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APPLICATIONS OF DERIVATIVE

IIT-MATHEMATICS

IN THIS CHAPTER ....

Approximations

Let $y = f(x)$ be a function such that x is an independent variable and y is the dependent variable. Let Δx be a small change in x and Δy be the corresponding change in y and given by $\Delta y = f(x + \Delta x) - f(x)$. Then,

- The differential of x, denoted by dx, is defined by $dx = \Delta x$.
- The differential of y, denoted by dy, is defined by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$.
- If $dx = \Delta x$ is relatively small, when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

Errors

- **Absolute Error** Δx is called an absolute error in x.
- **Relative Error** $\frac{\Delta x}{x}$ is called the relative error.
- **Percentage Error** $\left(\frac{\Delta x}{x} \times 100\right)$ is called the percentage error.

Example 1. *If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is*

(a) 47.66 (b) 57.66 (c) 67.66 (d) 77.66

Sol. (d) Consider $f(x) = 3x^2 + 15x + 5 \Rightarrow f'(x) = 6x + 15$

Let $x = 3$ and $\Delta x = 0.02$

Also, $f(x + \Delta x) \simeq f(x) + \Delta x\, f'(x)$.

$\Rightarrow f(x + \Delta x) \simeq (3x^2 + 15x + 5) + (6x + 15)\Delta x$

$\Rightarrow f(3.02) \simeq 3 \times 3^2 + 15 \times 3 + 5 + (6 \times 3 + 15)(0.02)$ [as $x = 3, \Delta x = 0.02$]

$= 27 + 45 + 5 + (18 + 15)(0.02)$

$= 77 + 33(0.02) = 77 + 0.66 \Rightarrow f(3.02) \simeq 77.66$

Hence, the approximate value of $f(3.02)$ is 77.66.

Example 2. *The approximate change in the volume of a cube of side x m caused by increasing the side by 3%.*

(a) $0.06x^3\ m^3$ (b) $0.6x^3\ m^3$ (c) $0.09x^3\ m^3$ (d) $0.9x^3\ m^3$

Sol. (c) We know that the volume V of a cube of side x is given by $V = x^3$

$\Rightarrow \quad \frac{dV}{dx} = 3x^2$

Let Δx be change in side $= 3\%$ of $x = 0.03x$

Now, change in volume, $\Delta V = \left(\frac{dV}{dx}\right)\Delta x = (3x^2)\Delta x = (3x^2)(0.03x)$

[as $\Delta x = 3\%$ of x is $0.03x$]

$= 0.09x^3\ m^3$

Hence, the approximate change in the volume of the cube is $0.09x^3\ m^3$.

Derivative as The Rate of Change

If a variable quantity y is some function of time t i.e. $y = f(t)$, then small change in time Δt have a corresponding change Δy in y.

Thus, average rate of change $= \frac{\Delta y}{\Delta t}$.

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to t at the instant t.

i.e. $$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

Hence, it is clear that the rate of change of any variable with respect to some other variable is derivative of first variable with respect to other variable.

Note $\frac{dy}{dt}$ is positive if y increases as t increase and it is negative if y decreases as t increase.

Derivative as the Rate of Change of Two Variables

Let two variables are varying with respect to another variable t, i.e. $y = f(t)$ and $x = g(t)$. Then, rate of change of y with respect to x is given by $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Example 3. *A balloon which always remains spherical on inflation, is being inflated by pumping in 900 cu cm of gas per second. The rate at which the radius of the balloon increases when the radius is 15 cm, is*

(a) $\frac{2}{\pi}$ cm/s (b) $\frac{1}{\pi}$ cm/s (c) $\frac{2}{\pi^2}$ cm/s (d) $\frac{1}{2\pi}$ cm/s

Sol. (b) At any instant of time t let the radius of the balloon be r and its volume be V, then

Volume of balloon $V = (4/3)\pi r^3$

The balloon is being inflated at 900 cu cm/s *i.e.*, the rate of change of volume with respect to time is 900 cm^3/s.

On differentiating w.r.t. t, we get

Rate of change of volume

$$\frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right)$$

Given, $r = 15$ cm

$\Rightarrow \quad 900 = \left(\frac{4}{3}\pi\right)\left\{3(15)^2\frac{dr}{dt}\right\}$

$\Rightarrow \quad \frac{dr}{dt} = \frac{900}{3\times(15)^2}\times\frac{3}{4\pi}$

Rate of change of radius r,

$$\frac{dr}{dt} = \frac{900}{4\pi\times(15)^2} = \frac{225}{\pi\times 225} = \frac{1}{\pi}\text{ cm/s}$$

Hence, the rate at which the radius of the balloon increases when the radius is 15 cm, is $\frac{1}{\pi}$ cm/s.

Example 4. *If the surface area of a cube is increasing at a rate of 3.6 cm^2/sec, retaining its shape; then the rate of change of its volume (in cm^3/sec), when the length of a side of the cube is 10 cm, is* ***(JEE Main 2020)***

(a) 18 (b) 10 (c) 9 (d) 20

Sol. (c) Since, surface area of cube, $A = 6a^2\ cm^2$.

It is given, $\frac{dA}{dt} = 3.6\ cm^2/sec$

$\Rightarrow \quad 12a\frac{da}{dt} = 3.6\ cm^2/sec \quad$ ...(i)

Now, as volume of cube, $v = a^3\ cm^3$

$\therefore \quad \frac{dv}{dt} = 3a^2\frac{da}{dt} = 3a^2\frac{3.6}{12a} \quad$ [from Eq. (i)]

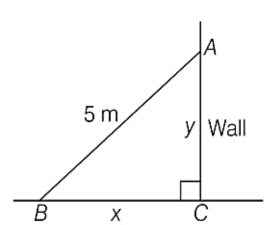
So, at $a = 10$ cm, $\frac{dv}{dt} = 0.9\times 10 = 9\ cm^3/sec$

Example 5. *A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?*

(a) $-\frac{8}{3}$ m/s (b) $\frac{8}{3}$ m/s (c) $\frac{4}{3}$ m/s (d) $-\frac{2}{3}$ m/s

Sol. (a) Let $AB = 5$ m be the ladder and y be the height of the wall at which the ladder touches.

Also, let the foot of the ladder be at B whose distance from the wall is x.



Given that the bottom of ladder is pulled along the ground at 2 cm/s, so $\frac{dx}{dt} = 2$ m/s.

As we know that ΔABC is right angled, so by Pythagoras theorem,

we have $$x^2 + y^2 = 5^2 \quad \text{...(i)}$$

when $x = 4$, then

$$y^2 = 5^2 - 4^2 \Rightarrow y = \sqrt{25 - 16}$$

$$\Rightarrow \quad y = 3 \text{ m}$$

On differentiating Eq. (i) w.r.t. time t on both sides, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad 4 \times 2 + 3 \times \frac{dy}{dt} = 0 \quad \left[\because x = 4 \text{ and } \frac{dx}{dt} = 2\right]$$

$\Rightarrow$ The rate of fall of height on the wall $\frac{dy}{dt} = -\frac{8}{3}$ m/s

[negative sign shows that height of ladder on the wall is decreasing at the rate of $\frac{8}{3}$ m/s]

Motion in a Straight Line

If x and v denotes the displacement and velocity of a particle at any instant t, then velocity is given by

$$v = \frac{dx}{dt}$$

and $$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

where, a is acceleration of particle. If the sign of acceleration is opposite to that of velocity, then the acceleration is called retardation which means decrease in magnitude of the velocity.

Example 6. *If $s = \frac{1}{2}t^3 - 6t$, then acceleration at the time when the velocity vanishes, is*

(a) 3 units/s^2 (b) 6 units/s^2
(c) 2 units/s^2 (d) None of these

Sol. (b) Given, $s = \frac{1}{2}t^3 - 6t$

$$\therefore \quad v = \frac{ds}{dt} = \left(\frac{3t^2}{2} - 6\right) \quad \text{...(i)}$$

and $$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 3t \quad \text{...(ii)}$$

when $v = 0$

$$\Rightarrow \quad \frac{3t^2}{2} - 6 = 0 \Rightarrow t^2 = 4 \Rightarrow t = 2$$

$\therefore$ Acceleration when velocity vanishes $= 3 \times 2 = 6$ units/s^2

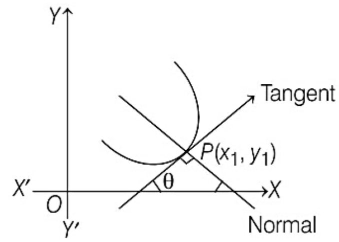
Tangents and Normals

A tangent is a straight line, which touches the curve $y = f(x)$ at a point.

A normal is a striaght line perpendicular to a tangent to the curve $y = f(x)$ intersecting at the point of contact.

Slope of Tangent

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be the point on it. Then, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is the slope of tangent to the curve $y = f(x)$ at a point P.



i.e. $\left(\frac{dy}{dx}\right)_P = \tan\theta =$ Slope of tangent at P

where, θ is the angle which the tangent at $P(x_1, y_1)$ makes with the positive direction of X-axis as shown in the figure.

Note If tangent is parallel to X-axis, then $\theta = 0°$

$$\Rightarrow \quad \tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

If tangent is perpendicular to X-axis (or parallel to Y-axis), then

$$\theta = 90° \Rightarrow \tan\theta = \infty \text{ or } \cot\theta = 0 \Rightarrow \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

Equation of Tangent

The equation of the tangent to the curve $y = f(x)$ at point $P(x_1, y_1)$ is given by $y - y_1 = \left(\frac{dy}{dx}\right)_{P(x_1, y_1)}(x - x_1)$.

Slope of Normal

We know that normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ and passing through P.

$\therefore$ Slope of the normal at $P = -\dfrac{1}{\text{Slope of the tangent at } P}$

$\Rightarrow$ Slope of normal at $P(x_1, y_1) = -\dfrac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$

$\Rightarrow$ Slope of normal at $P(x_1, y_1) = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$

Note If normal is parallel to X-axis.

$$\Rightarrow \quad -\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0 \text{ or } \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

If normal is perpendicular to X-axis (or parallel to Y-axis).

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

Equation of Normal

The equation of the normal to the curve $y = f(x)$ at point $P(x_1, y_1)$ is given by $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{P(x_1, y_1)}}(x - x_1)$.

Or

$$y - y_1 = -\left(\frac{dx}{dy}\right)(x - x_1)$$

Example 7. *The equation of the normal to the curve $y = (1+x)^{2y} + \cos^2(\sin^{-1}x)$ at $x = 0$ is* ***(JEE Main 2020)***

(a) $y + 4x = 2$ (b) $y = 4x + 2$ (c) $x + 4y = 8$ (d) $2y + x = 4$

Sol. (c) Equation of the given curve is

$$y = (1+x)^{2y} + \cos^2(\sin^{-1} x)$$

$$\Rightarrow \quad y = (1+x)^{2y} + 1 - \sin^2(\sin^{-1} x)$$

$$\Rightarrow \quad y = (1+x)^{2y} + 1 - x^2 \quad [\text{as } \sin(\sin^{-1} x) = x] \ldots\text{(i)}$$

So, at $x = 0$, $y = 2$.

Now, let a point $P(0, 2)$ on the curve.

On differentiating the Eq.(i) both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2y(1+x)^{2y-1} + 2(1+x)^{2y}(\log_e(1+x))\frac{dy}{dx} - 2x$$

So, at point $P(0, 2)$ $\left.\frac{dy}{dx}\right|_P = 4$

$\therefore$ Equation of normal to the curve at point 'P' is

$$y - 2 = \frac{1}{-4}(x - 0) \Rightarrow x + 4y = 8$$

Example 8. *If the tangent to the curve $y = x + \sin y$ at a point (a, b) is parallel to the line joining $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$, then* ***(JEE Main 2020)***

(a) $|b - a| = 1$ (b) $|a + b| = 1$ (c) $b = a$ (d) $b = \frac{\pi}{2} + a$

Sol. (a) Given curve is $y = x + \sin y$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 1 + \cos y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{1 - \cos y} \quad \ldots\text{(i)}$$

$\because$ Tangent at point (a, b) at given curve is parallel to line joining $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$.

So, $\left.\frac{dy}{dx}\right|_{(a, b)} = \frac{2 - (3/2)}{(1/2) - 0} = 1$

$$\Rightarrow \quad \frac{1}{1 - \cos b} = 1$$

$$\Rightarrow \quad 1 = 1 - \cos b \Rightarrow \cos b = 0$$

$$\Rightarrow \quad \sin b = \pm 1$$

Now, as point (a, b) on the given curve.

So, $b = a + \sin b \Rightarrow b - a = \sin b$

$$\Rightarrow \quad |b - a| = |\sin b| \Rightarrow |b - a| = 1$$

Example 9. *Which of the following points lies on the tangent to the curve $x^4e^y + 2\sqrt{y+1} = 3$ at the point $(1, 0)$?* ***(JEE Main 2020)***

(a) $(2, 2)$ (b) $(2, 6)$
(c) $(-2, 6)$ (d) $(-2, 4)$

Sol. (c) Equation of the given curve is $x^4e^y + 2\sqrt{y+1} = 3$

On differentiating w.r.t. 'x', we get

$$e^y(4x^3) + x^4e^y\frac{dy}{dx} + \frac{1}{\sqrt{y+1}}\frac{dy}{dx} = 0$$

$\therefore$ At point $P(1, 0)$,

$$e^0(4 \times 1) + 1 \cdot e^0\frac{dy}{dx} + \frac{1}{\sqrt{0+1}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_P = -2$$

$\therefore$ Equation of tangent at point $P(1,0)$ is

$$y = -2(x-1) \Rightarrow 2x + y = 2 \quad \ldots\text{(i)}$$

From the option point $(-2,6)$ contain by the tangent (i).

Example 10. *If the tangent to the curve, $y = x^3 + ax - b$ at the point $(1, -5)$ is perpendicular to the line, $-x + y + 4 = 0$, then which one of the following points lies on the curve ?* ***(JEE Main 2019)***

(a) $(-2, 2)$ (b) $(2, -2)$ (c) $(-2, 1)$ (d) $(2, -1)$

Sol. (b) Given curve is $y = x^3 + ax - b$ ...(i)

passes through point $P(1, -5)$.

$$\therefore \quad -5 = 1 + a - b \Rightarrow b - a = 6 \quad \ldots\text{(ii)}$$

and slope of tangent at point $P(1, -5)$ to the curve (i), is

$$m_1 = \left.\frac{dy}{dx}\right|_{(1, -5)} = [3x^2 + a]_{(1, -5)} = a + 3$$

$\because$ The tangent having slope $m_1 = a + 3$ at point $P(1, -5)$ is perpendicular to line $-x + y + 4 = 0$, whose slope is $m_2 = 1$.

$$\therefore \quad a + 3 = -1 \Rightarrow a = -4 \quad [\because m_1m_2 = -1]$$

Now, on substituting $a = -4$ in Eq. (ii), we get $b = 2$

On putting $a = -4$ and $b = 2$ in Eq. (i), we get

$$y = x^3 - 4x - 2$$

Now, from option $(2, -2)$ is the required point which lie on it.

Parametric Form

If the equation(s) of the curve $y = f(x)$ are represented in the form of the parameter t, such that

$$x = f_x(t) \text{ and } y = f_y(t)$$

(i) **Slope of Tangent**

$$m = \left(\frac{dy}{dx}\right) \Rightarrow m = \frac{dy}{dt} \cdot \frac{dt}{dx} \Rightarrow m = \frac{f_y'}{f_x'}$$

(ii) **Slope of Normal**

$$M = -\frac{1}{m} = -\frac{f_x'}{f_y'}$$

(iii) **Equation of Tangent**

$$y - f_y(t) = \frac{f_y'}{f_x'}[x - f_x(t)]$$

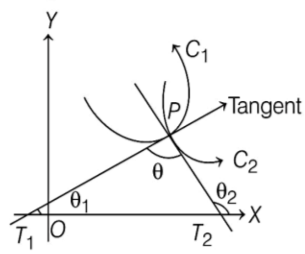
(iv) **Equation of Normal**

$$y - f_y(t) = -\frac{f_x'}{f_y'}[x - f_x(t)]$$

Angle of Intersection of Two Curves

The angle of intersection of two curves is the angle subtended between the tangents at their point of intersection.

Let C_1 and C_2 be two curves having equations $y = f(x)$ and $y = g(x)$, respectively.



Let PT_1 and PT_2 be tangents to the curves C_1 and C_2 at their point of intersection.

Let θ be the angle between the two tangents PT_1 and PT_2 and θ_1 and θ_2 are the angles made by tangents with the positive direction of X-axis in anti-clockwise sense.

Then, $$m_1 = \tan\theta_1 = \left(\frac{dy}{dx}\right)_{C_1}$$

and $$m_2 = \tan\theta_2 = \left(\frac{dy}{dx}\right)_{C_2}$$

From the figure it follows,

$$\theta = \theta_2 - \theta_1$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\Rightarrow \quad \tan\theta = \left|\frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}\right| = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

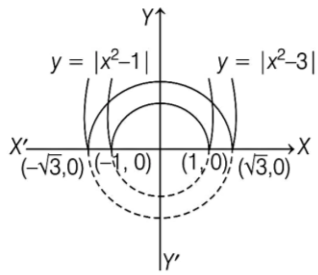
Angle of intersection of these curves is defined as acute angle between the tangents.

Example 11. *The acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersections is*

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ (c) $\frac{\pi}{6}$ (d) $\tan^{-1}\left(\frac{4}{7}\right)$

Sol. (*b*) The points of intersection are $(\pm\sqrt{2}, 1)$.

Since, the curves are symmetrical about Y-axis, the angle of intersection at $(-\sqrt{2}, 1)$ = The angle of intersection at $(\sqrt{2}, 1)$



At $(\sqrt{2}, 1)$, $m_1 = 2x = 2\sqrt{2}, m_2 = -2x = -2\sqrt{2}$.

$$\therefore \quad \tan\theta = \left|\frac{4\sqrt{2}}{1-8}\right| = \frac{4\sqrt{2}}{7} \Rightarrow \theta = \tan^{-1}\frac{4\sqrt{2}}{7}$$

Orthogonal Curves

If the angle of intersection of two curves is right angle, then two curves are said to be orthogonal curves.

If the curves are orthogonal, then $\theta = \frac{\pi}{2}$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1$$

$$\Rightarrow \quad m_1 m_2 = -1$$

If two curves touches each other, then $m_1 = m_2$.

Example 12. *The curves $x = y^2$ and $xy = k$ cut at right angle (orthogonally), if $8k^2$ is equal to*

(a) 1 (b) 3

(c) $\frac{1}{2}$ (d) None of these

Sol. (a) When the curve cut at right angle, their tangents at the point of intersection are also perpendicular *i.e.*, the product of their slopes is equal to –1.

The equation of the given curves are

$$x = y^2 \quad \text{...(i)}$$

and $$xy = k \quad \text{...(ii)}$$

The two curves meet where $\frac{k}{y} = y^2$

[eliminating x between Eqs. (i) and (ii)]

$$\Rightarrow \quad y^3 = k \Rightarrow y = k^{1/3}$$

On substituting this value of y in Eq. (i), we get

$$x = (k^{1/3})^2 = k^{2/3}$$

$\therefore$ Eq. (i) and Eq. (ii) intersect at the point $(k^{2/3}, k^{1/3})$.

On differentiating Eq. (i), w.r.t. x, we get

$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$\therefore$ Slope of the tangent to the first curve Eq. (i) at $(k^{2/3}, k^{1/3})$

$$= \frac{1}{2k^{1/3}} \quad \text{...(iii)}$$

From Eq. (ii), $y = \frac{k}{x} \Rightarrow \frac{dy}{dx} = -\frac{k}{x^2}$

$\therefore$ Slope of the tangent to the second curve Eq. (ii) at $(k^{2/3}, k^{1/3})$

$$= -\frac{k}{(k^{2/3})^2} = -\frac{1}{k^{1/3}} \quad \text{...(iv)}$$

We know that two curves intersect at right angles, if the tangents to the curves at the point of intersection *i.e.*, at $(k^{2/3}, k^{1/3})$ are perpendicular to each other.

This implies that we should have the product of the slope of the tangents

$$= -1$$

$$\Rightarrow \left(\frac{1}{2k^{1/3}}\right)\left(-\frac{1}{k^{1/3}}\right) = -1$$

$$\Rightarrow 1 = 2k^{2/3} \Rightarrow 1^3 = (2k^{2/3})^3 \Rightarrow 1 = 8k^2$$

Hence, the given two curves cut at right angles, if $8k^2 = 1$

Length of Tangent, Normal, Subtangent, Subnormal

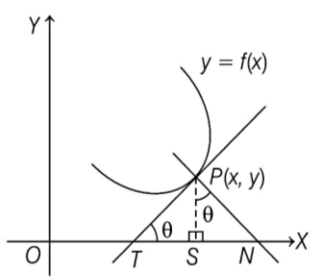
Let $y = f(x)$ be the equation of the given curve and $P(x, y)$ be any general point on it. Through P, draw a tangent so that it intersects the X-axis at point T. Also, PN is the normal line to the same curve at the same point P (see figure).

Now, PT and PN be the length of tangent and normal respectively and the projections of PT and PN, i.e. ST and SN are the lengths of sub-tangent and sub-normal, respectively.

Length of Tangent (PT)

In right angled ΔPTS, right angled at S, we have

$$\sin\theta = \frac{|PS|}{|PT|}$$



$$\Rightarrow |PT| = |PS| \operatorname{cosec}\theta$$

$$\Rightarrow |PT| = |PS|\sqrt{1+\cot^2\theta}$$

$$\Rightarrow |PT| = y\sqrt{1+\frac{1}{y'^2}}$$

$$\therefore \text{Length of tangent} = \frac{y\sqrt{1+y'^2}}{y'}$$

Length of Normal (PN)

In right angled ΔPSN right angled at S, we have

$$\cos\theta = \frac{|PS|}{|PN|} \Rightarrow |PN| = |PS|\sec\theta$$

$$\Rightarrow |PN| = |PS|\sqrt{1+\tan^2\theta}$$

$$\therefore \text{Length of normal} = y\sqrt{1+y'^2}$$

Length of Subtangent (ST)

In right angled ΔPST, right angled at S, we have

$$\tan\theta = \frac{|PS|}{|ST|} \Rightarrow |ST| = \frac{y}{y'}$$

$$\therefore \text{Length of subtangent} = \frac{y}{y'}$$

Length of Subnormal (SN)

In right angled ΔPSN, right angled at S, we have

$$\cot\theta = \frac{|PS|}{|SN|} \Rightarrow |SN| = y\tan\theta \Rightarrow |SN| = yy'$$

$$\therefore \text{Length of subnormal} = yy'$$

Condition for a Given Line to Touch the Given Curve

Let the line $ax + by + c = 0$ be a tangent to the given curve at (x_1, y_1), then write the equation of the tangent as

$$y - y_1 = \left(\frac{dy}{dx}\right)(x - x_1)$$

On comparing the equation with the given equation, we get

$$ax_1 + by_1 + c = 0$$

And also, slope of line = slope of tangent to the curve at (x_1, y_1). Eliminating, x_1, y_1, we will get desired values for which the line touches the curve.

Example 13. *In the curve $x^{m+n} = a^{m-n}y^{2n}$, the mth power of the subtangent varies as the kth power of subnormal, then k is*

(a) m (b) n (c) $1/n$ (d) $1/m$

Sol. (b) Given, $x^{m+n} = a^{m-n}y^{2n}$

$$\Rightarrow (m+n)\log x = (m-n)\log a + 2n\log y$$

On differentiating w.r.t. x, we get

$$\frac{(m+n)}{x} = 0 + \frac{2n}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{(m+n)}{2n}\left(\frac{y}{x}\right)$$

Now, $$\frac{(\text{subtangent})^m}{(\text{subnormal})^n} = \frac{\left(y\frac{dx}{dy}\right)^m}{\left(y\frac{dy}{dx}\right)^n} = \frac{y^{m-n}}{\left(\frac{dy}{dx}\right)^{m+n}}$$

$$= \frac{x^{m+n}}{\left(\frac{m+n}{2n}\right)^{m+n} y^{2n}} = \left(\frac{2n}{m+n}\right)^{m+n} \times a^{m-n}$$

= Constant (independent of x and y)

$$\Rightarrow (\text{Subtangent})^m \propto (\text{Subnormal})^n$$

Example 14. *The equation of family of curves for which the length of normal at any point P is equal to the distance of P from origin, is*

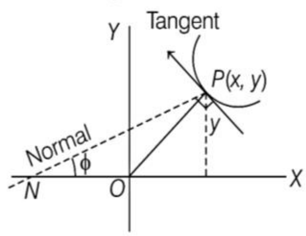
(a) $x^2 = -y + C$ (b) $y^2 = \pm x^2 + C$

(c) $x = \pm y + C$ (d) $2x^2 = \pm y^2 + C$

Sol. (b), Let $P(x, y)$ be the point on the curve.

$$OP = \text{Radius vector} = \sqrt{x^2 + y^2}$$

$$PN = \text{Length of normal}$$



Now, $\tan\phi = -\dfrac{1}{\left(\dfrac{dy}{dx}\right)}$

$$PN = \frac{y}{\sin\phi}$$

It is given $OP = PN$

$\Rightarrow \sqrt{x^2 + y^2} = y\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} \Rightarrow x^2 + y^2 = y^2\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]$

$\Rightarrow x^2 = y^2\left(\dfrac{dy}{dx}\right)^2 \Rightarrow \dfrac{dy}{dx} = \pm\dfrac{x}{y}$

$\Rightarrow y\,dy = \pm x\,dx$ integrating both sides,

$y^2 = \pm x^2 + C$ is the required family of curves.

Increasing and Decreasing Functions

Increasing Function

These functions are of two types

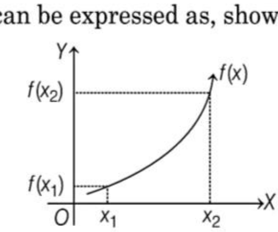
Strictly Increasing Function

A function $f(x)$ is known as strictly increasing function in its domain, if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Therefore, for the smaller input, we have smaller output and for higher value of input we have higher output.

Graphically it can be expressed as, shown in the figure.



Here, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Thus, $f(x)$ is strictly increasing function.

In the graph, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

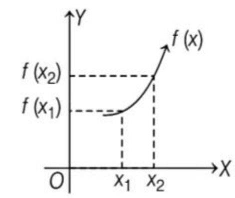
As $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Thus, $f(x) < f(x+h)$

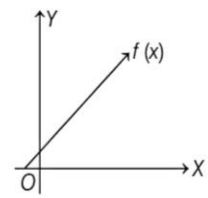
$$f'(x) = \lim_{h \to 0} \frac{\text{positive}}{\text{positive}} \text{ i.e. } f'(x) > 0$$

Thus, $f(x)$ will be strictly increasing, if $f'(x) > 0$, $\forall\, x \in$ domain.

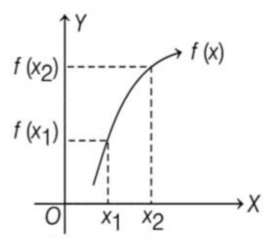
Classification of Strictly Increasing Function



Concave up
when $f'(x) > 0$
and $f''(x) > 0, \forall x \in$ domain



Neither concave up nor concave down
when $f'(x) > 0$
and $f''(x) = 0$, $\forall x \in$ domain



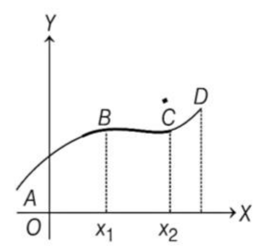
Concave down
when $f'(x) > 0$ and $f''(x) < 0, \forall x \in$ domain

Non-decreasing Function

A function $f(x)$ is said to be non-decreasing, if $x_1 < x_2$

$\Rightarrow \quad f(x_1) \le f(x_2)$

As shown in figure.



For AB and CD portion,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

and for BC, $\quad x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$

Hence, as a whole we can say that for non-decreasing function (or increasing function),

if $\quad x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

Obviously, for this $f'(x) \ge 0$, where equality holds for horizontal path of the graph i.e. in the interval of BC.

Decreasing Function

These functions are also of two types

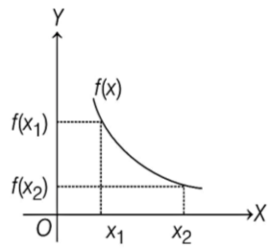
Strictly Decreasing Function

A function $f(x)$ is known as strictly decreasing function in its domain, if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

Therefore, for the smaller input we have higher output and for higher value of input we have smaller output.

Graphically it can be expressed as shown in the figure.



Here, $\quad x_1 < x_2$

$\Rightarrow f(x_1) > f(x_2)$ thus, $f(x)$ is strictly decreasing.

In graph, $\quad f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

As $\quad x_1 < x_2$

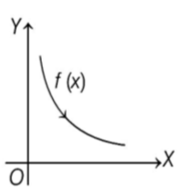
$\Rightarrow \quad f(x_1) > f(x_2)$

Thus, $\quad f(x+h) < f(x)$

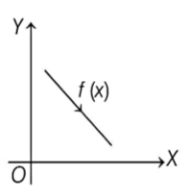
$\Rightarrow \quad f'(x) = \lim_{h \to 0} \dfrac{-\text{ve}}{+\text{ve}}$ i.e. $f'(x) < 0$

Thus, $f(x)$ will be strictly decreasing, if $f'(x) < 0 \; \forall \; x \in$ domain.

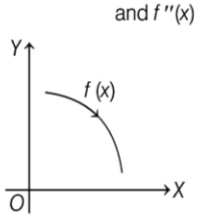
Classification of Strictly Decreasing Function



Concave up
when $f'(x) < 0$
and $f''(x) > 0$,
$\forall x \in$ domain



Neither concave up nor concave down
when $f'(x) < 0$
and $f''(x) = 0, \forall x \in$ domain



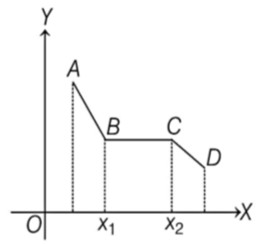
Concave down
when $f'(x) < 0$
and $f''(x) < 0, \forall x \in$ domain

Non-Increasing Function

A function $f(x)$ is said to be non-increasing, if for $x_1 < x_2$

$\Rightarrow \quad f(x_1) \geq f(x_2)$

As shown in figure.



For AB and CD portion, $x_1 < x_2$

$\Rightarrow \quad f(x_1) > f(x_2)$

and for BC, $\quad x_1 < x_2$

$\Rightarrow \quad f(x_1) = f(x_2)$

Hence, as a whole we can say that for non-increasing function (or decreasing function), if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

Obviously, for this $f'(x) \leq 0$, where equality holds for horizontal path of the graph i.e. in the interval of BC.

Example 15. *The function $f(x) = \cot^{-1} x + x$ increases in the interval*

(a) $(1, \infty)$ (b) $(-1, \infty)$ (c) $(-\infty, \infty)$ (d) $(0, \infty)$

Sol. (c) Since, $\quad f(x) = \cot^{-1} x + x$

On differentiating w.r.t. x, we get

$$f'(x) = -\frac{1}{1+x^2} + 1 = \frac{x^2}{1+x^2} \geq 0$$

Hence, $f(x)$ is increasing function for all $x \in (-\infty, \infty)$.

Example 16. *The function, $f(x) = (3x-7)\,x^{2/3}$, $x \in R$, is increasing for all x lying in* ***(JEE Main 2020)***

(a) $(-\infty, 0) \cup \left(\dfrac{14}{15}, \infty\right)$ (b) $(-\infty, 0) \cup \left(\dfrac{3}{7}, \infty\right)$

(c) $\left(-\infty, \dfrac{14}{15}\right)$ (d) $\left(-\infty, -\dfrac{14}{15}\right) \cup (0, \infty)$

Sol. (a) Since, the given function $f(x) = (3x-7)x^{2/3}$ is increasing for $x \in R$.

$\therefore \quad f'(x) \geq 0$

$\Rightarrow \quad \dfrac{2}{3} x^{-1/3}(3x-7) + x^{2/3}(3) \geq 0, \; x \neq 0$

$\Rightarrow \quad \dfrac{2(3x-7) + 9x}{x^{1/3}} \geq 0, \; x \neq 0 \quad \Rightarrow \quad \dfrac{15x - 14}{x^{1/3}} \geq 0$

$\Rightarrow \quad x \in (-\infty, 0) \cup \left(\dfrac{14}{15}, \infty\right)$

$\because \quad (-\infty, 0) \cup \left(\dfrac{14}{15}, \infty\right) \subset (-\infty, 0) \cup \left(\dfrac{14}{15}, \infty\right)$

Example 17. *Let* $f(x) = x\cos^{-1}(-\sin|x|), x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *then which of the following is true?* **(JEE Main 2020)**

(a) f' is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$

(b) f' is increasing in $\left(-\frac{\pi}{2}, 0\right)$ and decreasing in $\left(0, \frac{\pi}{2}\right)$

(c) f is not differentiable at $x = 0$

(d) $f'(0) = -\frac{\pi}{2}$

Sol. (a) Given function

$$f(x) = x\cos^{-1}(-\sin|x|), x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$= x(\pi - \cos^{-1}(\sin|x|)) \qquad [\because \cos^{-1}(-x) = \pi - \cos^{-1}x]$$

$$= x\left[\pi - \left(\frac{\pi}{2} - \sin^{-1}\sin|x|\right)\right] \quad \left[\because \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x\right]$$

$$= x\left[\frac{\pi}{2} + |x|\right] \qquad [\because \sin^{-1}\sin x = x]$$

$$= \begin{cases} x\left(\frac{\pi}{2} - x\right), & x \in \left(-\frac{\pi}{2}, 0\right) \\ x\left(\frac{\pi}{2} + x\right), & x \in \left(0, \frac{\pi}{2}\right) \end{cases}$$

So, $f'(x) = \begin{cases} \frac{\pi}{2} - 2x, & x \in \left(-\frac{\pi}{2}, 0\right) \\ \frac{\pi}{2} + 2x, & x \in \left(0, \frac{\pi}{2}\right) \end{cases}$

$\Rightarrow f''(x) = \begin{cases} -2, & x \in \left(-\frac{\pi}{2}, 0\right) \\ 2, & x \in \left(0, \frac{\pi}{2}\right) \end{cases}$

$\therefore$ f' is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$.

Example 18. *Let* $f : (-1, \infty) \to R$ *be defined by* $f(0) = 1$ *and* $f(x) = \frac{1}{x}\log_e(1+x), x \neq 0$. *Then, the function* f **(JEE Main 2020)**

(a) decreases in $(-1, 0)$ and increases in $(0, \infty)$

(b) increases in $(-1, \infty)$

(c) increases in $(-1, 0)$ and decreases in $(0, \infty)$

(d) decreases in $(-1, \infty)$

Sol. (d) Given function

$$f(x) = \begin{cases} \frac{1}{x}\log_e(1+x), & x \neq 0 \\ 1, & x = 0 \end{cases}, \text{ for } x \in (-1, \infty)$$

Now, $f'(x) = \frac{1}{x(1+x)} - \frac{\log_e(1+x)}{x^2}$,

for $x \in (-1, \infty) - \{0\} = \frac{x - (1+x)\log_e(1+x)}{x^2(1+x)}$

Let another function

$$g(x) = x - (1+x)\log_e(1+x)$$

$\because \quad g'(x) = 1 - 1 - \log_e(1+x) = -\log_e(1+x)$

Since, for $x \in (-1, 0)$, $g'(x) > 0$.

So, $g(x)$ is increasing function for $x \in (-1, 0)$ but as

$$g(x) < g(0), \forall\, x \in (-1, 0)$$

$\therefore \quad g(x) < 0, \ \forall\, x \in (-1, 0)$

$\therefore \quad f'(x) = \frac{g(x)}{x^2} < 0$

$\Rightarrow f(x)$ is decreasing function for $x \in (-1, 0)$.

Similarly, for $x \in (0, \infty)$, $g'(x) < 0$, so $g(x)$ is decreasing function for $x \in (0, \infty)$.

So, $g(x) < g(0) \Rightarrow g(x) < 0, \forall\, x \in (0, \infty)$

$\therefore \quad f'(x) = \frac{g(x)}{x^2} < 0$

$\Rightarrow f(x)$ is decreasing function for $x \in (0, \infty)$.

$\therefore$ The given function $f(x)$ is decreasing function for $(-1, \infty)$.

Monotonic Function

A function f is said to be monotonic or monotone in an interval I. If it is either increasing or decreasing in the interval I. $f(x) = \ln x$, $f(x) = 2^x$, $f(x) = -2x + 3$ are monotonic functions. $f(x) = x^2$ is monotonic in $(-\infty, 0)$ or $(0, \infty)$ but is not monotonic in R.

Properties of Monotonic Functions

(i) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \leq 0 \{f'(c) < 0\}$ for each $c \in (a, b)$, then $f(x)$ is monotonically (or strictly) decreasing function on $[a, b]$.

(ii) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \geq 0$ $[f'(c) > 0]$ for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) increasing function on $[a, b]$.

(iii) If $f(x)$ and $g(x)$ are monotonically (or strictly) increasing (or decreasing) functions on $[a, b]$, then $gof(x)$ is a monotonically (or strictly) increasing function on $[a, b]$.

(iv) If one of the two functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decreasing on $[a, b]$.

(v) If $f(x)$ is strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

Operations of Monotonic Functions

I : Increasing, D : Decreasing, λ : Neither increasing nor decreasing

$f(x)$	I	D	I	D
$g(x)$	I	I	D	D
$-f(x)$	D	I	D	I
$-g(x)$	D	D	I	I
$f(x)+g(x)$	I	I or D or λ	I or D or λ	D
$f(x)-g(x)$	I or D or λ	D	I	I or D or λ
$f(x)\cdot g(x)$	I	I or D or λ	I or D or λ	D
$\frac{f(x)}{g(x)}$	I or D or λ	D	I	I or D or λ
$\frac{1}{f(x)}$	D	I	D	I
$\frac{1}{g(x)}$	D	D	I	I
$(fog)x$	I	D	D	I

Use of Monotonicity for Proving Inequalities

Comparision of two functions $f(x)$ and $g(x)$ can be done by analysing the monotonic behaviour of $h(x) = f(x) - g(x)$,

If $f(a) = g(a)$ and $f'(x) \geq g'(x) \, \forall \, x \geq a$

$\Rightarrow f(x) \geq g(a) \, \forall \, x \geq a$.

Example 19. *Which of the following statements is/are true?*

(a) $\log(1+x) > x - \frac{x^2}{2} \forall \, x \in (0, \infty)$

(b) $\log(1+x) < x - \frac{x^2}{2} \forall \, x \in (0, \infty)$

(c) $\sin x < x < \tan x \forall \, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(d) $\sin x > x > \tan x \forall \, x \in \left(0, \frac{\pi}{2}\right)$

Sol. (a) Consider the function $f(x) = \ln(1+x) - x + \frac{x^2}{2}, x \in (0, \infty)$

Then, $f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x} > 0 \quad \forall \, x \in (0, \infty)$

$\Rightarrow$ $f(x)$ increases in $(0, \infty)$

$\Rightarrow$ $f(x) > f(0^+) = 0$

i.e. $\ln(1+x) > x - \frac{x^2}{2}$

which is the desired result.

Concavity and Point of Inflexion

A function $f(x)$ is concave up in (a, b), if tangent drawn at every point $(x_0, f(x_0))$, for $x_0 \in (a, b)$ lie below the curve, $f(x)$ is concave down in (a, b) if tangent drawn at each point $(x_0, f(x_0)), x_0 \in (a, b)$ lie above the curve.

A point $\{c, f(c)\}$ of the graph $y = f(x)$ is said to be a point of inflection of the graph. If $f(x)$ is concave up in $(c-\delta, c)$ and concave down in $(c, c+\delta)$ (or *vice versa*), for some $\delta \in R^+$.

Results

(i) If $f''(x) > 0, \forall \, x \in (a, b)$, then the curve $y = f(x)$ is concave up in (a, b).

(ii) If $f''(x) < 0, \forall \, x \in (a, b)$, then the curve $y = f(x)$ is concave down in (a, b).

(iii) If f is continuous at $x = c$ and $f''(x)$ has opposite signs on either sides of c, then the point $\{c, f(c)\}$ is a point of inflexion of the curve.

(iv) If $f''(c) = 0$ and $f'''(c) \neq 0$, then the point $\{c, f(c)\}$ is a point of inflexion.

Example 20. *If the graph of the function $f(x) = 3x^4 + 2x^3 + ax^2 - x + 2$ is concave up for all real value x, then values of a is*

(a) $a > \frac{1}{2}$ (b) $a > 0$ (c) $a < -2$ (d) $a < \frac{1}{3}$

Sol. (a) Given, $f(x) = 3x^4 + 2x^3 + ax^2 - x + 2$

$\Rightarrow \quad f'(x) = 12x^3 + 6x^2 + 2ax - 1$

and $\quad f''(x) = 36x^2 + 12x + 2a$

Since, graph of concave upward for all real x.

$f''(x) > 0 \Rightarrow 36x^2 + 12x + 2a > 0 \, \forall \, x \in x$

$\therefore \quad 12^2 - 4(36)(2a) < 0 = 1 - 2a < 0 = a > \frac{1}{2}$

Example 21. *The points of inflection of the curve $f(x) = e^{-x^2}$ are*

(a) $(\pm\sqrt{2}, \sqrt{e})$ (b) $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$

(c) $\left(\pm\frac{1}{\sqrt{3}}, \sqrt{e}\right)$ (d) $\left(\sqrt{5}, \frac{1}{\sqrt{e}}\right)$

Sol. (b) Given, $f(x) = e^{-x^2}$

$\Rightarrow \quad f'(x) = -2xe^{-x^2}$

$\Rightarrow \quad f''(x) = -2(e^{-x^2} \cdot 1 + x \cdot e^{-x^2}(-2x))$

$= 2e^{-x^2}(2x^2 - 1)$

For points of inflection $f''(x) = 0$

$\Rightarrow \quad 2e^{-x^2}(2x^2 - 1) = 0$

$\Rightarrow \quad 2x^2 - 1 = 0 \quad [\because e^{-x^2} \neq 0]$

$\Rightarrow \quad x = \pm\frac{1}{2}$

When $x = \pm\frac{1}{\sqrt{2}}$, then $y = e^{-1/2} = \frac{1}{\sqrt{e}}$

$\therefore$ Points of inflection are $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$.

Mean Value Theorem

Rolle's Theorem

If a function $f(x)$

(i) continuous in the closed interval $[a, b]$, i.e. continuous at each point in the interval $[a, b]$

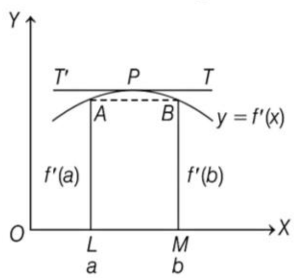
(ii) differentiable in an open interval (a, b) i.e. differentiable at each point in the open interval (a, b)

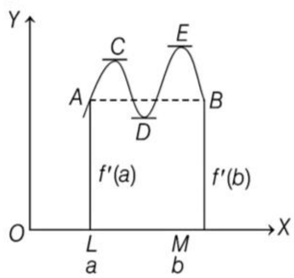
(iii) $f(a) = f(b)$

Then, there will be atleast one point c in the interval (a, b) such that $f'(c) = 0$.

Geometrical Meaning of Rolle's Theorem

In the graph of a function $y = f(x)$ be continuous at each point from the point $A\{a, f(a)\}$ to the point $B\{b, f(b)\}$ and tangent at each point between point A and B is unique, i.e. tangent at each point between A and B exists and ordinates, i.e. y-coordinates of points A and B are equal, then, there will be atleast one point P on the curve between A and B at which tangent will be parallel to X-axis.





In fig. (i) there is only one such point P, where tangent is parallel to X-axis, but in Fig. (ii) there are more than one such point, where tangents are parallel to X-axis.

Note Converse of Rolle's theorem is not true.

Lagrange's Mean Value Theorem

If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$, i.e. continuous at each point in the interval $[a, b]$.

(ii) differentiable in an open interval (a, b), i.e. differentiable at each point in the interval (a, b).

Then, there will be atleast one point c, where $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Another Form of Lagrange's Mean Value Theorem

If a function $f(x)$ is

(i) continuous in the closed interval $[a, a + h]$

(ii) differentiable in the open interval $(a, a + h)$

Then, there exists atleast one value $\theta, 0 < \theta < 1$ such that

$$f(a + h) = f(a) + hf'(a + \theta h)$$

Geometrical Meaning of Lagrange's Mean Value Theorem

Let $A\{a, f(a)\}$ and $B\{b, f(b)\}$ be two points on the curve $y = f(x)$.

Then, $OL = a$, $\quad OM = b$,

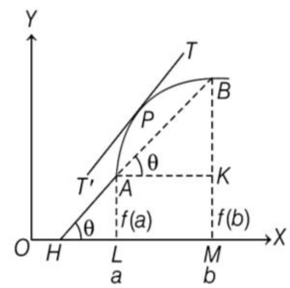
$Al = f(a)$, $\quad BM = f(b)$

Now, slope of chord AB

$$\tan\theta = \frac{BK}{AK} = \frac{f(b) - f(a)}{b - a} \quad \text{...(i)}$$

By Lagrange's mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c) = \text{Slope of tangent at point } P\{c, f(c)\}$$



From Eq. (i),

$$\tan\theta = \text{Slope of tangent at } P$$

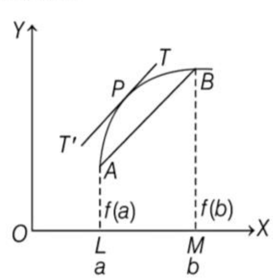
Slope of chord AB = Slope of tangent at P

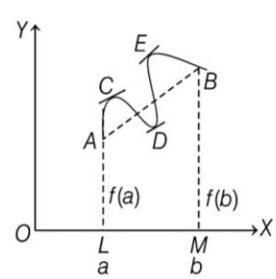
Hence, chord $AB \parallel$ tangent PT

Thus, geometrical meaning of mean value theorem is as follows

In the graph of a curve $y = f(x)$ be continuous at each point from the point $A\{a, f(a)\}$ to the point $B\{b, f(b)\}$ and tangent at each point between A and B exists, i.e. tangent is unique, then there will be atleast one point P on the curve between A and B, where tangent will be parallel to chord AB.

In Fig. (i) there is only one such point P where tangent is parallel to chord AB but in Fig. (ii) there are more than one such points where tangents are parallel to cord AB.





Example 20. *A value of C for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is*

(a) $2\log_3 e$ (b) $\frac{1}{2}\log_e 3$

(c) $\log_3 e$ (d) $\log_e 3$

Sol. (a) Using mean value theorem,

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} \qquad \left[\because f'(c) = \frac{f(b) - f(a)}{b - a}\right]$$

$$\Rightarrow \quad \frac{1}{c} = \frac{\log_e 3 - \log_e 1}{2}$$

$$\therefore \quad c = \frac{2}{\log_e 3} = 2\log_3 e$$

Example 21. *The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0, 1]$ is* **(JEE Main 2020)**

(a) $\frac{\sqrt{7} - 2}{3}$ (b) $\frac{2}{3}$ (c) $\frac{4 - \sqrt{5}}{3}$ (d) $\frac{4 - \sqrt{7}}{3}$

Sol. (d) Given function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0, 1]$ is a continuous function in interval $x \in [0, 1]$ and differentiable in interval $x \in (0, 1)$, so according to Lagrange's mean value theorem for $x = c \in (0, 1)$

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow (3x^2 - 8x + 8)_{x=c} = \frac{(1 - 4 + 8 + 11) - 11}{1 - 0}$$

$$\Rightarrow \quad 3c^2 - 8c + 8 = 5 \Rightarrow 3c^2 - 8c + 3 = 0$$

$$\Rightarrow \quad c = \frac{8 - \sqrt{64 - 36}}{6}$$

$$= \frac{4 - \sqrt{7}}{3} \qquad [\because c \in (0, 1)]$$

Example 22. *If the tangent to the curve, $y = f(x) = x\log_e x$, $(x > 0)$ at a point $(c, f(c))$ is parallel to the line segment joining the point (1, 0) and (e, e) then c is equal to* **(JEE Main 2020)**

(a) $\frac{e - 1}{e}$ (b) $e^{\left(\frac{1}{e-1}\right)}$ (c) $e^{\left(\frac{1}{1-e}\right)}$ (d) $\frac{1}{e - 1}$

Sol. (b) Equation of given curve,

$$y = f(x) = x\log_e x, (x > 0)$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=c} = f'(c) = 1 + \log_e c$$

$\because$ The tangent to the given curve $y = f(x)$ at point $x = c$ is parallel to line segment joining points (1, 0) and (e, e).

$$\text{So, } 1 + \log_e c = \frac{e}{e - 1}$$

$$\Rightarrow \quad \log_e c = \frac{e}{e - 1} - 1 = \frac{1}{e - 1}$$

$$\Rightarrow \quad c = e^{\left(\frac{1}{e-1}\right)} \text{ is positive.}$$

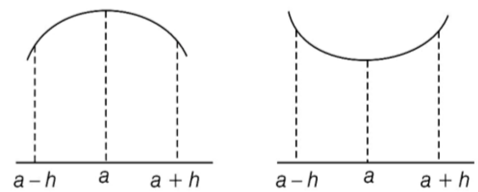
Maxima and Minima/Extremum

Concept of Local Maxima and Local Minima

Let $y = f(x)$ be a function defined at $x = a$ and also in the vicinity of the point $x = a$.

Then, $f(x)$ is said to have a local maximum at $x = a$, if the value of the function at $x = a$ is greater than the value of the function at the neighbouring points of $x = a$.

Mathematically, $f(a) > f(a - h)$ and $f(a) > f(a + h)$, where $h > 0$. (very small quantity).



Similarly, $f(x)$ is said to have a local minimum at $x = a$, if the value of the function at $x = a$ is less than the value of the function at the neighbouring points of $x = a$.

Mathematically, $f(a) < f(a - h)$ and $f(a) < f(a + h)$, where $h > 0$.

A local maximum or a local minimum is also called a local extremum.

Critical Points

It is a collection of points for which,

(i) $f(x)$ does not exists

(ii) $f'(x)$ does not exists or

(iii) $f'(x) = 0$

All the values of x obtained from above conditions are said to be critical points.

It should be noted that critical points are the interior points of an interval.

Example 23. *The number of the critical points for $f(x) = (x - 2)^{2/3}(2x + 1)$ is*

(a) 0 (b) 1 (c) 2 (d) 3

Sol. (c) Given, $f(x) = (x - 2)^{2/3}(2x + 1)$

$$f'(x) = \frac{2}{3}(x - 2)^{-1/3}(2x + 1) + (x - 2)^{2/3} \cdot 2$$

$$\Rightarrow \quad f'(x) = 2\left[\frac{(2x + 1)}{3(x - 2)^{1/3}} + \frac{(x - 2)^{2/3}}{1}\right]$$

Clearly, $f'(x)$ is not defined at $x = 2$, so $x = 2$ is a critical point.

Another critical point is given by,

$$f'(x) = 0$$

$$\Rightarrow \quad 2\left[\frac{(2x + 1) + 3(x - 2)}{3(x - 2)^{1/3}}\right] = 0$$

$$\Rightarrow \quad 5x - 5 = 0 \Rightarrow x = 1$$

Hence, $x = 1$ and $x = 2$ are two critical points of $f(x)$.

Test for Local Maximum/Minimum

We have two cases to consider

(a) Test for Local Maximum/Minimum at $x = a$, if $f(x)$ is Differentiable at $x = a$

Test for local maximum/minimum is the most important topic of this chapter generally question seen from this topic. The level of question is from moderate to typical.

If $f(x)$ is differentiable at $x = a$ and if it is a critical point of the function (*i.e.*, $f'(a) = 0$) , then we have the following three tests to decide whether $f(x)$ has a local maximum or local minimum or neither at $x = a$.

(i) First Derivative Test

If $f'(a) = 0$ and $f'(x)$ changes its sign while passing through the point $x = a$, then

(a) $f(x)$ would have a local maximum at $x = a$, if $f'(a-0) > 0$ and $f'(a+0) < 0$. It means that $f'(x)$ should change its sign from positive to negative.

(b) $f(x)$ would have local minimum at $x = a$, if $f'(a-0) < 0$ and $f'(a+0) > 0$. It means that $f'(x)$ should change its sign from negative to positive.

(c) If $f'(x)$ does not change its sign while passing through $x = a$, then $f(x)$ would have neither a maximum nor minimum at $x = a$

$\therefore$ At $x = -1$, we have local maximum $\Rightarrow f(x) = -2$ and at $x = 1$, we have local minimum $\Rightarrow f_{\min}(x) = 2$

Example 24. *Let $f(x) = x + \frac{1}{x}$, $x \neq 0$, then at which $f(x)$ assumes maximum and minimum are respectively*

(a) −1, 1 (b) 1, −1

(c) 0, 1 (d) None of these

Sol. (a) Here, $f'(x) = 1 - \frac{1}{x^2}$

$$\Rightarrow \quad f'(x) = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

Sign scheme for $f'(x)$

+ −1 − 1 +

Using number line rule, we have maximum at $x = -1$ and minimum at $x = 1$.

$\therefore$ At $x = -1$, we have local maximum $\Rightarrow f_{\max}(x) = -2$ and at $x = 1$, we have local minimum $\Rightarrow f_{\min}(x) = 2$.

(ii) Second Derivative Test

First we find the roots of $f'(x) = 0$. Suppose $x = a$ is one of the roots of $f'(x) = 0$.

Now, find $f''(x)$ at $x = a$.

(a) If $f''(a) =$ negative, then $f(x)$ is maximum at $x = a$.

(b) If $f''(a) =$ positive, then $f(x)$ is minimum at $x = a$.

(c) If $f''(a) =$ zero, then we find $f'''(x)$ at $x = a$.

If $f'''(a) \neq 0$, then $f(x)$ has neither maximum nor minimum (inflexion point) at $x = a$.

But, if $f'''(a) = 0$, then find $f^{iv}(a)$.

If $f^{iv}(a) =$ positive, then $f(x)$ is minimum at $x = a$.

If $f^{iv}(a) =$ negative, then $f(x)$ is maximum at $x = a$.

and so on, process is repeated till point is discussed.

Example 25. *Locate the position and nature of any turning points of the function $y = x^3 - 3x + 2$ is*

(a) Maxima at (1, 0)

(b) Minima at (1, 0)

(c) Maxima at (−1, 0)

(d) None of the above

Sol. (*b*) We need to find where the turning points are and whether we have maximum or minimum points.

First of all, we carry out the differentiation and set $\frac{dy}{dx}$ equal to zero. This will enable us to look for any stationary points. Including any turning points.

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3$$

At stationary points, $\frac{dy}{dx} = 0$ and so

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0 \qquad \text{[factorising]}$$

$$3(x-1)(x+1) = 0$$

[factorising the difference of two squares]

It follows that either $x - 1 = 0$ or $x + 1 = 0$

and so either $x = 1$ or $x = -1$

We have found the *x*-coordinates of the points on the graph, where $\frac{dy}{dx} = 0$, that is the stationary points. We need the *y*-coordinates which are found by substituting the *x* values in the original function $y = x^3 - 3x + 2$.

When $x = 1$, $\quad y = 1^3 - 3(1) + 2 = 0$

When $x = -1$, $\quad y = (-1)^3 - 3(-1) + 2 = 4$

To summarise, we have located two stationary points and these occur at (1,0) and (−1, 4).

Next, we need to determine whether, we have maximum or minimum points, or possibly points such as C in which are neither maxima nor minima.

We have seen that the first derivative $\frac{dy}{dx} = 3x^2 - 3$.

Differentiating this we can find the second derivative.

$$\frac{d^2y}{dx^2} = 6x$$

We now take each point in turn and use our test.

When $x = 1$, $\frac{d^2y}{dx^2} = 6x = 6(1) = 6$.

We are not really interested in this value. What is important is its sign. Because it is positive. We know, we are dealing with a minimum point.

When $x = -1$, $\frac{d^2y}{dx^2} = 6x = 6(-1) = -6$

Again, what is important its sign. Because it is negative we have a maximum point.

*n*th Derivative Test

It is nothing but the general version of the second derivative test, it says that if, $f'(a) = f''(a) = f'''(a) = \dots f^n(a) = 0$ and $f^{n+1}(a) \neq 0$ (all derivatives of the function up to order n vanishes and $(n+1)$th order derivative does not vanish at $x = a$, then $f(x)$ would have a local maximum or local minimum at $x = a$, if n is odd natural number and that $x = a$ would be a point of local maxima, if $f^{n+1}(a) < 0$ and would be a point of local minima, if $f^{n+1}(a) > 0$.

However if n is even, then f has neither a maxima nor a minima at $x = a$. It is clear that the last two tests are basically the Mathematical representation of the first derivative test. But that should not diminish the importance of these tests. Because at that times it becomes very difficult to decide whether $f'(x)$ changes it's sign or not while passing through point $x = a$ and the remaining tests may come handy in these kind of situations.

Note It must be remembered that this method is not applicable to those critical points, where $f'(x)$ remains undefined.

$\therefore$ at $x = -1$, we have local maximum $\Rightarrow f_{max}(x) = -2$ and at $x = 1$, we have local minimum $\Rightarrow f_{min}(x) = 2$.

(b) Test for Local Maximum/Minimum at $x = a$, if $f(x)$ is not Differentiable at $x = a$.

***Case* I** When $f(x)$ is continuous at $x = a$ and $f'(a-h)$ and $f'(a+h)$ exists and are non-zero, then $f(x)$ has a local maximum or minimum at $x = a$, if $f'(a-h)$ and $f'(a+h)$ are of opposite signs.

If $f'(a-h) > 0$ and $f'(a+h) < 0$, then $x = a$ will be a point of local maximum.

If $f'(a-h) < 0$ and $f'(a+h) > 0$, then $x = a$ will be a point of local minimum.

***Case* II** When $f(x)$ is continuous and $f'(a-h)$ and $f'(a+h)$ exist but one of them is zero, we should infer the following about the existence of local maxima/minima from the basic definition of local maxima/ minima.

***Case* III** If $f(x)$ is not continuous at $x = a$ and $f'(a-h)$ and/ or $f'(a+h)$ are not finite, then compare the values of $f(x)$ at the neighbouring points of $x = a$.

Remark It is advisable to draw the graph of the function in the vicinity of the point $x = a$, because the graph would give us the clear picture about the existence of local maxima/minima at $x = a$.

Example 26. *Let*

$$f(x) = \begin{cases} x^3 + x^2 + 10x, & x < 0 \\ -3\sin x, & x \geq 0 \end{cases}, \text{ then at } x = 0 \ f(x) \text{ is}$$

(a) local minimum (b) local maximum
(c) Neither maximum nor minimum
(d) None of the above

Sol. (b) Clearly, $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$ as $f(0) = f(0-0) = f(0+0) = 0$

$$f'-(0) = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h\to 0} \frac{-h^3 + h^2 - 10h - 0}{-h} = 10$$

But $$f'_+(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{-h} = \lim_{h\to 0} \frac{-3\sin h}{h} = -3$$

Since, $f'_-(0) > 0$ and $f'_+(0) < 0$, $x = 0$ is the point of local maximum.

Example 27. *If $p(x)$ be a polynomial of degree three that has a local maximum value 8 at $x = 1$ and a local minimum value 4 at $x = 2$; then $p(0)$ is equal to* **(JEE Main 2020)**

(a) -24 (b) 6 (c) 12 (d) -12

Sol. (d) Since, $p'(x) = 0$ at $x = 1$ and $x = 2$ and $p(x)$ is cubic polynomial.

So, $p'(x) = a(x-1)(x-2) = a(x^2 - 3x + 2)$

$$\therefore \quad p(x) = a\left(\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right) + b$$

According to the question,

$$p(1) = 8 \Rightarrow a\left(\frac{1}{3} - \frac{3}{2} + 2\right) + b = 8$$

$$\Rightarrow a\left(\frac{1}{3} + \frac{1}{2}\right) + b = 8 \Rightarrow 5a + 6b = 48 \quad \dots (i)$$

and $$p(2) = 4 \Rightarrow a\left(\frac{8}{3} - 6 + 4\right) + b = 4$$

$$\Rightarrow \quad 2a + 3b = 12 \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$a = 24, b = -12$$

$$\because \quad p(0) = b = -12$$

Example 28. *If $x = 1$ is a critical point of the function $f(x) = (3x^2 + ax - 2 - a)e^x$, then* **(JEE Main 2020)**

(a) $x = 1$ and $x = -\frac{2}{3}$ are local minima of f

(b) $x = 1$ and $x = -\frac{2}{3}$ are local maxima of f

(c) $x = 1$ is a local maxima and $x = -\frac{2}{3}$ is a local minima of f

(d) $x = 1$ is a local minima and $x = -\frac{2}{3}$ is a local maxima of f

Sol. (d) It is given that $x = 1$ is a critical point of the function

$$f(x) = (3x^2 + ax - 2 - a)e^x$$

So, $f'(1) = e^x(6x + a) + e^x(3x^2 + ax - 2 - a)|_{x=1} = 0$

$\Rightarrow \quad 6 + a + 3 + a - 2 - a = 0 \Rightarrow a = -7$

$\therefore \quad f'(x) = e^x[3x^2 - x - 2] = 0 \Rightarrow x = 1 \text{ or } \left(-\frac{2}{3}\right)$

and $f''(x) = e^x(6x - 1 + 3x^2 - x - 2) = e^x(3x^2 + 5x - 3)$

$\because \quad f''(1) = 5e > 0$

$\Rightarrow x = 1$ is the point of local minima.

and $f''\left(-\frac{2}{3}\right) = e^{-2/3}\left(\frac{4}{3} - \frac{10}{3} - 3\right) = -5e^{-2/3} < 0$

$\Rightarrow \quad x = -\frac{2}{3}$ is the point of local maxima.

Example 29. *If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively, such that $p^2 = q$, then a is equal to*

(a) 3 (b) 1 (c) 2 (d) 1/2

Sol. (c) $\because \quad f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$\therefore \quad f'(x) = 6x^2 - 18ax + 12a^2$

For maxima or minima, put $f'(x) = 0$

$\therefore \quad 6(x^2 - 3ax + 2a^2) = 0$

$\Rightarrow \quad x^2 - 3ax + 2a^2 = 0$

$\Rightarrow \quad x^2 - 2ax - ax + 2a^2 = 0$

$\Rightarrow \quad x(x - 2a) - a(x - 2a) = 0$

$\Rightarrow \quad (x - a)(x - 2a) = 0$

$\Rightarrow \quad x = a, \; x = 2a$

Now, $f''(x) = 12x - 18a$

At $x = a$, $f''(x) = 12a - 18a = -6a$

So, $f(x)$ will be maximum at $x = a$. i.e. $p = a$

Again, at $x = 2a$, $f''(x) = 24a - 18a = 6a$

So, $f(x)$ will be minimum at $x = 2a$. i.e. $q = 2a$

Given, $p^2 = q \Rightarrow a^2 = 2a$

$\therefore \quad a = 2$

Concept of Global (Absolute) Maximum and Minimum

Let $y = f(x)$ be a given function with domain D. Let $[a, b] \subseteq D$. Global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in [a, b].

Global maximum and minimum in [a, b] would occur at critical point of $f(x)$ within $[a, b]$ or at the end points of the interval.

Global (Absolute) Maxima or Minima in $[a, b]$

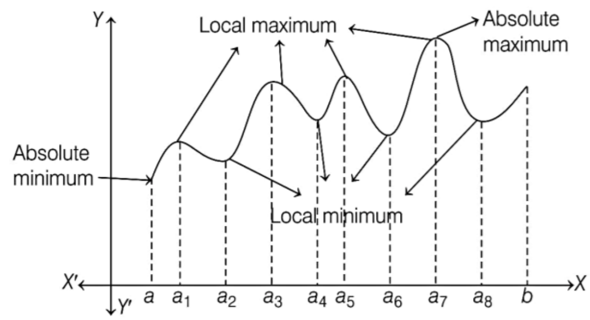
***Step* I** Find out all the critical points of $f(x)$ in (a, b). Let $c_1, c_2, \ldots, c_n$ be the different critical points.

***Step* II** Find the value of the function at these critical points and also at the end points of the domain. Let the values are $f(c_1), f(c_2), \ldots, f(c_n)$.

***Step* III** Find $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$ and $M_2 = \min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

Now, M_1 is the maximum value of $f(x)$ in $[a, b]$, so M_1 is absolute maximum and M_2 is the minimum value of $f(x)$ in $[a, b]$, so M_2 is absolute minimum.

Let $y = f(x)$ be the function defined on $[a, b]$ in the graph, then



(i) $f(x)$ has local maximum values at $x = a_1, a_3, a_5, a_7$

(ii) $f(x)$ has local minimum values at

$$x = a_2, a_4, a_6, a_8$$

(iii) The absolute maximum value of the function is $f(a_7)$ and absolute minimum value is $f(a)$.

Note • Between two local maximum values, there is a local minimum value and *vice-versa*.

• A local minimum value may be greater than a local maximum value. In the above graph, local minimum at a_6 is greater than local maximum at a_1.

Absolute Maxima or Minima in (a, b)

To find the absolute maxima and minima in (a, b) step I and step II are same. Now,

Step III Find $M_1 = \max\{f(c_1), f(c_2), \ldots, f(c_n)\}$

and $M_2 = \min\{f(c_1), f(c_2), \ldots, f(c_n)\}$

Now, if $\lim_{\substack{x \to a^+ \\ \text{or } x \to b^-}} f(x) > M_1$ or $\lim_{\substack{x \to a^+ \\ \text{or } x \to b^-}} f(x) < M_2$, then

$f(x)$ would not have absolute maximum or absolute minimum in (a, b)

and if $\lim_{x \to a^+ \text{ and } x \to b^-} f(x) < M_1$

and $\lim_{x \to a^+ \text{ and } x \to b^-} f(x) > M_2$,

then M_1 and M_2 would respectively be the absolute maximum and absolute minimum of $f(x)$ in (a, b).

Example 30. *Let $f(x) = 2x^3 - 9x^2 + 12x + 6$, then absolute maxima of $f(x)$ in [0, 2] and (1, 3) are respectively*

(a) 0, 2 (b) 1, 2
(c) 2, 2 (d) None of these

Sol. (b) $f(x) = 2x^3 - 9x^2 + 12x + 6$

$f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2)$

In [0, 2], Critical point of $f(x)$ in $[0, 2]$ is $x = 1$.

$\therefore \quad f(0) = 6, f(1) = 11, f(2) = 10$

Thus, $x = 0$ is the point of absolute minimum and $x = 1$ is the point of absolute maximum of $f(x)$ in [0, 2].

In (1, 3), Critical point of $f(x)$ in (1, 3) is $x = 2$.

$\therefore \quad f(2) = 10, \quad \lim_{x \to 1^+} f(x) = 11$

and $\quad \lim_{x \to 3^-} f(x) = 15$

Thus, $x = 2$ is the point of absolute minimum in (1, 3) and absolute maximum in (1, 3) does not exist.

Points of Inflection

Consider function $f(x) = x^3$. At $x = 0$. $f'(x) = 0$. Also, $f''(x) = 0$ at $x = 0$.

Such point is called point of inflection, where 2nd derivatives is zero.

Consider another function $f(x) = \sin x$, $f''(x) = -\sin x$.

Now, $f''(x) = 0$ when $x = n\pi$, then these points are called points of inflection.

At point of inflection

(i) It is not necessary that Ist derivative is zero.

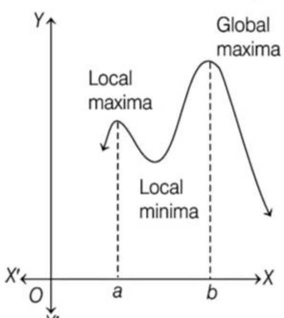
(ii) 2nd derivative must be zero or 2nd derivative changes sign in the neighbourhood of point of inflection.

(iii) Graph of curve changes its concavity.

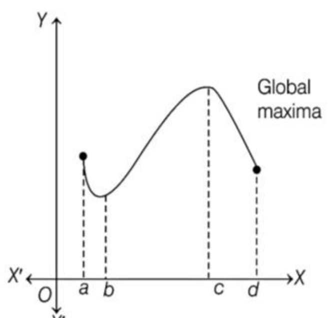
(iv) If $f''(x) > 0$ graph is concave towards negative Y-axis and if $f''(x) < 0$, graph is concave towards positive Y-axis.

Note • For a continous function maximum and minimum value occurs alternately.

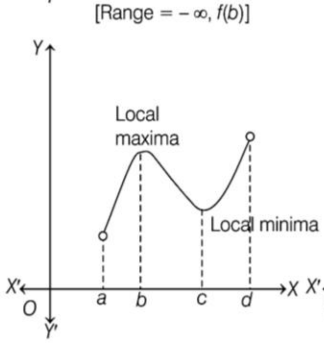
• If a function is discontinuous at a point $x = a$, it may have maximum value although it decreases on the left and increases on the right side of $x = a$.

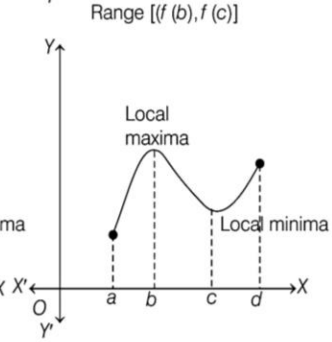


[Range $= -\infty, f(b)$]



Range $[(f(b), f(c)]$

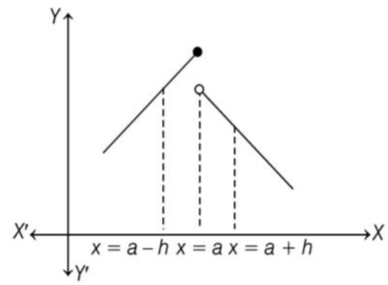




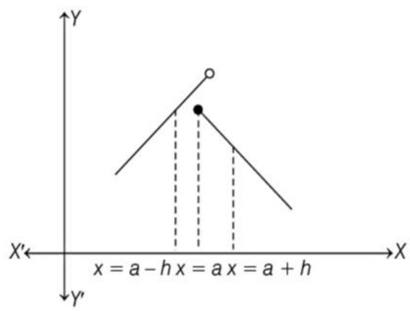
Function has no global maxima and minima.

Range $(f(a), f(d))$ Range $[f(a), f(d)]$

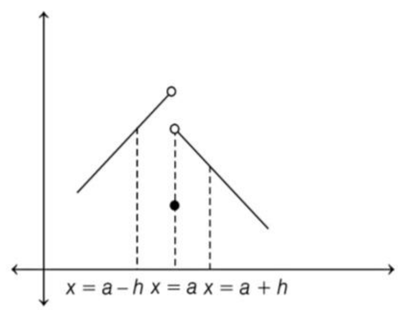
Some Cases of Extremum of Discontinuous Functions



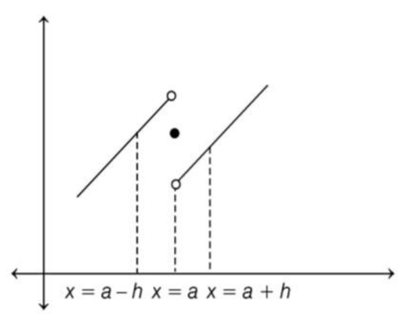
Clearly, $x = a$ is point of maxima as $f(a) > f(a - h)$ and $f(a) > f(a + h)$



Clearly, $x = a$ is not a point of extremum as $f(a) > f(a + h)$ and $f(a) < f(a - h)$



Clearly, $x = a$ is a point of minima as $f(a) < f(a - h)$ and $f(a) < f(a + h)$.



Clearly, $x = a$ is not a point of extremum as $f(a) < f(a - h)$ and $f(a) > f(a + h)$.

Application of Maxima and Minima to Problems (Mensuration and Geometry)

For solving this type of problem, we follows these steps.

***Step* I** First, we read the given problem very carefully for an objective function.

***Step* II** If objective function is of two parameter, then we converts it in terms of one parameter with the help of other given condition in the problem.

***Step* III** Now, we proceed the second derivative test for maxima/minima and get the desired result.

Example 31. *The height of the cylinder of maximum volume that can be inscribed in a sphere of radius R and the volume of the largest cylinder inscribed in a sphere of radius R are*

(a) $\frac{2R}{\sqrt{3}}, \frac{4\pi R^3}{3\sqrt{3}}$ (b) $\frac{2R}{3}, \frac{4\pi R^3}{3}$ (c) $\frac{2R}{3\sqrt{3}}, \frac{4\pi R^3}{\sqrt{3}}$ (d) None of these

Sol. (a) Let r be the radius and h the height of the inscribed cylinder $ABCD$. Let V be its volume.

Then, $V = \pi r^2 h$ ...(i)

Clearly, $AC = 2R$

Aslo, $AC^2 = AB^2 + BC^2$

$\Rightarrow (2R)^2 = (2r)^2 + h^2 \Rightarrow r^2 = \frac{1}{4}(4R^2 - h^2)$ ...(ii)

Using Eqs. (i) and (ii), we get

$$V = \frac{\pi h}{4}(4R^2 - h^2)$$

$$\Rightarrow \frac{dV}{dh} = \left(\pi R^2 - \frac{3}{4}\pi h^2\right)$$

and $$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi h$$

For a maxima or minima, we have $(dV/dh) = 0$

Now, $\frac{dV}{dh} = 0 \Rightarrow \pi R^2 - \frac{3}{4}\pi h^2 = 0 \Rightarrow h = \frac{2R}{\sqrt{3}}$

$$\left[\frac{d^2V}{dh^2}\right]_{h=(2R/\sqrt{3})} = -\frac{3}{2}\pi \times \frac{2R}{\sqrt{3}} = \pi R\sqrt{3} < 0$$

So, V is maximum when $h = \frac{2R}{\sqrt{3}}$

Hence, the height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$.

Largest volume of the cylinder $= \pi \times \frac{1}{4}\left[4R^2 - \frac{4R^2}{3}\right] \times \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$

Example 32. *An open box is to be made out of a piece of paper of cardboard measuring (24 cm × 24 cm) by cutting off equal squares from the corners and turning up the sides, then the height of the box when it has maximum volume is*

(a) 2 cm (b) 4 cm (c) 6 cm (d) 8 cm

Sol. (*b*) Let the length of the side of each square cut off from the corners be x cm. Then, height of the box $= x$ cm.

$\therefore V = (24 - 2x)^2 \times (x) = 4x^3 - 96x^2 + 576x$

$\Rightarrow \frac{dV}{dx} = 12(x^2 - 16x + 48)$ and $\frac{d^2V}{dx^2} = 24(x - 8)$

Now, $\frac{dV}{dx} = 0 \Rightarrow x^2 - 16x + 48 = 0$

i.e. $(x - 12)(x - 4) = 0 \Rightarrow x = 4$ $[\because x \neq 12]$

$$\left[\frac{d^2V}{dx^2}\right]_{x=4} = -96 < 0$$

$\therefore$ V is maximum at $x = 4$.

Hence, the volume of the box is maximum when its height is 4 cm.

Practice Exercise

ROUND I Topically Divided Problems

Derivatives at the Rate of change

1. The position of a moving car at time t is given by $f(t) = at^2 + bt + c, t > 0$, where a, b and c are real numbers greater than 1. Then, average speed of the car over the time interval $[t_1, t_2]$ is attained at the point *(JEE Main 2020)*

(a) $(t_2 - t_1)/2$ (b) $a(t_2 - t_1) + b$
(c) $(t_1 + t_2)/2$ (d) $2a(t_1 + t_2) + b$

2. A water tank has the shape of an inverted right circular cone, whose semi-vertical angle is $\tan^{-1}\left(\frac{1}{2}\right)$. Water is poured into it at a constant rate of 5 cu m/min. Then, the rate (in m/min) at which the level of water is rising at the instant when the depth of water in the tank is 10 m is *(JEE Main 2019)*

(a) $\frac{2}{\pi}$ (b) $\frac{1}{5\pi}$ (c) $\frac{1}{15\pi}$ (d) $\frac{1}{10\pi}$

3. A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate (in cm/ min.) at which of the thickness of ice decreases, is *(JEE Main 2020)*

(a) $\frac{5}{6\pi}$ (b) $\frac{1}{54\pi}$ (c) $\frac{1}{36\pi}$ (d) $\frac{1}{18\pi}$

4. The radius of the base of a cone is increasing at the rate of 3 cm/min and the altitude is decreasing at the rate of 4 cm/min. The rate of change of lateral surface when the radius is 7 cm and altitude is 24 cm is

(a) 50 cm^2/min (b) 54 π cm^2/min
(c) 62 π cm^2/min (d) 66 π cm^2/min

5. A spherical balloon is filled with 4500π cu m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu m / min, then the rate (in m/min) at which the radius of the balloon decreases 49 min after the leakage began is

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$ (c) $\frac{2}{9}$ (d) $\frac{9}{2}$

6. x and y are the sides of two squares such that $y = x - x^2$. The rate of the change of the area of second square with respect to the first square is

(a) $2x^2 - 3x + 1$ (b) $x^2 - 4$
(c) $x^2 - x + 1$ (d) $3x^2 + 2x + 3$

7. A lizard, at an initial distance of 21 cm behind an insect, moves from rest with an acceleration of 2 cm/s^2 and pursues the insect which is crawling uniformly along a straight line at a speed of 20 cm/s. Then, the lizard will catch the insect after

(a) 24 s (b) 21 s (c) 1 s (d) 20 s

8. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa, is

(a) (2, 4) (b) (2, – 4) (c) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

9. The sides of an equilateral triangle are increasing at the rate of 2 cm/s. The rate at which the area increases, when the side is 10 cm, is

(a) $\sqrt{3}$ cm^2/s (b) 10 cm^2/s
(c) $10\sqrt{3}$ cm^2/s (d) $\frac{10}{\sqrt{3}}$ cm^2/s

10. If the volume of a sphere is increasing at a constant rate, then the rate at which its radius is increasing, is

(a) a constant
(b) proportional to the radius
(c) inversely proportional to the radius
(d) inversely proportional to the surface area

11. Moving along the X-axis there are two points with $x = 10 + 6t$, $x = 3 + t^2$. The speed with which they are reaching from each other at the time of encounter is (x is in centimetre and t is in seconds)

(a) 16 cm/s (b) 20 cm/s (c) 8 cm/s (d) 12 cm/s

12. Gas is being pumped into a spherical balloon at the rate of 30 ft^3 / min. Then, the rate at which the radius increases when it reaches the value 15 ft, is

(a) $\frac{1}{30\pi}$ ft/min (b) $\frac{1}{15\pi}$ ft/min
(c) $\frac{1}{20}$ ft/min (d) $\frac{1}{15}$ ft/min

13. An object is moving in the clockwise direction around the unit circle $x^2 + y^2 = 1$. As it passes through the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, its y-coordinate is decreasing at the rate of 3 units per second. The rate at which the x-coordinate changes at this point is (in unit per second)
(a) 2 (b) $3\sqrt{3}$ (c) $\sqrt{3}$ (d) $2\sqrt{3}$

14. The position of a point in time 't' is given by $x = a + bt - ct^2$, $y = at + bt^2$. Its acceleration at time 't' is
(a) $b - c$ (b) $b + c$ (c) $2b - 2c$ (d) $2\sqrt{b^2 + c^2}$

15. Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm^2/s in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4 cm, the rate of decrease of the slant height of water, is
(a) $\frac{\sqrt{2}}{4\pi}$ cm/s (b) $\frac{1}{4\pi}$ cm/s
(c) $\frac{1}{\pi\sqrt{2}}$ cm/s (d) None of these

Errors and Its Approximations

16. If the radius of a sphere is measured as 7 m with an error of 0.02 m, then the approximate error in calculating its volume is
(a) $3.12\,\pi$ m^3 (b) $3.92\,\pi$ m^3
(c) $3.56\,\pi$ m^3 (d) $4.01\,\pi$ m^3

17. The approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$, is
(a) – 34.995 (b) – 33.995
(c) – 33.335 (d) – 35.993

18. If the error committed in measuring the radius of the circle is 0.05%, then the corresponding error in calculating the area is
(a) 0.05% (b) 0.0025% (c) 0.25% (d) 0.1%

19. If $1° = \alpha$ radius, then the approximate value of $\cos(60°1')$ is
(a) $\frac{1}{2} - \frac{\sqrt{3}}{120}$ (b) $\frac{3}{4} - \frac{4}{\sqrt{21}}$
(c) $\frac{1}{3} - \frac{\sqrt{2}}{100}$ (d) $\frac{4}{3} - \frac{\sqrt{5}}{121}$

20. If there is 2% error in measuring the radius of sphere, then the percentage error in the surface area is
(a) 3% (b) 1% (c) 4% (d) 2%

21. The approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, is
(a) $0.0180\,\pi$ cm^3 (b) $0.023\,\pi$ cm^3
(c) $0.0540\,\pi$ cm^3 (d) $0.0432\,\pi$ cm^3

Tangent and Normals

22. For the curve $y = 4x^3 - 2x^5$, the points at which the tangent passes through the origin are
(a) (0, 0), (1, 2) and (–1, –2) (b) (0, 0), (2, 4) and (–1, –3)
(c) (0, 0), (2, 3) and (–3, –1) (d) None of these

23. The point at which the tangent to the curve $y = 2x^2 - x + 1$ is parallel to $y = 3x + 9$, will be
(a) (2, 1) (b) (1, 2) (c) (3, 9) (d) (–2, 1)

24. The length of subtangent to the curve $x^2y^2 = a^4$ at the point $(-a, a)$ is
(a) $3a$ (b) $2a$ (c) a (d) $4a$

25. For the curve $xy = c^2$, the subnormal at any point varies
(a) x^2 (b) x^3 (c) y^2 (d) y^3

26. The abscissa of the points, where the tangent to the curve $y = x^3 - 3x^2 - 9x + 5$ is parallel to X-axis, are
(a) $x = 0$ and 0 (b) $x = 1$ and –1
(c) $x = 1$ and –3 (d) $x = -1$ and 3

27. The slope of the tangent to the curve $x = 3t^2 + 1$, $y = t^3 - 1$, at $x = 1$ is
(a) 0 (b) $\frac{1}{2}$ (c) ∞ (d) –2

28. If tangent to the curve $x = at^2$, $y = 2at$ is perpendicular to X-axis, then its point of contact is
(a) (a, a) (b) $(0, a)$ (c) (0, 0) (d) $(a, 0)$

29. The equation of the tangent to the curve $(1 + x^2)y = 2 - x$, where it crosses the X-axis, is
(a) $x + 5y = 2$ (b) $x - 5y = 2$
(c) $5x - y = 2$ (d) $5x + y - 2 = 0$

30. The point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at which the normal is parallel to the X-axis, is
(a) (0, 0) (b) $(0, a)$ (c) $(a, 0)$ (d) (a, a)

31. Coordinates of a point on the curve $y = x \log x$ at which the normal is parallel to the line $2x - 2y = 3$, are
(a) (0, 0) (b) (e, e)
(c) $(e^2, 2e^2)$ (d) $(e^{-2}, -2e^{-2})$

32. The length of the normal at point 't' of the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is

(a) $a \sin t$ (b) $2a \sin^3 (t/2) \sec (t/2)$
(c) $2a \sin (t/2) \tan (t/2)$ (d) $2a \sin (t/2)$

33. The tangent drawn at the point (0, 1) on the curve $y = e^{2x}$, meets X-axis at the point

(a) $\left(\frac{1}{2}, 0\right)$ (b) $\left(-\frac{1}{2}, 0\right)$ (c) (2, 0) (d) (0, 0)

34. The tangent to the curve $y = 2x^2 - x + 1$ at a point P is parallel to $y = 3x + 4$, then the coordinates of P are

(a) (2, 1) (b) (1, 2) (c) (–1,2) (d) (2, –1)

35. The product of the lengths of subtangent and subnormal at any point of a curve is

(a) square of the abscissa (b) square of the ordinate
(c) constant (d) None of these

36. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{2}$ with the positive X-axis, then $f'(3)$ is equal to

(a) -1 (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

37. Tangent of the angle at which the curves $y = a^x$ and $y = b^x$ $(a \neq b > 0)$ intersect, is given by

(a) $\frac{\log ab}{1 + \log ab}$ (b) $\frac{\log \frac{a}{b}}{1 + (\log a)(\log b)}$
(c) $\frac{\log ab}{1 + (\log a)(\log b)}$ (d) None of these

38. The equation of tangent to the curve $y = be^{-x/a}$ at the point where it crosses Y-axis, is

(a) $ax + by = 1$ (b) $ax - by = 1$
(c) $\frac{x}{a} - \frac{y}{b} = 1$ (d) $\frac{x}{a} + \frac{y}{b} = 1$

39. At what point on the curve $x^3 - 8a^2 y = 0$, the slope of the normal is $-\frac{2}{3}$?

(a) (a, a) (b) $(2a, -a)$
(c) $(2a, a)$ (d) None of these

40. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point

(a) (1, 0) (b) at no point
(c) (0, 1) (d) (0, 0)

41. The point (s) on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical (parallel to Y-axis), is (are)

(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\frac{\sqrt{11}}{3}, 1\right)$ (c) (0, 0) (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

42. If $y = 4x - 5$ is tangent to the curve $y^2 = px^3 + q$ at (2, 3), then (p, q) is

(a) (2, 7) (b) (–2, 7)
(c) (–2, –7) (d) (2, –7)

43. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is

(a) -1 (b) 3 (c) -3 (d) 1

44. If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again at (x_2, y_2), then

(a) $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$ (b) $\frac{x_2}{y_1} + \frac{x_1}{y_2} = -1$
(c) $\frac{x_1}{x_2} + \frac{y_1}{y_2} = -1$ (d) $\frac{x_2}{x_1} + \frac{y_2}{y_1} = 1$

45. The angle between the curves $y = \sin x$ and $y = \cos x$ is

(a) $\tan^{-1}(2\sqrt{2})$ (b) $\tan^{-1}(3\sqrt{2})$
(c) $\tan^{-1}(3\sqrt{3})$ (d) $\tan^{-1}(5\sqrt{2})$

46. The condition for the curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, xy = c^2$ to intersect orthogonally, is

(a) $a^2 + b^2 = 0$ (b) $a^2 - b^2 = 0$
(c) $a = b$ (d) None of these

47. $y = \log(1 + x) - \frac{2x}{2 + x}, x > -1$, is an increasing function of x throughout in,

(a) $x > -1$ (b) $x > 1$ (c) $x < 0$ (d) $x > 0$

48. If the tangent to the curve $y = \frac{x}{x^2 - 3}, x \in R$, $(x \neq \pm\sqrt{3})$, at a point $(\alpha, \beta) \neq (0, 0)$ on it is parallel to the line $2x + 6y - 11 = 0$, then *(JEE Main 2019)*

(a) $|6\alpha + 2\beta| = 19$ (b) $|6\alpha + 2\beta| = 9$
(c) $|2\alpha + 6\beta| = 19$ (d) $|2\alpha + 6\beta| = 11$

49. Let S be the set of all values of x for which the tangent to the curve $y = f(x) = x^3 - x^2 - 2x$ at (x, y) is parallel to the line segment joining the points $(1, f(1))$ and $(-1, f(-1))$, then S is equal to *(JEE Main 2019)*

(a) $\left\{\frac{1}{3}, -1\right\}$ (b) $\left\{\frac{1}{3}, 1\right\}$
(c) $\left\{-\frac{1}{3}, 1\right\}$ (d) $\left\{-\frac{1}{3}, -1\right\}$

50. If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan \theta|$ is equal to *(JEE Main 2019)*

(a) $\frac{7}{17}$ (b) $\frac{8}{15}$ (c) $\frac{4}{9}$ (d) $\frac{8}{17}$

51. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$

(a) cut at right angle (b) touch each other
(c) cut at an angle $\frac{\pi}{3}$ (d) cut at an angle $\frac{\pi}{4}$

52. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, i.e. parallel to X-axis, is

(a) $y = 0$ (b) $y = 1$ (c) $y = 2$ (d) $y = 3$

53. The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point

(a) $(a, 0)$ (b) $(0, a)$ (c) $(0, 0)$ (d) (a, a)

54. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that

(a) it is at a constant distance from the origin
(b) it passes through $\left(\frac{a\pi}{2}, -a\right)$
(c) it makes angle $\frac{\pi}{2} - \theta$ with the X-axis
(d) it passes through the origin

55. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

56. The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1) *(JEE Main 2015)*

(a) does not meet the curve again
(b) meets the curve again in the second quadrant
(c) meets the curve again in the third quadrant
(d) meets the curve again in the fourth quadrant

57. The intercepts on X-axis made by tangents to the curve, $y = \int_0^x |t|\,dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to *(JEE Main 2013)*

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

58. For which of the following curves, the line $x + \sqrt{3}y = 2\sqrt{3}$ is the tangent at the point $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$? *(JEE Main 2021)*

(a) $x^2 + 9y^2 = 9$ (b) $2x^2 - 18y^2 = 9$
(c) $y^2 = \frac{1}{6\sqrt{3}}x$ (d) $x^2 + y^2 = 7$

59. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point, where the curve intersects the Y-axis passes through the point *(JEE Main 2017)*

(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (c) $\left(\frac{1}{2}, -\frac{1}{3}\right)$ (d) $\left(\frac{1}{2}, \frac{1}{3}\right)$

60. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is *(JEE Main 2018)*

(a) 6 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{9}{2}$

61. The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive direction of X-axis, is *(JEE Main 2019)*

(a) $y = x\tan\theta - 2\cot\theta$ (b) $x = y\cot\theta + 2\tan\theta$
(c) $y = x\tan\theta + 2\cot\theta$ (d) $x = y\cot\theta - 2\tan\theta$

62. The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point *(JEE Main 2019)*

(a) $\left(\frac{1}{4}, \frac{7}{2}\right)$ (b) $\left(\frac{7}{2}, \frac{1}{4}\right)$ (c) $\left(-\frac{1}{8}, 7\right)$ (d) $\left(\frac{1}{8}, -7\right)$

63. The tangent to the curve, $y = xe^{x^2}$ passing through the point $(1, e)$ also passes through the point *(JEE Main 2019)*

(a) $\left(\frac{4}{3}, 2e\right)$ (b) $(3, 6e)$ (c) $(2, 3e)$ (d) $\left(\frac{5}{3}, 2e\right)$

64. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$.

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point *(JEE Main 2016)*

(a) (0, 0) (b) $\left(0, \frac{2\pi}{3}\right)$ (c) $\left(\frac{\pi}{6}, 0\right)$ (d) $\left(\frac{\pi}{4}, 0\right)$

Increasing and Decreasing Function

65. The function x^x is increasing, when

(a) $x > \frac{1}{e}$ (b) $x < \frac{1}{e}$
(c) $x < 0$ (d) for all real x

66. If $g(x) = \min(x, x^2)$, where x is real number, then

(a) $g(x)$ is an increasing function
(b) $g(x)$ is a decreasing function
(c) $f(x)$ is a constant function
(d) $g(x)$ is a continuous function except at $x = 0$

67. The function $f(x) = \frac{x}{1 + |x|}$ is

(a) strictly increasing
(b) strictly decreasing
(c) neither increasing nor decreasing
(d) not differential at $x = 0$

68. The function $f(x) = x + \cos x$ is

(a) always increasing
(b) always decreasing
(c) increasing for certain range of x
(d) None of the above

69. $2x^3 - 6x + 5$ is an increasing function, if
(a) $0 < x < 1$ (b) $-1 < x < 1$
(c) $x < -1$ or $x > 1$ (d) $-1 < x < -\frac{1}{2}$

70. The length of the longest interval, in which the function $3\sin x - 4\sin^3 x$ is increasing, is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

71. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is
(a) increasing on $\left[-\frac{1}{2}, 1\right]$ (b) decreasing on R
(c) increasing on R (d) decreasing on $\left[-\frac{1}{2}, 1\right]$

72. Let f be a real valued function, defined on $R - \{-1, 1\}$ and given by
$$f(x) = 3\log_e\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1}$$
Then in which of the following intervals, function $f(x)$ is increasing? *(JEE Main 2021)*
(a) $(-\infty, -1) \cup \left(\left[\frac{1}{2}, \infty\right) - \{1\}\right)$ (b) $(-\infty, \infty) - \{-1, 1\}$
(c) $\left[-1, \frac{1}{2}\right)$ (d) $\left[-\infty, \frac{1}{2}\right) - \{-1\}$

73. $f(x) = \cos\frac{\pi}{x}$ increases in
(a) $\left(\frac{1}{2n+1}, \frac{1}{2n}\right)$ (b) $\left(\frac{1}{2n+2}, \frac{1}{2n+1}\right)$
(c) $(0, 2n)$ (d) R

74. The function $f(x) = \sin^4 x + \cos^4 x$ increases, if
(a) $0 < x < \frac{\pi}{8}$ (b) $\frac{\pi}{4} < x < \frac{3\pi}{8}$
(c) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (d) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

75. The function 'g' defined by $g(x) = f(x^2 - 2x + 8) + f(14 + 2x - x^2)$, where $f(x)$ is twice differentiable function, $f''(x) \geq 0$ for all real numbers x. The function $g(x)$ is increasing in the interval
(a) $[-1, 1] \cup [2, \infty)$ (b) $(\infty, -1] \cup [1, 3]$
(c) $[-1, 1] \cup [3, \infty)$ (d) $(\infty, -2] \cup [1, \infty)$

76. The function $f(x) = \log(\cos x)$ is strictly decreasing and strictly increasing in
(a) $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
(c) $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ (d) $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{3\pi}{2}\right)$

77. The function which is neither decreasing nor increasing in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, is
(a) $\operatorname{cosec} x$ (b) $\tan x$ (c) x^2 (d) $|x-1|$

78. The value of x for which the polynomial $2x^3 - 9x^2 + 12x + 4$ is a decreasing function of x, is
(a) $-1 < x < 1$ (b) $0 < x < 2$ (c) $x > 3$ (d) $1 < x < 2$

79. If $f(x) = \frac{1}{x+1} - \log(1+x)$, $x > 0$, then f is
(a) an increasing function
(b) a decreasing function
(c) both increasing and decreasing function
(d) None of the above

80. If $f(x) = \sin x - \cos x$, the interval in which function is decreasing in $0 \leq x \leq 2\pi$, is
(a) $\left[\frac{5\pi}{6}, \frac{3\pi}{4}\right]$ (b) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
(c) $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ (d) None of these

81. The function $f(x) = x^{1/x}$ is
(a) increasing in $(1, \infty)$
(b) decreasing in $(1, \infty)$
(c) increasing in $(1, e)$ and decreasing in (e, ∞)
(d) decreasing in $(1, e)$ and increasing in (e, ∞)

82. The range of $a \in R$ for which of the function
$$f(x) = (4a - 3)(x + \log_e 5) + 2(a - 7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)$$
$x + 2n\pi$, $n \in N$, has critical points, is *(JEE Main 2021)*
(a) $(-3, 1)$ (b) $\left(-\frac{4}{3}, 2\right]$
(c) $[1, \infty)$ (d) $(-\infty, -1]$

83. $g(x) = 2f\left|\frac{x^2}{2}\right| + f(6 - x^2)$ for all $x \in R$. It is given that $f''(x) > 0$ for all $x \in R$, then $g(x)$ decreases for
(a) $(-\infty, \sqrt{3}) \cup (0, \sqrt{3})$ (b) R
(c) $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ (d) $(-\infty, -2) \cup (0, 2)$

84. Function $f(x) = \frac{\lambda\sin x + 6\cos x}{2\sin x + 3\cos x}$ is monotonic increasing, if
(a) $\lambda > 1$ (b) $\lambda < 1$ (c) $\lambda < 4$ (d) $\lambda > 4$

85. Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $\forall x \in R$. Then, the set of all $x \in R$, where the function $h(x) = (fog)(x)$ is increasing, is *(JEE Main 2019)*
(a) $\left[0, \frac{1}{2}\right] \cup [1, \infty)$ (b) $\left[-1, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$
(c) $[0, \infty)$ (d) $\left[\frac{-1}{2}, 0\right] \cup [1, \infty)$

86. Let $f:[0, 2] \to R$ be a twice differentiable function such that $f''(x) > 0$, for all $x \in (0, 2)$. If $\phi(x) = f(x) + f(2-x)$, then ϕ is *(JEE Main 2019)*

(a) increasing on (0, 1) and decreasing on (1, 2)
(b) decreasing on (0, 2)
(c) decreasing on (0, 1) and increasing on (1, 2)
(d) increasing on (0, 2)

87. Let $f(x) = \dfrac{x}{\sqrt{a^2+x^2}} - \dfrac{d-x}{\sqrt{b^2+(d-x)^2}}, x \in R$, where a, b and d are non-zero real constants. Then, *(JEE Main 2019)*

(a) f is an increasing function of x
(b) f' is not a continuous function of x
(c) f is a decreasing function of x
(d) f is neither increasing nor decreasing function of x

88. Let $f: R \to R$ be a positive increasing function with $\lim_{x\to\infty} \dfrac{f(3x)}{f(x)} = 1$. Then, $\lim_{x\to\infty} \dfrac{f(2x)}{f(x)}$ is equal to

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 3

89. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$ (c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Mean Value Theorem

90. If the function $f(x) = x^3 - 6x^2 + ax + b$ satisfies Rolle's theorem in the interval [1, 3] and $f'\left(\dfrac{2\sqrt{3}+1}{\sqrt{3}}\right) = 0$, then

(a) $a = -11$ (b) $a = -6$ (c) $a = 6$ (d) $a = 11$

91. $f(x)$ satisfies the conditions of Rolle's theorem in [1, 2] and $f(x)$ is continuous in [1, 2], then $\int_1^2 f'(x)\,dx$ is equal to

(a) 3 (b) 0 (c) 1 (d) 2

92. If $f(x)$ satisfies the conditions for Rolle's theorem in [3, 5], then $\int_3^5 f(x)\,dx$ is equal to

(a) 2 (b) -1 (c) 0 (d) $-\frac{4}{3}$

93. In the mean value theorem $\dfrac{f(b)-f(a)}{b-a} = f'(c)$, if $a = 0, b = \dfrac{1}{2}$ and $f(x) = x(x-1)(x-2)$, then value of c is

(a) $1 - \frac{\sqrt{15}}{6}$ (b) $1 + \sqrt{15}$ (c) $1 - \frac{\sqrt{21}}{6}$ (d) $1 + \sqrt{21}$

94. If Rolle's theorem holds for the function $f(x) = x^3 - ax^2 + bx - 4, x \in [1, 2]$ with $f'\left(\frac{4}{3}\right) = 0$, then ordered pair (a, b) is equal to *(JEE Main 2021)*

(a) $(-5, 8)$ (b) $(5, 8)$ (c) $(5, -8)$ (d) $(-5, -8)$

95. Rolle's theorem holds for the function $x^3 + bx^2 + cx$, $1 \le x \le 2$ at the point $\frac{4}{3}$, the values of b and c are

(a) $b = 8, c = -5$ (b) $b = -5, c = 8$
(c) $b = 5, c = -8$ (d) $b = -5, c = -8$

96. Suppose the cubic equation $x^3 - px + q = 0$ has three distinct real roots, where $p > 0$ and $q > 0$. Then, which one of the following holds?

(a) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$
(b) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$
(c) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$
(d) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

97. If $2a + 3b + 6c = 0$, then the equation $ax^2 + bx + c = 0$ has atleast one real root in

(a) (0, 1) (b) $\left(0, \frac{1}{2}\right)$ (c) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (d) (−1, 1)

98. For all twice differentiable functions $f: R \to R$, with $f(0) = f(1) = f'(0) = 0$ *(JEE Main 2020)*

(a) $f''(x) \ne 0$ at every point $x \in (0, 1)$
(b) $f''(x) = 0$ at every point $x \in (0, 1)$
(c) $f''(0) = 0$
(d) $f''(x) = 0$ at some point $x \in (0, 1)$

99. If c is a point at which Rolle's theorem holds for the function, $f(x) = \log_e\left(\dfrac{x^2+\alpha}{7x}\right)$ in the interval [3, 4], where $\alpha \in R$, then $f''(c)$ is equal to *(JEE Main 2020)*

(a) $-\frac{1}{24}$ (b) $-\frac{1}{12}$ (c) $\frac{1}{12}$ (d) $\frac{\sqrt{3}}{7}$

Critical Points and Test for Local Maximum/Minimum

100. The critical points of the function

$$f(x) = 2\sin^2\left(\frac{x}{6}\right) + \sin\left(\frac{x}{3}\right) - \left(\frac{x}{3}\right)$$

whose coordinates satisfy the inequality $x^2 - 10 < -19.5x$, is

(a) -6π (b) 6π (c) $\frac{9\pi}{2}$ (d) -4π

101. The points of extrema of $f(x)=\int_0^x \frac{\sin t}{t}\,dt$ in the domain $x>0$ are

(a) $(2n+1)\frac{\pi}{2}, n=1,2,\ldots$ (b) $(4n+1)\frac{\pi}{2}, n=1,2,\ldots$

(c) $(2n+1)\frac{\pi}{4}, n=1,2,\ldots$ (d) $n\pi,\ n=1,2,\ldots$

102. The number of critical points of

$f(x)=|x|(x-1)(x-2)(x-3)$ is

(a) 1 (b) 2 (c) 3 (d) 4

103. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

(a) $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$ (b) $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$

(c) $x=\frac{-\pi}{4}$ and $x=\frac{-3\pi}{4}$ (d) None of these

104. It is given that at $x=1$, the function x^4-62x^2+ax+9 attains its maximum value, on the interval $[0, 2]$. Find the value of a.

(a) 30 (b) 40 (c) 75 (d) 120

105. The maximum and minimum values of $x+\sin 2x$ on $[0, 2\pi]$ is

(a) 2π and 0 (b) π and $\frac{1}{2}$

(c) $\frac{\pi}{2}$ and -1 (d) None of these

106. The maximum value of $[x(x-1)+1]^{1/3}, 0\le x\le 1$ is

(a) $\left(\frac{1}{3}\right)^{1/3}$ (b) $\frac{1}{2}$ (c) 1 (d) zero

107. The function $f(x)=4x^3-18x^2+27x-7$ has

(a) one local maxima
(b) one local minima
(c) one local maxima and two local minima
(d) neither maxima nor minima

108. All the points of local maxima and local minima of the function $f(x)=\frac{-3}{4}x^4-8x^3-\frac{45}{2}x^2+105$ are

(a) $x=0,-5$ and $x=-3$ (b) $x=1,3$ and $x=-2$
(c) $x=0,-3$ and $x=-2$ (d) None of these

109. The value of 'a' for which the function $f(x)=a\sin x+\frac{1}{3}\sin 3x$ has an extremum at $x=\frac{\pi}{3}$ is

(a) 1 (b) -1 (c) 2 (d) 0

110. If $f(x)=x+1, \forall x\in R$ and $g(x)=e^x, \forall x\in[-2, 0]$, then the maximum value of $f(|x|)-g(x)$ is

(a) $3+\frac{1}{e}$ (b) $3+\frac{1}{e^2}$ (c) $-3-\frac{1}{e^2}$ (d) $3-\frac{1}{e^2}$

111. On the interval $[0,1]$, the function $x^{25}(1-x)^{75}$ takes its maximum value at the point

(a) 0 (b) 1/4 (c) 1/2 (d) 1/3

112. The function $f(x)=x^{-x}, (x>0)$ attains a maximum value at x which is

(a) 2 (b) 3 (c) $\frac{1}{e}$ (d) 1

113. The function $y=a(1-\cos x)$ is maximum when x is equal to

(a) π (b) $\pi/2$ (c) $-\pi/2$ (d) $-\pi/6$

114. The function $f(x)=x+\sin x$ has

(a) a minimum but no maximum
(b) a maximum but no minimum
(c) neither maximum nor minimum
(d) both maximum and minimum

115. If a differential function $f(x)$ has a relative minimum at $x=0$, then the function $\phi(x)=f(x)+ax+b$ has a relative minimum at $x=0$ for

(a) all a and all b (b) all b, if $a=0$
(c) all $b>0$ (d) all $a>0$

116. The denominator of a fraction is greater than 16 of the square of numerator, then least value of fraction is

(a) $-1/4$ (b) $-1/8$
(c) 1/12 (d) 1/16

117. The function $f(x)=ax+\frac{b}{x}$, $b, x>0$ takes the least value at x equal to

(a) b (b) $\sqrt{a}$
(c) $\sqrt{b}$ (d) $\sqrt{\frac{b}{a}}$

118. In $(-4, 4)$ the function $f(x)=\int_{-10}^{x}(t^4-4)e^{-4t}\,dt$ has

(a) no extrema (b) one extremum
(c) two extrema (d) four extrema

119. The function $f(x)=a\cos x+b\tan x+x$ has extreme values at $x=0$ and $x=\frac{\pi}{6}$, then

(a) $a=-\frac{2}{3}, b=-1$ (b) $a=\frac{2}{3}, b=-1$
(c) $a=-\frac{2}{3}, b=1$ (d) $a=\frac{2}{3}, b=1$

120. The absolute maximum and minimum values of the function f given by $f(x)=\cos^2 x+\sin x, x\in[0, \pi]$

(a) 2.25 and 2 (b) 1.25 and 1
(c) 1.75 and 1.5 (d) None of these

121. The maximum value of $f(x)=\dfrac{x}{4+x+x^2}$ on $[-1,1]$ is

(a) $-\dfrac{1}{4}$ (b) $-\dfrac{1}{3}$ (c) $\dfrac{1}{6}$ (d) $\dfrac{1}{5}$

122. In interval $[1,e]$, the greatest value of $x^2\log x$ is

(a) e^2 (b) $\dfrac{1}{e}\log\dfrac{1}{\sqrt{e}}$

(c) $e^2\log\sqrt{e}$ (d) None of these

123. The function $f(x)=|px-q|+r|x|$, $x\in(-\infty,\infty)$, where $p>0, q>0, r>0$ assumes its minimum value only on one point, if

(a) $p\neq q$ (b) $r\neq q$ (c) $r\neq p$ (d) $p=q=r$

124. If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x)=9x^4+12x^3-36x^2+25$, $x\in R$, then *(JEE Main 2019)*

(a) $S_1=\{-2\};S_2=\{0,1\}$ (b) $S_1=\{-2,0\};S_2=\{1\}$

(c) $S_1=\{-2,1\};S_2=\{0\}$ (d) $S_1=\{-1\};S_2=\{0,2\}$

125. The set of all real values of λ for which the function $f(x)=(1-\cos^2 x).(\lambda+\sin x)$, $x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$, has exactly one maxima and exactly one minima, is *(JEE Main 2020)*

(a) $\left(-\dfrac{1}{2},\dfrac{1}{2}\right)-\{0\}$ (b) $\left(-\dfrac{3}{2},\dfrac{3}{2}\right)$

(c) $\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$ (d) $\left(-\dfrac{3}{2},\dfrac{3}{2}\right)-\{0\}$

126. Let $f(x)$ be a polynomial of degree 5 such that $x=\pm 1$ are its critical points. If $\lim\limits_{x\to 0}\left(2+\dfrac{f(x)}{x^3}\right)=4$, then which one of the following is not true? *(JEE Main 2020)*

(a) f is an odd function.

(b) $x=1$ is a point of minima and $x=-1$ is a point of maxima of f.

(c) $f(1)-4f(-1)=4$.

(d) $x=1$ is a point of maxima and $x=-1$ is a point of minimum of f.

127. Suppose $f(x)$ is a polynomial of degree four, having critical points at $-1, 0, 1$. If $T=\{x\in R\}\ f(x)=f(0)$, then the sum of squares of all the elements of T is *(JEE Main 2020)*

(a) 2 (b) 4 (c) 8 (d) 6

128. If $f(x)$ is a non-zero polynomial of degree four, having local extreme points at $x=-1,0,1$, then the set $S=\{x\in R: f(x)=f(0)\}$ contains exactly *(JEE Main 2019)*

(a) four rational numbers

(b) two irrational and two rational numbers

(c) four irrational numbers

(d) two irrational and one rational number

129. If m is the minimum value of k for which the function $f(x)=x\sqrt{kx-x^2}$ is increasing in the interval $[0,3]$ and M is the maximum value of f in the interval $[0,3]$ when $k=m$, then the ordered pair (m,M) is equal to *(JEE Main 2019)*

(a) $(4,3\sqrt{2})$ (b) $(4,3\sqrt{3})$ (c) $(3,3\sqrt{3})$ (d) $(5,3\sqrt{6})$

130. Let $f(x)=5-|x-2|$ and $g(x)=|x+1|$, $x\in R$. If $f(x)$ attains maximum value at α and $g(x)$ attains minimum value of β, then $\lim\limits_{x\to-\alpha\beta}\dfrac{(x-1)(x^2-5x+6)}{x^2-6x+8}$ is equal to *(JEE Main 2019)*

(a) 1/2 (b) $-3/2$ (c) $-1/2$ (d) 3/2

131. The maximum value of the function $f(x)=3x^3-18x^2+27x-40$ on the set $S=\{x\in R: x^2+30\le 11x\}$ is *(JEE Main 2019)*

(a) 122 (b) -122 (c) -222 (d) 222

132. Let $f(x)=x^2+\dfrac{1}{x^2}$ and $g(x)=x-\dfrac{1}{x}$, $x\in R-\{-1,0,1\}$. If $h(x)=\dfrac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is *(JEE Main 2018)*

(a) 3 (b) -3 (c) $-2\sqrt{2}$ (d) $2\sqrt{2}$

133. Let $f(x)$ be a polynomial of degree four having extreme values at $x=1$ and $x=2$. If $\lim\limits_{x\to 0}\left[1+\dfrac{f(x)}{x^2}\right]=3$, then $f(2)$ is equal to *(JEE Main 2015)*

(a) -8 (b) -4 (c) 0 (d) 4

134. If $x=-1$ and $x=2$ are extreme points of $f(x)=\alpha\log|x|+\beta x^2+x$, then *(JEE Main 2014)*

(a) $\alpha=-6, \beta=\dfrac{1}{2}$ (b) $\alpha=-6, \beta=-\dfrac{1}{2}$

(c) $\alpha=2, \beta=-\dfrac{1}{2}$ (d) $\alpha=2, \beta=\dfrac{1}{2}$

135. For $x\in\left(0,\dfrac{5\pi}{2}\right)$, define $f(x)=\int_0^x\sqrt{t}\sin t\,dt$. Then, f has

(a) local minimum at π and 2π

(b) local minimum at π and local maximum at 2π

(c) local maximum at π and local minimum at 2π

(d) local maximum at π and 2π

136. Let $f: R\to R$ be defined by $f(x)=\begin{cases}k-2x, & \text{if } x\le -1\\ 2x+3, & \text{if } x>-1\end{cases}$. If f has a local minimum at $x=-1$, then a possible value of k is

(a) 1 (b) 0 (c) $-\dfrac{1}{2}$ (d) -1

Point of Inflection and Application of Maxima and Minima

137. The point of inflexion for the curve $y = x^{5/2}$ is

(a) (1, 1) (b) (0, 0) (c) (1, 0) (d) (0, 1)

138. The two positive numbers x and y such that their sum is 35 and the product is x^2y^5 is maximum, are

(a) 15 and 20 (b) 10 and 25
(c) 5 and 30 (d) None of these

139. The two positive numbers whose sum is 16 and the sum of whose cubes is minimum, are

(a) 4 and 12 (b) 6 and 10
(c) 8 and 8 (d) None of these

140. A square piece of tin of side 18 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps of the box. What should be the side of the square to be cut-off so that the volume of the box is maximum possible?

(a) 3 cm (b) 4 cm (c) 5 cm (d) 9 cm

141. The closed right circular cylinder of given surface and maximum volume is such that its height is equal to

(a) the radius of the base
(b) the diameter of the base
(c) the twice of diameter of the base
(d) None of the above

142. The right circular cone of least curved surface area and given volume has an altitude equal to

(a) two times the radius of the base.
(b) $\sqrt{3}$ times the radius of the base.
(c) $\sqrt{2}$ times the radius of the base.
(d) None of the above

143. The semi-vertical angle of the cone of the maximum volume and of given slant height is

(a) $\tan^{-1}\sqrt{3}$ (b) $\tan^{-1}\sqrt{2}$
(c) $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (d) None of these

144. The semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

(a) $\sin^{-1}\left(\frac{1}{3}\right)$ (b) $\sin^{-1}\left(\frac{1}{2}\right)$
(c) $\sin^{-1}(\sqrt{3})$ (d) None of these

145. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(a) $(2\sqrt{2}, 4)$ (b) $(2\sqrt{2}, 0)$ (c) (0, 0) (d) (2, 2)

146. The maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

(a) $\frac{3}{4}ab$ sq unit (b) $\frac{3}{4}\sqrt{3}\,ab$ sq unit
(c) $\frac{\sqrt{3}}{4}ab$ sq unit (d) None of these

147. The sum of the perimeter of a circle and square is k, where k is some constant, then the sum of their areas is least when the side of square is

(a) equal to the radius of the circle
(b) double the radius of the circle
(c) triple the radius of the circle
(d) None of the above

148. A window is in the form of a rectangle surmounted by a semi-circle opening. The perimeter of the window is 10 m. The dimensions of the window to admit maximum light through the whole opening is

(a) length = breadth $= \frac{1}{\pi + 4}$
(b) length $= \frac{20}{\pi + 4}$ and breadth $= \frac{10}{\pi + 4}$
(c) length $= \frac{2}{\pi + 4}$ and breadth $= \frac{1}{\pi + 4}$
(d) None of the above

149. The altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is

(a) $\frac{r}{2}$ (b) $\frac{r}{3}$ (c) $\frac{3r}{4}$ (d) $\frac{4r}{3}$

150. The height of the cylinder of maximum volume that can be inscribed in a sphere of radius R and the maximum volume respectively

(a) $\frac{2R}{\sqrt{3}}$ and $\frac{4\pi R^3}{3\sqrt{3}}$ (b) $\frac{R}{\sqrt{3}}$ and $\frac{\pi R^3}{3\sqrt{3}}$
(c) $\frac{4R}{\sqrt{3}}$ and $\frac{2\pi R^3}{3\sqrt{3}}$ (d) None of these

151. The height of the cylinder of greatest volume which can be inscribed in a circular cone of height h and having semi-vertical angle α and the greatest volume of cylinder are respectively

(a) $\frac{2}{3}$ that of the cone, $\frac{2}{27}\pi h^3 \tan\alpha$
(b) $\frac{4}{3}$ that of the cone, $\frac{4}{27}\pi h^2 \tan^3\alpha$
(c) $\frac{1}{3}$ that of the cone, $\frac{4}{27}\pi h^3 \tan^2\alpha$
(d) $\frac{5}{3}$ that of the cone, $\frac{4}{27}\pi h \tan\alpha$

152. If the sum of the length of the hypotenuse and a side of a right angled triangle is given. Then, the area of the triangle is maximum when the angle between them is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

153. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12 (c) 16 (d) 32

154. If PQ and PR are the two sides of a triangle, then the angle between them which gives maximum area of the triangle, is

(a) π (b) $\pi/3$ (c) $\pi/4$ (d) $\pi/2$

155. If $ab = 2a + 3b$, $a > 0$, $b > 0$, then the minimum value of ab is

(a) 12 (b) 24
(c) $\frac{1}{4}$ (d) None of these

156. The minimum radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ is of length

(a) $a - b$ (b) $a + b$
(c) $2a + b$ (d) None of these

157. The perimeter of a sector is p. The area of the sector is maximum, when its radius is

(a) $\sqrt{p}$ (b) $\frac{1}{\sqrt{p}}$ (c) $\frac{p}{2}$ (d) $\frac{p}{4}$

158. If $xy = c^2$, then minimum value of $ax + by$ is

(a) $c\sqrt{ab}$ (b) $2c\sqrt{ab}$ (c) $-c\sqrt{ab}$ (d) $-2c\sqrt{ab}$

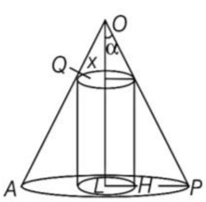
159. If $a^2x^4 + b^2y^4 = c^6$, then maximum value of xy is

(a) $\frac{c^2}{\sqrt{ab}}$ (b) $\frac{c^3}{ab}$ (c) $\frac{c^3}{\sqrt{2ab}}$ (d) $\frac{c^3}{2ab}$

160. A cone of maximum volume is inscribed in the given sphere, then ratio of the height of the cone to diameter of the sphere is

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

161. A given right circular cone has volume p and the largest right circular cylinder that can be inscribed in the cone has a volume q. Then, $p : q$ is



(a) 9 : 4 (b) 8 : 3
(c) 7 : 2 (d) None of these

162. The set of all values of the parameter a for which the points of minimum of the function $y = 1 + a^2x - x^3$ satisfy the inequality $\frac{x^2 + x + 2}{x^2 + 5x + 6} \le 0$, is

(a) an empty set
(b) $(-3\sqrt{3}, -2\sqrt{3})$
(c) $(2\sqrt{3}, 3\sqrt{3})$
(d) $(-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$

163. The tangent to the curve $y = x^3 - 6x^2 + 9x + 4$, $0 \le x \le 5$ has maximum slope at x which is equal to

(a) 2 (b) 3
(c) 4 (d) None of these

164. Let $P(h, k)$ be a point on the curve $y = x^2 + 7x + 2$, nearest to the line $y = 3x - 3$. Then, the equation of the normal to the curve at P is *(JEE Main 2020)*

(a) $x - 3y - 11 = 0$ (b) $x - 3y + 22 = 0$
(c) $x + 3y - 62 = 0$ (d) $x + 3y + 26 = 0$

165. The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 is *(JEE Main 2019)*

(a) $\sqrt{6}$ (b) $2\sqrt{3}$
(c) $\sqrt{3}$ (d) $\frac{2}{3}\sqrt{3}$

166. The maximum volume (in cum) of the right circular cone having slant height 3 m is *(JEE Main 2019)*

(a) $\frac{4}{3}\pi$ (b) $2\sqrt{3}\pi$ (c) $3\sqrt{3}\pi$ (d) 6π

167. A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \ge 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then, this nearest distance is *(JEE Main 2019)*

(a) $\frac{1}{3}\sqrt{\frac{7}{3}}$ (b) $\frac{\sqrt{5}}{6}$ (c) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (d) $\frac{1}{2}$

168. If 20 m of wire is available for fencing off a flower-bed in the form of a circular sector, then the maximum area (in sq m) of the flower-bed is *(JEE Main 2017)*

(a) 12.5 (b) 10 (c) 25 (d) 30

169. The radius of a circle having minimum area, which touches the curve $y = 4 - x^2$ and the lines $y = |x|$, is *(JEE Main 2017)*

(a) $2(\sqrt{2} + 1)$ (b) $2(\sqrt{2} - 1)$
(c) $4(\sqrt{2} - 1)$ (d) $4(\sqrt{2} + 1)$

170. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side $= x$ units and a circle of radius $= r$ units. If the sum of the areas of the square and the circle so formed is minimum, then *(JEE Main 2016)*

(a) $2x = (\pi + 4)r$ (b) $(4 - \pi)x = \pi r$
(c) $x = 2r$ (d) $2x = r$

ROUND II Mixed Bag

Only One Correct Option

1. Line joining the points (0, 3) and (5, – 2) is a tangent to the curve $y = \dfrac{ax}{1+x}$, then

(a) $a = 1 \pm \sqrt{3}$ (b) $a = \phi$
(c) $a = -1 \pm \sqrt{3}$ (d) $a = -2 \pm 2\sqrt{3}$

2. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$

(a) $f(x)$ is strictly increasing function
(b) $f(x)$ has a local maxima
(c) $f(x)$ is a strictly decreasing function
(d) $f(x)$ is unbounded

3. $f(x) = \int_0^x |\log_2 [\log_3\{\log_4 (\cos t + a)\}]|\, dt$. If $f(x)$ is increasing for all real values of x, then

(a) $a \in (-1, 1)$ (b) $a \in (1, 5)$
(c) $a \in (1, \infty)$ (d) $a \in (5, \infty)$

4. The parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect orthogonally at point $P(x_1, y_1)$, where $x_1, y_1 \neq 0$, then

(a) $b = a^2$ (b) $b = a^3$
(c) $b^3 = a^2$ (d) None of these

5. $y = f(x)$ is a parabola, having its axis parallel to y-axis. If the line $y = x$ touches this parabola at $x = 1$, then

(a) $f''(1) + f'(0) = 1$ (b) $f''(0) - f'(1) = 1$
(c) $f''(1) - f'(0) = 1$ (d) $f''(0) + f'(1) = 1$

6. If $f'(x) > 0$ and $f''(x) > 0$, $\forall x \in R$, then for any two real numbers x_1 and x_2, $(x_1 \neq x_2)$

(a) $f\left(\dfrac{x_1 + x_2}{2}\right) > \dfrac{f(x_1) + f(x_2)}{2}$

(b) $f\left(\dfrac{x_1 + x_2}{2}\right) < \dfrac{f(x_1) + f(x_2)}{2}$

(c) $f'\left(\dfrac{x_1 + x_2}{2}\right) > \dfrac{f'(x_1) + f'(x_2)}{2}$

(d) $f'\left(\dfrac{x_1 + x_2}{2}\right) < \dfrac{f'(x_1) + f'(x_2)}{2}$

7. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall x \in \left(0, \dfrac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then $g(x)$ is decreasing in

(a) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ (b) $\left(0, \dfrac{\pi}{4}\right)$
(c) $\left(0, \dfrac{\pi}{2}\right)$ (d) $\left(\dfrac{\pi}{6}, \dfrac{\pi}{2}\right)$

8. The slope of tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, – 1) is

(a) $\dfrac{22}{7}$ (b) $\dfrac{6}{7}$
(c) –6 (d) None of these

9. The values of a for which the function $(a+2)x^3 - 3ax^2 + 9ax - 1 = 0$ decreases monotonically throughout for all real x, are

(a) $a < -2$ (b) $a > -2$
(c) $-3 < a < 0$ (d) $-\infty < a \leq -3$

10. If $f(x) = \dfrac{x}{\sin x}$ and $g(x) = \dfrac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval

(a) both $f(x)$ and $g(x)$ are increasing functions
(b) both $f(x)$ and $g(x)$ are decreasing
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function

11. The function $\dfrac{a \sin x + b \cos x}{c \sin x + d \cos x}$ is decreasing, if

(a) $ad - bc > 0$ (b) $ad - bc < 0$
(c) $ab - cd > 0$ (d) $0 \leq x \leq -2$

12. The maximum slope of the curve $y = \dfrac{1}{2}x^4 - 5x^3 + 18x^2 - 19x$ occurs at the point **(JEE Main 2021)**

(a) (2, 9) (b) (2, 2) (c) $\left(3, \dfrac{21}{2}\right)$ (d) (0, 0)

13. The function $f(x) = x(x+3)e^{-(1/2)x}$ satisfies all the conditions of Rolle's theorem in $[-3, 0]$. The value of c is

(a) 0 (b) –1 (c) –2 (d) –3

14. Let $f(x)$ satisfy all the conditions of mean value theorem in [0, 2]. If $f(0) = 0$ and $|f'(x)| \leq \dfrac{1}{2}$ for all x, in [0, 2], then

(a) $f(x) < 2$
(b) $|f(x)| \leq 1$
(c) $f(x) = 2x$
(d) $f(x) = 3$ for atleast one x in [0, 2]

15. The function $f(x) = x^3 - 6x^2 + ax + b$ satisfy the conditions of Rolle's theorem in [1, 3]. The values of a and b are

(a) 11, –6 (b) –6, 11 (c) –11, 6 (d) 6, –11

16. The function $f(x) = (x-3)^2$ satisfies all the conditions of mean value theorem in [3, 4]. A point on $y = (x-3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1) is

(a) $\left(\dfrac{7}{2}, \dfrac{1}{2}\right)$ (b) $\left(\dfrac{7}{2}, \dfrac{1}{4}\right)$ (c) (1, 4) (d) (4, 1)

17. In the mean value theorem, $f(b)-f(a)=(b-a)f'(c)$, if $a=4$, $b=9$ and $f(x)=\sqrt{x}$, then the value of c is

(a) 8.00 (b) 5.25 (c) 4.00 (d) 6.25

18. Let $a+b=4$, $a<2$ and $g(x)$ be a monotonically increasing function of x. Then,

$$f(x)=\int_0^a g(x)\,dx+\int_0^b g(x)\,dx$$

(a) increases with increase in $(b-a)$
(b) decreases with increase in $(b-a)$
(c) increases with decreases in $(b-a)$
(d) None of the above

19. The sum of intercepts on coordinate axes made by tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ is

(a) a (b) $2a$
(c) $2\sqrt{a}$ (d) None of these

20. Let $f'(x)=e^{x^2}$ and $f(0)=10$. If $A<f(1)<B$ can be concluded from the mean value theorem, then the largest value of $(A-B)$ is equals

(a) e (b) $1-e$
(c) $e-1$ (d) $1+e$

21. The function $f(x)=\cos x-2px$ is monotonically decreasing for

(a) $p<\frac{1}{2}$ (b) $p>\frac{1}{2}$
(c) $p<2$ (d) $p>2$

22. The abscissa of the points of the curve $y=x^3$ in the interval [–2, 2], where the slope of the tangents can be obtained by mean value theorem for the interval [–2, 2], are

(a) $\pm\frac{2}{\sqrt{3}}$ (b) $+\sqrt{3}$ (c) $\pm\frac{\sqrt{3}}{2}$ (d) 0

23. The equation of the tangent to the curve $y=1-e^{x/2}$ at the point of intersection with the Y-axis is

(a) $x+2y=0$ (b) $2x+y=0$
(c) $x-y=2$ (d) None of these

24. The chord joining the points, where $x=p$ and $x=q$ on the curve $y=ax^2+bx+c$ is parallel to the tangent at the point on the curve whose abscissa is

(a) $\frac{p+q}{2}$ (b) $\frac{p-q}{2}$
(c) $\frac{pq}{2}$ (d) None of these

25. If $f(x)=(ab-b^2-2)x+\int_0^x(\cos^4\theta+\sin^4\theta)d\theta$ is decreasing function of x for all $x\in R$ and $b\in R$, b being independent of x, then

(a) $a\in(0,\sqrt{6})$ (b) $a\in(-\sqrt{6},\sqrt{6})$
(c) $a\in(-\sqrt{6},0)$ (d) None of these

26. The values of a in order that $f(x)=\sqrt{3}\sin x-\cos x-2ax+b$ decreases for all real values of x, is given by

(a) $a<1$ (b) $a\geq 1$
(c) $a\leq\sqrt{2}$ (d) $a<\sqrt{2}$

27. In [0, 1], Lagrange's mean value theorem is not applicable to

(a) $f(x)=\begin{cases}\frac{1}{2}-x, & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2, & x\geq\frac{1}{2}\end{cases}$

(b) $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0\\ 1, & x=0\end{cases}$

(c) $f(x)=x|x|$
(d) $f(x)=|x|$

28. The tangent to the curve $y=e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$

(a) on the left of $x=c$ (b) on the right of $x=c$
(c) at no point (d) at all points

29. The tangent at (1, 7) to the curve $x^2=y-6$ touches the circle $x^2+y^2+16x+12y+c=0$ at

(a) (6, 7) (b) (–6, 7) (c) (6, – 7) (d) (–6, – 7)

30. If m be the slope of a tangent to the curve $e^y=1+x^2$, then

(a) $|m|>1$ (b) $m<1$ (c) $|m|<1$ (d) $|m|\leq 1$

31. If the line joining the points (0, 3) and (5, – 2) is a tangent to the curve $y=\frac{c}{x+1}$, then the value of c is

(a) 1 (b) –2
(c) 4 (d) None of these

32. The total number of parallel tangents of $f_1(x)=x^2-x+1$ and $f_2(x)=x^3-x^2-2x+1$ are

(a) 2 (b) 0 (c) 1 (d) infinite

33. The angle of intersection of curves, $y=[|\sin x|+|\cos x|]$ and $x^2+y^2=5$, where $[\cdot]$ denotes greatest integral function is

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$
(c) $\tan^{-1}(2)$ (d) None of these

34. The area of the triangle formed by the coordinate axes and a tangent to the curve $xy=a^2$ at the point (x_1, y_1) on it is

(a) $\frac{a^2x_1}{y_1}$ (b) $\frac{a^2x_2}{x_1}$
(c) $2a^2$ (d) $4a^2$

35. The equation of tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at (a, b) is

(a) $\frac{x}{a}+\frac{y}{b}=2$ (b) $\frac{x}{a}+\frac{y}{b}=\frac{1}{2}$

(c) $\frac{x}{b}-\frac{y}{a}=2$ (d) $ax+by=2$

36. If the line $ax+by+c=0$ is a normal to the curve $xy=1$, then

(a) $a>0, b>0$ (b) $a>0, b<0$

(c) $a<0, b<0$ (d) None of these

37. The interval in which $f(x)=3\cos^4 x+10\cos^3 x+6\cos^3 x-3$ decreases is $x\in[0, \pi]$

(a) $\left(0, \frac{\pi}{2}\right)\cup\left(\frac{2\pi}{3}, \pi\right)$ (b) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

(c) $\left(0, \frac{\pi}{3}\right)\cup\left(\frac{2\pi}{3}, \pi\right)$ (d) None of these

38. Function $f(x)=2x^2-\log|x|, x\neq 0$ monotonically increases in

(a) $\left(-\infty, -\frac{1}{2}\right)\cup\left(0, \frac{1}{2}\right)$ (b) $\left(-\frac{1}{2}, 0\right)\cup\left(\frac{1}{2}, \infty\right)$

(c) $(-\infty, 0)\cup\left(\frac{1}{2}, \infty\right)$ (d) None of these

39. Let $f(x)$ and $g(x)$ be two continuous functions defined from $R\to R$, such that $f(x_1)>f(x_2)$ and $g(x_1)<g(x_2), \forall x_1>x_2$, then solution set of $f\{g(\alpha^2-2\alpha)\}>f\{g(3\alpha-4)\}$ is

(a) R (b) ϕ (c) $(1, 4)$ (d) $R-[1, 4]$

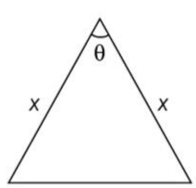
40. The function $f(x)=\frac{\ln(\pi+x)}{\ln(e+x)}$ is

(a) increasing on $(0, \infty)$
(b) decreasing in $(0, \infty)$
(c) increasing on $(0, \pi/e)$, decreasing on $(\pi/e, \infty)$
(d) decreasing on $(0, \pi/e)$, increasing on $(\pi/e, \infty)$

41. Assuming the petrol burnt (per hour) in driving a motorboat varies as the cube of its velocity, then the most economical speed when going against a current of c miles per hour

(a) $\frac{c}{2}$ mph (b) $\frac{2c}{3}$ mph (c) $\frac{3c}{2}$ mph (d) $2c$ mph

42. Let x be the length of one of the equal sides of an isosceles triangle and let θ be the angle between them.



If x is increasing at the rate $\left(\frac{1}{12}\right)$ mph and θ is increasing at the rate of $\frac{\pi}{180}$ radius/h, then the rate in m^2/h at which the area of the triangle is increasing when $x=12$ m and $\theta=\frac{\pi}{4}$

(a) $2^{1/2}\left(1+\frac{2\pi}{5}\right)$ (b) $\frac{73}{2}\cdot 2^{1/2}$

(c) $\frac{3^{1/2}}{2}+\frac{\pi}{5}$ (d) $2^{1/2}\left(\frac{1}{2}+\frac{\pi}{5}\right)$

43. $f(x)=x^2-4|x|$ and

$$g(x)=\begin{cases}\min\{f(t):-6\le t\le x\}, x\in[-6, 0]\\ \max\{f(t):0<t\le x\}, x\in(0, 6]\end{cases}$$, then $g(x)$ has

(a) exactly one point of local minima
(b) exactly one point of local maxima
(c) no point to local maxima but exactly one point of local minima
(d) neither a point of local maxima nor minima

44. The function $f(x)=\frac{x^2-2}{x^2-4}$ has

(a) no point of local minima
(b) no point of local maxima
(c) exactly one point of local minima
(d) exactly one point of local maxima

45. If $f(x)=\int_0^x (t^2-1)\cos t\, dt, x\in(0, 2\pi)$. Then, $f(x)$ attains local maximum value at

(a) $x=\frac{\pi}{2}$ (b) $x=1$ (c) $x=\frac{3\pi}{2}$ (d) None of these

46. $$f(x)=\begin{cases}4x-x^3+\log(a^2-3a+3), & 0\le x<3\\ x-18, & x\ge 3\end{cases}$$

Complete the set of values of a such that $f(x)$ has a local maxima at $x=3$, is

(a) $[-1, 2]$ (b) $(-\infty, 1)\cup(2, \infty)$

(c) $[1, 2]$ (d) $(-\infty, -1)\cup(2, \infty)$

47. The function $f(x)=\frac{ax+b}{(x-1)(x-4)}$ has a local maxima at $(2, -1)$, then

(a) $b=1, a=0$ (b) $a=1, b=0$

(c) $b=-1, a=0$ (d) $a=-1, b=0$

48. If $f(x)=\frac{x}{1+x\tan x}, x\in\left(0, \frac{\pi}{2}\right)$, then

(a) $f(x)$ has exactly one point of minima
(b) $f(x)$ has exactly one point of maxima
(c) $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$
(d) $f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

49. A straight line is drawn through the point $P(3, 4)$ meeting the positive direction of coordinate axes at the points A and B. If O is the origin, then minimum area of ΔOAB is equal to

(a) 12 sq units (b) 6 sq units
(c) 24 sq units (d) 48 sq units

50. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC. A parallelogram $AFDE$ is drawn with D, E and F on the line segment BC, CA and AB, respectively. Then, maximum area of such a parallelogram is

(a) $\frac{1}{2}$ (area of ΔABC) (b) $\frac{1}{4}$ (area of ΔABC)
(c) $\frac{1}{6}$ (area of ΔABC) (d) $\frac{1}{8}$ (area of ΔABC)

51. Let $y = f(x)$ be a parametrically defined expression such that $x = 3t^2 - 18t + 7$ and $y = 2t^3 - 15t^2 + 24t + 10, \forall, x \in [0, 6]$. Then, the minimum and maximum values of $y = f(x)$ are

(a) 36, 3 (b) 46, 6 (c) 40, –6 (d) 46, –6

52. Let $f(x) = \begin{cases} |x^2 - 2|, & -1 \le x < \sqrt{3} \\ \frac{x}{\sqrt{3}}, & \sqrt{3} \le x < 2\sqrt{3}, \\ 3 - x, & 2\sqrt{3} \le x \le 4 \end{cases}$ then the points, where $f(x)$ takes maximum and minimum values, are

(a) 1, 4 (b) 0, 4
(c) 2, 4 (d) None of these

53. The value of a so that the sum of the squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume the least value is

(a) 2 (b) 1 (c) 3 (d) 0

54. A minimum value of $\int_0^x te^{-t^2}\,dt$ is

(a) 1 (b) 2 (c) 3 (d) 0

55. If $y = a\log x + bx^2 + x$ has its extremum value at $x = 1$ and $x = 2$, then (a, b) is equal to

(a) $\left(1, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 2\right)$ (c) $\left(2, \frac{-1}{2}\right)$ (d) $\left(\frac{-2}{3}, \frac{-1}{6}\right)$

56. If $P = (1, 1)$, $Q = (3, 2)$ and R is a point on x-axis, then the value of $PR + RQ$ will be minimm at

(a) $\left(\frac{5}{3}, 0\right)$ (b) $\left(\frac{1}{3}, 0\right)$ (c) $(3, 0)$ (d) $(1, 0)$

57. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a is equal to

(a) 3 (b) 1 (c) 2 (d) $\frac{1}{2}$

58. Let $f(x) = 1 + 2x^2 + 2^2x^4 + \ldots + 2^{10}x^{20}$, then $f(x)$ has

(a) more than one minimum
(b) exactly one minimum
(c) atleast one maximum
(d) None of the above

59. The point in the interval $[0, 2\pi]$, where $f(x) = e^x \sin x$ has maximum slope is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) None of these

60. Let $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$, then number of points [where, $f(x)$ attains its minimum value] is

(a) 1 (b) 2
(c) 3 (d) infinite many

61. If $f(x) = a\log_e|x| + bx^2 + x$ has extremum at $x = 1$ and $x = 3$, then

(a) $a = -3/4, b = -1/8$ (b) $a = 3/4, b = -1/8$
(c) $a = -3/4, b = 1/8$ (d) None of these

62. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is

(a) 0 (b) 1
(c) 2 (d) 3

63. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that minimum $f(x) > g(x)$, then the relation between b and c is

(a) $0 < c < b\sqrt{2}$
(b) $|c| < |b|\sqrt{2}$
(c) $|c| > |b|\sqrt{2}$
(d) no real values of b and c

64. The curved surface of the cone inscribed in a given sphere is maximum, if

(a) $h = \frac{4R}{3}$ (b) $h = \frac{R}{3}$
(c) $h = \frac{2R}{3}$ (d) None of these

65. The minimum intercepts made by the axes on the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is

(a) 25 (b) 7
(c) 1 (d) None of these

66. Maximum value of $\left(\frac{1}{x}\right)^x$ is

(a) $(e)^e$ (b) $(e)^{1/e}$

(c) $(e)^{-e}$ (d) $\left(\frac{1}{e}\right)^e$

67. The largest term in the sequence $a_n = \frac{n^2}{n^3+200}$ is given by

(a) $\frac{529}{49}$ (b) $\frac{8}{89}$

(c) $\frac{49}{543}$ (d) None of these

68. If $A>0, B>0$ and $A+B=\frac{\pi}{3}$, then the maximum value of $\tan A \cdot \tan B$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{3}$

(c) 3 (d) $\sqrt{3}$

69. All possible values of the parameter a so that the function $f(x)=x^3-3(7-a)x^2-3(9-a^2)x+2$ has a negative point of local minimum are

(a) all real values

(b) no real values

(c) $(0,\infty)$

(d) $(-\infty,0)$

70. $f(x)=\begin{cases}\cos\frac{\pi x}{2}, x>0\\ x+a, x\le 0\end{cases}$,

then $x=0$ will be a point of local maxima for $f(x)$, if

(a) $a\in(-1,1)$ (b) $a\in(0,1)$

(c) $a\le 0$ (d) $a\ge 1$

71. Let (h, k) be a fixed point, where $h>0, k>0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Which of the following is the minimum area of the $\Delta OPQ, O$ being the origin?

(a) hk (b) $2hk$

(c) $\frac{1}{2}hk$ (d) None of these

72. The coordinates of a point of the parabola $y=x^2+7x+2$ which is closest to the straight line $y=3x-3$ is

(a) $(-2,8)$ (b) $(-2,-8)$

(c) $(2,-8)$ (d) None of these

Numerical Value Type Questions

73. If the tangent to the curve, $y=e^x$ at a point (c, e^c) and the normal to the parabola, $y^2=4x$ at the point (1, 2) intersect at the same point the X-axis, then the value of c is .... *(JEE Main 2020)*

74. Let $f(x)$ be a polynomial of degree 3 such that $f(-1)=10$, $f(1)=-6$, $f(x)$ has a critical point at $x=-1$ and $f'(x)$ has a critical point at $x=1$. The $f(x)$ has a local minima at $x=$............ *(JEE Main 2020)*

75. The x-coordinate of the point on the curve $9y^2=x^3$, where the normal to the curve makes equal intercepts with the axes is ....

76. The curve $y=ax^3+bx^2+cx+5$ touches the X-axis at $P(-2, 0)$ and cuts the Y-axis at a point Q, where its gradient is 3. Then, the value of $c-4a-12b$ is ....

77. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. Then, the length of the side of the square to be cut-off so that the volume of the box is maximum is ....

78. The minimum value of $f(x)=|3-x|+|2+x|+|5-x|$ is ....

79. If $S=4t+\frac{1}{t}$ is the equation of motion of a particle, then its acceleration when velocity vanishes is ....

80. Let $f(x)=x^3+ax^2+bx+5\sin^2 x$ be an increasing function on the set R and if $a^2-3b+\lambda<0$, then the value of λ is ....

Answers...

Round I

1. (c)	2. (b)	3. (d)	4. (b)	5. (c)	6. (a)	7. (b)	8. (d)	9. (c)	10. (d)
11. (c)	12. (a)	13. (b)	14. (d)	15. (a)	16. (b)	17. (a)	18. (d)	19. (a)	20. (c)
21. (a)	22. (a)	23. (b)	24. (c)	25. (d)	26. (d)	27. (a)	28. (c)	29. (a)	30. (b)
31. (d)	32. (c)	33. (b)	34. (b)	35. (b)	36. (d)	37. (b)	38. (d)	39. (c)	40. (a)
41. (d)	42. (d)	43. (c)	44. (a)	45. (a)	46. (b)	47. (a)	48. (a)	49. (c)	50. (b)
51. (a)	52. (d)	53. (a)	54. (a)	55. (a)	56. (d)	57. (a)	58. (a)	59. (b)	60. (d)
61. (b)	62. (d)	63. (a)	64. (b)	65. (a)	66. (a)	67. (a)	68. (a)	69. (c)	70. (a)
71. (a)	72. (a)	73. (a)	74. (b)	75. (c)	76. (c)	77. (a)	78. (d)	79. (b)	80. (d)
81. (c)	82. (b)	83. (d)	84. (d)	85. (a)	86. (c)	87. (a)	88. (a)	89. (b)	90. (d)
91. (b)	92. (d)	93. (c)	94. (b)	95. (b)	96. (c)	97. (a)	98. (b)	99. (c)	100. (b)
101. (d)	102. (d)	103. (a)	104. (d)	105. (a)	106. (c)	107. (d)	108. (a)	109. (c)	110. (d)
111. (b)	112. (c)	113. (a)	114. (a)	115. (b)	116. (b)	117. (d)	118. (c)	119. (a)	120. (b)
121. (c)	122. (a)	123. (c)	124. (c)	125. (d)	126. (d)	127. (b)	128. (d)	129. (b)	130. (a)
131. (a)	132. (d)	133. (c)	134. (c)	135. (c)	136. (c)	137. (b)	138. (b)	139. (c)	140. (a)
141. (b)	142. (c)	143. (b)	144. (a)	145. (a)	146. (b)	147. (b)	148. (b)	149. (d)	150. (a)
151. (c)	152. (c)	153. (b)	154. (d)	155. (b)	156. (b)	157. (d)	158. (b)	159. (c)	160. (a)
161. (a)	162. (d)	163. (d)	164. (d)	165. (b)	166. (b)	167. (c)	168. (c)	169. (c)	170. (c)

Round II

1. (b)	2. (a)	3. (d)	4. (d)	5. (a)	6. (b)	7. (b)	8. (b)	9. (d)	10. (c)
11. (b)	12. (b)	13. (c)	14. (b)	15. (a)	16. (b)	17. (d)	18. (a)	19. (a)	20. (b)
21. (b)	22. (a)	23. (a)	24. (a)	25. (b)	26. (b)	27. (a)	28. (a)	29. (d)	30. (d)
31. (c)	32. (d)	33. (c)	34. (c)	35. (a)	36. (b)	37. (a)	38. (b)	39. (c)	40. (b)
41. (c)	42. (d)	43. (d)	44. (d)	45. (a)	46. (c)	47. (b)	48. (b)	49. (c)	50. (a)
51. (d)	52. (b)	53. (b)	54. (d)	55. (d)	56. (a)	57. (c)	58. (b)	59. (b)	60. (a)
61. (a)	62. (c)	63. (c)	64. (a)	65. (b)	66. (b)	67. (c)	68. (b)	69. (b)	70. (d)
71. (b)	72. (b)	73. (4)	74. (3)	75. (4)	76. (14)	77. (5)	78. (7)	79. (16)	80. (15)

Solutions...

Round I

1. The average speed of the car, for time interval $[t_1, t_2]$ is

$$\frac{f(t_2)-f(t_1)}{t_2-t_1}=\frac{a\,(t_2^2-t_1^2)+b\,(t_2-t_1)}{t_2-t_1}=\frac{d(f(t))}{dt}$$

$$\therefore \quad 2at+b=a\,(t_2+t_1)+b$$

$$\Rightarrow \quad t=\frac{t_1+t_2}{2}$$

$\therefore$ The average speed of the car over the time interval $[t_1, t_2]$ is attained at the point $\frac{t_1+t_2}{2}$.

2. Given, semi-vertical angle of right circular cone

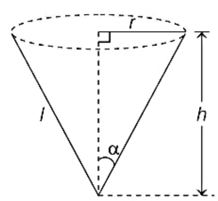
$$=\tan^{-1}\left(\frac{1}{2}\right)$$

Let $\quad \alpha=\tan^{-1}\left(\frac{1}{2}\right)$

$$\Rightarrow \quad \tan\alpha=\frac{1}{2}$$

$$\Rightarrow \quad \frac{r}{h}=\frac{1}{2} \qquad \left[\text{from figure } \tan\alpha=\frac{r}{h}\right]$$

$$\Rightarrow \quad r=\frac{1}{2}h \qquad \ldots\text{(i)}$$



$\because$ Volume of cone is $(V)=\frac{1}{3}\pi r^2 h$

$$\therefore \quad V=\frac{1}{3}\pi\left(\frac{1}{2}h\right)^2(h)=\frac{1}{12}\pi h^3 \qquad \text{[from Eq. (i)]}$$

On differentiating both sides w.r.t. 't', we get

$$\frac{dV}{dt}=\frac{1}{12}\pi(3h^2)\frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt}=\frac{4}{\pi h^2}\frac{dV}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt}=\frac{4}{\pi h^2}\times 5 \qquad \left[\because \text{ given } \frac{dV}{dt}=5\text{ m}^3\text{/min}\right]$$

Now, at $h=10$ m, the rate at which height of water level is rising $=\left.\frac{dh}{dt}\right|_{h=10}=\frac{4}{\pi(10)^2}\times 5=\frac{1}{5\pi}$ m/min

3. It is given that, a spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness, let the thickness is 'x' cm, then volume of the ball is

$$V=\frac{4}{3}\pi(10+x)^3$$

On differentiating w.r.t. 't', we get

$$\frac{dV}{dt}=4\pi(10+x)^2\frac{dx}{dt}, \qquad \ldots\text{(i)}$$

where t is time in min.

It is given, the $\frac{dV}{dt}=-50\text{ cm}^3/\text{min}$,

Now when x is 5 cm, then

$$-50=4\pi(10+5)^2\frac{dx}{dt} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{dx}{dt}=-\frac{50}{4\pi(225)}=-\frac{1}{18\pi}\text{ cm/min}$$

Negative sign indicates the thickness of ice layer decreases with time. Hence, option (d) is correct.

4. Let r, l and h be the radius, slant height and height of cone respectively at any time t. Then,

$$l^2=r^2+h^2$$

$$\Rightarrow \quad 2l\frac{dl}{dt}=2r\frac{dr}{dt}+2h\frac{dh}{dt}$$

$$\Rightarrow \quad l\frac{dl}{dt}=r\frac{dr}{dt}+h\frac{dh}{dt}=7(3)+24(-4)$$

$$=-75$$

$$\left[\because \frac{dh}{dt}=-4 \text{ and } \frac{dr}{dt}=3\right]$$

Now, $l^2=7^2+24^2$

$\Rightarrow \quad l^2=625 \Rightarrow l=25$

$$\therefore \quad \frac{dl}{dt}=-3$$

$$\therefore \quad \frac{ds}{dt}=\frac{d}{dt}(\pi rl)=\pi\left[l\frac{dr}{dt}+r\frac{dl}{dt}\right]$$

$$=\pi\,[25\times 3+7\times(-3)]$$

$$=\pi\,(54)$$

$$=54\,\pi\text{ cm}^2\text{/min}$$

5. Given

(i) Volume ($V=4500\,\pi$ m^3/min) of the helium gas filled in a spherical balloon.

(ii) Due to a leak, the gas escapes the balloon at the rate of $72\,\pi$ m^3/min.

$\therefore$ Rate of decrease of volume of the balloon is

$$\frac{dV}{dt}=-72\,\pi\text{ m}^3\text{/min}$$

To find The rate of decrease of the radius of the balloon 49 min after the leakage started.

i.e. $\quad \frac{dr}{dt}$ at $t=49$ min

[assuming that the leakage started at time $t=0$]

Now, the balloon is spherical in shape, hence the volume of the balloon is $V = \frac{4}{3}\pi r^3$.

On differentiating both sides w.r.t. t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \times \frac{dr}{dt}\right)$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} \quad \text{...(i)}$$

Now, to find $\frac{dr}{dt}$ at $t = 49$ min, we require $\frac{dV}{dt}$ and the radius (r) at that stage,

$$\frac{dV}{dt} = -72\pi \text{ m}^3/\text{min}$$

$\therefore$ Amount of volume lost in 49 min $= 72\pi \times 49 \text{ m}^3$

$\therefore$ Final volume at the end of 49 min

$$= (4500\pi - 3528\pi)\text{ m}^3 = 972\pi \text{ m}^3$$

If r is the radius at the end of 49 min, then

$$\frac{4}{3}\pi r^3 = 972\pi \Rightarrow r^3 = 729 \Rightarrow r = 9$$

$\therefore$ Radius of the balloon at the end of 49 min = 9 m

Hence, from Eq. (i), we get

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$$

$$\Rightarrow \quad \left(\frac{dr}{dt}\right)_{t=49} = \frac{(dV/dt)_{t=49}}{4\pi(r^2)_{t=49}}$$

$$= \frac{72\pi}{4\pi(9^2)} = \frac{2}{9} \text{ m/min}$$

6. Since, x and y are the sides of two squares.

Thus, the area of two squares is x^2 and y^2.

$$\therefore \quad \frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x}\frac{dy}{dx} \quad \text{...(i)}$$

Given, $y = x - x^2$

$$\Rightarrow \quad \frac{dy}{dx} = 1 - 2x$$

$$\therefore \quad \frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1-2x) \quad \text{[using Eq. (i)]}$$

$$= \frac{(x - x^2)(1-2x)}{x}$$

$$= (2x^2 - 3x + 1)$$

Thus, the rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$.

7. Let lizard catch the insect C.

And distance covered by insect $= S$

Time taken by insect, $t = \frac{S}{20}$ ...(i)

Distance covered by lizard $= 21 + S$

$$\therefore \quad 21 + S = \frac{1}{2}(2)\cdot t^2 \quad \text{...(ii)}$$

$[\because S = ut + \frac{1}{2}at^2$; here $u = 0$, $a = 2 \text{ cm/s}^2$ and $S = 20t]$

$$\Rightarrow \quad 21 + 20t = t^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad t^2 - 20t - 21 = 0$$

$$\Rightarrow \quad t^2 - 21t + t - 21 = 0$$

$$\Rightarrow \quad t(t-21) + 1(t-21) = 0$$

$$\Rightarrow \quad (t+1)(t-21) = 0$$

$$\Rightarrow \quad t = -1, 21$$

$$\therefore \quad t = 21 \text{ s} \quad [\because \text{neglecting } t = -1]$$

8. Equation of parabola is $y^2 = 18x$.

On differentiating w.r.t. t, we get

$$2y\frac{dy}{dt} = 18\frac{dx}{dt}$$

$$\Rightarrow \quad 2 \cdot 2y = 18 \quad \left[\because \frac{dy}{dt} = 2\frac{dx}{dt}, \text{ given}\right]$$

$$\Rightarrow \quad y = \frac{9}{2}$$

From equation of parabola,

$$\left(\frac{9}{2}\right)^2 = 18x \Rightarrow \frac{81}{4} = 18x$$

$$\Rightarrow \quad x = \frac{9}{8}$$

Hence, required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$.

9. If x is the side of an equilateral triangle and A is its area, then

$$A = \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{4}2x\frac{dx}{dt}$$

Here, $x = 10$ cm and $\frac{dx}{dt} = 2$ cm/s

$$\therefore \quad A = \frac{\sqrt{3}}{4}2(10)2 = 10\sqrt{3} \text{ cm}^2/\text{s}$$

10. Given that, $\frac{dV}{dt} = k$ (say)

$$\because \quad V = \frac{4}{3}\pi R^3$$

$$\Rightarrow \quad \frac{dV}{dt} = 4\pi R^2\frac{dR}{dt} \Rightarrow \frac{dR}{dt} = \frac{k}{4\pi R^2} = \frac{k}{S}$$

Rate of increasing radius is inversely proportional to its surface area.

11. They will encounter, if

$$10 + 6t = 3 + t^2$$

$$\Rightarrow \quad t^2 - 6t - 7 = 0 \Rightarrow t = 7$$

At $t = 7$ s, moving in a first point

$$v_1 = \frac{d}{dt}(10 + 6t) = 6 \text{ cm/s}$$

At $t = 7$ s, moving in a second point

$$v_2 = \frac{d}{dt}(3 + t^2) = 2t = 2 \times 7 = 14 \text{ cm/s}$$

$\therefore$ Resultant velocity $= v_2 - v_1 = 14 - 6 = 8$ cm/s

12. Given that, $\frac{dv}{dt} = 30\ \text{ft}^3/\text{min}$ and $r = 15$ ft

Volume of spherical balloon

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \quad 30 = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{30}{4\times\pi\times15\times15} = \frac{1}{30\pi}\ \text{ft/min}$$

13. The equation of given circle is

$$x^2 + y^2 = 1$$

On differentiating w.r.t. t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

But we have, $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$ and $\frac{dy}{dt} = -3$, then

$$\frac{1}{2}\frac{dx}{dt} + \frac{\sqrt{3}}{2}(-3) = 0 \Rightarrow \frac{dx}{dt} = 3\sqrt{3}$$

14. Given point is $x = a + bt - ct^2$

Acceleration in x direction $= \frac{d^2x}{dt^2} = -2c$

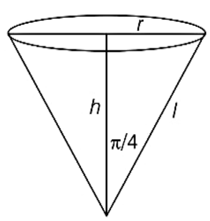
and acceleration in y direction $= \frac{d^2y}{dt^2} = 2b$

$$\therefore \text{ Resultant acceleration} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

$$= \sqrt{(-2c)^2 + (2b)^2} = 2\sqrt{b^2 + c^2}$$

15. If S represents the surface area, then

$$\frac{dS}{dt} = 2\ \text{cm}^2/\text{s}$$



$$S = \pi r l = \pi l \cdot \sin\frac{\pi}{4}\, l = \frac{\pi}{\sqrt{2}} l^2$$

Therefore, $\frac{dS}{dt} = \frac{2\pi}{\sqrt{2}} l \cdot \frac{dl}{dt} = \sqrt{2}\pi l \cdot \frac{dl}{dt}$

when $l = 4$ cm, $\frac{dl}{dt} = \frac{l}{\sqrt{2}\pi\cdot 4}\cdot 2 = \frac{1}{2\sqrt{2}\pi} = \frac{\sqrt{2}}{4\pi}$ cm/s

16. Use the relation between volume V and radius r

i.e. $V = \frac{4}{3}\pi r^3$, then differentiate it w.r.t. r and use

$\Delta V = \left(\frac{dV}{dr}\right)\Delta r$, to find the approximate change in volume V.

Let r be the radius of the sphere and Δr be the error in measuring radius.

Then, $r = 7$ m and $\Delta r = 0.02$ m

Now, volume of a sphere is given by $V = \frac{4}{3}\pi r^3$

On differentiate w.r.t. r, we get $\frac{dV}{dr} = \left(\frac{4}{3}\pi\right)(3r^2) = 4\pi r^2$

$$\therefore \Delta V = \left(\frac{dV}{dr}\right)\Delta r = (4\pi r^2)\Delta r = 4\pi \times 7^2 \times 0.02 = 3.92\,\pi\ \text{m}^3$$

Hence, the approximate error in calculating the volume is $3.92\,\pi\,\text{m}^3$.

17. Firstly, break the number 5.001 as $x = 5$ and $\Delta x = 0.001$ and use the relation $f(x + \Delta x) \simeq f(x) + \Delta x\, f'(x)$.

Consider $\quad f(x) = x^3 - 7x^2 + 15$

$\Rightarrow \quad f'(x) = 3x^2 - 14x$

Let $\quad x = 5$

and $\quad \Delta x = 0.001$

Also, $\quad f(x + \Delta x) \simeq f(x) + \Delta x\, f'(x)$

Therefore, $\quad f(x + \Delta x) \simeq (x^3 - 7x^2 + 15) + \Delta x\,(3x^2 - 14x)$

$$\Rightarrow \quad f(5.001) \simeq (5^3 - 7\times5^2 + 15) + (3\times5^2 - 14\times5)(0.001)$$

[as $x = 5$, $\Delta x = 0.001$]

$$= 125 - 175 + 15 + (75 - 70)(0.001)$$
$$= -35 + (5)(0.001)$$
$$= -35 + 0.005 = -34.995$$

18. We know that, area of circle, $A = \pi r^2$

Taking log on both sides, we get

$$\log A = \log\pi + 2\log r$$

$$\therefore \quad \frac{\Delta A}{A}\times 100 = 2\times\frac{\Delta r}{r}\times 100 = 2\times 0.05 = 0.1\%$$

19. Let $y = \cos x$

$$\therefore \frac{dy}{dx} = -\sin x$$

$$\Delta y = \cos(x + \Delta x) - \cos x$$
$$= \cos(60°1') - \cos 60°$$

Since, dy is approximately equal to Δy and it is given by

$$\Delta y = \left(\frac{dy}{dx}\right)_{x = 60°}\Delta x$$

$$= \frac{-\sqrt{3}}{2}\times 1' = \frac{-\sqrt{3}}{2}\times\frac{1}{60}$$

$$\therefore \cos(60°1') = \frac{1}{2} - \frac{\sqrt{3}}{120}$$

20. Given, $\frac{\delta r}{r}\times 100 = 2 \Rightarrow \delta r = \frac{2r}{100}$

Surface area, $S = 4\pi r^2$

$$\Rightarrow \delta S = 4\pi\cdot 2r\cdot\delta r = 8\pi r\cdot\frac{2r}{100} = \frac{16\pi r^2}{100}$$

Now, percentage error in surface area

$$= \frac{\delta S}{S} \times 100 = \frac{16\pi r^2}{100} \times \frac{1}{4\pi r^2} \times 100 = 4\%$$

21. Volume of hollow, spherical shell,

$$V = \frac{4}{3}\pi\,[(3.0005)^3 - (3)^3]$$

Now, $(3.0005)^3 = y + \Delta y, x = 3$ and $\Delta x = 0.0005$

Let $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$

$\therefore\ \Delta y = \frac{dy}{dx} \times \Delta x = 3x^2 \times 0.0005$

$$= 3 \times 3^2 \times 0.0005$$
$$= 0.0135$$

$\therefore\ (3.0005)^3 = y + \Delta y = 3^3 + 0.0135$

$$= 27.0135$$

$\therefore\quad V = \frac{4}{3}\pi\,[27.0135 - 27.000]$

$$= \frac{4}{3}\pi\,[0.0135]$$
$$= 4\pi \times (0.0045)$$
$$= 0.0180\,\pi\ \text{cm}^3$$

22. Find the slope of the tangent to the curve and write the general equation for tangent. Since, tangent passes through the origin, so (0, 0) will satisfy it.

The equation of the given curve is $y = 4x^3 - 2x^5$

$$\frac{dy}{dx} = 12x^2 - 10x^4$$

Therefore, the slope of the tangent at point (x, y) is $12x^2 - 10x^4$.

The equation of the tangent at (x, y) is given by

$$Y - y = (12x^2 - 10x^4)(X - x) \qquad \text{...(i)}$$

When the tangent passes through the origin (0, 0), then $X = Y = 0$

Therefore, Eq.(i) reduced to

$$-y = (12x^2 - 10x^4)(-x) \Rightarrow \quad y = 12x^3 - 10x^5$$

Also, we have $y = 4x^3 - 2x^5$

$\therefore\quad 12x^3 - 10x^5 = 4x^3 - 2x^5$

$\Rightarrow\quad 8x^5 - 8x^3 = 0$

$\Rightarrow\quad x^5 - x^3 = 0$

$\Rightarrow\quad x^3(x^2 - 1) = 0$

$\Rightarrow\quad x = 0, \pm 1$

When $x = 0$, $y = 4(0)^3 - 2(0)^5 = 0$. When $x = 1$,

$$y = 4(1)^3 - 2(1)^5 = 2.$$

When $x = -1$, $y = 4(-1)^3 - 2(-1)^5 = -2$

Hence, the required points are (0, 0), (1, 2) and (–1, –2).

23. Given curve is $y = 2x^2 - x + 1$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 4x - 1$$

Since, this is parallel to the given curve $y = 3x + 9$.

$\therefore$ These slopes are equal.

$\Rightarrow\quad 4x - 1 = 3 \Rightarrow x = 1$

At $x = 1$, $\quad y = 2(1)^2 - 1 + 1 \Rightarrow y = 2$

Thus, the point is (1, 2).

24. Equation of the curve is $x^2y^2 = a^4$.

On differentiating w.r.t. x, we get

$$x^2 2y\frac{dy}{dx} + y^2 2x = 0$$

$\Rightarrow\quad \frac{dy}{dx} = \frac{-y}{x}$

$\Rightarrow\quad \left(\frac{dy}{dx}\right)_{(-a,\,a)} = -\left(\frac{a}{-a}\right) = 1$

Therefore, length of subtangent at the point $(-a, a)$

$$= \frac{y}{\left(\frac{dy}{dx}\right)} = \frac{a}{1} = a$$

25. Given curve is $\quad xy = c^2 \qquad$ ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

Length of subnormal $= y\frac{dy}{dx} = \frac{y \times (-c)^2}{x^2} = \frac{-yc^2}{\left(\frac{c^2}{y}\right)^2}$

$$\left[\text{from Eq. (i)}, x = \frac{c^2}{y}\right]$$

$$= \frac{-yc^2y^2}{c^4} = \frac{-y^3}{c^2}$$

$\therefore$ Subnormal varies as y^3.

26. $\because\quad y = x^3 - 3x^2 - 9x + 5$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Since, tangent is parallel to X-axis.

$\therefore\quad \frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow\quad (x+1)(x-3) = 0 \Rightarrow x = -1, 3$

27. Given curve is $x = 3t^2 + 1, y = t^3 - 1$

For $x = 1$, $\quad 3t^2 + 1 = 1 \Rightarrow t = 0$

$\therefore\quad \frac{dx}{dt} = 6t, \frac{dy}{dt} = 3t^2$

Now, $\quad \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{3t^2}{6t} = \frac{t}{2}$

$\therefore\quad \left(\frac{dy}{dx}\right)_{(t=0)} = \frac{0}{2} = 0$

28. We have, $x = at^2 \Rightarrow \frac{dx}{dt} = 2at$

and $\quad y = 2at \Rightarrow \frac{dy}{dt} = 2a$

$\therefore$ Slope of tangent $= \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

$\Rightarrow \quad \frac{1}{t} = \infty$

$\Rightarrow \quad t = 0 \Rightarrow$ Point of contact is (0, 0).

29. The given curve is $(1 + x^2)\, y = 2 - x$ ...(i)

It meets X-axis, where $y = 0 \Rightarrow 0 = 2 - x \Rightarrow x = 2$

So, Eq. (i) meets x-axis at the point (2, 0).

Also, from Eq. (i), $\quad y = \frac{2 - x}{1 + x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1 + x^2)(-1) - (2 - x)(2x)}{(1 + x^2)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 - 4x - 1}{(1 + x^2)^2}$$

$\therefore$ Slope of tangent at (2, 0) $= \frac{2^2 - 4(2) - 1}{(1 + 2^2)^2}$

$$= \frac{4 - 8 - 1}{(1 + 4)^2} = -\frac{5}{25} = -\frac{1}{5}$$

$\therefore$ Equation of tangent at (2, 0) with slope $-\frac{1}{5}$ is

$$y - 0 = -\frac{1}{5}(x - 2)$$

$$\Rightarrow \quad 5y = -x + 2$$

$$\Rightarrow \quad x + 5y = 2$$

30. The equation of given curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

$$\therefore \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

The normal is parallel to X-axis, if

$$\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0 \Rightarrow x_1 = 0$$

$\therefore$ From equation of curve, $y_1 = a$

$\therefore$ Required point is $(0, a)$.

31. Given curve is $y = x \log x$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 1 + \log x$$

The slope of the normal $= -\frac{1}{(dy/dx)} = \frac{-1}{1 + \log x}$

The slope of the given line $2x - 2y = 3$ is 1.

Since, these lines are parallel.

$$\therefore \quad \frac{-1}{1 + \log x} = 1$$

$$\Rightarrow \quad \log x = -2 \Rightarrow x = e^{-2}$$

and $\quad y = -2e^{-2}$

$\therefore$ Coordinates of the point are $(e^{-2}, -2e^{-2})$.

32. Given curve is

$$x = a(t + \sin t),\ y = a(1 - \cos t)$$

$$\Rightarrow \quad \frac{dx}{dt} = a(1 + \cos t), \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(\sin t)}{a(1 + \cos t)}$$

$$= \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$$

Length of the normal $= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$= a(1 - \cos t)\sqrt{1 + \tan^2\left(\frac{t}{2}\right)}$$

$$= a(1 - \cos t)\sec\left(\frac{t}{2}\right)$$

$$= 2a\sin^2\left(\frac{t}{2}\right)\sec\left(\frac{t}{2}\right)$$

$$= 2a\sin\left(\frac{t}{2}\right)\tan\left(\frac{t}{2}\right)$$

33. Given curve is $y = e^{2x}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2e^{2x} \Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = 2e^0 = 2$$

Equation of tangent at (0, 1) with slope 2 is

$$y - 1 = 2(x - 0) \Rightarrow y = 2x + 1$$

This tangent meets X-axis.

$$\therefore \quad y = 0$$

$$\Rightarrow \quad 0 = 2x + 1 \Rightarrow x = -\frac{1}{2}$$

$\therefore$ Coordinates of the point on X-axis is $\left(-\frac{1}{2}, 0\right)$.

34. Given curve is $y = 2x^2 - x + 1$.

Let the coordinate of P are (h, k).

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 4x - 1$$

At the point (h, k), the slope $= \left(\frac{dy}{dx}\right)_{(h,k)} = 4h - 1$

Since, the tangents is parallel to the given line $y = 3x + 4$.

$$\Rightarrow \quad 4h - 1 = 3 \Rightarrow h = 1, k = 2$$

$\therefore$ Coordinates of point P are (1, 2).

35. Length of subtangent $= y\dfrac{dx}{dy}$

and length of subnormal $= y\dfrac{dy}{dx}$

$\therefore$ Product $= y^2$

$\Rightarrow$ Required product is the square of the ordinate.

36. Slope of the curve at an angle $\theta = \dfrac{3\pi}{4}$ is

$$\frac{dy}{dx} = \tan\frac{3\pi}{4} = -1$$

Slope of the normal $= \dfrac{-1}{dy/dx}$

$$\therefore \left(\frac{dy}{dx}\right)_{(3,4)} = 1$$

$$\Rightarrow f'(3) = 1$$

37. The equations of given curves are

$$y = a^x \quad \ldots(i)$$

and $$y = b^x \quad \ldots(ii)$$

From Eq. (i), $m_1 = \dfrac{dy}{dx} = a^x \log a$

and from Eq. (ii), $m_2 = \dfrac{dy}{dx} = b^x \log b$

From Eqs. (i) and (ii), we get

$$a^x = b^x$$

$$\Rightarrow x = 0$$

Let α be the angle at which the two curves intersect.

$$\therefore \tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{a^x \log a - b^x \log b}{1 + a^x b^x (\log a)(\log b)}$$

$$= \frac{\log \frac{a}{b}}{1 + (\log a)(\log b)}$$

38. Given equation of curve is $y = be^{-x/a}$ ...(i)

Since, the curve crosses Y-axis *i.e.,* $x = 0$.

$$\Rightarrow y = be^{-0}$$

$$\Rightarrow y = b$$

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$$

At point $(0, b)$, $\left(\dfrac{dy}{dx}\right)_{(0,b)} = \dfrac{-b}{a} e^{-0/a} = \dfrac{-b}{a}$

$\therefore$ Required equation of tangent is

$$y - b = \frac{-b}{a}(x - 0)$$

$$\Rightarrow \frac{y}{b} - 1 = -\frac{x}{a}$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

39. Given curve is $x^3 - 8a^2 y = 0$.

On differentiating w.r.t. x, we get

$$3x^2 - 8a^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{8a^2}$$

$\therefore$ Slope of the normal $= -\dfrac{1}{\left(\dfrac{dy}{dx}\right)} = -\dfrac{1}{\dfrac{3x^2}{8a^2}} = -\dfrac{8a^2}{3x^2}$

Given, $\dfrac{-8a^2}{3x^2} = \dfrac{-2}{3} \Rightarrow x^2 = 4a^2$

$$\Rightarrow x = \pm 2a$$

At $x = \pm 2a$, $y = \pm a$

$$(x, y) = (2a, a)$$

40. Given equation is

$$y - e^{xy} + x = 0 \Rightarrow e^{xy} = x + y$$

On taking log both sides, we get

$$\log(x + y) = xy$$

On differentiating w.r.t. x, we get

$$\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y+x} - x\right) = y - \frac{1}{y+x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(y+x) - 1}{1 - x(y+x)}$$

Since, the curve has a vertical tangent.

$$\therefore \frac{dy}{dx} = \infty \Rightarrow 1 - x(x+y) = 0$$

which is satisfied by the point $(1, 0)$.

41. Given curve is $y^3 + 3x^2 = 12y$

On differentiating w.r.t. y, we get

$$3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(3y^2 - 12) + 6x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x}{12 - 3y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{12 - 3y^2}{6x}$$

Since, tangent is parallel to Y-axis.

$$\frac{dx}{dy} = 0 \Rightarrow 12 - 3y^2 = 0$$

$$\Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Then, at $y = 2$, $x = \pm\dfrac{4}{\sqrt{3}}$

At $y = -2$, x cannot be real.

$\therefore$ The required point is $\left(\pm\dfrac{4}{\sqrt{3}}, 2\right)$.

42. Curve is $y^2 = px^3 + q$

$\therefore \quad 2y\dfrac{dy}{dx} = 3px^2$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{(2,3)} = \dfrac{3p\cdot 4}{2\cdot 3}$

$\Rightarrow \quad 4 = 2p$

$\Rightarrow \quad p = 2$

Also, curve is passing through (2, 3).

$\therefore \quad 9 = 8p + q$

$\Rightarrow \quad q = -7$

$\therefore \quad (p, q)$ is $(2, -7)$.

43. Given curve is $y = f(x) = x^2 + bx - b$.

On differentiating, we get $\dfrac{dy}{dx} = 2x + b$

The equation of the tangent at (1, 1) is

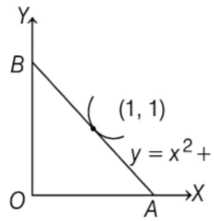
$$y - 1 = \left(\frac{dy}{dx}\right)_{(1,1)}(x-1)$$

$\Rightarrow \quad y - 1 = (b+2)(x-1)$

$\Rightarrow \quad (2+b)x - y = 1 + b$

$\Rightarrow \quad \dfrac{x}{\dfrac{(1+b)}{(2+b)}} - \dfrac{y}{1+b} = 1$

So, $OA = \dfrac{1+b}{2+b}$ and $OB = -(1+b)$



Now, area of $\Delta AOB = \dfrac{1}{2}\cdot\dfrac{1+b}{2+b}\cdot[-(1+b)] = 2$ (given)

$\Rightarrow \quad 4(2+b) + (1+b)^2 = 0$

$\Rightarrow \quad 8 + 4b + 1 + b^2 + 2b = 0$

$\Rightarrow \quad b^2 + 6b + 9 = 0$

$\Rightarrow \quad (b+3)^2 = 0$

$\Rightarrow \quad b = -3$

44. $\dfrac{dy}{dx} = -\dfrac{x_1^2}{y_1^2}$

The tangents cuts the curve again at (x_2, y_2)

$\therefore$ Slope of tangent $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$\therefore \quad \dfrac{-x_1^2}{y_1^2} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Also, $x_1^3 + y_1^3 = a^3$ and $x_2^3 + y_2^3 = a^3$

$\Rightarrow \quad x_1^3 + y_1^3 = x_2^3 + y_2^3$

$\Rightarrow \quad \dfrac{y_2^3 - y_1^3}{x_1^3 - x_2^3} = 1$

$\Rightarrow \quad \dfrac{y_2 - y_1}{x_1 - x_2} = \dfrac{x_1^2 + x_2^2 + x_1x_2}{y_1^2 + y_2^2 + y_1y_2}$

$\Rightarrow \quad \dfrac{x_1^2}{y_1^2} = \dfrac{x_1^2 + x_2^2 + x_1x_2}{y_1^2 + y_2^2 + y_1y_2}$

$\Rightarrow \quad x_1^2y_1^2 + x_1^2y_2^2 + x_1^2y_1y_2 = y_1^2x_1^2 + x_2^2y_1^2 + y_1^2x_1x_2$

$\Rightarrow \quad x_2^2y_1^2 - y_2^2x_1^2 = x_1y_1(x_1y_2 - x_2y_1)$

$\Rightarrow \quad x_2y_1 + y_2x_1 = -x_1y_1$

$\Rightarrow \quad \dfrac{x_2}{x_1} + \dfrac{y_2}{y_1} = -1$

45. Equation of given curves are

$$y = \sin x \qquad \text{...(i)}$$

and $$y = \cos x \qquad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$x = \frac{\pi}{4}$$

$\therefore$ Point of intersection of curves is $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$.

For $y = \sin x$, $\quad \dfrac{dy}{dx} = \cos x$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{x=\pi/4} = \dfrac{1}{\sqrt{2}} = m_1$ (say)

For $y = \cos x$, $\quad \dfrac{dy}{dx} = -\sin x$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{x=\pi/4} = -\dfrac{1}{\sqrt{2}} = m_2$ (say)

$\therefore \quad \tan\theta = \dfrac{m_1 - m_2}{1 + m_1m_2} = \dfrac{\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}}{1 - \dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{\sqrt{2}}} = \dfrac{\dfrac{2}{\sqrt{2}}}{\dfrac{1}{2}}$

$\Rightarrow \quad \tan\theta = 2\sqrt{2} \Rightarrow \theta = \tan^{-1}(2\sqrt{2})$

46. Let the curves intersect at (x_1, y_1).

Therefore, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} - \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{b^2x}{a^2y}$

$\Rightarrow$ Slope of tangent at the point of intersection

$$(m_1) = \frac{b^2x_1}{a^2y_1}$$

Again, $xy = c^2 \Rightarrow x\dfrac{dy}{dx} + y = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{-y}{x}$

$\Rightarrow \quad m_2 = \dfrac{-y_1}{x_1}$

For orthogonality, $m_1 \times m_2 = -1$

$\Rightarrow \quad \dfrac{b^2}{a^2} = 1$ or $a^2 - b^2 = 0$

47. y is an increasing function throughout its domain, if $y' > 0$ throughout the domain.

Given, $y = \log(1+x) - \dfrac{2x}{(2+x)}$.

On differentiating, we get $\dfrac{dy}{dx} = \dfrac{d}{dx}\left[\log(1+x) - \dfrac{2x}{2+x}\right]$

$$= \frac{1}{1+x} - \frac{(2+x)\frac{d}{dx}(2x) - 2x\frac{d}{dx}(2+x)}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4+2x-2x}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2} = \frac{4+x^2+4x-4-4x}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2}$$

When, $x \in (-1, \infty)$, then $\dfrac{x^2}{(2+x)^2} > 0$ and $(1+x) > 0$

$\therefore$ $y' > 0$ when $x > -1$

Hence, y is an increasing function throughout $(x > -1)$ its domain.

48. Equation of given curve is

$$y = \frac{x}{x^2-3},\ x \in R,\ (x \neq \pm\sqrt{3}) \quad \text{...(i)}$$

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2-3) - x(2x)}{(x^2-3)^2} = \frac{(-x^2-3)}{(x^2-3)^2}$$

It is given that tangent at a point $(\alpha, \beta) \neq (0,0)$ on it is parallel to the line $2x + 6y - 11 = 0$.

$\therefore$ Slope of this line $= -\dfrac{2}{6} = \left.\dfrac{dy}{dx}\right|_{(\alpha, \beta)}$

$$\Rightarrow \quad -\frac{\alpha^2+3}{(\alpha^2-3)^2} = -\frac{1}{3}$$

$$\Rightarrow \quad 3\alpha^2 + 9 = \alpha^4 - 6\alpha^2 + 9$$

$$\Rightarrow \quad \alpha^4 - 9\alpha^2 = 0 \Rightarrow \alpha = 0, -3, 3$$

$$\Rightarrow \quad \alpha = 3 \text{ or } -3, \qquad [\because \alpha \neq 0]$$

Now, from Eq. (i), $\beta = \dfrac{\alpha}{\alpha^2-3}$

$$\Rightarrow \quad \beta = \frac{3}{9-3} \text{ or } \frac{-3}{9-3} = \frac{1}{2} \text{ or } -\frac{1}{2}$$

According to the options, $|6\alpha + 2\beta| = 19$ at

$$(\alpha, \beta) = \left(\pm 3, \pm \frac{1}{2}\right)$$

49. Given curve is $y = f(x) = x^3 - x^2 - 2x$ ...(i)

So, $f(1) = 1 - 1 - 2 = -2$

and $f(-1) = -1 - 1 + 2 = 0$

Since, slope of a line passing through (x_1, y_1) and (x_2, y_2) is given by

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

$\therefore$ Slope of line joining points $(1, f(1))$ and $(-1, f(-1))$ is

$$m = \frac{f(1) - f(-1)}{1-(-1)} = \frac{-2-0}{1+1} = -1$$

Now, $\dfrac{dy}{dx} = 3x^2 - 2x - 2$

[differentiating Eq. (i), w.r.t. 'x']

According to the question,

$$\frac{dy}{dx} = m \Rightarrow 3x^2 - 2x - 2 = -1$$

$$\Rightarrow \quad 3x^2 - 2x - 1 = 0$$

$$\Rightarrow \quad (x-1)(3x+1) = 0 \Rightarrow x = 1, -\frac{1}{3}$$

Therefore, set $S = \left\{-\dfrac{1}{3}, 1\right\}$.

50. Given equation of curves are

$$y = 10 - x^2 \quad \text{...(i)}$$

and $$y = 2 + x^2 \quad \text{...(ii)}$$

For point of intersection, consider

$$10 - x^2 = 2 + x^2$$

$$\Rightarrow \quad 2x^2 = 8$$

$$\Rightarrow \quad x^2 = 4 \Rightarrow x = \pm 2$$

Clearly, when $x = 2$, then $y = 6$ [using Eq. (i)]

and when $x = -2$, then $y = 6$

Thus, the point of intersection are $(2, 6)$ and $(-2, 6)$.

Let m_1 be the slope of tangent to the curve (i) and m_2 be the slope of tangent to the curve (ii)

For curve (i) $\dfrac{dy}{dx} = -2x$ and for curve (ii) $\dfrac{dy}{dx} = 2x$

$\therefore$ At $(2, 6)$, slopes $m_1 = -4$ and $m_2 = 4$, and in that case

$$|\tan\theta| = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{4+4}{1-16}\right| = \frac{8}{15}$$

At $(-2, 6)$, slopes $m_1 = 4$ and $m_2 = -4$ and in that case

$$|\tan\theta| = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-4-4}{1-16}\right| = \frac{8}{15}$$

51. The equations of two curves are

$$x^3 - 3xy^2 + 2 = 0 \quad \text{...(i)}$$

and $$3x^2y - y^3 - 2 = 0 \quad \text{...(ii)}$$

On differentiating Eqs. (i) and (ii) w.r.t. x, we get

$$\left(\frac{dy}{dx}\right)_{C_1} = \frac{x^2 - y^2}{2xy}$$

and $$\left(\frac{dy}{dx}\right)_{C_2} = \frac{-2xy}{x^2 - y^2}$$

Now, $$\left(\frac{dy}{dx}\right)_{C_1} \times \left(\frac{dy}{dx}\right)_{C_2} = \left(\frac{x^2-y^2}{2xy}\right)\left(\frac{-2xy}{x^2-y^2}\right) = -1$$

Hence, the two curves cut at right angle.

52. We have, $y = x + \frac{4}{x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 1 - \frac{8}{x^3}$$

Since, the tangent is parallel to X-axis, therefore

$$\frac{dy}{dx} = 0 \Rightarrow x^3 = 8$$

$\Rightarrow$ $x = 2$ and $y = 3$

53. Given that, $x = a(1 + \cos\theta)$, $y = a\sin\theta$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(-\sin\theta) \text{ and } \frac{dy}{d\theta} = a\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{-\cos\theta}{\sin\theta}$$

$\therefore$ Slope of normal $= \frac{-1}{(-\cos\theta/\sin\theta)}$

Equation of normal at the given points is

$$y - a\sin\theta = \frac{\sin\theta}{\cos\theta}[x - a(1 + \cos\theta)]$$

It is clear that in the given options, normal passes through the point $(a, 0)$.

54. Given that, $x = a(\cos\theta + \theta\sin\theta)$

and $y = a(\sin\theta - \theta\cos\theta)$

On differentiating w.r.t. θ respectively, we get

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\theta \quad \text{...(i)}$$

and $$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$$

$$\Rightarrow \frac{dy}{d\theta} = a\theta\sin\theta \quad \text{...(ii)}$$

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{dy}{dx} = \tan\theta$$

Since, slope of normal $= -\frac{dx}{dy} = -\cot\theta$

So, equation of normal is

$$y - a\sin\theta + a\theta\cos\theta = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\cos\theta\sin\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta = a$$

So, it is always at a constant distance 'a' from origin.

55. $\because$ $y = x^2 - 5x + 6$

$$\therefore \frac{dy}{dx} = 2x - 5$$

Now, $m_1 = \left(\frac{dy}{dx}\right)_{(2,\,0)} = 4 - 5 = -1$

and $m_2 = \left(\frac{dy}{dx}\right)_{(3,\,0)} = 6 - 5 = 1$

Now, $m_1 m_2 = -1 \times 1 = -1$

Hence, angle between the tangents is $\frac{\pi}{2}$.

56. Given equation of curve is

$$x^2 + 2xy - 3y^2 = 0 \quad \text{...(i)}$$

On differentiating w.r.t. x, we get

$$2x + 2xy' + 2y - 6yy' = 0$$

$$\Rightarrow y' = \frac{x + y}{3y - x}$$

At $x = 1$, $y = 1$, $y' = 1$

i.e. $\left(\frac{dy}{dx}\right)_{(1,\,1)} = 1$

Equation of normal at $(1, 1)$ is

$$y - 1 = -\frac{1}{1}(x - 1)$$

$$\Rightarrow y - 1 = -(x - 1)$$

$$\Rightarrow x + y = 2 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii) simultaneously, we get

$$x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$$

$$\Rightarrow x^2 + 4x - 2x^2 - 3(4 + x^2 - 4x) = 0$$

$$\Rightarrow -x^2 + 4x - 12 - 3x^2 + 12x = 0$$

$$\Rightarrow -4x^2 + 16x - 12 = 0$$

$$\Rightarrow 4x^2 - 16x + 12 = 0$$

$$\Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow (x - 1)(x - 3) = 0$$

$$\Rightarrow x = 1, 3$$

Now, when $x = 1$, then $y = 1$

and when $x = 3$, then $y = -1$

$\therefore$ $P = (1, 1)$ and $Q = (3, -1)$

Hence, normal meets the curve again at $(3, -1)$ in fourth quadrant.

57. Given, $y = \int_0^x |t|\, dt$

$$\therefore \frac{dy}{dx} = |x|$$

Since, tangent to the curve is parallel to line $y = 2x$.

$$\Rightarrow \frac{dy}{dx} = 2$$

$$\therefore x = \pm 2$$

$\therefore$ Points, $y = \int_0^{\pm 2} |t|\, dt = \pm 2$

$\therefore$ Equation of tangents are

$$y - 2 = 2(x - 2)$$

or $$y + 2 = 2(x + 2)$$

For x-intercept, put $y = 0$, we get

$$0 - 2 = 2(x - 2)$$

or $$0 + 2 = 2(x + 2)$$

$$\Rightarrow x = \pm 1$$

58. Tangent to $x^2+9y^2=9$ at point $\left(\frac{3\sqrt{3}}{2},\frac{1}{2}\right)$ is

$$x\left(\frac{3\sqrt{3}}{2}\right)+9y\left(\frac{1}{2}\right)=9$$

$$3\sqrt{3}x+9y=18$$

$$x+\sqrt{3}y=2\sqrt{3}$$

59. Given curve is

$$y(x-2)(x-3)=x+6 \quad \ldots(i)$$

Put $x=0$ in Eq. (i), we get

$$y(-2)(-3)=6 \Rightarrow y=1$$

So, point of intersection is (0, 1).

Now, $y=\frac{x+6}{(x-2)(x-3)}$

$$\Rightarrow \frac{dy}{dx}=\frac{1(x-2)(x-3)-(x+6)(x-3+x-2)}{(x-2)^2(x-3)^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)}=\frac{6+30}{4\times 9}=\frac{36}{36}=1$$

$\therefore$ Equation of normal at (0, 1) is given by

$$y-1=\frac{-1}{1}(x-0) \Rightarrow x+y-1=0$$

which passes through the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

60. We have, $y^2=6x \Rightarrow 2y\frac{dy}{dx}=6 \Rightarrow \frac{dy}{dx}=\frac{3}{y}$

Slope of tangent at (x_1, y_1) is $m_1=\frac{3}{y_1}$

Also, $9x^2+by^2=16$

$$\Rightarrow 18x+2by\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-9x}{by}$$

Slope of tangent at (x_1, y_1) is

$$m_2=\frac{-9x_1}{by_1}$$

Since, these are intersection at right angle.

$$\therefore m_1m_2=-1 \Rightarrow \frac{27x_1}{by_1^2}=1$$

$$\Rightarrow \frac{27x_1}{6bx_1}=1 \quad [\because y_1^2=6x_1]$$

$$\Rightarrow b=\frac{9}{2}$$

61. Given parabola is $x^2=8y$ ...(i)

Now, slope of tangent at any point (x, y) on the parabola (i) is

$$\frac{dy}{dx}=\frac{x}{4}=\tan\theta$$

[$\because$ tangent is making an angle θ with the positive direction of X-axis]

So, $x=4\tan\theta \Rightarrow 8y=(4\tan\theta)^2$

[on putting $x=4\tan\theta$ in Eq. (i)]

$$\Rightarrow y=2\tan^2\theta$$

Now, equation of required tangent is

$$y-2\tan^2\theta=\tan\theta\,(x-4\tan\theta)$$

$$\Rightarrow y=x\tan\theta-2\tan^2\theta$$

$$\Rightarrow x=y\cot\theta+2\tan\theta$$

62. The given curve is $y=x^2-5x+5$ ...(i)

Now, slope of tangent at any point (x, y) on the curve is

$$\frac{dy}{dx}=2x-5 \quad \ldots(ii)$$

[on differentiating Eq. (i) w.r.t. x]

$\because$ It is given that tangent is parallel to line

$$2y=4x+1$$

So, $\frac{dy}{dx}=2$ [$\because$ slope of line $2y=4x+1$ is 2]

$$\Rightarrow 2x-5=2 \Rightarrow 2x=7 \Rightarrow x=\frac{7}{2}$$

On putting $x=\frac{7}{2}$ in Eq. (i), we get

$$y=\frac{49}{4}-\frac{35}{2}+5=\frac{69}{4}-\frac{35}{2}=-\frac{1}{4}$$

Now, equation of tangent to the curve (i) at point $\left(\frac{7}{2},-\frac{1}{4}\right)$ and having slope 2, is

$$y+\frac{1}{4}=2\left(x-\frac{7}{2}\right) \Rightarrow y+\frac{1}{4}=2x-7$$

$$\Rightarrow y=2x-\frac{29}{4} \quad \ldots(iii)$$

On checking all the options, we get the point $\left(\frac{1}{8},-7\right)$ satisfy the line (iii).

63. Given equation of curve is $y=xe^{x^2}$ ...(i)

Note that $(1, e)$ lie on the curve, so the point of contact is (1, e).

Now, slope of tangent, at point $(1, e)$, to the curve (i) is

$$\left.\frac{dy}{dx}\right|_{(1,e)}=(x(2x)e^{x^2}+e^{x^2})_{(1,e)}=2e+e=3e$$

Now, equation of tangent is given by

$$(y-y_1)=m(x-x_1)$$

$$y-e=3e(x-1) \Rightarrow y=3ex-2e$$

On checking all the options, the option $\left(\frac{4}{3},2e\right)$ satisfy the equation of tangent.

64. We have, $f(x)=\tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}}, x\in\left(0,\frac{\pi}{2}\right)$

$$\Rightarrow f(x)=\tan^{-1}\sqrt{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)$$

$$\left[\because \cos\frac{x}{2} > \sin\frac{x}{2} \text{ for } 0 < \frac{x}{2} < \frac{\pi}{4}\right]$$

$$= \tan^{-1}\left(\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = \frac{\pi}{4} + \frac{x}{2}$$

$$\Rightarrow \quad f'(x) = \frac{1}{2} \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Now, equation of normal at $x = \frac{\pi}{6}$ is given by

$$\left[y - f\left(\frac{\pi}{6}\right)\right] = -2\left(x - \frac{\pi}{6}\right) \Rightarrow \left(y - \frac{\pi}{3}\right) = -2\left(x - \frac{\pi}{6}\right)$$

$$\left[\because f\left(\frac{\pi}{6}\right) = \frac{\pi}{4} + \frac{\pi}{12} = \frac{4\pi}{12} = \frac{\pi}{3}\right]$$

which passes through $\left(0, \frac{2\pi}{3}\right)$.

65. Let $y = x^x$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = x^x(1 + \log x)$$

For increasing function, $\frac{dy}{dx} > 0$

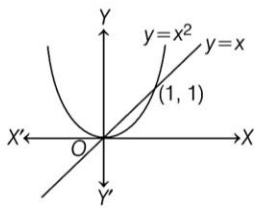
$$\Rightarrow \quad x^x(1 + \log x) > 0 \Rightarrow 1 + \log x > 0$$

$$\Rightarrow \quad \log_e x > \log_e \frac{1}{e} \Rightarrow x > \frac{1}{e}$$

Function is increasing when $x > \frac{1}{e}$.

66. $\because g(x) = \min(x, x^2)$

$$g(x) = \begin{cases} x, x \le 0 \\ x^2, 0 < x \le 1 \\ x, x > 1 \end{cases}$$



Clearly, $g(x)$ is an increasing function.

67. Given, $f(x) = \frac{x}{1 + |x|}$

$$\therefore \quad f'(x) = \frac{(1 + |x|)\cdot 1 - x\cdot\frac{|x|}{x}}{(1 + |x|)^2}$$

$$= \frac{1}{(1 + |x|)^2} > 0, \forall x \in R$$

$\Rightarrow$ $f(x)$ is strictly increasing.

68. $\because$ $f(x) = x + \cos x$

On differentiating, we get $f'(x) = 1 - \sin x$

$f'(x) > 0$ for all values of x

[$\because$ $\sin x$ is lying between -1 to $+1$]

$\therefore$ $f(x)$ is always increasing.

69. Let $f(x) = 2x^3 - 6x + 5$

On differentiating w.r.t. x, we get

$$f'(x) = 6x^2 - 6$$

Since, it is increasing function.

$$\Rightarrow \quad 6x^2 - 6 > 0$$

$$\Rightarrow \quad (x - 1)(x + 1) > 0$$

$$\Rightarrow \quad x > 1 \quad \text{or} \quad x < -1$$

70. Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

Since, $\sin x$ is increasing in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \quad -\frac{\pi}{2} \le 3x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{6} \le x \le \frac{\pi}{6}$$

Thus, length of interval $= \left|\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right| = \frac{\pi}{3}$

71. $\because f(x) = xe^{x(1-x)}$

On differentiating w.r.t. x, we get

$$f'(x) = e^{x(1-x)} + x\cdot e^{x(1-x)}\cdot(1 - 2x)$$

$$= e^{x(1-x)}\{1 + x(1 - 2x)\}$$

$$= e^{x(1-x)}\cdot(-2x^2 + x + 1)$$

It is clear that $e^{x(1-x)} > 0$ for all x.

Now, by sign rule for $-2x^2 + x + 1$

$f'(x) > 0$, if $x \in \left[-\frac{1}{2}, 1\right]$.

So, $f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$.

72. $f(x) = 3\ln(x - 1) - 3\ln(x + 1) - \frac{2}{x - 1}$

$$f'(x) = \frac{3}{x - 1} - \frac{3}{x + 1} + \frac{2}{(x - 1)^2}$$

$$f'(x) = \frac{4(2x - 1)}{(x - 1)^2(x + 1)} \Rightarrow f'(x) \ge 0$$

$$\Rightarrow \quad x \in (-\infty, -1) \cup \left[\frac{1}{2}, 1\right) \cup (1, \infty)$$

73. $f'(x) = -\sin\left(\frac{\pi}{2}\right)\left[\frac{-\pi}{x^2}\right]$

$$= \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)$$

$$f'(x) > 0$$

$$\frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right) > 0 \Rightarrow \sin\frac{\pi}{x} > 0$$

$$2n\pi < \frac{\pi}{x} < (2n + 1)\pi$$

$$\frac{1}{2n + 1} < x < \frac{1}{2n}$$

74. $f(x) = (\sin^2 x + \cos^2 x) - 2\sin^2 x \cos^2 x$

$$= 1 - \frac{1}{2}\sin^2 2x$$

Now, evaluate $f'(x)$ and decide monotonicity.

$\Rightarrow \quad f'(x) = -\sin 2x \cos 2x (2) = -\sin 4x$

f is increasing when $f'(x) > 0$

$\Rightarrow \quad -\sin 4x > 0 \Rightarrow \sin 4x < 0$

$\Rightarrow \quad \pi < 4x < 2\pi \Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{4} < x < \frac{3\pi}{8}$

75. $f''(x) \geq 0$

$\Rightarrow \quad f'(x)$ is increasing function and

$g'(x) = (2x - 2)\,[f'(x^2 - 2x + 8) - f'(14 + 2x - x^2)]$

For $g'(x) > 0$

(i) $x > 1$ and $x^2 - 2x + 8 > 14 + 2x - x^2 \Rightarrow x \in (3, \infty)$

or (ii) $x < 1$ and $x^2 - 2x + 8 < 14 + 2x - x^2 \Rightarrow x \in (-1, 3)$

$\therefore$ From (i) and (ii) $x \in [-1, 1] \cup [3, \infty)$

76. Given, $f(x) = \log(\cos x)$

$\Rightarrow f'(x) = \frac{1}{\cos x}.(-\sin x) = -\tan x$ [differentiate w.r.t. x]

In interval $\left(0, \frac{\pi}{2}\right)$, $\tan x > 0$ [$\because \tan x$ is in Ist quadrant]

$\Rightarrow \quad -\tan x < 0$ [$\because \tan x$ is in Ist quadrant]

$\therefore \quad f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, f is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$.

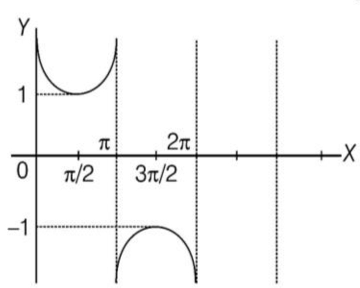
Also, in interval $\left(\frac{\pi}{2}, \pi\right)$, $\tan x < 0 \Rightarrow -\tan x > 0$

[$\because \tan x$ is in IInd quadrant]

$\therefore \quad f'(x) > 0$ in $\left(\frac{\pi}{2}, \pi\right)$

Hence, f is strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$.

77. The graph of cosec x is opposite in interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.



78. Let $\quad f(x) = 2x^3 - 9x^2 + 12x + 4$

$\Rightarrow \quad f'(x) = 6x^2 - 18x + 12$

$f'(x) < 0$ for function to be decreasing

$\Rightarrow \quad 6(x^2 - 3x + 2) < 0$

$\Rightarrow \quad (x^2 - 2x - x + 2) < 0$

$\Rightarrow \quad (x-2)(x-1) < 0 \Rightarrow 1 < x < 2$

79. Given curve is $f(x) = \frac{1}{x+1} - \log(1+x)$

On differentiating w.r.t. x, we get

$$f'(x) = -\frac{1}{(x+1)^2} - \frac{1}{1+x}$$

$\Rightarrow \quad f'(x) = -\left[\frac{1}{x+1} + \frac{1}{(x+1)^2}\right]$

$\Rightarrow \quad f'(x) = -$ve, when $x > 0$

$\therefore \quad f(x)$ is a decreasing function.

80. $\because \quad f(x) = \sin x - \cos x$

On differentiating w.r.t. x, we get

$$f'(x) = \cos x + \sin x$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right)$$

$$= \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right)\right]$$

For decreasing, $f'(x) < 0$

$\frac{\pi}{2} < \left(x - \frac{\pi}{4}\right) < \frac{3\pi}{2}$ [within $0 \leq x \leq 2\pi$]

$\Rightarrow \quad \frac{\pi}{2} + \frac{\pi}{4} < \left(x - \frac{\pi}{4} + \frac{\pi}{4}\right) < \frac{3\pi}{2} + \frac{\pi}{4}$

$\Rightarrow \quad \frac{3\pi}{4} < x < \frac{7\pi}{4}$

81. Let $y = x^{1/x}$

On taking log on both sides, we get

$$\log y = \frac{1}{x}\log x$$

$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{\log x}{x^2} = \frac{1 - \log x}{x^2}$

$\Rightarrow \quad \frac{dy}{dx} = x^{1/x}\left(\frac{1 - \log x}{x^2}\right)$

Now, $x^{1/x} > 0$ for all positive values of x

and $\quad \frac{1 - \log x}{x^2} > 0$ in $(1, e)$

and $\quad \frac{1 - \log x}{x^2} < 0$ in (e, ∞)

$\therefore \quad f(x)$ is increasing in $(1, e)$ and decreasing in (e, ∞).

82. $f(x) = (4a-3)(x+\log_e 5) + (a-7)\sin x$

$f'(x) = (4a-3)(1) + (a-7)\cos x = 0$

$\Rightarrow \quad \cos x = \dfrac{3-4a}{a-7}$

$\Rightarrow \quad \dfrac{3-4a}{a-7} \ge -1$

$\Rightarrow \quad \dfrac{3-4a}{a-7} + 1 \ge 0$

$\Rightarrow \quad \dfrac{3-4a+a-7}{a-7} \ge 0$

$\Rightarrow \quad \dfrac{-3a-4}{a-7} \ge 0$

and $\quad \dfrac{3-4a}{a-7} < 1$

$\Rightarrow \quad \dfrac{3-4a}{a-7} - 1 < 0$

$\Rightarrow \quad \dfrac{3-4a-a+7}{a-7} < 0$

$\Rightarrow \quad \dfrac{-5a+10}{a-7} < 0$

$\Rightarrow \quad \dfrac{5a-10}{a-7} > 0$

$\Rightarrow \quad \dfrac{5(a-2)}{a-7} > 0$

$\therefore \ \alpha \in \left[-\dfrac{4}{3}, 2\right)$

Check end point $\left[-\dfrac{4}{3}, 2\right)$

83. $g'(x) = 2x\left\{f'\left(\dfrac{x^2}{2}\right) - f'(6-x^2)\right\}$

***Case* I** $\dfrac{x^2}{2} > 6 - x^2 \Rightarrow x^2 > 4$

$\Rightarrow \quad x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow \quad f'\left(\dfrac{x^2}{2}\right) > f'(6-x^2)$

$\Rightarrow \quad f'\left(\dfrac{x^2}{2}\right) - f'(6-x^2) > 0$

if $\quad x > 0 \Rightarrow g'(x) > 0 \Rightarrow x \in (2, \infty)$

if $\quad x < 0 \Rightarrow g'(x) < 0 \Rightarrow x \in (-\infty, -2)$

***Case* II** $\dfrac{x^2}{2} < 6 - x^2 \Rightarrow x^2 < 4 \Rightarrow x \in (-2, 2)$

$\Rightarrow \quad f'\left(\dfrac{x^2}{2}\right) - f'(6-x^2) < 0$

if $\quad x > 0 \Rightarrow g'(x) < 0 \Rightarrow x \in (0, 2)$

if $\quad x < 0 \Rightarrow g'(x) > 0 \Rightarrow x \in (-2, 0)$

For $g(x)$ decreasing $x \in (-\infty, -2) \cup (0, 2)$.

84. $\because \quad f(x) = \dfrac{\lambda \sin x + 6\cos x}{2\sin x + 3\cos x}$ ...(i)

On differentiating w.r.t. x, we get

$$f'(x) = \frac{\begin{bmatrix}(2\sin x + 3\cos x)(\lambda\cos x - 6\sin x) \\ -(\lambda \sin x + 6\cos x)(2\cos x - 3\sin x)\end{bmatrix}}{(2\sin x + 3\cos x)^2}$$

The function is monotonic increasing, if $f'(x) > 0$

$\Rightarrow \quad 3\lambda(\sin^2 x + \cos^2 x) - 12(\sin^2 x + \cos^2 x) > 0$

$\Rightarrow \quad 3\lambda - 12 > 0 \quad [\because \sin^2 x + \cos^2 x = 1]$

$\Rightarrow \quad \lambda > 4$

85. The given functions are

$f(x) = e^x - x$ and $g(x) = x^2 - x, \forall x \in R$

Then, $\quad h(x) = (fog)(x) = f(g(x))$

Now, $\quad h'(x) = f'(g(x)) \cdot g'(x) = (e^{g(x)} - 1) \cdot (2x-1)$

$= (e^{(x^2-x)} - 1)(2x-1)$

$= (e^{x(x-1)} - 1)(2x-1)$

$\because$ It is given that $h(x)$ is an increasing function, so

$h'(x) \ge 0 \Rightarrow (e^{x(x-1)} - 1)(2x-1) \ge 0$

***Case* I** $(2x-1) \ge 0$ and $(e^{x(x-1)} - 1) \ge 0$

$\Rightarrow \quad x \ge \dfrac{1}{2}$ and $x(x-1) \ge 0$

$\Rightarrow x \in [1/2, \infty)$ and $x \in (-\infty, 0] \cup [1, \infty)$, so $x \in [1, \infty)$

***Case* II** $(2x-1) \le 0$ and $[e^{x(x-1)} - 1] \le 0$

$\Rightarrow \quad x \le \dfrac{1}{2}$ and $x(x-1) \le 0$

$\Rightarrow \quad x \in \left(-\infty, \dfrac{1}{2}\right]$ and $x \in [0, 1]$, so $x \in \left[0, \dfrac{1}{2}\right]$

From the above cases, $x \in \left[0, \dfrac{1}{2}\right] \cup [1, \infty)$.

86. Given, $\quad \phi(x) = f(x) + f(2-x), \forall x \in (0, 2)$

$\Rightarrow \quad \phi'(x) = f'(x) - f'(2-x)$ ...(i)

Also, we have $f''(x) > 0 \ \forall x \in (0, 2)$

$\Rightarrow \quad f'(x)$ is a strictly increasing function $\forall x \in (0, 2)$.

Now, for $\phi(x)$ to be increasing,

$\phi'(x) \ge 0$

$\Rightarrow \quad f'(x) - f'(2-x) \ge 0$ [using Eq. (i)]

$\Rightarrow \quad f'(x) \ge f'(2-x) \Rightarrow x > 2 - x$

[$\because f'$ is a strictly increasing function]

$\Rightarrow \quad 2x > 2 \Rightarrow x > 1$

Thus, $\phi(x)$ is increasing on (1, 2).

Similarly, for $\phi(x)$ to be decreasing,

$\phi'(x) \le 0$

$\Rightarrow \quad f'(x) - f'(2-x) \le 0$ [using Eq. (i)]

$\Rightarrow \quad f'(x) \le f'(2-x) \Rightarrow x < 2 - x$

[$\because f'$ is a strictly increasing function]

$\Rightarrow \quad 2x < 2 \Rightarrow x < 1$

Thus, $\phi(x)$ is decreasing on (0, 1).

87. We have, $f(x)=\dfrac{x}{(a^2+x^2)^{1/2}}-\dfrac{(d-x)}{(b^2+(d-x)^2)^{1/2}}$

Differentiating above w.r.t. x, we get

$$f'(x)=\frac{(a^2+x^2)^{1/2}-x\frac{1}{2}\frac{2x}{(a^2+x^2)^{1/2}}}{(a^2+x^2)}-\frac{(b^2+(d-x)^2)^{1/2}(-1)-(d-x)\frac{2(d-x)(-1)}{2(b^2+(d-x)^2)^{1/2}}}{(b^2+(d-x)^2)}$$

[by using quotient rule of derivative]

$$=\frac{a^2+x^2-x^2}{(a^2+x^2)^{3/2}}+\frac{b^2+(d-x)^2-(d-x)^2}{(b^2+(d-x)^2)^{3/2}}$$

$$=\frac{a^2}{(a^2+x^2)^{3/2}}+\frac{b^2}{(b^2+(d-x)^2)^{3/2}}>0,\ \forall\ x\in R$$

Hence, $f(x)$ is an increasing function of x.

88. Since, $f(x)$ is a positive increasing function.

$\Rightarrow\quad 0<f(x)<f(2x)<f(3x)$

$\Rightarrow\quad 0<1<\dfrac{f(2x)}{f(x)}<\dfrac{f(3x)}{f(x)}$

$\Rightarrow\quad \lim\limits_{x\to\infty}1\le\lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)}\le\lim\limits_{x\to\infty}\dfrac{f(3x)}{f(x)}$

By Sandwich theorem, $\lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)}=1$

89. Since, $f(x)=\tan^{-1}(\sin x+\cos x)$

On differentiating w.r.t. x, we get

$$f'(x)=\frac{1}{1+(\sin x+\cos x)^2}(\cos x-\sin x)$$

$$=\frac{\sqrt{2}\left\{\cos x\cdot\cos\frac{\pi}{4}-\sin x\cdot\sin\frac{\pi}{4}\right\}}{1+(\sin x+\cos x)^2}$$

$$=\frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

For $f(x)$ to be increasing,

$$-\frac{\pi}{2}<x+\frac{\pi}{4}<\frac{\pi}{2}\Rightarrow-\frac{3\pi}{4}<x<\frac{\pi}{4}$$

Hence, option (b) is correct, which lies in the above interval.

90. $\because f(x)=x^3-6x^2+ax+b$

On differentiating w.r.t. x, we get

$f'(x)=3x^2-12x+a$

By the definition of Rolle's theorem

$$f'(c)=0\ \Rightarrow\ f'\left(2+\frac{1}{\sqrt{3}}\right)=0$$

$\Rightarrow\quad 3\left(2+\dfrac{1}{\sqrt{3}}\right)^2-12\left(2+\dfrac{1}{\sqrt{3}}\right)+a=0$

$\Rightarrow\quad 3\left(4+\dfrac{1}{3}+\dfrac{4}{\sqrt{3}}\right)-12\left(2+\dfrac{1}{\sqrt{3}}\right)+a=0$

$\Rightarrow\quad 12+1+4\sqrt{3}-24-4\sqrt{3}+a=0\Rightarrow a=11$

91. $\int_1^2 f'(x)\,dx=[f(x)]_1^2=f(2)-f(1)=0$

[$\because f(x)$ satisfies the conditions of Rolle's theorem]

$\therefore\quad f(2)=f(1)$

92. Since, $f(x)$ satisfies all the conditions of Rolle's theorem in [3, 5].

Let $\quad f(x)=(x-3)(x-5)$

$\qquad=x^2-8x+15$

Now, $\quad\int_3^5 f(x)\,dx=\int_3^5(x^2-8x+15)\,dx$

$$=\left[\frac{x^3}{3}-\frac{8x^2}{2}+15x\right]_3^5$$

$$=\left(\frac{125}{3}-100+75\right)-(9-36+45)$$

$$=\frac{50}{3}-18=-\frac{4}{3}$$

93. From mean value theorem $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Given, $\quad a=0\ \Rightarrow\ f(a)=0$

and $\quad b=\dfrac{1}{2}\ \Rightarrow\ f(b)=\dfrac{3}{8}$

Now, $\quad f'(x)=(x-1)(x-2)+x(x-2)+x(x-1)$

$\therefore\quad f'(c)=(c-1)(c-2)+c(c-2)+c(c-1)$

$\qquad=c^2-3c+2+c^2-2c+c^2-c$

$\Rightarrow\quad f'(c)=3c^2-6c+2$

By definition of mean value theorem

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$\Rightarrow\quad 3c^2-6c+2=\dfrac{\left(\frac{3}{8}\right)-0}{\left(\frac{1}{2}\right)-0}=\dfrac{3}{4}$

$\Rightarrow\quad 3c^2-6c+\dfrac{5}{4}=0$

This is a quadratic equation in c.

$\therefore\quad c=\dfrac{6\pm\sqrt{36-15}}{2\times3}=\dfrac{6\pm\sqrt{21}}{6}=1\pm\dfrac{\sqrt{21}}{6}$

Since, c lies between $\left[0,\dfrac{1}{2}\right]$.

$\therefore\quad c=1-\dfrac{\sqrt{21}}{6}\qquad\left[\text{neglecting } c=1+\dfrac{\sqrt{21}}{6}\right]$

94. $f(1)=f(2)$

$\Rightarrow 1-a+b-4=8-4a+2b-4$

$3a-b=7$

$f'(x)=3x^2-2ax+b$

$\Rightarrow\quad f'\left(\dfrac{4}{3}\right)=0\Rightarrow 3\times\dfrac{16}{9}-\dfrac{8}{3}a+b=0$

$\Rightarrow\quad -8a+3b=-16$

$a=5,\ b=8$

95. $f(1) = f(2)$

$\Rightarrow \quad 1 + b + c = 8 + 4b + 2c \Rightarrow \quad 3b + c = -7$

$f'(4/3) = 0 \quad \Rightarrow \quad 3 \cdot \frac{16}{9} + 2b\,\frac{4}{3} + c = 0$

On solving both, we get $b = -5, c = 8$.

96. As $f(x) = 0$ has three real and distinct zero, so $f'(x) = 3x^2 - p = 0$ has two real and distinct zeroes by Rolle's theorem

$$x = \pm\sqrt{\frac{p}{3}}$$

$$f'(x) > 0 \text{ if } x < -\sqrt{\frac{p}{3}}$$

$$f'(x) < 0 \text{ if } -\sqrt{\frac{p}{3}} < x < \sqrt{\frac{p}{3}}$$

and $\quad f'(x) > 0 \text{ if } x > \sqrt{\frac{p}{3}}$

Thus, $f(x)$ has a local maximum at $x = -\sqrt{\frac{p}{3}}$ and local minimum at $x = \sqrt{\frac{p}{3}}$.

97. Let $f'(x) = ax^2 + bx + c$

$\Rightarrow \quad f(x) = \frac{2ax^3 + 3bx^2 + 6cx + 6d}{6}$

$\Rightarrow \quad f(0) = f(1) = \frac{d}{6}$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. So, there exists at least one $\alpha \in (0, 1)$ for which $f'(\alpha) = 0$, so $ax^2 + bx + c = 0$ has at least one root in $(0, 1)$.

98. Given function $f: R \rightarrow R$ with $f(0) = f(1) = f'(0) = 0$.

So, by Rolle's theorem, for some $c \in (0, 1)$ $f'(c) = 0$.

And as $f'(0) = 0$ and function 'f' is twice differential.

So, again for some $x \in (0, 1)$.

$f''(x) = 0 \qquad$ [by Rolle's theorem]

99. The given function $f(x) = \log_e\left(\frac{x^2 + \alpha}{7x}\right)$ holds the Rolle's theorem for the interval [3, 4].

So, $\quad f(3) = f(4)$

$\Rightarrow \quad \log_e\left(\frac{9 + \alpha}{21}\right) = \log_e\left(\frac{16 + \alpha}{28}\right)$

$\Rightarrow \quad \frac{9 + \alpha}{3} = \frac{16 + \alpha}{4}$

$\Rightarrow \quad 36 + 4\alpha = 48 + 3\alpha$

$\Rightarrow \quad \alpha = 12 \qquad$ ...(i)

and $f'(c) = 0$, for some $c \in (3, 4)$

$\Rightarrow \quad \left[\frac{7x}{x^2 + \alpha} \times \frac{7x(2x + 0) - (x^2 + \alpha)7}{(7x)^2}\right]_{x = c} = 0$

$\Rightarrow \quad \frac{c(2c) - (c^2 + 12)}{(c^2 + 12)c} = 0$

$\Rightarrow \quad c^2 - 12 = 0 \quad [\because c \in (3, 4)]$

$\Rightarrow \quad c = \sqrt{12} \quad [\because c \in (3, 4)]$

$\therefore \; f''(c = \sqrt{12}) = \frac{c(c^2 + 12)(2c) - (c^2 - 12)(3c^2 + 12)}{((c^2 + 12)c)^2}$

$$= \frac{(2 \times 12 \times 24) - (0 \times 48)}{(24)^2(12)} = \frac{1}{12}$$

100. $\because \; f'(x) = 2\left(\frac{1}{3}\right)\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right) + \left(\frac{1}{3}\right)\cos\frac{x}{3} - \left(\frac{1}{3}\right)$

$$= \left(\frac{1}{3}\right)\left[2\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right) - 2\sin^2\left(\frac{x}{6}\right)\right]$$

$$= \left(\frac{2}{3}\right)\sin\left(\frac{x}{6}\right)\left[\cos\left(\frac{x}{6}\right) - \sin\left(\frac{x}{6}\right)\right]$$

Put $f'(x) = 0 \quad \Leftrightarrow \quad \sin\left(\frac{x}{6}\right) = 0 \Rightarrow \tan\left(\frac{x}{6}\right) = 1$

$\Rightarrow \quad \frac{x}{6} = k\pi, \; k \in I$ or $\frac{x}{6} = n\pi + \frac{\pi}{4}, n \in I$

$x^2 - 10 < -19.5x$

$\Leftrightarrow \quad (x + 9.75)^2 < 105.0625$

$\Leftrightarrow \quad (x - 0.5)(x + 20) < 0$

$\Leftrightarrow \quad -20 < x < 0.5$

So, the critical points satisfying the last inequality will be $0, 6\pi, -\frac{9\pi}{2}$.

101. Given curve is $f(x) = \int_0^x \frac{\sin t}{t}\,dt$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{\sin x}{x}$$

For point of extrema, put $f'(x) = 0$

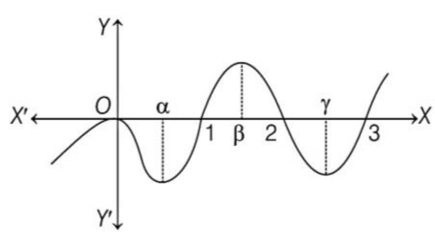
$\Rightarrow \quad \frac{\sin x}{x} = 0$

$\Rightarrow \quad \sin x = 0$

$\Rightarrow \quad x = n\pi, \; n = 1, 2, 3, \ldots$

102. We have, $f(x) = |x|(x-1)(x-2)(x-3)$

$$= \begin{cases} x(x-1)(x-2)(x-3), & x \geq 0 \\ -x(x-1)(x-2)(x-3), & x < 0 \end{cases}$$



It is clear from the figure that, there are four critical points *i.e.*, $0, \alpha, \beta, \gamma$.

103. Let $f(x)=\sin 2x \Rightarrow f'(x)=2\cos 2x$

For maxima or minima put $f'(x)=0$

$\Rightarrow \quad 2\cos 2x=0 \Rightarrow \cos 2x=0$

$\Rightarrow \quad 2x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}$

$\Rightarrow \quad x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

Thus, we evaluate the values of f at critical points $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$ and at the end points of the interval $[0,2\pi]$.

At $x=0$, $f(0)=\sin(2\times 0)=0$

At $x=2\pi$, $f(2\pi)=\sin(2\times 2\pi)=0$

At $x=\frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right)=\sin\left(2\times\frac{\pi}{4}\right)=\sin\frac{\pi}{2}=1$

At $x=\frac{3\pi}{4}$, $f\left(\frac{3\pi}{4}\right)=\sin\left(2\times\frac{3\pi}{4}\right)=\sin\frac{3\pi}{2}$

$=\sin\left(\pi+\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}=-1$

At $x=\frac{5\pi}{4}$, $f\left(\frac{5\pi}{4}\right)=\sin\left(2\times\frac{5\pi}{4}\right)=\sin\frac{5\pi}{2}$

$=\sin\left(2\pi+\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1$

At $x=\frac{7\pi}{4}$, $f\left(\frac{7\pi}{4}\right)=\sin\left(2\times\frac{7\pi}{4}\right)=\sin\frac{7\pi}{2}$

$=\sin\left(2\pi+\frac{3\pi}{2}\right)=\sin\frac{3\pi}{2}=-1$

Hence, we can conclude that absolute maximum value of f on $[0,2\pi]$ is 1 occurring at $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$

104. Let $f(x)=x^4-62x^2+ax+9$

$\Rightarrow f'(x)=4x^3-124x+a$

It is given that function f attains its maximum value on the interval $[0,2]$ at $x=1$.

$\therefore \quad f'(1)=0$

$\Rightarrow \quad 4\times 1^3-124\times 1+a=0$

$\Rightarrow \quad 4-124+a=0 \Rightarrow a=120$

Hence, the value of a is 120.

105. Let $f(x)=x+\sin 2x$, $f'(x)=1+2\cos 2x$

For maxima or minima put $f'(x)=0$

$\Rightarrow \quad 1+2\cos 2x=0 \Rightarrow \cos 2x=-\frac{1}{2}$

$\Rightarrow \quad \cos 2x=-\cos\frac{\pi}{3}=\cos\frac{2\pi}{3}$

$\Rightarrow \quad \cos 2x=\cos\left(\pi-\frac{\pi}{3}\right),\cos\left(\pi+\frac{\pi}{3}\right),$

$\cos\left(3\pi-\frac{\pi}{3}\right),\cos\left(3\pi+\frac{\pi}{3}\right)$

[$\because$ we know that $\cos x$ is negative in second and third quadrant]

Then, $2x=2n\pm\frac{2\pi}{3}, n\in Z \Rightarrow 2x=\frac{2\pi}{3},\frac{4\pi}{3},\frac{8\pi}{3},\frac{10\pi}{3}$

$\Rightarrow \quad x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}\in[0,2\pi]$

Then, we evaluate the value of f at critical points

$x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$

and at the end points of the interval $[0,2\pi]$.

At $x=0$, $f(0)=0+\sin 0=0$

At $x=\frac{\pi}{3}$, $f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}+\sin\frac{2\pi}{3}=\frac{\pi}{3}+\sin\frac{\pi}{3}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$

At $x=\frac{2\pi}{3}$, $f\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}+\sin\frac{4\pi}{3}=\frac{2\pi}{3}-\sin\frac{\pi}{3}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

At $x=\frac{4\pi}{3}$, $f\left(\frac{4\pi}{3}\right)=\frac{4\pi}{3}+\sin\frac{8\pi}{3}=\frac{4\pi}{3}+\sin\frac{2\pi}{3}=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$

At $x=\frac{5\pi}{3}$, $f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sin\frac{10\pi}{3}=\frac{5\pi}{3}-\sin\frac{2\pi}{3}$

$=\frac{5\pi}{3}-\frac{\sqrt{3}}{2}$

At $x=2\pi$, $f(2\pi)=2\pi+\sin 4\pi=2\pi+0=2\pi$

Thus, maximum value is 2π at $x=2\pi$ and minimum value is 0 at $x=0$.

106. Let $f(x)=[x(x-1)+1]^{1/3}$, $0\le x\le 1=(x^2-x+1)^{1/3}$

On differentiating w.r.t. x, we get

$$f'(x)=\frac{1}{3}(x^2-x+1)^{\frac{1}{3}-1}(2x-1)=\frac{1(2x-1)}{3(x^2-x+1)^{2/3}}$$

Now, put $f'(x)=0 \Rightarrow 2x-1=0 \Rightarrow x=\frac{1}{2}\in[0,1]$.

So, $x=\frac{1}{2}$ is a critical point.

Now, we evaluate the value of f at critical point $x=\frac{1}{2}$ and at the end points of the interval [0, 1].

At $x=0$, $f(0)=(0-0+1)^{1/3}=1$

At $x=1$, $f(1)=(1-1+1)^{1/3}=1$

At $x=\frac{1}{2}$, $f\left(\frac{1}{2}\right)=\left(\frac{1}{4}-\frac{1}{2}+1\right)^{1/3}=\left(\frac{3}{4}\right)^{1/3}$

$\therefore$ Maximum value of $f(x)$ is 1 at $x=0,1$. Hence, (c) is the correct option.

107. $f(x)=4x^3-18x^2+27x-7$

$f'(x)=12x^2-36x+27=3(4x^2-12x+9)=3(2x-3)^2$

$f'(x)=0 \Rightarrow x=\frac{3}{2}$ (critical point)

Since, $f'(x)>0$ for all $x<\frac{3}{2}$ and for all $x>\frac{3}{2}$

Hence, $x=\frac{3}{2}$ is a point of inflexion i.e. neither a point of maxima nor a point of minima. $x=\frac{3}{2}$ is the only critical point and f has neither maxima nor minima.

108. $f'(x) = -3x^3 - 24x^2 - 45x$

$= -3x(x^2 + 8x + 15) = -3x(x+5)(x+3)$

$f'(x) = 0 \Rightarrow x = -5, x = -3, x = 0$

$f''(x) = -9x^2 - 48x - 45 = -3(3x^2 + 16x + 15)$

$f''(0) = -45 < 0$. Therefore, $x = 0$ is point of local maxima

$f''(-3) = 18 > 0$. Therefore, $x = -3$ is point of local minima

$f''(-5) = -30 < 0$. Therefore, $x = -5$ is point of local maxima

109. $f'\left(\frac{\pi}{3}\right) = 0 \Rightarrow \frac{a}{2} - 1 = 0 \Rightarrow a = 2$

110. $f(|x|) = |x| + 1 = \begin{cases} -x+1; x<0 \\ x+1; x>0 \end{cases}, g(x) = \{e^x, -2 \le x \le 0$

$f(|x|) - g(x) = \{-x + 1 - e^x, -2 \le x \le 0$

Say $H(x) = x + 1 - e^x, -2 \le x \le 0$

$H'(x) = -1 - e^x < 0$ for all $x \in [-2, 0)$.

So, $H(x)$ is maximum at $x = -2$.

So, its maximum value at $x = -2$,

$H(x) = -(-2) + 1 - e^{-2} = 3 - \frac{1}{e^2}$

111. $f'(x) = 25x^{24}(1-x)^{75} - 75x^{25}(1-x)^{74}$

$= 25x^{24}(1-x)^{74}[(1-x) - 3x]$

$= 25x^{24}(1-x)^{74}(1-4x)$;

$f'(x)$ changes sign about $x = \frac{1}{4}$ only

112. Given, $f(x) = x^{-x} \Rightarrow \log f(x) = -x \log x$

On differentiating w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = -\log x - 1$$

$\Rightarrow \quad f'(x) = -f(x)(1 + \log x)$

Put $f'(x) = 0 \Rightarrow \log x = -1 \Rightarrow x = e^{-1}$

$\therefore \quad f''(x) = -f'(x)(1 + \log x) - f(x)\frac{1}{x}$

$= f(x)(1 + \log x)^2 - \frac{f(x)}{x}$

At $x = \frac{1}{e}$, $f''(x) = -ef\left(\frac{1}{e}\right) < 0$, maxima

Hence, at $x = \frac{1}{e}$, $f(x)$ is maximum.

113. $\because y = a(1 - \cos x)$

On differentiating w.r.t. x, we get

$y' = a \sin x$ ...(i)

Put $y' = 0$ for maxima or minima,

$\sin x = 0 \Rightarrow x = 0, \pi$

Again, differentiating w.r.t. x of Eq. (i), we get

$y'' = a \cos x \Rightarrow y''(0) = a$ and $y''(\pi) = -a$

Hence, y is maximum when $x = \pi$.

114. Given function is $f(x) = x + \sin x$

On differentiating w.r.t. x, we get

$$f'(x) = 1 + \cos x$$

For maxima or minima put $f'(x) = 0$

$\Rightarrow \quad 1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow x = \pi$

Again, differentiating w.r.t. x, we get

$f''(x) = -\sin x$, at $x = \pi$, $f''(\pi) = 0$

Again, differentiating w.r.t. x, we get

$$f'''(x) = -\cos x, \quad f'''(\pi) = 1$$

At $x = \pi$, $f(x)$ is minimum.

115. $\phi'(x) = f'(x) + a$

$\because \phi'(0) = 0 \Rightarrow f'(0) + a = 0 \Rightarrow a = 0$ $[\because f'(0) = 0]$

Also, $\phi''(0) > 0$ $[\because f''(0) > 0]$

$\Rightarrow \phi(x)$ has relative minimum at $x = 0$ for all b, if $a = 0$

116. Let the number be x, then $f(x) = \frac{x}{x^2 + 16}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{(x^2+16)\cdot 1 - x(2x)}{(x^2+16)^2}$$

$$= \frac{x^2 + 16 - 2x^2}{(x^2+16)^2}$$

$$= \frac{16 - x^2}{(x^2+16)^2} \quad \text{...(i)}$$

Put $f'(x) = 0$ for maxima or minima

$f'(x) = 0 \Rightarrow 16 - x^2 = 0 \Rightarrow x = 4, -4$

Again, on differentiating w.r.t. x, we get

$$f''(x) = \frac{(x^2+16)^2(-2x) - (16 - x^2)\,2(x^2+16)\,2x}{(x^2+16)^4}$$

At $x = 4$, $f''(x) < 0$

$\therefore f(x)$ is maximum at $x = 4$.

and at $x = -4$, $f''(x) > 0$, $f(x)$ is minimum.

$\therefore$ Least value of $f(x) = \frac{-4}{16+16} = -\frac{1}{8}$

117. Given, $f(x) = ax + \frac{b}{x}$

On differentiating w.r.t. x, we get

$$f'(x) = a - \frac{b}{x^2}$$

For maxima or minima, put $f'(x) = 0 \Rightarrow x = \sqrt{\frac{b}{a}}$

Again, differentiating w.r.t. x, we get

$$f''(x) = \frac{2b}{x^3}$$

At $x = \sqrt{\frac{b}{a}}$, $f''(x) =$ positive

$\Rightarrow f(x)$ is minimum at $x = \sqrt{\frac{b}{a}}$.

$\therefore f(x)$ has the least value at $x = \sqrt{\frac{b}{a}}$.

118. Given, $f(x) = \int_{-10}^{x} (t^4 - 4)\, e^{-4t}\, dt$

On differentiating w.r.t. x, we get

$$f'(x) = (x^4 - 4)\, e^{-4x}$$

For maxima or minima, put $f'(x) = 0 \Rightarrow x = \pm\sqrt{2}, \pm\sqrt{2}$

Again, on differentiating w.r.t. x, we get

$$f''(x) = -4(x^4 - 4)\, e^{-4x} + 4x^3 e^{-4x}$$

At $x = \sqrt{2}$ and $x = -\sqrt{2}$, the given function has two extreme values.

119. $f'(x) = -a \sin x + b \sec^2 x + 1$

Now, $f'(0) = 0$ and $f'\left(\frac{\pi}{6}\right) = 0$

$\Rightarrow \quad b + 1 = 0$ and $-\frac{a}{2} + \frac{4b}{3} + 1 = 0 \Rightarrow b = -1,\ a = -\frac{2}{3}$

120. Given, $f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$

Now, $f'(x) = 2\cos x(-\sin x) + \cos x = -2\sin x \cos x + \cos x$

For maximum or minimum put $f'(x) = 0$

$\Rightarrow \quad -2\sin x \cos x + \cos x = 0$

$\Rightarrow \quad \cos x(-2\sin x + 1) = 0 \Rightarrow \cos x = 0$ or $\sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{2}$

For absolute maximum and absolute minimum, we have to evaluate

$$f(0),\ f\left(\frac{\pi}{6}\right),\ f\left(\frac{\pi}{2}\right),\ f(\pi)$$

At $x = 0$, $\quad f(0) = \cos^2 0 + \sin 0 = 1^2 + 0 = 1$

At $x = \frac{\pi}{6}$, $f\left(\frac{\pi}{6}\right) = \cos^2\left(\frac{\pi}{6}\right) + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4} = 1.25$

At $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{2}\right) + \sin\frac{\pi}{2} = 0^2 + 1 = 1$

At $x = \pi$, $f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$

Hence, the absolute maximum value of f is 1.25 occurring at $x = \frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at $x = 0, \frac{\pi}{2}$ and π.

121. $\because \quad f(x) = \frac{x}{4 + x + x^2}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{4 + x + x^2 - x(1 + 2x)}{(4 + x + x^2)^2}$$

For maximum, put $f'(x) = 0 \Rightarrow \frac{4 - x^2}{(4 + x + x^2)^2} = 0$

$\Rightarrow \quad x = 2, -2$

Both the values of x are not in the interval $[-1, 1]$.

$\therefore \quad f(-1) = \frac{-1}{4 - 1 + 1} = \frac{-1}{4}$

$$f(1) = \frac{1}{4 + 1 + 1} = \frac{1}{6} \text{ (maximum)}$$

122. Given, $f(x) = x^2 \log x$

On differentiating w.r.t. x, we get

$$f'(x) = (2\log x + 1)\, x$$

For a maximum, put $f'(x) = 0$

$\Rightarrow \quad (2\log x + 1)\, x = 0 \quad \Rightarrow \quad x = e^{-1/2}, 0$

$\because \quad 0 < e^{-1/2} < 1$

None of these critical points lies in the interval $[1, e]$.

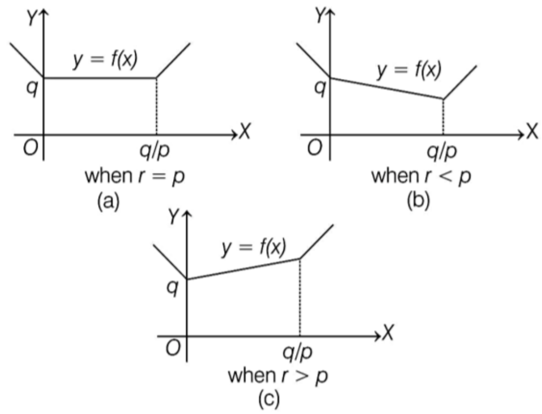
So, we only compute the value of $f(x)$ at the end points 1 and e.

We have, $\quad f(1) = 0, \quad f(e) = e^2$

Hence, greatest value of $f(x) = e^2$

123. We have, $f(x) = |px - q| + r|x|, x \in (-\infty, \infty)$

$$= \begin{cases} -px + q - rx, & x \le 0 \\ -px + q + rx, & 0 < x < q/p \\ px - q + rx, & q/p < x \end{cases}$$



Thus, f has infinite points of minimum, if $r = p$.

In case, $p \ne r$, then $x = 0$ is point of minimum, if $r > p$ and $x = \frac{q}{p}$ is point of minimum, if $r < p$.

124. Given function is $f(x) = 9x^4 + 12x^3 - 36x^2 + 25 = y$ (let)

For maxima or minima put $\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = 36x^3 + 36x^2 - 72x = 0$

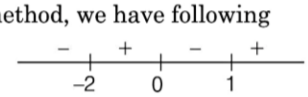
$\Rightarrow x^3 + x^2 - 2x = 0 \Rightarrow x[x^2 + x - 2] = 0$

$\Rightarrow \quad x[x^2 + 2x - x - 2] = 0$

$\Rightarrow \quad x[x(x + 2) - 1(x + 2)] = 0$

$\Rightarrow \quad x(x - 1)(x + 2) = 0 \Rightarrow x = -2, 0, 1$

By sign method, we have following



Since, $\frac{dy}{dx}$ changes it's sign from negative to positive at $x =$ '–2' and '1', so $x = -2, 1$ are points of local minima.

Also, $\frac{dy}{dx}$ changes it's sign from positive to negative at $x = 0$, so $x = 0$ is point of local maxima.

$\therefore \quad S_1 = \{-2, 1\}$ and $S_2 = \{0\}$.

130. Given functions are $f(x) = 5 - |x-2|$

and $\quad g(x) = |x+1|$, where $x \in R$.

Clearly, maximum of $f(x)$ occurred at $x = 2$, so $\alpha = 2$.

and minimum of $g(x)$ occurred at $x = -1$, so $\beta = -1$.

$\Rightarrow \quad \alpha\beta = -2$

Now, $\lim\limits_{x \to -\alpha\beta} \dfrac{(x-1)(x^2-5x+6)}{x^2-6x+8}$

$= \lim\limits_{x \to 2} \dfrac{(x-1)(x-3)(x-2)}{(x-4)(x-2)} \quad [\because \alpha\beta = -2]$

$= \lim\limits_{x \to 2} \dfrac{(x-1)(x-3)}{(x-4)}$

$= \dfrac{(2-1)(2-3)}{(2-4)} = \dfrac{1 \times (-1)}{(-2)} = \dfrac{1}{2}$

131. We have, $f(x) = 3x^3 - 18x^2 + 27x - 40$

$\Rightarrow \quad f'(x) = 9x^2 - 36x + 27$

$= 9(x^2 - 4x + 3) = 9(x-1)(x-3) \quad \ldots(i)$

Also, we have

$S = \{x \in R : x^2 + 30 \le 11x\}$

Clearly, $\quad x^2 + 30 \le 11x$

$x^2 - 11x + 30 \le 0$

$\Rightarrow \quad (x-5)(x-6) \le 0 \Rightarrow x \in [5, 6]$

So, $\quad S = [5, 6]$

Note that $f(x)$ is increasing in $[5, 6]$

$[\because f'(x) > 0 \text{ for } x \in [5, 6]]$

$\therefore f(6)$ is maximum, where

$f(6) = 3(6)^3 - 18(6)^2 + 27(6) - 40 = 122$

132. We have,

$f(x) = x^2 + \dfrac{1}{x^2}$ and $g(x) = x - \dfrac{1}{x} \Rightarrow h(x) = \dfrac{f(x)}{g(x)}$

$\therefore \quad h(x) = \dfrac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \dfrac{\left(x - \frac{1}{x}\right)^2 + 2}{x - \frac{1}{x}}$

$\Rightarrow \quad h(x) = \left(x - \dfrac{1}{x}\right) + \dfrac{2}{x - \frac{1}{x}}$

$x - \dfrac{1}{x} > 0, \left(x - \dfrac{1}{x}\right) + \dfrac{2}{x - \frac{1}{x}} \in [2\sqrt{2}, \infty)$

$x - \dfrac{1}{x} < 0, \left(x - \dfrac{1}{x}\right) + \dfrac{2}{x - \frac{1}{x}} \in (-\infty, 2\sqrt{2}]$

$\therefore$ Local minimum value is $2\sqrt{2}$.

133. Since, the function have extreme values at $x = 1$ and $x = 2$.

$\therefore \quad f'(x) = 0$ at $x = 1$ and $x = 2$

$\Rightarrow \quad f'(1) = 0$ and $f'(2) = 0$

Also it is given that

$\lim\limits_{x \to 0} \left[1 + \dfrac{f(x)}{x^2}\right] = 3$

$\Rightarrow \quad 1 + \lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = 3 \Rightarrow \lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = 2$

$\Rightarrow \quad f(x)$ will be of the form

$ax^4 + bx^3 + 2x^2$

$[\because f(x)$ is of four degree polynomial$]$

Let $\quad f(x) = ax^4 + bx^3 + 2x^2$

$\Rightarrow \quad f'(x) = 4ax^3 + 3bx^2 + 4x$

$\Rightarrow \quad f'(1) = 4a + 3b + 4 = 0 \quad \ldots(i)$

and $\quad f'(2) = 32a + 12b + 8 = 0$

$\Rightarrow \quad 8a + 3b + 2 = 0 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$a = \dfrac{1}{2}, b = -2$

$\therefore \quad f(x) = \dfrac{x^4}{2} - 2x^3 + 2x^2$

$\Rightarrow \quad f(2) = 8 - 16 + 8 = 0$

134. Here, $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then

$f'(x) = \dfrac{\alpha}{x} + 2\beta x + 1$

$\therefore f'(-1) = -\alpha - 2\beta + 1 = 0 \quad \ldots(i)$

[at extreme point, $f'(x) = 0$]

and $\quad f'(2) = \dfrac{\alpha}{2} + 4\beta + 1 = 0 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$\alpha = 2, \beta = -\dfrac{1}{2}$

135. Here, $f(x) = \int_0^x \sqrt{t} \sin t \, dt$, where $x \in \left(0, \dfrac{5\pi}{2}\right)$

$f'(x) = \{\sqrt{x} \sin x - 0\} \quad \ldots(i)$

[using Newton-Leibnitz formula]

$\therefore \quad f'(x) = \sqrt{x} \sin x = 0 \Rightarrow \sin x = 0$

$\therefore \quad x = \pi, 2\pi$

$f''(x) = \sqrt{x} \cos x + \dfrac{1}{2\sqrt{x}} \sin x$

$f''(\pi) = -\sqrt{\pi} < 0$

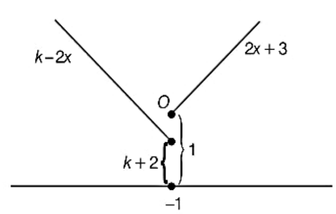
So, $f(x)$ has local maximum at $x = \pi$.

$f''(2\pi) = \sqrt{\pi} > 0$

So, $f(x)$ has local minimum at $x = 2\pi$.

136. $\because \quad k + 2 \le 1$

$\therefore \quad k \le -1$



137. Given, $y = x^{5/2}$

$\therefore \quad \frac{dy}{dx} = \frac{5}{2}x^{3/2}, \quad \frac{d^2y}{dx^2} = \frac{15}{4}x^{1/2}$

At $x=0$, $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}=0$ and $\frac{d^3y}{dx^3}$ is not defined,

when $x=0$, $y=0$

$\therefore (0, 0)$ is a point of inflection.

138. Let the numbers be x and y and $P = x^2y^5$, then $x + y = 35$

$\Rightarrow \quad x = 35 - y$

$\therefore \quad P = (35-y)^2 y^5$

On differentiating twice w.r.t. y, we get

$$\frac{dP}{dy} = (35-y)^2 5y^4 + y^5\, 2(35-y)(-1)$$
$$= y^4(35-y)[5(35-y)-2y]$$
$$= y^4(35-y)(175-5y-2y)$$
$$= y^4(35-y)(175-7y) = (35y^4 - y^5)(175-7y)$$

and $\frac{d^2P}{dy^2} = (35y^4 - y^5)(-7) + (175-7y)(4\times 35\times y^3 - 5y^4)$

$$= -7y^4(35-y) + 7(25-y)\times 5y^3(28-y)$$
$$= -7y^4(35-y) + 35y^3(25-y)(28-y)$$

For maxima put $\frac{dP}{dy} = 0$

$\Rightarrow \quad y^4(35-y)(175-7y) = 0 \Rightarrow y = 0, 35 - y = 0,$

$175 - 7y = 0 \Rightarrow y = 0, y = 25, y = 35$

When $y = 0$, $x = 35 - 0 = 35$ and the product x^2y^5 will be 0.

When $y = 35$ and $x = 35 - 35 = 0$. This will make the product x^2y^5 equal to 0.

$\therefore$ $y = 0$ and $y = 35$ cannot be the possible value of y.

When $y = 25$,

$$\left(\frac{d^2P}{dy^2}\right)_{y=25} = -7\times(25)^4\times(35-25) + 35\times(25)^3 \times (25-25)(28-25)$$
$$= -7\times 390625\times 10 + 35\times 15625\times 0\times 3$$
$$= -27343750 + 0$$
$$= -27343750 < 0$$

$\therefore$ By second derivative test, P will be the maximum when $y = 25$ and $x = 35 - 25 = 10$.

Hence, the required numbers are 10 and 25.

139. Let one number is x. Then, the other number will be $(16 - x)$.

Let the sum of the cubes of these numbers be denoted by S.

Then, $\quad S = x^3 + (16-x)^3$

On differentiating w.r.t. x, we get

$$\frac{dS}{dx} = 3x^2 + 3(16-x)^2(-1) = 3x^2 - 3(16-x)^2$$

$\Rightarrow \quad \frac{d^2S}{dx^2} = 6x + 6(16-x) = 96$

For minima put $\frac{dS}{dx} = 0$

$\Rightarrow \quad 3x^2 - 3(16-x)^2 = 0$

$\Rightarrow \quad x^2 - (256 + x^2 - 32x) = 0$

$\Rightarrow \quad 32x = 256 \Rightarrow x = 8$

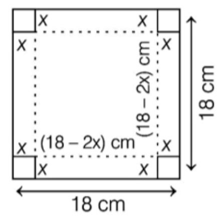
At $x = 8$, $\quad \left(\frac{d^2S}{dx^2}\right)_{x=8} = 96 > 0$

$\therefore$ By second derivative test, $x = 8$ is the point of local minima of S.

Thus, the sum of the cubes of the numbers is the minimum when the numbers are 8 and $16 - 8 = 8$.

Hence, the required numbers are 8 and 8.

140. First of all, draw the figure of square piece. Establish a relation between volume V and the side of the box formed, differentiate it. Put $\frac{dV}{dx} = 0$. Find the critical points and apply the second derivative test to find the required side of the square.



Let the side of the square to be cut-off be x cm $(0 < x < 9)$. Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is x cm.

Let V the volume of the open box formed by folding up the flaps, then

$$V = x(18-2x)(18-2x)$$
$$= 4x(9-x)^2 = 4x(81 + x^2 - 18x)$$
$$= 4(x^3 - 18x^2 + 81x)$$

On differentiating twice w.r.t. x, we get

$$\frac{dV}{dx} = 4(3x^2 - 36x + 81) = 12(x^2 - 12x + 27)$$

and $\quad \frac{d^2V}{dx^2} = 12(2x - 12) = 24(x - 6)$

For maxima, put $\frac{dV}{dx} = 0 \Rightarrow 12(x^2 - 12x + 27) = 0$

$\Rightarrow \quad x^2 - 12x + 27 = 0$

$\Rightarrow \quad (x-3)(x-9) = 0 \Rightarrow x = 3, 9$

But $x = 9$ is not possible.

$\because \quad 2x = 2\times 9 = 18$

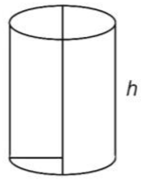
which is equal to the side of square piece.

At $x = 3$, $\quad \left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3-6) = -72 < 0$

$\therefore$ By second derivative test, $x = 3$ is the point of maxima.

Hence, if we cut-off the side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

141. Hence, we have two independent variable r and h, so we eliminate one variable. For this, find the value of h in terms of r and S and put in volume, then use the second derivative test. Let r and h be the radius and height of the cylinder respectively.



Then, the surface area S of the cylinder is

$$S = 2\pi r^2 + 2\pi rh \quad \text{(given)}$$

$$\Rightarrow \quad 2\pi rh = S - 2\pi r^2 \Rightarrow h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{...(i)}$$

Also, $$V = \pi r^2 h \quad \text{...(ii)}$$

On putting the value of h from Eq. (i) in Eq. (ii), we get

$$V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{Sr}{2} - \pi r^3 \quad \text{...(iii)}$$

On differentiating Eq. (iii) w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$$

For maxima or minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow \quad \frac{S}{2} - 3\pi r^2 = 0 \Rightarrow S = 6\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi} \quad \text{...(iv)}$$

Now, $$\frac{d^2V}{dr^2} = -6\pi r$$

At $r^2 = \frac{S}{6\pi}$, $$\left(\frac{d^2V}{dr^2}\right)_{r=\sqrt{\frac{S}{6\pi}}} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0$$

By second derivative test, the volume is maximum when $r^2 = \frac{S}{6\pi}$.

when $$r^2 = \frac{S}{6\pi} \text{ or } S = 6\pi r^2$$

Then, $$h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r$$

Hence, the volume is maximum when the height is twice the radius i.e. when the height is equal to the diameter.

142. Let r be the radius of the base, h be the height, V be the volume and S be the curved surface area of the cone.

Then, $$V = \frac{1}{3}\pi r^2 h \Rightarrow 3V = \pi r^2 h$$

$$\Rightarrow \quad 9V^2 = \pi^2 r^4 h^2 \Rightarrow h^2 = \frac{9V^2}{\pi^2 r^4} \quad \text{...(i)}$$

and $$S = \pi r l \Rightarrow S = \pi r\sqrt{r^2 + h^2}$$

$$[\because l = \sqrt{h^2 + r^2}]$$

$$\Rightarrow \quad S^2 = \pi^2 r^2 (r^2 + h^2) = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$$

[using Eq. (i)]

$$\Rightarrow \quad S^2 = \frac{9V^2}{r^2} + \pi^2 r^4 \quad \text{...(ii)}$$

when S is least, S^2 is also least.

Now, $$\frac{d}{dr}(S^2) = -\frac{18V^2}{r^3} + 4\pi^2 r^3 \quad \text{...(iii)}$$

For minima, put $$\frac{d}{dr}(S^2) = 0$$

$$\Rightarrow \quad -\frac{18V^2}{r^3} + 4\pi^2 r^3 = 0 \Rightarrow 18V^2 = 4\pi^2 r^6$$

$$\Rightarrow \quad 9V^2 = 2\pi^2 r^6 \quad \text{...(iv)}$$

Again, differentiating Eq. (iii) w.r.t. r, we get

$$\frac{d^2}{dr^2}(S^2) = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

At $$9V^2 = 2\pi^2 r^6, \quad \frac{d^2}{dr^2}(S^2) = \frac{54}{r^4}\left(\frac{2\pi^2 r^6}{9}\right) + 12\pi^2 r^2$$

$$= \frac{12\pi^2 r^6}{r^4} + 12\pi^2 r^2 = 24\pi^2 r^2 > 0$$

Hence, S^2 and therefore S is minimum when $9V^2 = 2\pi^2 r^6$

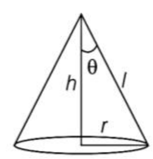
On putting $9V^2 = 2\pi^2 r^6$ in Eq. (i), we get

$$2\pi^2 r^6 = \pi^2 r^4 h^2 \Rightarrow 2r^2 = h^2 \Rightarrow h = \sqrt{2}r$$

Hence, altitude of right circular cone is $\sqrt{2}$ times the radius of the base.

143. Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left(0, \frac{\pi}{2}\right)$.



Let r, h and l be the radius, height and the slant height of the cone respectively.

The slant height of the cone is given *i.e.*, consider as constant.

Now, $$r = l\sin\theta$$

and $$h = l\cos\theta$$

Let V be the volume of the cone; $V = \frac{\pi}{3}r^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi(l^2\sin^2\theta)(l\cos\theta) = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

On differentiating w.r.t. θ, we get

$$\frac{dV}{d\theta} = \frac{l^3\pi}{3}[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)]$$

$$= \frac{l^3\pi}{3}(-\sin^3\theta + 2\sin\theta\cos^2\theta)$$

and $$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta)$$

$$= \frac{l^3\pi}{3}(2\cos^3\theta - 7\sin^2\theta\cos\theta)$$

For maxima, put $\dfrac{dV}{d\theta}=0$

$\Rightarrow \quad \sin^3\theta = 2\sin\theta\cos^2\theta$

$\Rightarrow \quad \tan^2\theta = 2$

$\Rightarrow \quad \tan\theta = \sqrt{2}$

$\Rightarrow \quad \theta = \tan^{-1}\sqrt{2}$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then

$$\tan^2\theta = 2 \text{ or } \sin^2\theta = 2\cos^2\theta$$

Then, we have

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 14\cos^3\theta)$$

$$= -4\pi l^3\cos^3\theta < 0 \text{ for } \theta \in \left(0, \frac{\pi}{2}\right)$$

$\therefore$ By second derivative test, the volume V is maximum when $\theta = \tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1}\sqrt{2}$.

144. With usual notation, given that total surface area

$$S = \pi r l + \pi r^2$$

$\Rightarrow \quad S = \pi r\sqrt{r^2+h^2} + \pi r^2 \quad \left[\because l = \sqrt{r^2+h^2}\right]$

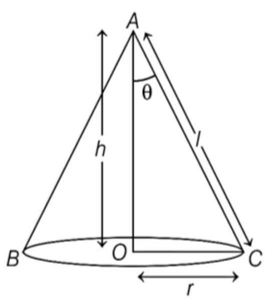
$\Rightarrow \quad \dfrac{S}{\pi r} - r = \sqrt{r^2+h^2} \Rightarrow \dfrac{S^2}{\pi^2 r^2} - \dfrac{2S}{\pi} = h^2$

$\Rightarrow \quad h = \sqrt{\dfrac{S^2}{\pi^2r^2} - \dfrac{2S}{\pi}}, \quad \left(\because \dfrac{S^2}{\pi^2r^2} > \dfrac{2S}{\pi}\right) \quad \ldots(i)$

and volume $\quad V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi r^2\sqrt{\dfrac{S^2}{\pi^2r^2} - \dfrac{2S}{\pi}}$

$\Rightarrow \quad V = \dfrac{r}{3}\sqrt{S^2 - 2S\pi r^2}, \; r^2 < \dfrac{S}{2\pi}$

i.e. $\quad 0 < r < \sqrt{\dfrac{S}{2\pi}}$



Since, V is maximum, then V^2 is maximum.

Now, $\quad V^2 = \dfrac{S^2r^2}{9} - \dfrac{2S\pi r^4}{9}, 0 < r < \sqrt{\dfrac{S}{2\pi}}$

$\therefore \quad \dfrac{d}{dr}(V^2) = \dfrac{2rS^2}{9} - \dfrac{8S\pi r^3}{9}$

and $\quad \dfrac{d^2}{dr^2}(V^2) = \dfrac{2S^2}{9} - \dfrac{24S\pi r^2}{9}$

For maxima, put $\dfrac{dV}{dr} = 0$

$\Rightarrow \quad \dfrac{2rS^2}{9} - \dfrac{8S\pi r^3}{9} = 0 \Rightarrow r^2 = \dfrac{S}{4\pi} \Rightarrow r = \sqrt{\dfrac{S}{4\pi}}$

Here, $\quad \dfrac{d^2(V^2)}{dr^2} < 0 \quad$ for $\quad r = \sqrt{\dfrac{S}{4\pi}}$

So, V^2 and hence V is maximum, when $r = \sqrt{\dfrac{S}{4\pi}}$

From Eq. (i), $h = \sqrt{\dfrac{S^2}{\pi^2r^2} - \dfrac{2S}{\pi}} = \sqrt{\dfrac{S^2(4\pi)}{\pi^2 S} - \dfrac{2S}{\pi}} = \sqrt{\dfrac{2S}{\pi}}$

If θ is the semi-vertical angle of the cone when the volume is maximum, then

in right ΔAOC,

$$\sin\theta = \frac{r}{\sqrt{r^2+h^2}} = \frac{\sqrt{\dfrac{S}{4\pi}}}{\sqrt{\dfrac{S}{4\pi} + \dfrac{2S}{\pi}}} = \frac{1}{\sqrt{1+8}} \text{ i.e., } \theta = \sin^{-1}\left(\frac{1}{3}\right)$$

145. Firstly, consider any point (x, y) on the curve and use the formulas of distance between two points and adjust them in one variable and simplify it.

Let d be the distance of the point (x, y) on $x^2 = 2y$ from the point $(0, 5)$, then

$$d = \sqrt{(x-0)^2 + (y-5)^2} = \sqrt{x^2 + (y-5)^2} \quad \ldots(i)$$

$$= \sqrt{2y + (y-5)^2} \quad [\text{putting } x^2 = 2y]$$

$$= \sqrt{y^2 - 8y + 25} = \sqrt{y^2 - 8y + 4^2 + 9} = \sqrt{(y-4)^2 + 9}$$

d is least when $(y-4)^2 = 0$ i.e. when $y = 4$

When $y = 4$, then $x^2 = 2 \times 4 \Rightarrow x = \pm\sqrt{8} = \pm 2\sqrt{2}$

$\therefore$ The points $(2\sqrt{2}, 4)$ and $(-2\sqrt{2}, 4)$ on the given curve are nearest to the point $(0, 5)$. So, (a) is the correct option.

146. Let the equation of an ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then any point P on the ellipse is $(a\cos\theta, b\sin\theta)$.

From P, draw $PM \perp OX$ and produce it to meet the ellipse at Q, then APQ is an isosceles triangle, let S be its area, then

$$S = 2 \times \frac{1}{2} \times AM \times MP = (OA - OM) \times MP$$

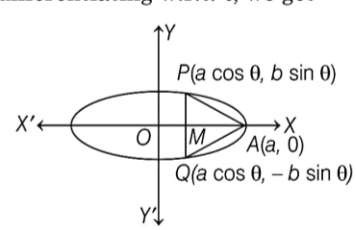
$$= (a - a\cos\theta) \cdot b\sin\theta$$

$\Rightarrow \quad S = ab(\sin\theta - \sin\theta\cos\theta) = ab\left(\sin\theta - \dfrac{1}{2}\sin 2\theta\right)$

On differentiating w.r.t. θ, we get

$$\frac{dS}{d\theta} = ab(\cos\theta - \cos 2\theta)$$

Again, differentiating w.r.t. θ, we get



$$\frac{d^2S}{d\theta^2} = ab\,(-\sin\theta + 2\sin 2\theta)$$

For maxima or minima, put $\frac{dS}{d\theta} = 0 \Rightarrow \cos\theta = \cos 2\theta$

$$\Rightarrow \quad 2\theta = 2\pi - \theta \Rightarrow \theta = \frac{2\pi}{3}$$

At $\theta = \frac{2\pi}{3}$, $\left(\frac{d^2S}{d\theta^2}\right)_{\theta=\frac{2\pi}{3}} = ab\left[-\sin\frac{2\pi}{3} + 2\sin\left(2\times\frac{2\pi}{3}\right)\right]$

$$= ab\left[-\sin\left(\pi - \frac{\pi}{3}\right) + 2\sin\left(\pi + \frac{\pi}{3}\right)\right]$$

$$= ab\left(-\sin\frac{\pi}{3} - 2\sin\frac{\pi}{3}\right) \quad \left[\because \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3},\ \sin\left(\pi + \frac{\pi}{3}\right) = \frac{-\sin\pi}{3}\right]$$

$$= ab\left(-\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right) = ab\left(\frac{-3\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}ab}{2} < 0$$

$\therefore$ S is maximum, when $\theta = \frac{2\pi}{3}$

and maximum value of

$$S = ab\left(\sin\frac{2\pi}{3} - \frac{1}{2}\cdot 2\sin\frac{2\pi}{3}\cos\frac{2\pi}{3}\right)$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

$$= ab\left[\sin\left(\pi - \frac{\pi}{3}\right) - \sin\left(\pi - \frac{\pi}{3}\right)\cos\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= ab\left[\sin\frac{\pi}{3} - \sin\frac{\pi}{3}\times\left(-\cos\frac{\pi}{3}\right)\right]$$

$$= ab\left(\sin\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\frac{\pi}{3}\right) = ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\times\frac{1}{2}\right)$$

$$= ab\left(\frac{2\sqrt{3} + \sqrt{3}}{4}\right) = \frac{3\sqrt{3}}{4}\,ab \text{ sq unit}$$

Thus, maximum area of isosceles triangle is $\frac{3\sqrt{3}}{4}\,ab$ sq unit.

147. Let x be the side of square and r be the radius of circle.

Perimeter of a circle $= 2\pi r$ and perimeter of a square $= 4x$

Given, $2\pi r + 4x = k \Rightarrow x = \frac{k - 2\pi r}{4}$ ...(i)

$$\therefore \quad A = x^2 + \pi r^2 = \left[\frac{k - 2\pi r}{4}\right]^2 + \pi r^2$$

$$= \left(\frac{1}{16}\right)(k^2 - 4k\pi r + 4\pi^2 r^2) + \pi r^2$$

On differentiating w.r.t. r, we get

$$\frac{dA}{dr} = \left(\frac{1}{16}\right)(-4k\pi + 8\pi^2 r) + 2\pi r$$

Again, differentiating w.r. t. r, we get

$$\frac{d^2A}{dr^2} = \frac{1}{16}[0 + 8\pi^2] + 2\pi = 2\pi + \frac{\pi^2}{2} > 0$$

For maximum or minimum, put $\frac{dA}{dr} = 0$

$$\Rightarrow \quad 2\pi r - \frac{4k\pi}{16} + \frac{8\pi^2 r}{16} = 0 \Rightarrow r\left(2\pi + \frac{\pi^2}{2}\right) = \frac{k\pi}{4}$$

$$\Rightarrow \quad r = \frac{\left(\frac{k\pi}{4}\right)}{2\pi + \frac{\pi^2}{2}} = \frac{k}{8 + 2\pi} \quad ...(ii)$$

Now, $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{k}{8+2\pi}}$ = positive

$\therefore$ A is least, when $r = \frac{k}{8 + 2\pi}$ and put this value in Eq. (i), we get

$$x = \frac{k - 2\pi r}{4} = \frac{1}{4}\left(k - 2\pi\times\frac{k}{8 + 2\pi}\right) = \frac{1}{4}\left[\frac{8k + 2\pi k - 2\pi k}{8 + 2\pi}\right]$$

$$= \frac{2k}{8 + 2\pi} = 2\left(\frac{k}{8 + 2\pi}\right) = 2r \quad \text{[using Eq. (ii)]}$$

Hence, S is least when side of the square is double the radius of the circle.

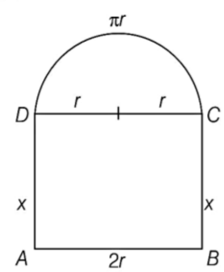
148. Let radius of semi-circle $= r$

$\therefore$ One side of rectangle $= 2r$

Let other side $= x$

$\therefore$ P = Perimeter = 10 (given)

$$\Rightarrow \quad 2x + 2r + \frac{1}{2}(2\pi r) = 10 \Rightarrow 2x = 10 - r(\pi + 2) \quad ...(i)$$



Let A be area of the figure, then

$$A = \text{Area of semi-circle} + \text{Area of rectangle}$$

$$= \frac{1}{2}\pi r^2 + 2rx$$

$$\Rightarrow \quad A = \frac{1}{2}(\pi r^2) + r[10 - r(\pi + 2)] \quad \text{[using Eq. (i)]}$$

$$= \frac{1}{2}(\pi r^2) + 10r - r^2\pi - 2r^2 = 10r - \frac{\pi r^2}{2} - 2r^2$$

On differentiating twice w.r.t. r, we get

$$\frac{dA}{dr} = 10 - \pi r - 4r \quad ...(ii)$$

and $\frac{d^2A}{dr^2} = -\pi - 4$ ...(iii)

For maxima or minima, put $\frac{dA}{dr} = 0 \Rightarrow 10 - \pi r - 4r = 0$

$$\Rightarrow \quad 10 = (4 + \pi)\,r \Rightarrow r = \frac{10}{4 + \pi}$$

On putting $r = \frac{10}{4 + \pi}$ in Eq. (iii), we get $\frac{d^2A}{dr^2}$ = negative

Thus A has local maximum when $r = \frac{10}{4 + \pi}$ ...(iv)

$\therefore$ Radius of semi-circle $= \dfrac{10}{4+\pi}$

and one side of rectangle $= 2r = \dfrac{2 \times 10}{4+\pi} = \dfrac{20}{4+\pi}$

and other side of rectangle i.e. x from Eq. (i) is given by

$$x = \frac{1}{2}[10 - r(\pi+2)] = \frac{1}{2}\left[10 - \left(\frac{10}{\pi+4}\right)(\pi+2)\right]$$ [from Eq. (iv)]

$$= \frac{10\pi + 40 - 10\pi - 20}{2(\pi+4)} = \frac{20}{2(\pi+4)} = \frac{10}{\pi+4}$$

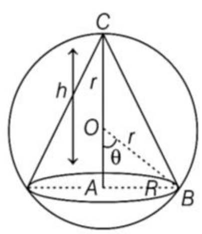
Light is maximum when area is maximum.

So, dimensions of the window are length

$$= 2r = \frac{20}{\pi+4}, \text{ breadth} = x = \frac{10}{\pi+4}.$$

149. Let R be the radius and h be the height of cone.

$\therefore \quad OA = h - r$



In ΔOAB, $r^2 = R^2 + (h-r)^2 \Rightarrow r^2 = R^2 + h^2 + r^2 - 2rh$

$\Rightarrow \quad R^2 = 2rh - h^2$

The volume V of the cone is given by

$$V = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi h(2rh - h^2) = \frac{1}{3}\pi(2rh^2 - h^3)$$

On differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \frac{1}{3}\pi(4rh - 3h^2)$$

For maximum or minimum, put $\dfrac{dV}{dh} = 0$

$\Rightarrow \quad 4rh = 3h^2 \quad \Rightarrow \quad 4r = 3h$

$\therefore \quad h = \dfrac{4r}{3}$ $\quad [h \neq 0]$

Now, $\quad \dfrac{d^2V}{dh^2} = \dfrac{1}{3}\pi(4r - 6h)$

At $h = \dfrac{4r}{3}$, $\quad \left(\dfrac{d^2V}{dh^2}\right)_{h=\frac{4r}{3}} = \dfrac{1}{3}\pi\left(4r - 6 \times \dfrac{4r}{3}\right)$

$$= \frac{\pi}{3}(4r - 8r)$$

$$= \frac{-4r\pi}{3} < 0$$

$\Rightarrow V$ is maximum when $h = \dfrac{4r}{3}$.

Hence, volume of the cone is maximum when $h = \dfrac{4r}{3}$, which is the altitude of cone

150. Draw the diagram of a cylinder of height h and radius r inscribed in a sphere of radius R. Now, express volume V in terms of R and h and then apply second derivative test to prove the required results.

Radius of the sphere $= R$

Let h be the height and x be the diameter of the base of the inscribed cylinder. Then,

$$h^2 + x^2 = (2R)^2 \Rightarrow h^2 + x^2 = 4R^2 \quad \text{...(i)}$$

Volume of the cylinder $= \pi\,(\text{radius})^2 \times \text{height}$

$$\Rightarrow V = \pi\left(\frac{x}{2}\right)^2 \cdot h = \frac{1}{4}\pi x^2 h \Rightarrow V = \frac{1}{4}\pi h(4R^2 - h^2) \quad \text{...(ii)}$$

[from Eq. (i), $x^2 = 4R^2 - h^2$]

$$\Rightarrow \quad V = \pi R^2 h - \frac{1}{4}\pi h^3$$

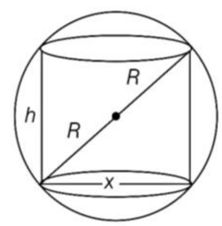
On differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \pi R^2 - \frac{3}{4}\pi h^2 = \pi\left(R^2 - \frac{3}{4}h^2\right)$$

Put $\quad \dfrac{dV}{dh} = 0 \Rightarrow R^2 = \dfrac{3}{4}h^2 \Rightarrow h = \dfrac{2R}{\sqrt{3}}$

Also, $\quad \dfrac{d^2V}{dh^2} = -\dfrac{3}{4} \times 2\pi h$

At $h = \dfrac{2R}{\sqrt{3}}$, $\dfrac{d^2V}{dh^2} = -\dfrac{3}{4} \times 2\pi\left(\dfrac{2R}{\sqrt{3}}\right) = -\sqrt{3}\pi R = -$ ve



$\Rightarrow V$ is maximum at $h = \dfrac{2R}{\sqrt{3}}$

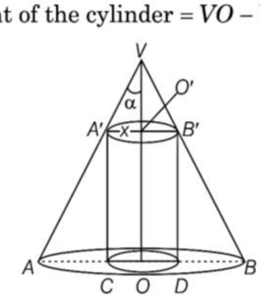
Maximum volume at $h = \dfrac{2R}{\sqrt{3}}$ is

$$V = \frac{1}{4}\pi\left(\frac{2R}{\sqrt{3}}\right)\left(4R^2 - \frac{4R^2}{3}\right) \quad \text{[using Eq. (ii)]}$$

$$= \frac{\pi R}{2\sqrt{3}}\left(\frac{8R^2}{3}\right) = \frac{4\pi R^3}{3\sqrt{3}} \text{ sq unit}$$

Thus, volume of the cylinder is maximum when $h = \dfrac{2R}{\sqrt{3}}$

151. Let VAB be the cone of height h, semi-vertical angle α and let x be the radius of the base of the cylinder $A'B'DC$ which is inscribed in the cone VAB. Then, OO' is the height of the cylinder $= VO - VO' = h - x\cot\alpha$



Volume of the cylinder, $V = \pi x^2 (h - x\cot\alpha)$ ...(i)

On differentiating w.r.t. x, we get

$$\frac{dV}{dx} = 2\pi xh - 3\pi x^2 \cot\alpha$$

For maxima or minima, put $\frac{dV}{dx} = 0$

$\Rightarrow$ $2\pi xh - 3\pi x^2 \cot\alpha = 0$

$\Rightarrow$ $x = \frac{2h}{3}\tan\alpha$ $[\because x \neq 0]$

Now, $\frac{d^2V}{dx^2} = 2\pi h - 6\pi x \cot\alpha$

At $x = \frac{2h}{3}\tan\alpha$, $\frac{d^2V}{dx^2} = \pi(2h - 4h) = -2\pi h < 0$

$\Rightarrow V$ is maximum, when $x = \frac{2h}{3}\tan\alpha$.

Now, $OO' = h - x\cot\alpha = h - \frac{2h}{3} = \frac{h}{3}$

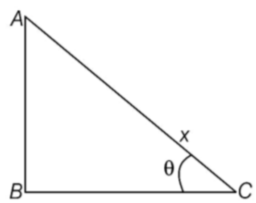
$\therefore$ The maximum volume of the cylinder is

$$V = \pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(h - \frac{2h}{3}\right) = \frac{4}{27}\pi h^3 \tan^2\alpha$$

152. Given, $AC + BC = \text{constant} = k$ ...(i)

Let $\angle ACB = \theta$

and $AC = x$, then $BC = x\cos\theta$ and $AB = x\sin\theta$



Let y be the area of ΔABC.

Then, $y = \frac{1}{2} BC \cdot AB = \frac{1}{2} x\cos\theta \cdot x\sin\theta$

$= \frac{1}{2} x^2 \sin\theta\cos\theta$ ...(ii)

From Eq. (i), $x + x\cos\theta = k \Rightarrow x = \frac{k}{1+\cos\theta}$ ...(iii)

On putting the value of x in Eq. (ii), we get

$$y = \frac{k^2}{2} \cdot \frac{\sin\theta\cos\theta}{(1+\cos\theta)^2} \quad \text{...(iv)}$$

On differentiating w.r.t. θ, we get

$$\frac{dy}{dx} = \frac{k^2}{2} \frac{(1+\cos\theta)^2(\cos^2\theta - \sin^2\theta) - \sin\theta\cos\theta \cdot 2(1+\cos\theta)(-\sin\theta)}{(1+\cos\theta)^4}$$

$$= \frac{k^2}{2} \frac{(1+\cos\theta)[(1+\cos\theta)(\cos^2\theta - \sin^2\theta) + 2\sin^2\theta\cos\theta]}{(1+\cos\theta)^4}$$

$$= \frac{k^2}{2} \frac{\cos^2\theta - \sin^2\theta + \cos^3\theta - \cos\theta\sin^2\theta + 2\cos\theta\sin^2\theta}{(1+\cos\theta)^3}$$

$$= \frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta - 1 + \cos^3\theta + \cos\theta\sin^2\theta)$$

$[\because \sin^2\theta = 1 - \cos^2\theta]$

$$= \frac{k^2}{2(1+\cos\theta)^3}[2\cos^2\theta - 1 + \cos\theta(\cos^2\theta + \sin^2\theta)]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1) \quad \text{...(v)}$$

$[\because \cos^2\theta + \sin^2\theta = 1]$

Since, $0 < \theta < \frac{\pi}{2}$

$\therefore$ $\frac{k^2}{2(1+\cos\theta)^3} > 0$

Therefore, sign scheme for $\frac{dy}{dx}$ will be same as that of $2\cos^2\theta + \cos\theta - 1$.

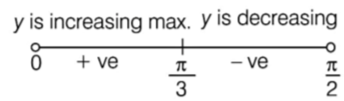
Now, $2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow$ $(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow$ $\cos\theta = \frac{1}{2}$ $[\because \cos\theta \neq -1]$

$\Rightarrow$ $\theta = \frac{\pi}{3}$ $\left[\because 0 < \theta < \frac{\pi}{2}\right]$

Since, sign scheme for $\frac{dy}{dx}$ i.e. for $(2\cos^2\theta + \cos\theta - 1)$ is



Thus, y has maximum value when $\theta = \frac{\pi}{3}$.

153. Let $f(x) = -x^3 + 3x^2 + 9x - 27$

The slope of this curve $f'(x) = -3x^2 + 6x + 9$

Let $g(x) = f'(x) = -3x^2 + 6x + 9$

On differentiating w.r.t. x, we get

$$g'(x) = -6x + 6$$

For maxima or minima put $g'(x) = 0 \Rightarrow x = 1$

Now, $g''(x) = -6 < 0$ and hence, at $x = 1$, $g(x)$ (slope) will have maximum value.

$\therefore$ $[g(1)]_{max} = -3 \times 1 + 6(1) + 9 = 12$

154. Let $PQ = a$ and $PR = b$, then $\Delta = \frac{1}{2} ab\sin\theta$

$\because$ $-1 \leq \sin\theta \leq 1$

$\therefore$ Area is maximum when $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

155. Given,

$ab = 2a + 3b \Rightarrow (a-3)b = 2a \Rightarrow b = \frac{2a}{a-3}$

Now, let $z = ab = \frac{2a^2}{a-3}$

On differentiating w.r.t. x, we get

$$\frac{dz}{da} = \frac{2[(a-3)2a - a^2]}{(a-3)^2} = \frac{2[a^2 - 6a]}{(a-3)^2}$$

For a minimum, put $\frac{dz}{da} = 0$

$\Rightarrow \quad a^2 - 6a = 0 \Rightarrow a = 0, 6$

At $a = 6$, $\quad \dfrac{d^2z}{da^2} = \text{positive}$

When $\quad a = 6, \quad b = 4$

$\therefore \quad (ab)_{\min} = 6 \times 4 = 24$

156. Let radius vector is r.

$\therefore \quad r^2 = x^2 + y^2$

$\Rightarrow \quad r^2 = \dfrac{a^2y^2}{y^2 - b^2} + y^2 \qquad \left[\because \dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1\right]$

For minimum value of r,

$$\frac{d(r^2)}{dy} = 0$$

$\Rightarrow \quad \dfrac{-2yb^2a^2}{(y^2 - b^2)^2} + 2y = 0$

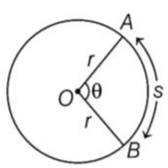
$\Rightarrow \quad y^2 = b(a + b)$

$\therefore \quad x^2 = a(a + b)$

$\Rightarrow \quad r^2 = (a + b)^2$

$\Rightarrow \quad r = a + b$

157. $\because$ Perimeter of a sector $= p$



Let AOB be the sector with radius r.

If angle of the sector be θ radians, then area of sector,

$$A = \frac{1}{2}r^2\theta \qquad \ldots(i)$$

and length of arc, $\quad s = r\theta \Rightarrow \theta = \dfrac{s}{r}$

$\therefore$ Perimeter of the sector

$$p = r + s + r = 2r + s \qquad \ldots(ii)$$

On substituting $\theta = \dfrac{s}{r}$ in Eq. (i), we get

$$A = \left(\frac{1}{2}r^2\right)\left(\frac{s}{r}\right) = \frac{1}{2}rs$$

$\Rightarrow \quad s = \dfrac{2A}{r}$

Now, on substituting the value of s in Eq. (ii), we get

$$p = 2r + \left(\frac{2A}{r}\right) \Rightarrow 2A = pr - 2r^2$$

On differentiating w.r.t. r, we get

$$2\frac{dA}{dr} = p - 4r$$

For the maximum area, put

$$\frac{dA}{dr} = 0 \Rightarrow p - 4r = 0 \Rightarrow r = \frac{p}{4}$$

158. Given curve is $\quad xy = c^2 \Rightarrow y = \dfrac{c^2}{x}$

Let $\quad f(x) = ax + by = ax + \dfrac{bc^2}{x}$

On differentiating w.r.t. x, we get

$$f'(x) = a - \frac{bc^2}{x^2}$$

For a maxima or minima, put $f'(x) = 0$

$\Rightarrow \quad ax^2 - bc^2 = 0$

$\Rightarrow \quad x^2 = \dfrac{bc^2}{a} \Rightarrow x = \pm c\sqrt{\dfrac{b}{a}}$

Again, on differentiating w.r.t. x, we get

$$f''(x) = \frac{2bc^2}{x^3}$$

At $x = c\sqrt{\dfrac{b}{a}}, \qquad f''(x) > 0$

$\therefore f(x)$ is minimum at $x = c\sqrt{\dfrac{b}{a}}$.

The minimum value at $x = c\sqrt{\dfrac{b}{a}}$ is

$$\therefore \quad f\left(c\sqrt{\frac{b}{a}}\right) = a \cdot c\sqrt{\frac{b}{a}} + \frac{bc^2}{c} \cdot \sqrt{\frac{a}{b}}$$

$$= \frac{abc + abc}{\sqrt{ab}} = \frac{2abc}{\sqrt{ab}} = 2c\sqrt{ab}$$

159. Given, $\quad a^2x^4 + b^2y^4 = c^6$

$\Rightarrow \quad y = \left(\dfrac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

and let $\quad f(x) = xy = x\left(\dfrac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

$\Rightarrow \quad f(x) = \left(\dfrac{c^6x^4 - a^2x^8}{b^2}\right)^{1/4}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{4}\left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{-3/4}\left(\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2}\right)$$

For maxima or minima, put $f'(x) = 0$

$\Rightarrow \quad \dfrac{4x^3c^6}{b^2} - \dfrac{8x^7a^2}{b^2} = 0$

$\Rightarrow \quad \dfrac{4x^3}{b^2}(c^6 - 2a^2x^4) = 0$

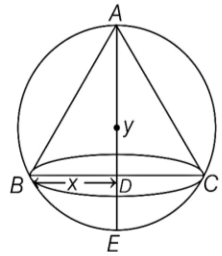
$\Rightarrow \quad x^4 = \dfrac{c^6}{2a^2} \Rightarrow \pm\dfrac{c^{3/2}}{2^{1/4}\sqrt{a}}$

At $x = \dfrac{c^{3/2}}{2^{1/4}\sqrt{a}}$, $f(x)$ will be maximum.

$$\therefore \quad f\left(\frac{c^{3/2}}{2^{1/4}\sqrt{a}}\right) = \left(\frac{c^{12}}{2a^2b^2} - \frac{c^{12}}{4a^2b^2}\right)^{1/4}$$

$$= \left(\frac{c^{12}}{4a^2b^2}\right)^{1/4} = \frac{c^3}{\sqrt{2ab}}$$

160. Let the diameter of the sphere is $AE = 2r$.



Let the radius of cone is x and height is y.

$\therefore \quad AD = y$

Since, $\quad BD^2 = AD \cdot DE$

$\Rightarrow \quad x^2 = y(2r - y) \quad \ldots$(i)

Volume of cone, $V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi y(2r - y)y$

$$= \frac{1}{3}\pi(2ry^2 - y^3)$$

On differentiating w.r.t. y, we get

$$\frac{dV}{dy} = \frac{1}{3}\pi(4ry - 3y^2)$$

For maxima and minima, put $\frac{dV}{dy} = 0$

$\Rightarrow \quad \frac{1}{3}\pi(4ry - 3y^2) = 0$

$\Rightarrow \quad y(4r - 3y) = 0$

$\Rightarrow \quad y = \frac{4}{3}r, 0$

Again, on differentiating w.r.t. y, we get

$$\frac{d^2V}{dy^2} = \frac{1}{3}\pi(4r - 6y)$$

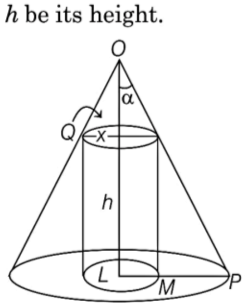
At $y = \frac{4}{3}r$, $\quad \frac{d^2V}{dy^2} = \frac{1}{3}\pi(4r - 8r) =$ negative

$\therefore$ Volume of cone is maximum at $y = \frac{4}{3}r$.

Now, required ratio $= \frac{\text{Height of cone}}{\text{Diameter of sphere}}$

$$= \frac{y}{2r} = \frac{\frac{4r}{3}}{2r} = \frac{2}{3}$$

161. Let H be the height of the cone and α be its semi-vertical angle. Suppose that x is the radius of the inscribed cylinder and h be its height.



$\therefore \quad h = QL = OL - OQ = H - x\cot\alpha$

$V =$ Volume of the cylinder

$= \pi x^2(H - x\cot\alpha)$

Also, $\quad p = \frac{1}{3}\pi(H\tan\alpha)^2 H \quad \ldots$(i)

$$\frac{dV}{dx} = \pi(2Hx - 3x^2\cot\alpha)$$

$\therefore \quad \frac{dV}{dx} = 0 \Rightarrow x = 0,$

$$x = \frac{2}{3}H\tan\alpha$$

$\Rightarrow \quad \left[\frac{d^2V}{dx^2}\right]_{x = \frac{2}{3}H\tan\alpha} = -2\pi H < 0$

$\therefore V$ is maximum when $x = \frac{2}{3}H\tan\alpha$

and $q = V_{\max} = \pi\frac{4}{9}H^2\tan^2\alpha\frac{1}{3}H = \frac{4}{9}p \quad$ [from Eq. (i)]

Hence, $\quad p : q = 9 : 4$

162. $\frac{dy}{dx} = a^2 - 3x^2 = 0 \Leftrightarrow x = \pm\frac{a}{\sqrt{3}}$

Since, $\frac{d^2y}{dx^2} = -6x$, so y is minimum for $x = -\frac{a}{\sqrt{3}}$.

Since, $x^2 + x + 2 > 0$ for all x, so for $\frac{x^2 + x + 2}{x^2 + 5x + 6} \le 0$,

we must have $x^2 + 5x + 6 < 0$.

If $x = -\frac{a}{\sqrt{3}}$, we have

$$\frac{a^2}{3} - \frac{5a}{\sqrt{3}} + 6 < 0$$

$\Rightarrow \quad a^2 - 5\sqrt{3}a + 18 < 0$

$\Rightarrow \quad (a - 2\sqrt{3})(a - 3\sqrt{3}) < 0$

If $\quad a > 0, a \in (2\sqrt{3}, 3\sqrt{3})$

If $\quad a < 0, a \in (-3\sqrt{3}, -2\sqrt{3})$

$\Rightarrow \quad a \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$

163. $\because \quad y = x^3 - 6x^2 + 9x + 4$

Now, $\frac{dy}{dx} = 3x^2 - 12x + 9$

Let $\quad u = \frac{dy}{dx} = 3x^2 - 12x + 9$

Now, $\frac{du}{dx} = 6x - 12$

Put $\frac{du}{dx} = 0$ for maximum or minimum

$\therefore \quad 6x - 12 = 0 \Rightarrow x = 2$

Now, at $x = 0$, $\quad u = 9$

At $x = 2$, $\quad u = -3$

and at $x = 5$, $\quad u = 24$

Thus, the maximum of $u(x)$, $0 \le x \le 5$ is $u(5)$.

Hence, $\quad x = 5$

164. As the point $P(h, k)$ is the nearest point on the curve $y = x^2 + 7x + 2$, to the line $y = 3x - 3$.

So, the tangent to the parabola $y = x^2 + 7x + 7$ at point $P(h, k)$ is parallel to the line $y = 3x - 3$

$$\therefore \quad \left.\frac{dy}{dx}\right|_p = 2h + 7 = 3 \Rightarrow h = -2 \quad \ldots \text{(i)}$$

and the point $p(h, k)$ on the curve, so

$$k = h^2 + 7h + 2 = (-2)^2 + 7(-2) + 2$$

$$\Rightarrow \quad k = 4 - 14 + 2 \Rightarrow k = -8$$

$\therefore$ Point $P(-2, -8)$

Now, equation of normal to the parabola $y = x^2 + 7x + 7$ at point $P(-2, -8)$ is

$$y + 8 = \frac{-1}{\left.\frac{dy}{dx}\right|_p}(x + 2)$$

$$\Rightarrow \quad y + 8 = -\frac{1}{3}(x + 2) \Rightarrow x + 3y + 26 = 0$$

165. Let a sphere of radius 3, which inscribed a right circular cylinder having radius r and height is h, so

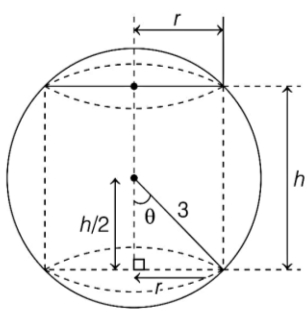
From the figure, $\frac{h}{2} = 3\cos\theta \Rightarrow h = 6\cos\theta$

and $\quad r = 3\sin\theta \quad \ldots$(i)

$\because$ Volume of cylinder $V = \pi r^2 h$

$$= \pi(3\sin\theta)^2(6\cos\theta)$$
$$= 54\pi\sin^2\theta\cos\theta$$

For maxima or minima, $\frac{dV}{d\theta} = 0$



$$\Rightarrow 54\pi[2\sin\theta\cos^2\theta - \sin^3\theta] = 0$$

$$\Rightarrow \quad \sin\theta[2\cos^2\theta - \sin^2\theta] = 0$$

$$\Rightarrow \quad \tan^2\theta = 2 \quad \left[\because \theta \in \left(0, \frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad \tan\theta = \sqrt{2}$$

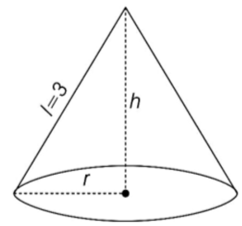
$$\Rightarrow \quad \sin\theta = \sqrt{\frac{2}{3}} \text{ and } \cos\theta = \frac{1}{\sqrt{3}} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$h = 6\frac{1}{\sqrt{3}} = 2\sqrt{3}$$

166. Let h = height of the cone, r = radius of circular base

$$= \sqrt{(3)^2 - h^2} \quad [\because l^2 = h^2 + r^2]$$
$$= \sqrt{9 - h^2} \quad \ldots\text{(i)}$$



Now, volume (V) of cone $= \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V(h) = \frac{1}{3}\pi(9 - h^2)h \quad \text{[from Eq. (i)]}$$
$$= \frac{1}{3}\pi[9h - h^3] \quad \ldots\text{(ii)}$$

For maximum volume $V'(h) = 0$ and $V''(h) < 0$.

Here, $\quad V'(h) = 0 \Rightarrow (9 - 3h^2) = 0$

$$\Rightarrow \quad h = \sqrt{3} \quad [\because h \not< 0]$$

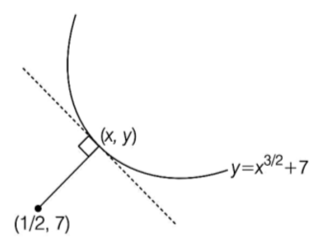
and $\quad V''(h) = \frac{1}{3}\pi(-6h) < 0$ for $h = \sqrt{3}$

Thus, volume is maximum when $h = \sqrt{3}$

Now, maximum volume

$$V(\sqrt{3}) = \frac{1}{3}\pi(9\sqrt{3} - 3\sqrt{3}) \quad \text{[from Eq. (ii)]}$$
$$= 2\sqrt{3}\pi$$

167. The helicopter is nearest to the soldier, if the tangent to the path, $y = x^{3/2} + 7$, $(x \geq 0)$ of helicopter at point (x, y) is perpendicular to the line joining (x, y) and the position of soldier $\left(\frac{1}{2}, 7\right)$.



$\because$ Slope of tangent at point (x, y) is

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} = m_1 \text{ (let)} \quad \ldots\text{(i)}$$

and slope of line joining (x, y) and $\left(\frac{1}{2}, 7\right)$ is

$$m_2 = \frac{y - 7}{x - \frac{1}{2}} \quad \ldots\text{(ii)}$$

Now, $\quad m_1 \cdot m_2 = -1$

$$\Rightarrow \quad \frac{3}{2}x^{1/2}\left(\frac{y - 7}{x - (1/2)}\right) = -1 \quad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad \frac{3}{2}x^{1/2}\frac{x^{3/2}}{x - \frac{1}{2}} = -1 \quad [\because y = x^{3/2} + 7]$$

$$\Rightarrow \quad \frac{3}{2}x^2 = -x + \frac{1}{2}$$

$$\Rightarrow \quad 3x^2 + 2x - 1 = 0$$

$\Rightarrow \quad 3x^2 + 3x - x - 1 = 0$

$\Rightarrow \quad 3x(x+1) - 1(x+1) = 0$

$\Rightarrow \quad x = \frac{1}{3}, -1$

$\because \quad x \geq 0$

$\therefore \quad x = \frac{1}{3}$

and so, $y = \left(\frac{1}{3}\right)^{3/2} + 7 \qquad [\because y = x^{3/2} + 7]$

Thus, the nearest point is $\left(\frac{1}{3}, \left(\frac{1}{3}\right)^{3/2} + 7\right)$

Now, the nearest distance

$$= \sqrt{\left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(7 - \left(\frac{1}{3}\right)^{3/2} - 7\right)^2}$$

$$= \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^3}$$

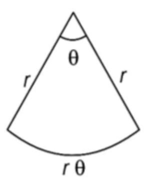
$$= \sqrt{\frac{1}{36} + \frac{1}{27}}$$

$$= \sqrt{\frac{3+4}{108}} = \sqrt{\frac{7}{108}}$$

$$= \frac{1}{6}\sqrt{\frac{7}{3}}$$

168. Total length $= 2r + r\theta = 20$

$\Rightarrow \quad \theta = \frac{20 - 2r}{r}$



Now, area of flower-bed,

$$A = \frac{1}{2} r^2 \theta$$

$\Rightarrow \quad A = \frac{1}{2} r^2 \left(\frac{20 - 2r}{r}\right)$

$\Rightarrow \quad A = 10r - r^2$

$\therefore \quad \frac{dA}{dr} = 10 - 2r$

For maxima or minima, put $\frac{dA}{dr} = 0$.

$\Rightarrow \quad 10 - 2r = 0$

$\Rightarrow \quad r = 5$

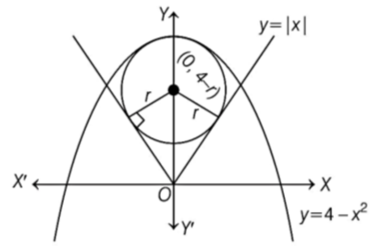
$\therefore \quad A_{max} = \frac{1}{2}(5)^2 \left[\frac{20 - 2(5)}{5}\right]$

$$= \frac{1}{2} \times 25 \times 2$$

$$= 25 \text{ sq m}$$

169. Let the radius of circle with least area be r.

Then, coordinates of centre $= (0, 4 - r)$.



Since, circle touches the line $y = x$ in first quadrant.

$\therefore \quad \left|\frac{0 - (4 - r)}{\sqrt{2}}\right| = r \Rightarrow r - 4 = \pm r\sqrt{2}$

$\Rightarrow \quad r = \frac{4}{\sqrt{2} + 1}$ or $\frac{4}{1 - \sqrt{2}}$

But $\quad r \neq \frac{4}{1 - \sqrt{2}} \qquad \left[\because \frac{4}{1 - \sqrt{2}} < 0\right]$

$\therefore \quad r = \frac{4}{\sqrt{2} + 1} = 4(\sqrt{2} - 1)$

170. According to given information, we have

Perimeter of square + Perimeter of circle = 2 units

$\Rightarrow \quad 4x + 2\pi r = 2$

$\Rightarrow \quad r = \frac{1 - 2x}{\pi} \qquad \text{...(i)}$

Now, let A be the sum of the areas of the square and the circle. Then,

$$A = x^2 + \pi r^2$$

$$= x^2 + \pi \frac{(1 - 2x)^2}{\pi^2}$$

$\Rightarrow \quad A(x) = x^2 + \frac{(1 - 2x)^2}{\pi}$

Now, for minimum value of $A(x)$,

$$\frac{dA}{dx} = 0$$

$\Rightarrow \quad 2x + \frac{2(1 - 2x)}{\pi} \cdot (-2) = 0$

$\Rightarrow \quad x = \frac{2 - 4x}{\pi}$

$\Rightarrow \quad \pi x + 4x = 2$

$\Rightarrow \quad x = \frac{2}{\pi + 4} \qquad \text{...(ii)}$

Now, from Eq. (i), we get

$$r = \frac{1 - 2 \cdot \frac{2}{\pi + 4}}{\pi}$$

$$= \frac{\pi + 4 - 4}{\pi(\pi + 4)} = \frac{1}{\pi + 4} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$x = 2r$$

Round II

1. Equation of line joining the points (0, 3) and (5, –2) is $y = 3 - x$. If this line is tangent to $y = \frac{ax}{(x+1)}$, then $(3 - x)(x + 1) = ax$ should have equal roots.

Thus, $(a-2)^2 + 12 = 0$

$\Rightarrow$ no value of $a \Rightarrow a \in \phi$.

2. We have, $f(x) = x^3 + bx^2 + cx + d$

$\Rightarrow \quad f'(x) = 3x^2 + 2bx + c$

Let D_1 be the discriminant of $f'(x) = 3x^2 + 2bx + c$.

Then, $\quad D_1 = 4b^2 - 12c = 4(b^2 - c) - 8c < 0$

$[\because b^2 < c \text{ and } c > 0]$

$\Rightarrow \quad f'(x) > 0$ for all $x \in (-\infty, \infty)$

$f(x)$ is strictly increasing function on $(-\infty, \infty)$.

3. $f'(x) = |\log_2[\log_3\{\log_4(\cos x + a)\}]|$

Clearly, $f(x)$ is increasing for all values of x, if

$\log_2[\log_3\{\log_4(\cos x + a)\}]$

is defined for all values of x.

$\Rightarrow \quad \log_3[\log_4(\cos x + a)] > 0, \forall x \in R$

$\Rightarrow \quad \log_4(\cos x + a) > 1, \forall x \in R$

$\Rightarrow \quad \cos x + a > 4, \forall x \in R$

$\Rightarrow \quad a > 5$

4. On solving, $y^2 = 4ax$ and $x^2 = 4by$, we get $x = 0$ or $x^3 = 64 - ab^2$. Slope of the curves at the common points are $\frac{2a}{y}$ and $\frac{x}{2b}$, respectively.

If these parabola intersect orthogonally, then

$$\frac{2a}{y} \cdot \frac{x}{2b} = -1$$

$\Rightarrow \quad ax + by = 0$

$\Rightarrow \quad ax + \frac{x^2}{4} = 0$

$\Rightarrow \quad x = -4a$

$\Rightarrow \quad -x^3 = 64a^3 \quad [\because x \neq 0]$

$\Rightarrow \quad 64ab^2 + 64a^3 = 0$

$\Rightarrow \quad a^2 + b^2 = 0$

which is not possible.

5. Let $y = f(x) = ax^2 + bx + c$, we have

$f(1) = 1 \Rightarrow a + b + c = 1$.

Also, $ax^2 + bx + c = x$ should have $x - 1$ as it's repeated root.

$\Rightarrow \quad ax^2 + (b-1)x + c = a(x-1)^2$

$\Rightarrow \quad 1 - b = 2a,\ a = c$

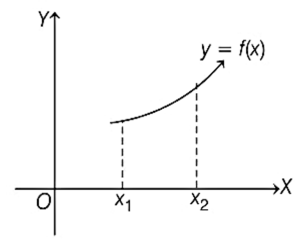
We have, $\quad f'(x) = 2ax + b,\ f''(x) = 2a$

$\Rightarrow \quad f''(1) = 2a,\ f'(0) = b$

$\Rightarrow \quad f''(1) + f'(0) = 1$

6. Let $A \equiv \{x_1, f(x_1)\}$ and $B \equiv \{x_2, f(x_2)\}$ be any two points on the graph of $y = f(x)$. Chord AB will be lie completely above the graph of $y = f(x)$.

Hence, $\frac{f(x_1) + f(x_2)}{2} > f\left(\frac{x_1 + x_2}{2}\right)$.



7. $g'(x) = f'(\sin x) \cdot \cos x - f'(\cos x) \cdot \sin x$

$\Rightarrow \quad g''(x) = -f'(\cos x) \cdot \sin x + \cos^2 x$

$f''(\sin x) + f''(\cos x) \cdot \sin^2 x - f'(\cos x) \cdot \cos x > 0,$

$\forall x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$. Also, $g'\left(\frac{\pi}{4}\right) = 0$

$\Rightarrow \quad g'(x) > 0, \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $g'(x) < 0, \forall x \in \left(0, \frac{\pi}{4}\right)$

Thus, $g(x)$ is decreasing in $\left(0, \frac{\pi}{4}\right)$.

8. $t = 2$, for the point $(2, -1)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3} = \frac{6}{7}$, for $t = 2$

9. Let $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \leq 0$ for all $x \in R$

$\Rightarrow \quad 3(a+2)x^2 - 6ax + 9a \leq 0$ for all $x \in R$

$\Rightarrow \quad (a+2)x^2 - 2ax + 3a \leq 0$ for all $x \in R$

$\Rightarrow \quad a + 2 < 0$ and discriminant ≤ 0

$\Rightarrow \quad a < -2, -8a^2 - 24a \leq 0$

$\Rightarrow \quad a < -2$ and $a(a+3) \geq 0$

$\Rightarrow \quad a < -2, a \leq -3$ or $a \geq 0$

$\Rightarrow \quad a \leq -3$

$\Rightarrow \quad -\infty < a \leq -3$

10. Now, $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = \frac{\cos x(\tan x - x)}{\sin^2 x}$

$\therefore \quad f'(x) > 0$ for $0 < x \leq 1$

$\therefore \quad f(x)$ is an increasing function.

Now, $\quad g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$

$= \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x}$

Now, $0 < 2x \leq 2$, for which $\sin 2x < 2x$

$\therefore \quad g'(x) < 0$

$\therefore \quad g(x)$ is decreasing.

11. Let $y=\dfrac{a\sin x+b\cos x}{c\sin x+d\cos x}$

The function will be decreasing, when $\dfrac{dy}{dx}<0$.

$$\frac{\left[\begin{array}{l}(c\sin x+d\cos x)(a\cos x-b\sin x)\\ \qquad -(a\sin x+b\cos x)(c\cos x-d\sin x)\end{array}\right]}{(c\sin x+d\cos x)^2}<0$$

$\Rightarrow\quad ac\sin x\cos x-bc\sin^2 x+ad\cos^2 x$
$-bd\sin x\cos x-ac\sin x\cos x+ad\sin^2 x$
$-bc\cos^2 x+bd\sin x\cos x<0$

$\Rightarrow\quad ad(\sin^2 x+\cos^2 x)-bc(\sin^2 x+\cos^2 x)<0$

$\Rightarrow\quad (ad-bc)<0$

12. $\dfrac{dy}{dx}=2x^3-15x^2+36x-19$

Let $f(x)=2x^3-15x^2+36x-19$

$\Rightarrow\quad f'(x)=6x^2-30x+36=0$

$\Rightarrow\quad x^2-5x+6=0$

$\Rightarrow\quad x=2,3$

$\Rightarrow\quad f''(x)=12x-30$

$\Rightarrow\quad f''(x)<0$ for $x=2$

At $\quad x=2$

$\Rightarrow\quad y=8-40+72-38$

$\Rightarrow\quad y=72-70=2$

$\Rightarrow\quad (2,2)$

13. To determine c in Rolle's theorem, $f'(c)=0$.

Here, $f'(x)=(x^2+3x)\,e^{-(1/2)x}\left(-\dfrac{1}{2}\right)+(2x+3)\,e^{-(1/2)x}$

$$=e^{-\left(\frac{1}{2}\right)x}\left\{-\frac{1}{2}(x^2+3x)+2x+3\right\}$$

$$=-\frac{1}{2}e^{-(x/2)}(x^2-x-6)$$

$\because\quad f'(c)=0$

$\Rightarrow\quad c^2-c-6=0$

$\Rightarrow\quad c=3,-2$

But $\quad c=3\notin[-3,0]$

14. Since, $\dfrac{f(2)-f(0)}{2-0}=f'(x)$

$\Rightarrow\quad \dfrac{f(2)-0}{2}=f'(x)\Rightarrow\dfrac{df(x)}{dx}=\dfrac{f(2)}{2}$

$\Rightarrow\quad f(x)=\dfrac{f(2)}{2}x+c$

$\because\quad f(0)=0\Rightarrow c=0$

$\therefore\quad f(x)=\dfrac{f(2)}{2}x$ ...(i)

Also, $\quad |f'(x)|\le\dfrac{1}{2}$

$\Rightarrow\quad \left|\dfrac{f(2)}{2}\right|\le\dfrac{1}{2}$ ...(ii)

From Eq. (i),

$$|f(x)|=\left|\frac{f(2)}{2}x\right|=\left|\frac{f(2)}{2}\right||x|\le\frac{1}{2}|x|$$

In interval $[0,2]$, for maximum x $(x=2)$

$$|f(x)|\le\frac{1}{2}\cdot 2$$

$\Rightarrow\quad |f(x)|\le 1$

15. Since, $\quad f(1)=f(3)$

$\Rightarrow\quad a+b-5=3a+b-27$

$\Rightarrow\quad a=11,$

Since, $f(1)=f(3)$ is independent of b, we have

$a=11$ and $b\in k$

which given in option (a) only.

16. Let the point be (x_1,y_1).

Therefore, $\quad y_1=(x_1-3)^2$ ...(i)

Now, slope at the tangent at (x_1,y_1) is $2(x_1-3)$ and it is equal to 1.

Therefore, $2(x_1-3)=1$

$\Rightarrow\quad x_1=\dfrac{7}{2}$

$\therefore\quad y_1=\left(\dfrac{7}{2}-3\right)^2=\dfrac{1}{4}$

Hence, the point is $\left(\dfrac{7}{2},\dfrac{1}{4}\right)$.

17. $f(x)=\sqrt{x}$

$\therefore\ f(a)=\sqrt{4}=2$

$f(b)=\sqrt{9}=3,\ f'(x)=\dfrac{1}{2\sqrt{x}}$

Also, $\quad f'(c)=\dfrac{f(b)-f(a)}{b-a}=\dfrac{3-2}{9-4}=\dfrac{1}{5}$

$\therefore\quad \dfrac{1}{2\sqrt{c}}=\dfrac{1}{5}\Rightarrow c=\dfrac{25}{4}=6.25$

18. $a+b=4\Rightarrow b=4-a$ and $b-a=4-2a=t$ (say)

Now, $\quad \displaystyle\int_0^a g(x)\,dx+\int_0^b g(x)\,dx=\int_0^a g(x)\,dx$

$$+\int_0^{4-a} g(x)\,dx=I(a)$$

$\Rightarrow\quad \dfrac{dI(a)}{da}=g(a)-g(4-a)$

As $a<2$ and $g(x)$ is increasing.

$\Rightarrow\quad 4-a>a$

$\Rightarrow\quad g(a)-g(4-a)<0$

$\Rightarrow\quad \dfrac{dI(a)}{da}<0$

Now, $\quad \dfrac{dI(a)}{d(a)}=\dfrac{dI(a)}{dt}\dfrac{dt}{da}=-2\cdot\dfrac{dI(a)}{dt}\Rightarrow\dfrac{dI(a)}{dt}>0$

Thus, $I(a)$ is an increasing function of t. Hence, the given expression increasing with $(b-a)$.

19. $\sqrt{x}+\sqrt{y}=\sqrt{a}$ ... (i)

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

Hence, tangent at (x, y) is $Y-y=-\frac{\sqrt{y}}{\sqrt{x}}(X-x)$.

$\Rightarrow \quad Y\sqrt{x}-y\sqrt{x}=-X\sqrt{y}+x\sqrt{y}$

$\Rightarrow \quad X\sqrt{y}+Y\sqrt{x}=x\sqrt{y}+y\sqrt{x}=\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y})$

$\Rightarrow \quad X\sqrt{y}+Y\sqrt{x}=\sqrt{axy}$ [from Eq. (i)]

$\Rightarrow \quad \frac{X}{\sqrt{a}\sqrt{x}}+\frac{Y}{\sqrt{a}\sqrt{y}}=1$

Clearly, its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

Sum of intercepts

$$=\sqrt{a}(\sqrt{x}+\sqrt{y})=\sqrt{a}\cdot\sqrt{a}=a$$

20. By LMVT in [0, 1] to the function $y=f(x)$, we get

$$f'(e)=\frac{f(1)-f(0)}{1-0} \text{ for some } c\in(0,1)$$

$\Rightarrow \quad e^{c^2}=\frac{f(1)-f(0)}{1}$

$\Rightarrow \quad f(1)-10=e^{c^2}$ for some $c\in(0,1)$

But $1<e^{c^2}<e$ in (0, 1)

$\therefore \quad 1<f(1)-10<e$

$=11<f(1)<10+e$

$\Rightarrow A=11, B=10+e$

$\therefore \quad A-B=1-e$

21. $f(x)$ will be monotonically decreasing, if $f(x)<0$.

$\Rightarrow \quad f'(x)=-\sin x-2p<0$

$\Rightarrow \quad \frac{1}{2}\sin x+p>0$

$\Rightarrow \quad p>\frac{1}{2}$ $[\because -1\le\sin x\le 1]$

22. Given that, equation of curve $y=x^3=f(x)$

So, $f(2)=8$ and $f(-2)=-8$

Now, $f'(x)=3x^2$

$\Rightarrow \quad f'(x)=\frac{f(2)-f(-2)}{2-(-2)}\Rightarrow\frac{8-(-8)}{4}=3x^2$

$\therefore \quad x=\pm\frac{2}{\sqrt{3}}$

23. For Y-axis, $x=0$

$\therefore \quad y=1-e^0=1-1=0$

$\Rightarrow \quad \frac{dy}{dx}=0-\frac{1}{2}e^{x/2}\Rightarrow\left(\frac{dy}{dx}\right)_{(0,0)}=-\frac{1}{2}$

$\therefore$ Equation of tangent is

$$y-0=-\frac{1}{2}(x-0)\Rightarrow x+2y=0$$

24. For $x=p$, $y=ap^2+bp+c$

And for $x=q$, $y=aq^2+bq+c$

$$\text{Slope}=\frac{aq^2+bq+c-ap^2-bp-c}{q-p}$$

$$=a(q+p)+b$$

$$\frac{dy}{dx}=2ax+b=a(q+p)+b$$

[according to the question]

$$\therefore \quad x=\frac{q+p}{2}$$

25. $f'(x)=(ab-b^2-2)+\cos^4 x+\sin^4 x<0$

$=ab-b^2-2+(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x<0$

$\Rightarrow \quad ab-b^2-1<\left(\frac{1}{2}\right)\sin^2 2x<\frac{1}{2}$

$\Rightarrow \quad 2ab-2b^2-2<1$

$\Rightarrow \quad 2b^2-2ab+3>0$

$\therefore \quad (-2a)^2-4\cdot2\cdot3<0$

$\Rightarrow \quad a^2<6$

$\Rightarrow \quad -\sqrt{6}<a<\sqrt{6}$

26. Given, $f'(x)<0, \forall x\in R$

$\Rightarrow \sqrt{3}\cos x+\sin x-2a<0, \forall x\in R$

$\Rightarrow \frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x<a, \forall x\in R$

$\Rightarrow \quad \sin\left(x+\frac{\pi}{3}\right)<a, \forall x\in R$

$\Rightarrow \quad a\ge 1 \quad \left[\because \sin\left(x+\frac{\pi}{3}\right)\le 1\right]$

27. There is only one function in option (a), whose critical point $\frac{1}{2}\in(0,1)$ but in other parts critical point $0\notin(0,1)$.

Then, we can say that functions in options (b), (c) and (d) are continuous on [0, 1] and differentiable in (0, 1).

Now, for $f(x)=\begin{cases}\frac{1}{2}-x, & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2, & x\ge\frac{1}{2}\end{cases}$

Here, $\quad Lf'\left(\frac{1}{2}\right)=-1$

and $\quad Rf'\left(\frac{1}{2}\right)=2\left(\frac{1}{2}-\frac{1}{2}\right)(-1)=0$

$\therefore \quad Lf'\left(\frac{1}{2}\right)\ne Rf'\left(\frac{1}{2}\right)$

$\Rightarrow$ f is non-differentiable at $x=\frac{1}{2}\in(0,1)$.

$\therefore$ Lagrange mean value theorem is not applicable to $f(x)$ in [0, 1].

28. The equation of the tangent to the curve $y = e^x$ at (c, e^c) is

$$y - e^c = e^c (x - c) \quad \text{...(i)}$$

Equation of the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$ is

$$y - e^{c-1} = e^c \cdot \frac{(e - e^{-1})}{2} \cdot [x - (c-1)]$$

$\Rightarrow$ $[x - (c-1)]\,[2 - (e - e^{-1})] = 2e^{-1}$ [from Eq. (i)]

$$\Rightarrow \quad x - c = \frac{e + e^{-1} - 2}{2 - (e - e^{-1})} < 2 \Rightarrow x < c$$

29. The tangent to the parabola $x^2 = y - 6$ at $(1, 7)$ is

$$x(1) = \frac{1}{2}(y + 7) - 6 \Rightarrow y = 2x + 5$$

which is also a tangent to the given circle.

i.e. $x^2 + (2x+5)^2 + 16x + 12(2x+5) + c = 0$

$\Rightarrow$ $(5x^2 + 60x + 85 + c = 0)$ must have equal roots.

Let the roots be $\alpha = \beta$.

$$\therefore \quad \alpha + \beta = -\frac{60}{5} \Rightarrow \alpha = -6$$

$$\therefore \quad x = -6 \text{ and } y = 2x + 5 = -7$$

30. On differentiating w.r.t. x, $e^y \cdot \dfrac{dy}{dx} = 2x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{1 + x^2} \qquad (\because\ e^y = 1 + x^2)$$

$$\Rightarrow \quad m = \frac{2x}{1 + x^2}$$

or $$|m| = \frac{2|x|}{1 + |x|^2}$$

But $1 + |x|^2 - 2|x| = (1 - |x|)^2 \geq 0$

$$\Rightarrow \quad 1 + |x|^2 \geq 2|x|$$

$$\therefore \quad |m| \leq 1$$

31. The equation of the line is $y - 3 = \dfrac{3+2}{0-5}(x - 0)$

i.e. $$x + y - 3 = 0$$

$$y = \frac{c}{x+1} \Rightarrow \frac{dy}{dx} = \frac{-c}{(x+1)^2}$$

Let the line touches the curve at (α, β).

$$\Rightarrow \quad \alpha + \beta - 3 = 0, \left[\frac{dy}{dx}\right]_{\alpha, \beta} = \frac{-c}{(\alpha+1)^2} = -1 \text{ and } \beta = \frac{c}{\alpha + 1}$$

$$\Rightarrow \quad \frac{c}{(c/\beta)^2} = 1 \quad \text{or} \quad \beta^2 = c \text{ or } (3 - \alpha)^2 = c = (\alpha+1)^2$$

$\Rightarrow$ $3 - \alpha = \pm(\alpha + 1)$ or $3 - \alpha = \alpha + 1 \Rightarrow \alpha = 1$

So, $c = (1+1)^2 = 4$

32. Here, $f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$

$\Rightarrow$ $f_1'(x_1) = 2x_1 - 1$ and $f_2'(x_1) = 3x_2^2 - 2x_2 - 2$

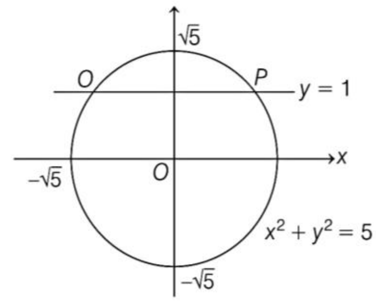
Let tangents drawn to the curves $y = f_1(x)$ and $y = f_2(x)$ is $\{x_1, f_1(x_1)\}$ and $\{x_2, f_2(x_2)\}$ be parallel.

$\Rightarrow$ $2x_1 - 1 = 3x_2^2 - 2x_2 - 2$ or $2x_1 = (3x_2^2 - 2x_2 - 1)$

which is possible for infinite numbers of ordered pairs.

$\Rightarrow$ Infinite solutions.

33. We know that, $1 \leq |\sin x| + |\cos x| \leq \sqrt{2}$



$$\Rightarrow \quad y = [\,|\sin x| + |\cos x|\,] = 1$$

Let P and Q be the points of intersection of given curves.

Clearly, the given curves meet at points where $y = 1$, so we get

$$x^2 + 1 = 5 \Rightarrow x = \pm 2$$

Now, $P(2, 1)$ and $Q(-2, 1)$

On differentiating $x^2 + y^2 = 5$ w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\left(\frac{dy}{dx}\right)_{(2,1)} = -2 \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{(-2,1)} = 2$$

Clearly, the slope of line $y = 1$ is zero and the slope of the tangents at P and Q are (-2) and (2), respectively.

Thus, the angle of intersection is $\tan^{-1}(2)$.

34. $y = \dfrac{a^2}{x} \quad \therefore \quad \dfrac{dy}{dx} = -\dfrac{a^2}{x^2}$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{a^2}{x_1^2}$$

$\Rightarrow$ Tangent at (x_1, y_1) to the curve $xy = a^2$ is

$$y - y_1 = -\frac{a^2}{x_1^2}$$

$(x - x_1)$ or $a^2 x + x_1^2 y = x_1(x_1 y_1 + a^2)$

But $x_1(x_1 y_1 + a^2) = x_1(a^2 + a^2) = 2a^2 x_1 \quad (\because\ x_1 y_1 = a^2)$

$\Rightarrow$ Tangent is $a^2 x^2 + x_1^2 y = 2a^2 x_1$

This meets the X-axis, where $y = 0$

$$\therefore \quad a^2 x = 2a^2 x_1$$

$$\therefore \quad x = 2x_1$$

$\therefore$ Point on the X-axis is $(2x_1, 0)$.

Again tangent meets the Y-axis, where $x = 0$

$$\therefore \quad x_1^2 = 2a^2 x_1$$

$$\therefore \quad y = \frac{2a^2}{x_1}$$

$\therefore$ Point on the Y-axis is $\left(0, \dfrac{2a^2}{x_1}\right)$.

$$\text{Required area} = \frac{1}{2}(2x_1)\left(\frac{2a^2}{x_1}\right) = 2a^2$$

35. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$\Rightarrow \quad n\left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{b}{a}\frac{(x/a)^{n-1}}{(y/b)^{n-1}}$

At (a, b), $\quad \frac{dy}{dx} = -\frac{b}{a} \cdot \frac{(a/a)^{n-1}}{(b/b)^{n-1}} = -\frac{b}{a}$

$\therefore$ Tangent at (a, b) is $y - b = -\frac{b}{a}(x - a)$

$\Rightarrow \quad ay - ab = -bx + ab$

$\Rightarrow \quad bx + ay = 2ab$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 2$

36. On differentiating the given curve $xy = 1$ w.r.t. x, we get

$$y + x\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$

Slope of normal at point (x_1, y_1) is $\frac{x_1}{y_1}$.

The given equation $ax + by + c = 0$ has slope $\frac{-a}{b}$.

According to the question, $\frac{x_1}{y_1} = \frac{-a}{b}$

Now, (x_1, y_1) lies on the curve $xy = 1$, which lies in first or third quadrant.

$\Rightarrow \quad x_1, y_1 > 0$ or $x_1, y_1 < 0 \Rightarrow \frac{-a}{b} > 0 \Rightarrow$ a and b have opposite sign.

37. Given, $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$

$\Rightarrow \quad f'(x) = 12\cos^3 x(-\sin x) + 30\cos^2 x(-\sin x) + 12\cos x(-\sin x)$

$$\overset{-}{}\underset{\pi/2}{\bullet}\overset{+}{}\underset{2\pi/3}{\bullet}\overset{-}{}$$

$\Rightarrow \quad f'(x) = -3\sin 2x(2\cos^2 x + 5\cos x + 2)$

$\Rightarrow \quad f'(x) = -3\sin 2x(2\cos x + 1)(\cos x + 2)$

When $f'(x) = 0 \Rightarrow \sin 2x = 0 \Rightarrow x = 0, \frac{\pi}{2}, \pi$

$\Rightarrow \quad 2\cos x + 1 = 0$

$\Rightarrow \quad x = \frac{2\pi}{3}$

as $\quad \cos x + 2 \neq 0$

Using sign scheme for $f'(x)$ in $[0, \pi]$ is shown in figure.

So, $f(x)$ decreases on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ and increases on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

38. The given function $y = 2x^2 - \log|x|, x \neq 0$

On differentiating w.r.t. x, we have

$$\frac{dy}{dx} = 4x - \frac{1}{x} = \frac{4}{x}\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right) : x \neq 0 \qquad \ldots(i)$$

$$\overset{-}{}\underset{-1/2}{\bullet}\overset{+}{}\underset{0}{\bullet}\overset{-}{}\underset{1/2}{\bullet}\overset{+}{}$$

Sign scheme of $f'(x)$

Hence, y is increasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

and y is decreasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

39. Obviously, f is increasing and g is decreasing in (x_1, x_2)

Hence, $f\{g(\alpha^2 - 2\alpha)\} > f\{g(3\alpha - 4)\}$ as f is increasing

$\Rightarrow \quad g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$

$\Rightarrow \quad \alpha^2 - 2\alpha < 3\alpha - 4$ as g is decreasing

$\Rightarrow \quad \alpha^2 - 5\alpha + 4 < 0$

$\Rightarrow \quad (\alpha - 1)(\alpha - 4) < 0 \Rightarrow \alpha \in (1, 4)$

40. We have, $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$

$$\Rightarrow f'(x) = \frac{\left(\frac{1}{\pi + x}\right)\ln(e + x) - \frac{1}{(e + x)}\ln(\pi + x)}{[\ln(e + x)]^2}$$

$$= \frac{(e + x)\ln(e + x) - (\pi + x)\ln(\pi + x)}{(e + x)(\pi + x)\{\ln(e + x)\}^2} < 0 \text{ as } (0, \infty)$$

Since, $1 < e < \pi$

$\therefore \quad f(x)$ decreases on $(0, \infty)$

41. Let the speed of the motorboat be v mph.

$\Rightarrow$ Velocity of the boat relative to the current $= (v - c)$ mph

If s miles is the distance covered, then the time taken to cover this distance is $t = \frac{s}{(v - c)}$ hours.

Since, the petrol burnt $= kv^3$ per hour

where, k is a constant.

$\Rightarrow \quad z =$ Total amount of petrol burnt for a distance of s miles $= kv^3 \cdot \frac{s}{(v - c)} \Rightarrow \frac{dz}{dv} = \frac{2ksv^2(v - 3c/2)}{(v - c)^2}$

For max or min of z, $\frac{dz}{dv} = 0 \Rightarrow v = \frac{3c}{2}$

If v is little less of little greater than $\frac{3c}{2}$, then the sign of $\frac{dz}{dv}$ changes from –ve to +ve. Hence, z is minimum when $v = \frac{3c}{2}$ mph.

Since, minima is the only extreme value, z is least at $v = \frac{3c}{2}$ i.e. the most economical speed is $\frac{3c}{2}$ mph.

42. $A=\frac{1}{2}x^2\sin\theta \;\Rightarrow\; 2A=x^2\sin\theta$

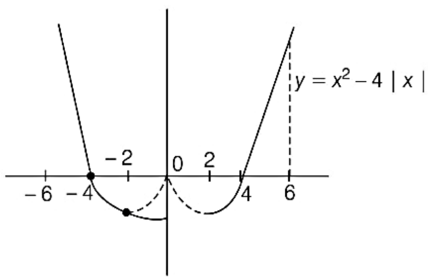
$$2\frac{dA}{dt}=x^2\cos\theta\frac{d\theta}{dt}+\sin\theta\,2x\frac{dx}{dt}$$

$$2\frac{dA}{dt}=(144)\left(\frac{1}{\sqrt{2}}\right)\frac{\pi}{180}+\frac{1}{\sqrt{2}}\cdot2\cdot12\cdot\frac{1}{12}$$

$$=\frac{12\pi}{15\sqrt{2}}+\frac{2}{\sqrt{2}}$$

$$\frac{dA}{dt}=\frac{2\pi}{5\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{\sqrt{2}\pi}{5}+\frac{\sqrt{2}}{2}=\sqrt{2}\left(\frac{\pi}{5}+\frac{1}{2}\right)$$

43. Bold line represents the graph of $y=g(x)$, clearly $g(x)$ has neither a point of local maxima nor a point of local minima.



44. For $y=\frac{x^2-2}{x^2-4}\Rightarrow\frac{dy}{dx}=\frac{-4x}{(x^2-4)^2}$

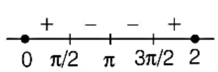
$\Rightarrow\quad \frac{dy}{dx}>0$ for $x<0$

and $\quad \frac{dy}{dx}<0$ for $x>0$

Thus, $x=0$ is the point of local maxima for y. Now, $(y)_{x=0}=\frac{1}{2}$ (positive). Thus, $x=0$ is also the point of local maximum for $y=\left|\frac{x^2-2}{x^2-4}\right|$.

45. $f'(x)=(x^2-1)\cos x$

Sign scheme of $f'(x)$, clearly $x=\frac{\pi}{2}$ is the point of local maxima.



46. $f'(x)=4-3x^2\;\forall\;x\in[0,3]$

$\Rightarrow\quad f'(x)<0$

$\Rightarrow\quad 4-3x^2<0$

$\Rightarrow\quad 3x^2>4$

$\Rightarrow\quad x>\frac{2}{\sqrt{3}}$

At $\quad x<3$

$f(x)$ is decreasing.

When $x\geq3$, $f'(x)=1$

$f(x)$ in increasing.

Clearly, $f(x)$ in increasing just before $x=3$ and decreasing after $x=3$. For $x=3$ to be the point of local maxima.

$$f(3)\geq f(3-0)$$

$\Rightarrow\quad -15\geq12-27+\log(a^2-3a+3)$

$\Rightarrow\quad 0<a^2-3a+3\leq1$

$\Rightarrow\quad 1\leq a\leq2$

47. Clearly, $f(2)=-1$

$\Rightarrow\quad -1=\frac{2a+b}{(2-1)(2-4)}$

$\Rightarrow\quad 2a+b=2$

Now, $\quad f'(x)=\frac{4a+5b-2bx-ax^2}{(x-1)^2(x-4)^2}$, $f'(2)=0$

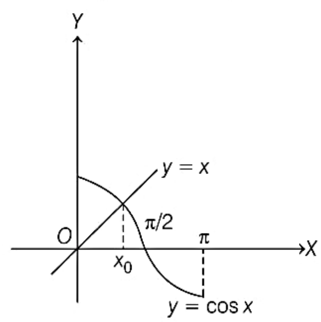
$\Rightarrow\quad b=0\Rightarrow a=1$

$\Rightarrow\quad f'(x)=-\frac{(x-2)(x+2)}{(x-1)^2(x-4)^2}$

Clearly, for $x>2$, $f'(x)<0$ and for $x<2$, $f'(x)>0$.

Thus, $x=2$ is indeed the point of local maxima for $y=f(x)$.

48. $f'(x)=\frac{1-x^2\sec^2x}{(1+x\tan x)^2}=\frac{\sec^2x(\cos x+x)(\cos x-x)}{(1+x\tan x)^2}$



Clearly, $\quad f'(x_0)=0$

and $f'(x)>0,\forall\,x\in(0,x_0)$, $f'(x)<0,\forall\,x\in\left(x_0,\frac{\pi}{2}\right)$

Thus, $x=x_0$ is the only point of local maxima for $y=f(x)$.

49. Let the equation of drawn line be $\frac{x}{a}+\frac{y}{b}=1$, where $a>3$, $b>4$, as the line passes through $(3,4)$ and meets the positive direction of coordinate axes.

We have, $\quad \frac{3}{a}+\frac{4}{b}=1\Rightarrow b=\frac{4a}{(a-3)}$

Now, area of ΔAOB,

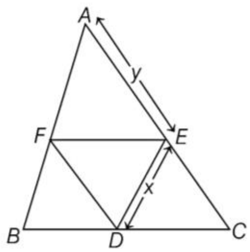
$$\Delta=\frac{1}{2}ab=\frac{2a^2}{(a-3)}$$

$$\frac{d\Delta}{da}=\frac{2a(a-6)}{(a-3)^2}$$

Clearly, $a=6$ is the point of minima for Δ.

Thus, $\Delta_{\min}=\frac{2\times36}{3}=24$ sq units

50. $AF \parallel DE$ and $AE \parallel FD$



Now, in ΔABC and ΔEDC,

$$\angle DEC = \angle BAC,\ \angle ACB \text{ is common.}$$

$\Rightarrow$ $\Delta ABC \cong \Delta\ EDC$

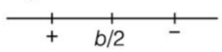
Now, $\dfrac{b-y}{b} = \dfrac{x}{c} \Rightarrow x = \dfrac{c}{b}(b-y)$

Now, S = Area of parallelogram $AFDE = 2$ (Area of ΔAEF)

$$\Rightarrow \quad S = 2\left(\frac{1}{2} \times y \sin A\right) = \frac{c}{b}(b-y)\sin A$$

$$\frac{dS}{dy} = \left(\frac{c}{b}\sin A\right)(b-2y)$$

Sign scheme of $\dfrac{dx}{dy}$,



Hence, S is maximum when $y = \dfrac{b}{2}$.

$$\therefore \quad S_{\max} = \frac{c}{b}\left(\frac{b}{2}\right) \times \frac{b}{2}\sin A$$

$$= \frac{1}{2}\left(\frac{1}{2}bc\sin A\right)$$

$$= \frac{1}{2}(\text{Area of } \Delta ABC)$$

51. We have, $\dfrac{dy}{dt} = 6t^2 - 30t + 24 = 6(t-1)(t-4)$

and $\dfrac{dx}{dt} = 6t - 18 = 6(t-3)$

Thus, $\dfrac{dy}{dx} = \dfrac{(t-1)(t-4)}{(t-3)}$

which indicates that $t = 1, 3$ and 4 are the critical points of $y = f(x)$.

$$\text{Now, } \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} = \frac{t^2-6t+11}{(t-3)^2} \times \frac{1}{6(t-3)}$$

At $(t=1)$, $\dfrac{d^2y}{dx^2} < 0 \Rightarrow t = 1$ is a point of local maxima.

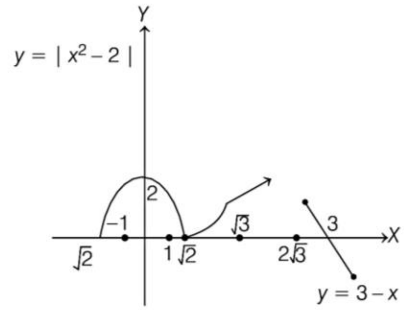
At $(t=4)$, $\dfrac{d^2y}{dx^2} > 0 \Rightarrow t = 4$ is a point of local minima.

At $(t=3)$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are not defined and change its sign.

$\dfrac{d^2y}{dx^2}$ is unknown in the vicinity of $t=3$, thus $t=3$ is a point of neither maxima nor minima.

Finally, maximum and minimum values of expression $y = f(x)$ are 46 and –6, respectively.

52. $f(x) = \begin{cases} |x^2-2|, -1 \le x < \sqrt{3} \\ \dfrac{x}{\sqrt{3}}, \sqrt{3} \le x < 2\sqrt{3} \\ 3 - x, 2\sqrt{3} \le x \le 4 \end{cases}$



From the above graph,

Maximum occurs at $x = 0$ and minimum at $x = 4$.

53. Let α and β be the roots of the equation

$$x^2 - (a-2)x - a + 1 = 0.$$

Then, $\alpha + \beta = a - 2,\ \alpha\beta = -a + 1$

Let $z = \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$= (a-2)^2 + 2(a-1) = a^2 - 2a + 2$$

$$\Rightarrow \quad \frac{dz}{da} = 2a - 2 \Rightarrow a = 1$$

Put $\dfrac{dz}{da} = 0$, then $\dfrac{d^2z}{da^2} = 2 > 0$

So, z has minima at $a = 1$.

So, $\alpha^2 + \beta^2$ has least value for $a = 1$. This is because, we have only one stationary value at which we have minima. Hence, $a = 1$.

54. Let $f(x) = \int_0^x te^{-t^2}dt$

$\Rightarrow$ $f'(x) = xe^{-x^2} = 0$

Put $f'(x) = 0 \Rightarrow x = 0$

Now, $f''(x) = e^{-x^2}(1 - 2x^2)$

$f''(0) = 1 > 0$

$\therefore$ Minimum value $f(0) = 0$

55. $\dfrac{dy}{dx} = \dfrac{a}{x} + 2bx + 1$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0 \Rightarrow a = -2b - 1$$

and $\left(\dfrac{dy}{dx}\right)_{x=2} = \dfrac{a}{2} + 4b + 1 = 0$

$$\Rightarrow \quad \frac{-2b-1}{2} + 4b + 1 = 0$$

$$\Rightarrow \quad -b + 4b + \frac{1}{2} = 0 \Rightarrow 3b = \frac{-1}{2}$$

$$\Rightarrow \quad b = \frac{-1}{6} \text{ and } a = \frac{1}{3} - 1 = \frac{-2}{3}$$

56. Let the coordinate of $R(x, 0)$.

Now, $PR + RQ = \sqrt{(x-1)^2 + (0-1)^2} + \sqrt{(x-3)^2 + (0-2)^2}$

$$= \sqrt{x^2 - 2x + 2} + \sqrt{x^2 - 6x + 13}$$

For minimum value of $PR + RQ$,

$$\frac{d}{dx}(PR + RQ) = 0$$

$$\Rightarrow \frac{d}{dx}\sqrt{(x^2 - 2x + 2)} + \frac{d}{dx}\sqrt{(x^2 - 6x + 13)} = 0$$

$$\Rightarrow \frac{(x-1)}{\sqrt{x^2 - 2x + 2}} = -\frac{(x-3)}{\sqrt{x^2 - 6x + 13}}$$

On squaring both sides, we get

$$\frac{(x-1)^2}{(x^2 - 2x + 2)} = \frac{(x-3)^2}{x^2 - 6x + 13}$$

$$\Rightarrow 3x^2 - 2x - 5 = 0$$

$$\Rightarrow (3x - 5)(x + 1) = 0$$

$$\Rightarrow x = \frac{5}{3}, -1$$

Also, $1 < x < 3$

$\therefore$ $R = (5/3, 0)$

57. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$$f'(x) = 6x^2 - 18ax + 12a^2$$

$$f''(x) = 12x - 18a$$

For maximum and minimum,

$$6x^2 - 18ax + 12a^2 = 0$$

$$\Rightarrow x^2 - 3ax + 2a^2 = 0$$

$x = a$ or $x = 2a$ at $x = a$ maximum and at $x = 2a$ minimum

$\because$ $p^2 = q$

$\therefore$ $a^2 = 2a$

$\Rightarrow$ $a = 2$ or $a = 0$

But $a > 0$, therefore $a = 2$

58. Given, $f(x) = 1 + 2x^2 + 2^2x^4 + 2^3x^6 + \ldots + 2^{10}x^{20}$

$$f'(x) = x(4 + 4 \cdot 2^2x^2 + \ldots + 20 \cdot 2^{10}x^{18})$$

Put $f(x) = 0 \Rightarrow x = 0$ only

Also, $f''(0) > 0$

59. (Slope) $f'(x) = e^x \cos x + \sin xe^x$

$$= e^x\sqrt{2}\sin(x + \pi/4)$$

$$f''(x) = \sqrt{2}e^x\{\sin(x + \pi/4) + \cos(x + \pi/4)\}$$

$$= 2e^x \cdot \sin(x + \pi/2)$$

For maximum slope, put $f''(x) = 0$

$\Rightarrow$ $\sin(x + \pi/2) = 0$

$\Rightarrow$ $\cos x = 0$

$\therefore$ $x = \pi/2, 3\pi/2$

$$f'''(x) = 2e^x\cos(x + \pi/2)$$

$$f'''(\pi/2) = 2e^x \cdot \cos\pi = -\text{ve}$$

Maximum slope is at $x = \pi/2$.

60. $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\left(\frac{1}{x}\right)\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Let $g(x) = x^3 + x^2 + 3x + \sin x$

$$g'(x) = 3x^2 + 2x + 3 + \cos x$$

$$= 3\left(x^2 + \frac{2x}{3} + 1\right) + \cos x$$

$$= 3\left\{\left(x + \frac{1}{3}\right)^2 + \frac{8}{9}\right\} + \cos x > 0$$

and $2 < 3 + \sin\left(\frac{1}{x}\right) < 4$

Hence, minimum value of $f(x)$ is 0 at $x = 0$.

Hence, number of points = 1

61. For $x > 0$ or $x < 0$

$$f'(x) = \frac{a}{x} + 2bx + 1$$

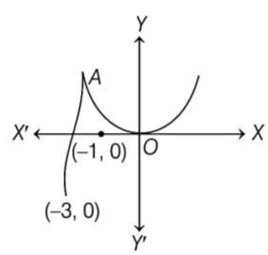
$\because$ $f'(1) = 0 \Rightarrow a + 2b + 1 = 0$ ...(i)

and $f'(3) = 0 \Rightarrow \frac{a}{3} + 6b + 1 = 0$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$a = -3/4,\ b = -1/8$$

62. $f'(x) = \begin{cases} 3(2 + x)^2, & -3 < x \le -1 \\ \frac{2}{3}x^{-1/3} & -1 < x < 2 \end{cases}$



Clearly, $f'(x)$ changes its sign at $x = -1$ from positive to negative and so $f(x)$ has local maxima at $x = -1$.

Also, $f'(0)$ does not exist but $f'(0^-) < 0$ and $f'(0^+) < 0$. It can only be inferred that $f(x)$ has a possibility of a minimum at $x = 0$.

Hence, it has one local maxima at $x = -1$ and one local minima at $x = 0$.

63. Minimum of $f(x) = -\frac{D}{4a} = \frac{-(4b^2 - 8c^2)}{4} = 2c^2 - b^2$

and maximum of $g(x) = -\frac{(4c^2 + 4b^2)}{4(-1)} = b^2 + c^2$

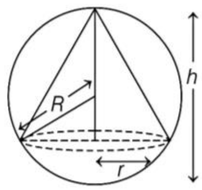
Since, $\min f(x) > \max g(x)$

$\Rightarrow$ $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow$ $c^2 > 2b^2$

$\Rightarrow$ $|c| > \sqrt{2}|b|$

64. Let S be the curved surface area of a cone.



$\therefore \quad S = \pi r l$
$= \pi(\sqrt{2Rh - h^2})(\sqrt{h^2 + r^2})$
$= (\pi\sqrt{2Rh - h^2})(\sqrt{2Rh})$

Let $\quad S^2 = P$

$\therefore \quad P = \pi^2 2R(2Rh^2 - h^3)$

$\because S$ is maximum, if P is maximum, then

$$\frac{dP}{dh} = 2\pi^2 R(4Rh - 3h^2) = 0$$

$\therefore \quad h = 0, \dfrac{4R}{3}$

Again, differentiating $\dfrac{dP}{dh}$, we get

$$\frac{d^2P}{dh^2} = 2\pi^2 R(4R - 6h)$$

$$\frac{d^2P}{dh^2} < 0 \text{ at } h = \frac{4R}{3}$$

65. Any tangent to the ellipse is $\dfrac{x}{4}\cos t + \dfrac{y}{3}\sin t = 1$, where the point of contact is $(4\cos t, 3\sin t)$

or $\quad \dfrac{x}{4\sec t} + \dfrac{y}{3\operatorname{cosec} t} = 1$

It means the axes $Q(4\sec t, 0)$ and $R(0, 3\operatorname{cosec} t)$.

$\therefore$ The distance of the line segment QR is

$$QR^2 = D = 16\sec^2 t + 9\operatorname{cosec}^2 t$$

So, the minimum value of D is $(4+3)^2$ or $QR = 7$.

66. $f(x) = \left(\dfrac{1}{x}\right)^x \Rightarrow f'(x) = \left(\dfrac{1}{x}\right)^x \left(\log\dfrac{1}{x} - 1\right)$

$$f'(x) = 0 \Rightarrow \log\frac{1}{x} = 1 = \log_e \Rightarrow \frac{1}{x} = e \Rightarrow x = \frac{1}{e}$$

Also, for $x < 1/e$, $f'(x)$ is positive and for $x > 1/e$, $f'(x)$ is negative.

Hence, $x = 1/e$ is point of maxima.

Therefore, maximum value of function is $e^{1/e}$.

67. Consider the function $f(x) = \dfrac{x^2}{(x^3 + 200)}$

$$f'(x) = x\frac{(400 - x^3)}{(x^3 + 200)^2} = 0$$

when $\quad x = (400)^{1/3}, \qquad (\because x \neq 0)$
$x = (400)^{1/3} - h \Rightarrow f'(x) > 0$
$x = (400)^{1/3} + h \Rightarrow f'(x) < 0$

$\therefore f(x)$ has maxima at $x = (400)^{1/3}$

Since, $7 < (400)^{1/3} < 8$, either a_7 or a_8 is the greatest term of the sequence.

$\because \quad a_7 = \dfrac{49}{543}$ and $a_8 = \dfrac{8}{89}$ and $\dfrac{49}{543} > \dfrac{8}{89}$

$\Rightarrow \quad a_7 = \dfrac{49}{543}$ is the greatest term.

68. Given, $A + B = 60° \Rightarrow B = 60° - A$

$\Rightarrow \quad \tan B = \tan(60° - A) = \dfrac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$

Now, $\quad z = \tan A \tan B$

or $\quad z = \dfrac{t(\sqrt{3} - t)}{1 + \sqrt{3}t} = \dfrac{\sqrt{3}t - t^2}{1 + \sqrt{3}t}$

where, $t = \tan A$

$$\frac{dz}{dt} = -\frac{(t + \sqrt{3})(\sqrt{3}t - 1)}{(1 + \sqrt{3}t)^2} = 0$$

$\Rightarrow \quad t = 1/\sqrt{3} \Rightarrow t = \tan A = \tan 30°$

The other value is rejected as both A and B are positive acute angles.

If $t < \dfrac{1}{\sqrt{3}}, \dfrac{dz}{dt}$ = positive and if $t > \dfrac{1}{\sqrt{3}}, \dfrac{dz}{dt}$ = negative

Hence, max when $t = \dfrac{1}{\sqrt{3}}$ and max value $= \dfrac{1}{3}$.

69. $f(x) = x^3 - 3(7-a)x^2 - 3(9 - a^2)x + 2$

$f'(x) = 3x^2 - 6(7-a)x - 3(9 - a^2)$

For real root $D \geq 0$,

$\Rightarrow \quad 49 + a^2 - 14a + 9 - a^2 \geq 0$

$\Rightarrow \quad a \leq \dfrac{58}{14}$

For local minimum $f''(x) = 6x - 6(7-a) > 0 \Rightarrow 7 - x$ has x must be negative

$\Rightarrow \quad 7 - a < 0 \Rightarrow a > 7$

Thus constradictory, i.e. for real roots $a \leq \dfrac{58}{14}$ and for negative point of local minimum $a > 7$.

No possible values of a.

70. Clearly $f(x)$ is increasing before $x = 0$ and starts decreasing after $x = 0$, $f(0) = a$.

For $x = 0$ to be the point of local maxima

$$f(0) \geq \lim_{x \to 0^+} f(x)$$

$\Rightarrow \quad a \geq 1$

71. Let the line in intercepts form be $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

It passes through $(h,k) \Rightarrow \dfrac{h}{a} + \dfrac{k}{b} = 1$

$\Rightarrow \quad \dfrac{k}{b} = 1 - \dfrac{h}{a} = \dfrac{a-h}{a}$

$\Rightarrow \quad b = \dfrac{ak}{a-h}$

$$\Delta = \frac{1}{2}ab = \frac{1}{2}a\cdot\frac{ak}{a-h} = \frac{1}{2}\frac{h}{\frac{a-h}{a^2}} \quad \text{...(i)}$$

Δ is min when $y = \frac{a-h}{a^2} = \frac{1}{a} - \frac{h}{a^2}$ is max

$$\Rightarrow \quad \frac{dy}{da} = -\frac{1}{a^2} + \frac{2h}{a^3} = 0 \Rightarrow a = 2h \quad \text{...(ii)}$$

$$\frac{d^2y}{da^2} = \frac{2}{a^3} - \frac{6h}{a^4} = \frac{2}{a^3} - \frac{3}{a^3} \quad \text{[by Eq. (ii)]}$$

$$\Rightarrow \quad \frac{d^2y}{da^2} = -\frac{1}{a^3} = \text{negative}$$

$\therefore$ Maximum

Now, put $a = 2h$ in Eq. (i), $\Delta = \frac{1}{2}\cdot 4h^2 \cdot \frac{k}{h} = 2hk$

72. Let (x,y) be one the parabola $y = x^2 + 7x + 2$

Its distance from the line $y = 3x - 3$ or $3x - y - 3 = 0$ is

$$D = \left|\frac{3x - y - 3}{\sqrt{10}}\right| = \left|\frac{3x - (x^2 + 7x + 2) - 3}{\sqrt{10}}\right|$$

$$= \left|\frac{-x^2 - 4x - 5}{\sqrt{10}}\right|$$

$$D = \left|\frac{x^2 + 4x + 5}{\sqrt{10}}\right| = \frac{(x+2)^2 + 1}{\sqrt{10}} = \frac{(x+2)^2 + 1}{\sqrt{10}}$$

as $\frac{\text{Numerator}}{\text{Denominator}}$ is positive

$$\frac{dD}{dx} = \frac{2(x+2)}{\sqrt{10}} = 0 \Rightarrow x = -2 \text{ and hence } y = -8,$$

i.e. Point is $(-2, -8)$

$$\frac{d^2D}{dx^2} = \frac{2}{\sqrt{10}} = \text{ Positive and hence min at } (-2, -8).$$

73. The equation of tangent to the curve, $y = e^x$ at a point (c, e^c) is

$$y - e^c = e^c(x - c) \quad \text{...(i)}$$

and equation of normal to the curve, $y^2 = 4x$ at the point (1, 2) is

$$y - 2 = -1(x - 1) \quad \text{...(ii)}$$

$\because$ The lines (i) and (ii) intersect at same point on the X-axis, so put $y = 0$ in both the equation and equate, we get

$$x = 3 = c - 1 \Rightarrow c = 4$$

74. Let a cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d$$

$\because \quad f(-1) = 10$

$\Rightarrow \quad -a + b - c + d = 10 \quad \text{...(i)}$

$\because \quad f(1) = -6$

$\Rightarrow \quad a + b + c + d = -6 \quad \text{...(ii)}$

$\because \quad f'(-1) = 0$

$\Rightarrow \quad 3a - 2b + c = 0 \quad \text{...(iii)}$

$\because \quad f''(1) = 0$

$\Rightarrow \quad 6a + 2b = 0$

$\Rightarrow \quad 3a + b = 0 \quad \text{...(iv)}$

From Eqs. (i) and (ii), we get

$$-2a - 2c = 16$$

$\Rightarrow \quad a + c = -8 \quad \text{...(v)}$

From Eqs. (iii), (iv) and (v), we get

$$3a - 2(-3a) + (-a - 8) = 0$$

$\Rightarrow \quad 8a - 8 = 0 \Rightarrow a = 1$

So, $\quad b = -3, \quad c = -9$ and $d = 5$

$\therefore \quad f(x) = x^3 - 3x^2 - 9x + 5$

$\because \quad f'(x) = 3x^2 - 6x - 9 = 0$

[for local maxima and minima]

$\Rightarrow \quad x^2 - 2x - 3 = 0 \Rightarrow x^2 - 3x + x - 3 = 0$

$\Rightarrow \quad (x+1)(x-3) = 0 \Rightarrow f'(x) = 0 \Rightarrow x = -1, 3$

$\because \quad f''(x) = 6x - 6$

$\because \quad f''(-1) = -12$ and $f''(3) = 12$

$\therefore x = 3$ is point of local minima.

75. Given curve is $9y^2 = x^3 \quad \text{...(i)}$

On differentiating, we get

$$18y\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{x^2}{6y} \quad \text{...(ii)}$$

Let (α, β) be a point on Eq. (i) at which normal makes equal intercepts on the axes, then

$$9\beta^2 = \alpha^3 \quad \text{...(iii)}$$

From Eq.(ii), slope of the normal at (α, β)

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{(\alpha,\beta)}} = \frac{-1}{\alpha^2/(6\beta)} = \frac{-6\beta}{\alpha^2} \quad \text{...(iv)}$$

Since, normal of the curve makes equal intercepts with the axes, so slope of normal

$$= \tan 45° \text{ or } \tan 135° = \pm 1 \quad \text{...(v)}$$

$\therefore$ From Eq. (iv), we get

$$\frac{-6\beta}{\alpha^2} = \pm 1 \Rightarrow \beta = \mp \frac{\alpha^2}{6}$$

On putting the value of β in Eq. (iii), we get

$$9\left(\mp\frac{\alpha^2}{6}\right)^2 = \alpha^3 \Rightarrow \alpha^4 = 4\alpha^3$$

$\Rightarrow \quad \alpha^3(\alpha - 4) = 0 \Rightarrow \alpha = 0$ or $\alpha = 4$

When $\alpha = 0, \beta = 0$, then normal passes through (0, 0) it mean that they do not intercepts.

Taking $\alpha = 4$, we get $\beta = \mp \frac{4^2}{6} = \mp \frac{8}{3}$.

76. Since, we have the curve $y = ax^3 + bx^3 + cx + 5$ touches X-axis at $P(-2, 0)$, then X-axis is the tangent at $(-2, 0)$. The curve meets Y-axis in $(0, 5)$.

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{0,5} = 0 + 0 + c = 3 \quad \text{(given)}$$

$\Rightarrow \quad c = 3 \quad \ldots$(i)

and $\left(\frac{dy}{dx}\right)_{(-2,\,0)} = 0$

$\Rightarrow \quad 12a - 4b + c = 0$

$\Rightarrow \quad 12a - 4b + 3 = 0$ [from Eq. (1)] ...(ii)

and $(-2, 0)$ lies on the curve, then

$$0 = -8a + 4b - 2c + 5$$

$\Rightarrow \quad 0 = -8a + 4b - 1 \quad [\because c = 3]$

$\Rightarrow \quad 8a - 4b + 1 = 0 \quad \ldots$(iii)

From Eqs. (ii) and (iii), we get

$$a = -\frac{1}{2} \text{ and } b = -\frac{3}{4}$$

$$\therefore c - 4a - 12b = 3 - 4\left(\frac{-1}{2}\right) - 12\left(\frac{-3}{4}\right) = 3 + 2 + 9 = 14$$

77. Let the side of the square to be cut off be x cm. Then, the height of the box is x, the length is $45 - 2x$ and the breadth is $24 - 2x$.

Let V be the corresponding volume of the box, then

$$V = x(24 - 2x)(45 - 2x)$$

$\Rightarrow \quad V = x(4x^2 - 138x + 1080) = 4x^3 - 138x^2 + 1080x$

On differentiating twice w.r.t. x, we get

$$\frac{dV}{dx} = 12x^2 - 276x + 1080$$

and $\frac{d^2V}{dx^2} = 24x - 276$

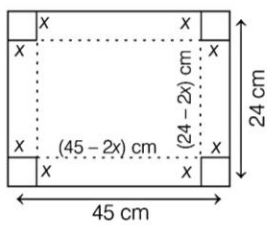
For maxima put $\frac{dV}{dx} = 0$

$\Rightarrow \quad 12x^2 - 276x + 1080 = 0$

$\Rightarrow \quad x^2 - 23x + 90 = 0$

$\Rightarrow \quad (x - 18)(x - 5) = 0 \Rightarrow x = 5, 18$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, x cannot be equal to 18.



At $x = 5$, $\left(\frac{d^2V}{dx^2}\right)_{x=5} = 24 \times 5 - 276$

$$= 120 - 276 = -156 < 0$$

$\therefore$ By second derivative test,

$x = 5$ is the point of maxima.

Hence, the side of the square to be cut-off to make the volume of the box maximum possible is 5 cm.

78. We have,

$$f(x) = \begin{cases} 6 - 3x & x < -2 \\ 10 - x & -2 \le x < 3 \\ x + 4 & 3 \le x < 5 \\ 3x - 6 & x \ge 5 \end{cases}$$

The function f is decreasing for $x \in (-\infty, 3)$ and increases on $(3, \infty)$.

Hence, $x = 3$ is a point of minimum and $f_{\min} = 7$.

79. We have, $S = 4t + \frac{1}{t}$

$\therefore \quad \frac{dS}{dt} = 4 - \frac{1}{t^2}$

and $\frac{d^2S}{dt^2} = \frac{2}{t^3}$

Now, velocity $= 0$

$\Rightarrow \quad \frac{dS}{dt} = 0 \Rightarrow 4 - \frac{1}{t^2} = 0 \Rightarrow t = \frac{1}{2}$

$\therefore$ Acceleration $\left(\text{at } t = \frac{1}{2}\right) = \left(\frac{d^2S}{dt^2}\right)_{t=\frac{1}{2}}$

$$= \frac{2}{\left(\frac{1}{2}\right)^3} = 16$$

80. $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is increasin on R.

$\Rightarrow f'(x) > 0$ for all $x \in R$

$\Rightarrow 3x^2 + 2ax + b\sin 2x > 0$ for all $x \in R$

$\Rightarrow 3x^2 + 2ax + (b - 5) > 0$ for all $x \in R$

$\Rightarrow (2a)^2 - 4 \times 3 \times (b - 5) < 0$

$\Rightarrow a^2 - 3b + 15 < 0$

$\therefore \lambda = 15$