^n\right]$$

$$\lim_{n\to\infty}\int_{1/2^n}^{1} f(x)dx = \frac{2}{3}$$

5. $\int_0^1 x^2(1-x)^9\,dx = -\int_1^0 (1-t)^2 t^9 dt \quad (1 - x = t)$

$$= \int_0^1 (1-t^2-2t)t^9 dt$$

$$= \frac{t^{10}}{10} + \frac{t^{12}}{12} - \frac{2t^{11}}{11}\Big|_0^1$$

$$= \frac{1}{10} + \frac{1}{12} - \frac{2}{11} = \frac{1}{660}$$

6. $\int_0^1 \max(e^x, e^{1-x})dx$

$$= \int_0^{1/2} \max(e^x, e^{1-x})dx + \int_{1/2}^1 \max(e^x, e^{1-x})dx$$

$$= \int_0^{1/2} e^{1-x}dx + \int_{1/2}^1 e^x dx$$

$$= -e^{1-x}\Big|_0^{1/2} + e - e^{1/2}$$

$$= -e^{1/2} + e + e - e^{1/2}$$

$$= 2(e - e^{1/2}).$$

7. $\int_0^{\pi/2} \min(\sin x, \cos x)dx$

$$= \int_0^{\pi/4} \min(\sin x, \cos x)dx + \int_{\pi/4}^{\pi/2} \min(\sin x, \cos x)dx$$

$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$$

$$= -\cos x\Big|_0^{\pi/4} + \sin x\Big|_{\pi/4}^{\pi/2}$$

$$= -\left[\frac{1}{\sqrt{2}} - 1\right] + \left[1 - \frac{1}{\sqrt{2}}\right] = 2 - \sqrt{2}$$

8. $f(x) = x^2\int_0^1 \frac{dt}{2-t} + \int_0^1 \frac{t^2}{2-t}dt$

$$= -x^2 \log|2-t|\Big|_0^1 + \int_0^1\left(-t-2+\frac{4}{2-t}\right)dt$$

$$y = x^2 \log 2 + \left(-\frac{1}{2} - 2 + 4\log 2\right)$$

which represent a parabola.

9. $\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)dx$

$$= \frac{(f'(x))^2}{2}\Big|_2^3 + f'(x)\Big|_1^3$$

$$= \frac{1}{2}\,[(f'(3))^2 - (f'(2))^2] + f'(3) - f'(1)$$

$$= \frac{1}{2}\left[\left(\tan\frac{\pi}{4}\right)^2 - \left(\tan\frac{\pi}{3}\right)^2\right] + \tan\frac{\pi}{4} - \tan\frac{\pi}{6}$$

$$= \frac{1}{2}\,[1 - 3] + 1 - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

10. For $x \in (1, 3)$, $2 < \sqrt{3+x^3} < \sqrt{30}$ so

$$2\int_1^3 dx < \int_1^3 \sqrt{3+x^3}\,dx < \sqrt{30}\int_1^3 dx$$

$$\Rightarrow 4 < \int_1^3 \sqrt{3+x^3}\,dx < 2\sqrt{30}$$

11. $\lim_{n\to\infty}\frac{1}{n}\left[1 + \frac{n^2}{n^2+1^2} + \frac{n^2}{n^2+2^2} + \cdots + \frac{n^2}{n^2+(n-1)^2}\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{n^2}{n^2+r^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4}$$

12. $\int_a^b f(x)dx = \int_a^b x\,|x|\,dx$

$$= \begin{cases} \int_a^b (-x^2)dx & b<0 \\ \int_a^0 -x^2dx + \int_0^b x^2dx, & a<0<b \\ \int_a^b x^2dx, & 0<a \end{cases}$$

$$= \begin{cases} -\frac{1}{3}(b^3-a^3), & b<0 \\ \frac{1}{3}(b^3+a^3), & a<0<b \\ \frac{1}{3}(b^3-a^3), & 0<a \end{cases}$$

$$= \frac{1}{3}\,(|b|^3 - |a|^3)$$

13. $\int_{\sqrt{\log 5}}^{\sqrt{\log 7}} \frac{x\cos x^2}{\cos(\log 35 - x^2) + \cos x^2}dx$

$$= \frac{1}{2}\int_{\log 5}^{\log 7} \frac{\cos t}{\cos(\log 35 - t) + \cos t}dt \quad (x^2 = t)$$

$$= \frac{1}{2}\; I'$$

$$I' = \int_{\log 5}^{\log 7} \frac{\cos(\log 7 + \log 5 - t)}{\cos t + \cos(\log 7 + \log 5 - t)} dt$$

$$= \int_{\log 5}^{\log 7} \frac{\cos(\log 35 - t)}{\cos t + \cos(\log 35 - t)} dt$$

$$2I' = I' + I' = \int_{\log 5}^{\log 7} \frac{\cos t + \cos(\log 35 - t)}{\cos t + \cos(\log 35 - t)} dt$$

$$= \log \frac{7}{5}$$

$$\Rightarrow I = \frac{1}{4} \log \frac{7}{5}$$

14. Same as Q 13 of AIEEE.

15. Let $I = \int_0^{1/2} \frac{e^x(2-x^2)}{(1-x)^{3/2}(1+x)^{1/2}} dx$

$$= \int_0^{1/2} e^x \left[\frac{(1-x^2)+1}{(1-x)\sqrt{1+x^2}} \right] dx$$

$$= \int_0^{1/2} e^x \left[\frac{\sqrt{1-x^2}}{1-x} + \frac{1}{(1-x)\sqrt{1-x^2}} \right] dx$$

As $\frac{d}{dx}\left(\frac{\sqrt{1-x^2}}{1-x}\right) = \frac{(1-x)\left(\frac{-x}{\sqrt{1-x^2}}\right) + \sqrt{1-x^2}}{(1-x)^2}$

$$= \frac{-x + x^2 + 1 - x^2}{(1-x)^2\sqrt{1-x^2}} = \frac{1}{(1-x)\sqrt{1-x^2}}$$

$$\therefore I = e^x \left.\left(\frac{\sqrt{1-x^2}}{1-x}\right)\right]_0^{1/2}$$

$$= \sqrt{e}\,\frac{\sqrt{3/4}}{2/2} - 1 = \sqrt{3e} - 1$$

16. $\int_0^2 [x^2] dx$

$$= \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] dx + \int_{\sqrt{3}}^{2} [x^2] dx$$

$$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\, dx + \int_{\sqrt{3}}^{2} 3\, dx$$

$$= 0 + (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$$

$$= -\sqrt{2} - \sqrt{3} + 6$$