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VECTORS

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VECTORS

1. **Scalars:** The physical quantities which have only magnitude and no direction are called scalars e.g., mass, length, time, speed, work, power, etc.

2. **Vectors:** The physical quantities which have both magnitude and direction are called vectors e.g., displacement, velocity, acceleration, force, momentum, etc.

3. **Representation of a vector:** A vector is represented by a straight line with an arrowhead over it. The length of the line gives the magnitude and the arrowhead gives the direction of the vector.

4. **Position vector:** A vector which gives position of an object with reference to the origin of a coordinate system is called position vector.

5. **Displacement vector:** It is that vector which tells how much and in which direction an object has changed its position in a given time internal.

6. **Polar vectors:** these are the vectors which have a starting point or a point of application e.g., displacement, force, velocity, etc.

7. **Axial vectors:** The vectors which represent rotational effect and act along the axis of rotation in accordance with right hand screw rule are called axial vectors e.g., torque, angular momentum, etc

8. **Equal vectors:** Two vectors are said to be equal if they have the same magnitude and direction.

9. **Negative vector:** The negative of a vector is defined as another vector having the same magnitude but having an opposite direction.

10. **Zero vector:** A vector having zero magnitude and an arbitrary direction is called a zero or null vector

11. **Collinear vectors:** The vectors which either act along the same line or along parallel lines are called collinear vectors.

12. **Coplanar vectors:** The vectors which act in the same plane are called coplanar vectors.

13. **Modulus of a vector:** The magnitude or length of a vector is called its modulus.
Modulus of vector = $\vec{A} = |\vec{A}| = \vec{A}$

13. **Fixed vector:** The vector whose initial vector is fixed is called a fixed vector or localised vector.

15. **Unit vector:** $\vec{A}$ unit vector is a vector of unit magnitude drawn in the direction of a given vector.
A unit vector in the direction of $\vec{A}$ is given by

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

16. **Free vector:** A vector whose initial point is not fixed is called a free vector or non-localised vector.

17. **Co-initial vectors:** The vectors which have the same initial point are called co-initial vectors.

18. **Co-terminus vectors:** the vectors which have the common terminal point are called co-terminus vectors.

19. **Properties of zero vector:** A zero vector has the following properties:

$\vec{A} + \vec{O} = \vec{A}$; $\lambda O = O$

$0\ A = O$

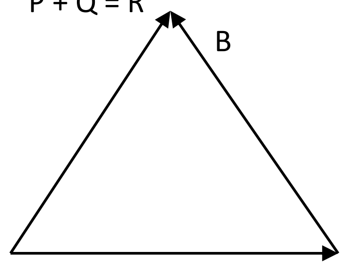
20. **Multiplication of vector by a real number:** When a vector $\vec{A}$ is multiplied by a real number λ, we get another vector $\lambda\vec{A}$. The magnitude of λA is λ times the magnitude of $\vec{A}$. If λ is positive, then the direction of λA is same as that of $\vec{A}$. If λ is negative then the direction of λA is opposite to that of A.

21. **Multiplication of a vector by a scalar:** When a vector $\vec{A}$ is multiplied by a scalar λ which has certain units, the units of $\lambda\vec{A}$ are obtained by multiplying the units of A by the units of λ.

22. **Composition of vectors:** The resultant of two or more vectors is that single vector which produces the same effect as the individual vectors together would produce. The process of adding two or more vectors is called the composition of vectors.

23. **Triangle law of vector addition:** If two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented completely both in magnitude and direction by the third side of the triangle taken in the opposite order. In Fig.

$\vec{OA} + \vec{AB} = \vec{OB}$ or $\vec{P} + \vec{Q} = R$



24. **Parallelogram law of vector addition:** If two vectors acting simultaneously at a point can be represented both in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented completely both in magnitude and direction by the diagonal of the parallelogram passing through that point. In Fig.

$\vec{OA} + \vec{OB} = \vec{OC}$ or $\vec{P} + \vec{Q} = \vec{R}$

The magnitude of the resultant R is given by

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

where θ is the angle between P and Q. If R makes angle β with P, then

$\tan\beta = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

25. **Polygon law of vector addition:** If a number of vectors are represented both in magnitude and direction by the sides of an open polygon taken in the same order, then their resultant is represented both in magnitude and direction by the closing side of the polygon taken in opposite order.

26. **Properties of vector addition:**

(i) Vectors representing physical quantities of same nature can only be added.

(ii) Vector addition is commutative.

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$

(iii) Vector addition is associative.

$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

27. **Subtraction of vectors:** The subtraction of a vector B from vector A is defined as the addition of vector – B to A. Thus

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

28. **Resolution of a vector:** The process of splitting a vector into two or more vectors is known as resolution of the vector. The vectors into which the given vector is splitted are called component vectors. A vector $\vec{A}$ can be resolved into components along two given vectors $\vec{a}$ and $\vec{b}$ lying in the same plane in one and only one way:

$$\vec{A} = \lambda \vec{a} + \mu \vec{b}$$

Where λ and μ are real numbers.

29. **Orthogonal triad of unit vectors:** Base vectors. The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are vectors of unit magnitude and point in the direction of the x-, y- and z-axes, respectively in a right-handed coordinate system. These are collectively known as the orthogonal triad of unit vectors or base vectors.

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$

30. **Rectangular components of a vector:** When a vector is resolved along two mutually perpendicular directions, components so obtained are called rectangular components of the given vector.

As shown in Fig. if $\vec{A}$ makes angle θ with X-axis and $\vec{A}_x$ and $\vec{A}_y$ are the rectangular components of $\vec{A}$ along X- and Y-axis respectively, then

$$\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{i} + A_y\hat{j}$$

Also, $A_x = A\cos\theta$, $A_y = A\sin\theta$

$$A = \sqrt{A_x^2 + A_y^2} \text{ and } \tan\theta = \frac{A_y}{A_x}$$

Any vector in three dimensions can be expressed in terms of its rectangular components as

$$A = \sqrt{A_x^2 + a_y^2 + A_z^2}$$

31. **Scalar or dot product:** The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of A and B and cosine of the angle θ between them. Thus

$$\vec{A} \cdot \vec{B} = |A|\,|B| \cos\theta = AB\cos\theta$$

It can positive, negative or zero depending upon the value of θ.

32. **Geometrical interpretation of scalar product:** The scalar product of two vectors can be interpreted as the product of magnitude of one vector and component of the other vector along the first vector.

33. **Properties of dot product of two vectors:**

(i) For parallel vectors,

$$\theta = 0°, \cos\theta = 1, \vec{A} \cdot \vec{B} = AB$$

(ii) For antiparallel vectors,

$$\theta = 180°, \cos\theta = -1, \vec{A} \cdot \vec{B} = -AB$$

(iii) For perpendicular vectors,

$$\theta = 90°, \quad \cos\theta = 0, \quad \vec{A} \cdot \vec{B} = 0$$

(iv) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ (Commutative law)

(v) $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ (Distributive law)

(vi) $A \cdot A = A^2$ (vii) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ (viii) $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

34. **Dot product in Cartesian co-ordinates. For**

$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{i}$ and $B = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, $A \cdot B = A_xB_x + A_yB_y + A_zB_z$

Angle θ between A and B is given by

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|}$$

$$= \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2} + \sqrt{B_x^2 + B_y^2 + B_z^2}}$$

35. **Examples of dot product:** (i) Work done, $W = \vec{F} \cdot \vec{S}$

(ii) Instantaneous power, $\vec{P} = \vec{F} \cdot \vec{v}$

36. **Vector or cross product:** For two vectors $\vec{A}$ and B inclined at an angle θ, the vector or cross product is defined as

$$\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$$

where n is a unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and its direction is that in which a right-handed screw advances when rotated from $\vec{A}$ to $\vec{B}$.

37. **Geometrical interpretation of vector product:** The magnitude of the vector product of two vectors is equal to (i) the area of the parallelogram formed by the two vectors as its adjacent sides and (ii) twice the area of the triangle formed by the two vectors as its adjacent sides.

38. **Properties of cross product of two vectors:**

(i) For parallel or antiparallel vectors, θ = 0° or 180°, $\vec{A} \times \vec{B} = 0$

(ii) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ [Anti-commutative law]

(iii) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$ [Distributive law]

(iv) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = k \times k = 0$ (v) $\hat{i} \times \hat{j} = k$, $\hat{j} \times k = \hat{i}$, $k \times \hat{i} = \hat{j}$

(vi) Unit vector perpendicular to the plane of A and B is given by

$$n = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

(vii) Angle θ between A and B is given by

$$\sin\theta = \frac{|\vec{A} \times \vec{B}|}{|A|\,|B|}$$

39. **Cross product in Cartesian co-ordinates:**

$$A \times B = (A_x \hat{i} + A_y \hat{j} + A_z k) \times (B_x \hat{i} + B_y \hat{j} + B_z k)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{i} - (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x) k.$$

40. **Examples of cross product:** (i) Moment of force or torque, $\vec{\tau} = r \times \vec{F}$

41. **Position and displacement vectors:** The position vector of an object in x – y plane is given by

$$r = x\hat{i} + y\hat{j}$$

and the displacement from position r to position r is given by

$$\Delta r = \vec{r} - r = (x' - x)\hat{i} = (y' - y)\hat{j} = \Delta x\hat{i} + \Delta y\hat{j}$$

42. **Velocity vector:** If an object undergoes a displacement Δr in time Δt, its average velocity is given by $\vec{v} = \frac{\Delta \vec{r}}{\Delta t}$

The velocity of an object at time t is the limiting value of the average velocity as Δt tends to zero. Thus

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

In component form, we have $v = v_x \hat{i} + v_y \hat{j} + v_z k$

where $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$, vs. $= \frac{dz}{dt}$ and

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

When position of a particle is plotted on a coordinate system, $\vec{v}$ is always tangent to the curve representing the path of the particle.

43. **Acceleration vector:** If the velocity of an object changes from $\vec{v}$ to $\vec{v'}$ in time Δt, then its average acceleration is given by

$$\bar{a} = \frac{\vec{v'} - \vec{v}}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}$$

The acceleration $\vec{a}$ at any time t is the limiting value of a as Δt tends to zero. So

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

In component form, we have $a = a_x\hat{i} + a_y\hat{j} + a_z k$

where $a_x = \frac{dv_x}{dt}$, $a_y = \frac{dv_y}{dt}$, $a_z = \frac{dv_z}{dt}$ and $|a| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

44. **Equations of motion in vector form**: For motion with constant acceleration,

(i) $\vec{v} = \vec{v}_0 + \vec{a}\,t$

(ii) $\vec{r} = \vec{r}_0 + \vec{v}_0 t + ½ \vec{a} t^2$

(iii) $v^2 – v_0^2 = 2\vec{a}\cdot(\vec{r} – \vec{r}_0)$

The motion in a plane with uniform acceleration can be treated as the superposition of two separate simultaneous one-dimensional motions along two perpendicular directions.

45. **Relative velocity**: The relative velocity of an object A with respect to object B, when both are in motion, is the rate of change of position of object A with respect to object B.

(i) If the two objects are moving with velocities v_A and v_B with respect to the ground, then

Relative velocity of A w.r.t. B, $\vec{v}_{AB} = \vec{v}_A – \vec{v}_B$

Relative velocity of B w.r.t. A, $\vec{v}_{BA} = \vec{v}_B – \vec{v}_A$

(ii) If the two objects A and B are moving with velocities $\vec{v}_A$ and $\vec{v}_B$ inclined at an angle θ, then magnitude of the relative velocity of A w.r.t. B is given by

$v_{AB} = \sqrt{|v_A|^2 + |–v_B|^2 + 2\,|v_A|\,|–v_B|\cos(180° – \theta)}$

$= \sqrt{v^2_A + v^2_B – 2v_Av_B\cos\theta}$

If the velocity v_{AB} makes angle β with the velocity v_A, then

$\tan\beta = \frac{|–v_B|\sin(180° – \theta)}{|v_A| + |–v_B|\cos(180° – \theta)}$

$= \frac{v_B\sin\theta}{v_A – v_B\cos\theta}$

46. **Projectile motion**: Anybody projected into space, such that it moves under the effect of gravity alone is called a projectile. The path followed by a projectile is called its trajectory which is always a parabola.

A projectile executes two independent motions simultaneously:

(i) Uniform horizontal motion and (ii) Uniform accelerated downward motion.

47. **Projectile fired horizontally**: Suppose a body is projected horizontally with velocity u from a height h above the ground. Let

it reaches the point (x, y) after time t. Then

(i) Position of the projectile after time t: $x = ut$, $y = ½ gt^2$

(ii) Equation of trajectory: $y = \frac{g}{2u^2} . x^2$

(iii) Velocity after time t:

$v = \sqrt{u^2 + g^2 t^2}$; $\beta = \tan^{-1}\frac{gt}{u}$

(iv) Time of flight, $T = \sqrt{\frac{2h}{g}}$

(v) Horizontal range, $R = u \times T = u\sqrt{\frac{2h}{g}}$

48. **Projectile fired at an angle with the horizontal**: Suppose a projectile is fired with velocity u at an angle θ with the horizontal. Let it reach the point (x, y) after time t. Then

(i) Components of initial velocity:

$u_x = u\cos\theta$, $u_y = u\sin\theta$

(ii) Components of acceleration at any instant:

$a_x = 0$, $a_y = – g$

(iii) Position after time t: $x = (u\cos\theta)\,t$, $y = (u\sin\theta)\,t – ½ gt^2$

(iv) Equation of trajectory:

$$y = x \tan\theta - \frac{g}{2u^2\cos^2\theta} \cdot x^2$$

(v) Maximum height, $H = \frac{u^2 \sin^2\theta}{2g}$

(vi) Time of flight, $T = \frac{2u \sin}{2g}$

(vi) Time of flight, $T = \frac{2u\sin\theta}{g}$

(vii) Horizontal range, $R = \frac{u^2 \sin 2\theta}{g}$

(viii) Maximum horizontal range is attained at $\theta = 45°$ and its value is $R_{max} = \frac{u^2}{g}$

(ix) Velocity after time t: $v_x = u\cos\theta$, $v_y = u\sin\theta - gt$

$\therefore \quad v = \sqrt{v^2x + v^2y}$ and $\tan\beta = \frac{v_y}{v_x}$

(x) The velocity with which the projectile reaches the horizontal plane through the point of projection is same as the velocity of projection.

49. **Uniform circular motion:** When a body moves along a circular path with uniform speed, its motion is said to uniform circular motion.

50. **Angular displacement:** It is the angle swept out by a radius vector in a given time interval.

$$\theta \text{ (rad)} = \frac{\text{Arc}}{\text{Radius}} = \frac{s}{r}$$

51. **Angular velocity:** The angle swept out by the radius vector per second is calle3d angular velocity.

$$\omega = \frac{\theta}{t} \quad \text{or} \quad \omega = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

52. **Time period and frequency:** Time taken for one complete revolution is called time period (T). The number of revolution completed per second is called frequency (v).

$$\omega = \frac{2\pi}{T} = 2\pi v$$

53. **Relationship between v and ω:** It is given by $v = r\omega$

i.e., Linear velocity = Radius × angular velocity.

54. **Angular acceleration and its relation with linear acceleration:** The rate of change of angular velocity is called angular acceleration. It is given by

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

Also, $a = r\alpha$

i.e., Linear acceleration = Radius × angular acceleration

55. **Centripetal acceleration:** A body moving along a circular path is acted upon by an acceleration directed towards
the centre along the radius. This acceleration is called centripetal acceleration. It is given by

$$a = \frac{v^2}{r} = r\omega^2$$

THEORY: --

All the measurable physical quantity can be divided into two classes: -- i) ***Scalar*** (ii) ***Vector quantities***

i) **Scalar Quantities**: - ***"Scalar quantities are those physical quantity which are characterized by magnitude only."***

.... **Scalar quantity are directionless quantities.**
.... **Scalar quantity obeys the ordinary laws of algebra**.
.... **Scalar quantity changes with change in magnitude only.**
...................**Represented by ordinary letter.**

ii) **Vector Quantities**: - **"Vector quantities are those physical quantities which are characterized by both magnitude and direction."**

- Since concept of vectors involves the idea of direction, therefore vectors do not follow the ordinary laws of Algebra.
- Vector quantities change with the change in magnitude or with the change in direction or with the change of both magnitude & direction.
- Represented by bold faced letter or letters having arrow over them i.e., $\vec{A}$ (read as vector A).

NEED FOR VECTORS

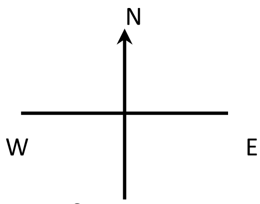
In one-dimensional motion, only two directions are possible. So, the directional aspect of the quantities like position, displacement, velocity and acceleration can be taken care of by using + and – signs. But in case of motion in two-dimensions (plane) or three dimensions (space), an object can have a large number of directions. In order to deal with such situations effectively, we need to introduce the concept of new physical quantities, called vectors, in which we taken care of both magnitudes the direction.

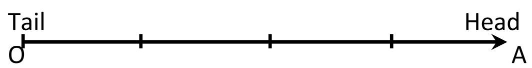
Points of differences between scalars and vectors.

Scalars	*Vectors*
1. Scalars have only magnitude.	Vectors have both magnitude and direction.
2. They change if their magnitude changes.	They change if either their magnitude, direction or both changes.
3. They can be added according to ordinary laws of algebra.	They can be added only by using special laws of vector addition.

□□□□REPRESENTATION OF A VECTOR

A vector quantity is represented by a straight line with an arrowhead over it. The length of the line gives the magnitude and the arrowhead gives the direction. Suppose a body has a velocity of 40 kmh^{-1} due east. If 1 cm is chosen to represent a velocity of 10 kmh^{-1} due east. If 1 cm is chosen to represent a velocity of 10 kmh^{-1}, a line OA, 4 cm in length and drawn towards east with arrowhead at A will completely represent the velocity of the body. The point A is called head or terminal point and point O is called tail or initial point of the vector $\vec{OA}$ [Fig.]





[Representation of a vector]

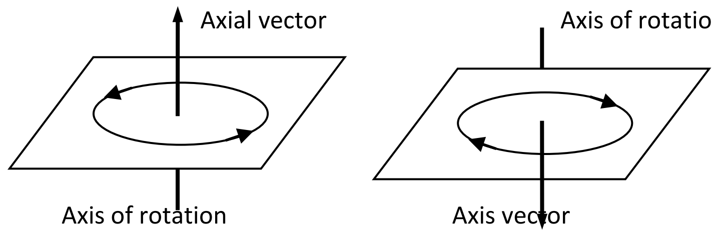
In a simpler notation, a vector is represented by single letter of alphabet either in bold face or with an arrow over it. For example, a force vector can be represented as $\vec{F}$ or F.

POLAR AND AXIAL VECTORS

Broadly speaking, vectors are of two types:

Polar vectors: The vectors which have a starting point or a point of application are called polar vectors.
Examples: Displacement, velocity, force, etc., are polar vectors.

Axial vectors: The vectors which represent rotational effect and act along the axis of rotation in accordance with right hand screw rule and called axial vectors.
Examples: Angular velocity, torque, angular momentum, etc. are axial vectors.



As shown in Fig., axial vector will have its direction along its axis of rotation depending on its anticlockwise or clockwise rotational effect.

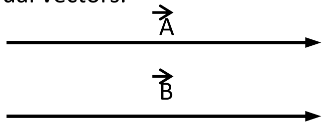
Knowledge +...............

- The Latin work vector means ***carrier***.
- The physical quantities which have no specified direction and have different values in different directions are called tensors. For examples, moment of inertia.

SOME DEFINITIONS IN VECTOR ALGEBRA

(i) Equal vectors **(ii) Negative of a vector** **(iii) Modulus of a vector** **(iv)Unit vector**
(v) Fixed vector **(vi) Free vector** **(vii) Collinear** **(viii) Coplanar vectors**
(ix) Co-initial vectors **(x) Co-terminus vectors.**

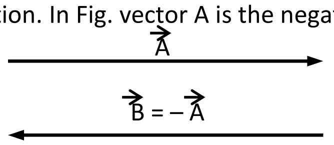
(i) **Equal vectors:** Two vectors are said to be equal if they have the same magnitude and same direction. In Fig. A and B are two equal vectors.



[Equal vectors]

□□ If a vector is displaced parallel to itself, it remains equal to itself.

(ii) **Negative of a vector:** The negative of a vector is defined as another vector having the same magnitude but having an opposite direction. In Fig. vector A is the negative of vector B or vice versa.



[Negative of a vector]

(iii) **Modulus of a vector:** The modulus of a vector means the length or the magnitude of that vector. It is a scalar quantity.

Modulus of vector $\vec{A} = |\vec{A}| = A$

(iv) **Unit vector:** A unit vector is a vector of unit magnitude drawn in the direction of a given vector.

A unit vector in the direction of a given vector is found by dividing the given vector by its modulus. Thus, a unit vector in the direction of vector, $\vec{A}$ is given by

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$

A unit vector in the direction of a given vector A is written as A and is pronounced as 'A carat' or 'A hat' or 'A cap'. Any vector can be expressed as the magnitude times the unit vector along its own direction.

$$\vec{A} = |\vec{A}|\,\hat{A}$$

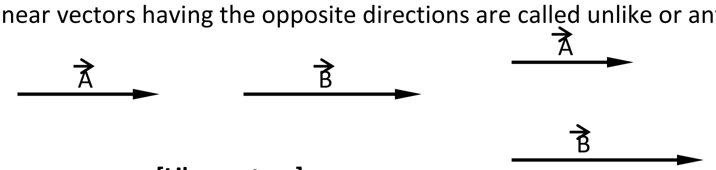
□□□ **The magnitude of a unit vector is unity.**
□□□ **It just gives the direction of a vector.**
□□□ **A unit vector has no units or dimensions**.

(iv) **Fixed vector:** The vector whose initial point is fixed is called a fixed vector or a localised vector. For example, the position vector of a particle is a fixed vector because it initial point lies at the origin.

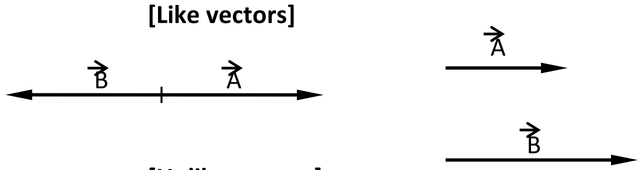
(v) **Free vector:** A vector whose initial point is not fixed is called a free vector of a non-localised vector. For example, the velocity vector of a particle moving along a straight line is a free vector.

(vi) **Collinear vectors:** The vectors which either act along the same line or along parallel lines are called collinear vectors.

□□ Two collinear vectors having the same direction (θ = 0°) are called like or parallel vectors.
□□ Two collinear vectors having the opposite directions are called unlike or antiparallel vectors.



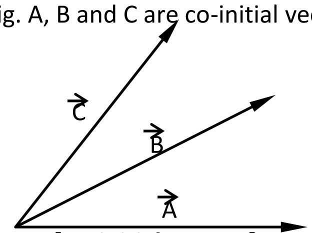
[Like vectors]



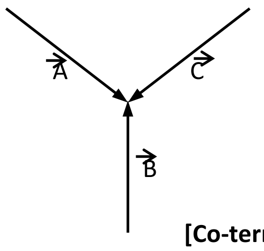
[Unlike vectors]

(vii) **Coplanar vector**: The vectors which act in the same plane are called coplanar vectors.

(viii) **Co-initial vectors**: The vectors which have the same initial point are called co-initial vectors. In Fig. A, B and C are co-initial vectors.



[Co-initial vectors]



[Co-terminus vectors]

(ix) **Co-terminus vectors**: The vectors which have the common terminal point are called co-terminus vectors. In Fig. A, B and C are co-terminus vectors.

ix) **Orthogonal unit vector**: "Three mutually ⊥ar lines meeting at a point to form a rectangular co-ordinate system."

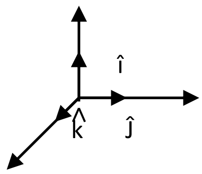
- The unit vector along X-axis, Y- axis & Z-axis are $\hat{i}$, $\hat{j}$, $\hat{k}$ resp.
- The unit vectors $\hat{i}$, $\hat{j}$, & $\hat{k}$ are mutually ⊥ar to each other & ***hence called orthogonal unit vectors***.
- The unit vector $\hat{i}$, $\hat{j}$, & $\hat{k}$ represents the direction of vectors in X , Y and Z axis.

Ex: - If A along x-axis

Then $\vec{A} = |\vec{A}|\,\hat{i}$

I| y $\vec{A} = |\vec{A}|\,\hat{j}$ (along Y axis)

and $\vec{A} = |\vec{A}|\,\hat{k}$ (along Z axis).



ZERO VECTOR AND ITS PROPERTIES

Zero vector: A zero or null vector is a vector that has zero magnitude and an arbitrary direction. It has represented by 0 (arrow over the number 0).

□□**Need of a zero vector**: The need of a zero vector arises from the following situations:

(i) If $\vec{A} = \vec{B}$, then what is $\vec{A} - \vec{B}$?

(ii) If $\mu = -\lambda$, then what is $(\lambda + \mu)\vec{A}$?

In all these cases the resultant has to be a vanishingly vector and not a scalar. Hence there is need for introducing the concept of zero vector.

(iii) When a vector is multiplied by zero, we get zero vector.

$0\,\vec{A} = \vec{0}$

(iv) If λ and μ are two different non-zero real numbers, then the relation can hold only if both $\vec{A}$ and $\vec{B}$ are zero vectors.

$\lambda\,\vec{A} = \mu\,\vec{B}$

□□**Physical examples of zero vector:**

(i) The position vector of a particle lying at the origin is a zero vector.

(ii) The velocity vector of a stationary object is a zero vector.

(iii) The acceleration vector of an object moving with uniform velocity is a zero vector.

Multiplication of a vector by Real No: - The multiplication of a vector by scalar quantity 'n' gives a new vector whose magnitude is 'n' times the magnitude of the given vector & direction is same as that of the given vector [where 'n' is a positive real no.]

Let a vector a be multiplied by a scalar quantity 'n'. Then new vector is $\vec{A} = n\,\vec{a}$

Ex: - If n = 5, then $\vec{A} = 5\,\vec{a}$

Here, Mag. of the new vector $\vec{A}$ is 5 time the mag. of the vector $\vec{a}$ but the direction remains the same as that of the vector $\vec{a}$.

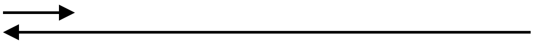
i.e., $\vec{a}$

$\vec{A} = 5\vec{a}$

► If n = - 5, then $\vec{A} = -5\vec{a}$

Here, magnitude of $\vec{A}$ is 5 times the magnitude of a but the direction of $\vec{A}$ is opposite to that of $\vec{a}$.

$\vec{a}$



$\vec{A} = -5\vec{a}$

If n = 0, then $\vec{A} = 0$ (null vector.)

* The unit of $\vec{A} = 4\vec{a}$ is same as that of $\vec{a}$.

Multiplication of a vector by a scalar: – When a vector $\vec{v}$ is multiplied by a scalar 'S', it becomes $\vec{v}$ where magnitude is S times of $\vec{v}$ and it acts along the direction of $\vec{v}$.

The unit of S v is different from the unit of v.

Ex: - If $\vec{v}$ = 100 N due east and S = 10 sec.

∴ $\vec{v}$ = 10 × 100 N sec due each = 10^3 N – sec due east.

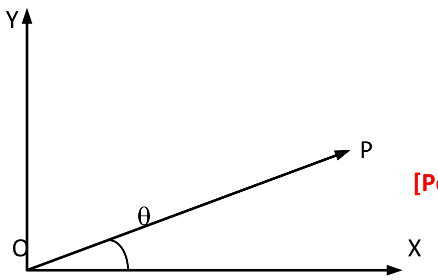
** **The units of a vector affected when it is multiplied by a scalar having units or dimensions.**

If λ is a pure number having no units or dimensions, then the units of $\lambda\vec{A}$ are the same as that of A. However, when a vector $\vec{A}$ is multiplied by a scalar λ which has certain units, the units of resultant $\lambda\vec{A}$ are obtained by multiplying the units of $\vec{A}$ by the units of λ. For example, when velocity vector is multiplied by mass (a scalar), we get momentum. The units of momentum are obtained by multiplying the units of velocity by units of mass.

◘ ◘ Position vector & Displacement vector:

"**A vector which gives the position of a point w. r. t the origin of the rectangular co-ordinate system is called position vector".**

Consider the motion of an object in X – Y plane with origin at O. Suppose an object is at point P at any instant t, as shown in Fig. Then OP is the position vector of the object at point P.



[Position vector]

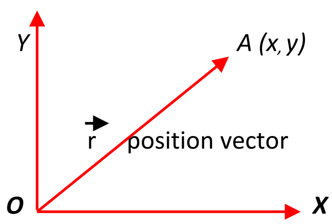
i.e., *A position provides two kinds of information................................*

☞ i) *It tells the straight-line distance of the object from the origin.*

☞ ii) It tells the direction of the object w. r. t. origin.

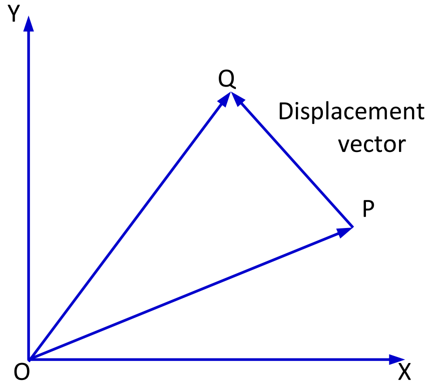
◘ Position vector is represented by $\vec{r}$.

◘ Magnitude of position vector is 'r' i.e., r = length of OA.



Displacement vector: - "**Displacement vector is a vector which gives the position of a point with reference to a point other them the origin of the co-ordinate system".**

Consider an object moving in the XY-plane. Suppose it at point P at any instant t and at point Q at any later instant t', as shown in Fig. Then vector PQ is the displacement vector of the object in time t to t'.



[Displacement vector]

Consider a moving particle in X – Y plane.
Let at any instant 't', its position is A.

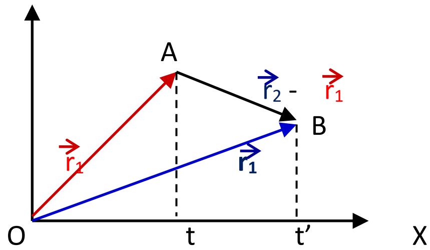
∴ Position vector, $\vec{OA} = \vec{r}_1$

After time 't' the particle moves to a new position B.

Now, **Position vector = $\vec{OB} = \vec{r}_2$**

∴ Displacement vector (in time interval t – t`)

$= \vec{AB} = \vec{r}_2 - \vec{r}_1$ (△ law of vector addition)



➢ **Position vector will be different for different position of the same particle.**

- The position vector r at any time 't' in term of 2 – D rectangular co-ordinate system is.

  $\vec{r} = \vec{x} + \vec{y}$, $\vec{r} = x\,\hat{i} + y\,\hat{j}$.

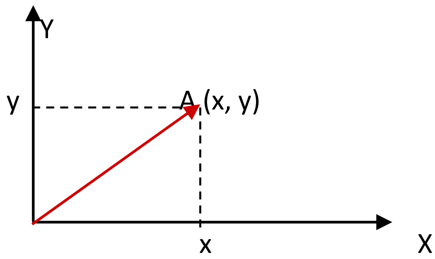
  In Magnitude, $|\vec{r}|$ or $r = \sqrt{x^2 + y^2}$

- If the position of a point A is chosen w. r. t. origin of the 3.D rectangular co-ordinate system,

  Then position vector is $\vec{r} = \vec{x} + \vec{y} + \vec{z}$

  or, $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

  Magnitude of $\vec{r} = |\,r\,|$ or $r = \sqrt{x^2 + y^2 + z^2}$



➢ <u>**Displacement vector is the difference of two position vector:**</u> -

Displacement (In 2 D).

If (x_1, y_1) and (x_2, y_2) are the co-ordinates of P & R resp.

Then $\vec{r}_1 = x_1\,\hat{i} + y_1\,\hat{j}$

$\vec{r}_2 = x_2\,\hat{i} + y_2\,\hat{j}$.

∴ , Displacement vector $\vec{PR} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$

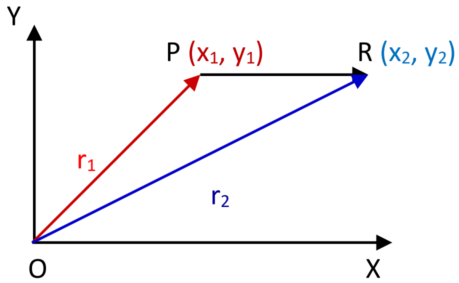
Mag. of PR = $|\,PR\,| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

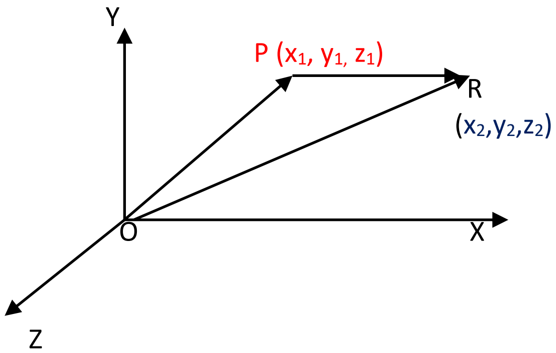
<u>In 3 D</u>

Displacement vector

$PR = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$

Mag. of PR = | PR | $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$





➢➢ <u>**Resultant vector**</u>: -

"The resultant of two or more vectors is a single vector which produces the same effects as the individual vectors do so"

Addition of V-E-C-T-O-R-S:

"The process of adding two or more than two vectors is called addition or composition of vectors."

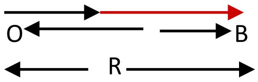
=> When two or more than two vectors are added, we get a single vector called Resultant vector.

► Case 1.: Addition of collinear vectors (when two vectors act in the same direction)

Consider two collinear vector $\vec{A}$ and $\vec{B}$. Now, $\vec{A}$ and $\vec{B}$ be added by drawing vector $\vec{A}$ and then displace B (without, changing its direction) in such away than the tails of $\vec{B}$ coincide with the head of A.

∴ Resultant vector $\vec{R} = \vec{A} + \vec{B}$

= O B

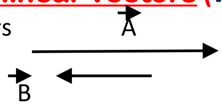


Ex: - Let a body be displaced 2 m due last and them it is further displaced through 3 m due to each.

∴ Resultant vector R = (2 + 3) = 5 m due east

► Case 2.: Addition of ANTI-collinear vectors (when two vectors act in the opposite direction).

Let $\vec{A}$ and $\vec{B}$ be two anti collinear vectors



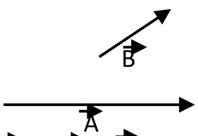
∴ Resultant vector $\vec{R} = \vec{A} - \vec{B}$

Ex: - Let a body be displaced 5 m due west and then 2 m due east.

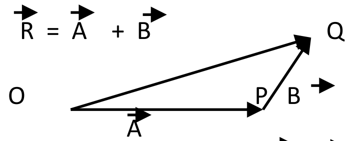
∴ Resultant vector, R = (5 – 3) = 2 m due west.

► Case 3: Addition of two vector pointing in different:

When the vectors are pointing in different directions, they can be added using the laws of vector addition. Consider two vector $\vec{A}$ and $\vec{B}$.



$\vec{R} = \vec{A} + \vec{B}$



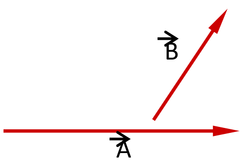
Resultant vector, $\vec{OQ} = \vec{OP} + \vec{PQ}$ $\vec{R} = \vec{A} + \vec{B}$

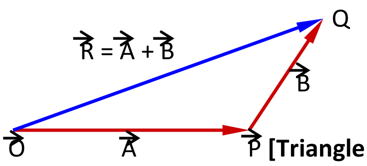
◘ ◘ LAWS OF VECTOR ADDITION:

- *i) Triangle law of vector addition.*
- *ii) Parallel law of vector addition.*
- *iii) Polygon law of vector addition.*

◙ **Triangle law of vector addition**: If two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented completely, both in magnitude and direction, by the third side of the triangle taken in the opposite order.

Suppose we wish to add two vectors A and B as shown in Fig. (a). Draw a vector OP equal and parallel to vector A, as shown in Fig. (b). From head P of OP, draw a vector PQ equal and parallel to vector B. Then the resultant vector is given by OQ which joins the tail of A and head of B.





[Triangle law of vector addition]

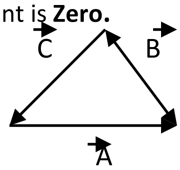
According to triangle law of vector addition:

$\vec{OQ} = \vec{OP} + \vec{PQ}$

or $\vec{R} = \vec{A} + \vec{B}$

► ☞ Triangle law of vector is applicable to Δ of any shape.

Corollary: - From the triangle law of vector if three vectors are represented by three sides of a △ taken in order, then, their resultant is **Zero.**



Then $\vec{A} + \vec{B} + \vec{C} = 0$

Resultant ($-\vec{C}$) of $\vec{A} + \vec{B}$ cancel the 3rd vector $\vec{C}$.

ANALYTICAL METHOD OF VECTOR - ADDITION

Let the two vectors $\vec{A}$ and $\vec{B}$ be represented both in magnitude and direction by the sides OP and PQ of ΔOPQ taken in the same order. Then according to the triangle law of vector addition, the resultant R is given by the closing side OQ taken in the reverse order, as shown in Fig.

Magnitude of the resultant R: From Q, drawn QN perpendicular to AB produced.

Then, $\angle QPN = \theta$, OA = A, PQ = B, OQ = R

From right angled Δ QNP, we have

$\frac{QN}{PQ} = \sin\theta$ or $QN = PQ \sin\theta = B \sin\theta$

and $\frac{PN}{PQ} = \cos\theta$ or $PN = PQ \cos\theta = B \cos$

Using Pythagoras theorem in right angled ΔONQ, we get

$OQ^2 = ON^2 + QN^2$

$= (OP + PN)^2 + QN^2$

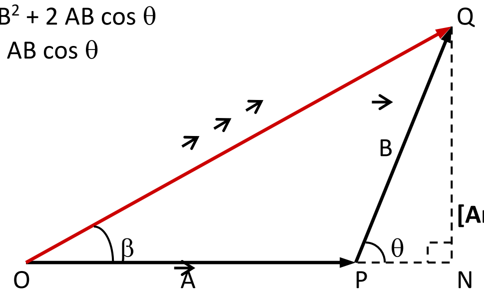
or $R^2 = (A + B\cos\theta)^2 + (B \sin\theta)^2$

$= A^2 + B^2 \cos^2\theta + 2AB\cos\theta + B^2 \sin^2\theta$

$= A^2 + B^2 (\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$

$= A^2 + B^2 + 2AB\cos\theta$

or $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$



[Analytical treatment of Δ law of vector addition]

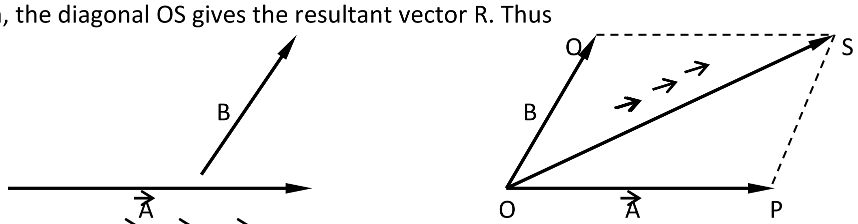
Direction of the resultant R: Let the resultant R make an angle β with the direction of A. Then from right angled Δ ONQ, we get $\tan\beta = \frac{QN}{ON} = \frac{QN}{OP + PN}$

or $\tan\beta = \frac{B\sin\theta}{A + B\cos\theta}$

$$\beta = \tan^{-1} \frac{B\sin\theta}{A + B\cos\theta}$$

Parallelogram law of vector addition: **If two vectors can be represented both in magnitude and direction by the two adjacent sides of a parallelogram drawn from a common point, then their resultant is completely represented, both in magnitude and direction, by the diagonal of the parallelogram passing through that point.**

Suppose wish to add two vectors A and B, as shown in Fig. (a). From a common point O, draw a vector OP equal and parallel to A and vector OQ equal and parallel to B. Complete the parallelogram OPSQ According to the parallelogram law of vector addition, the diagonal OS gives the resultant vector R. Thus



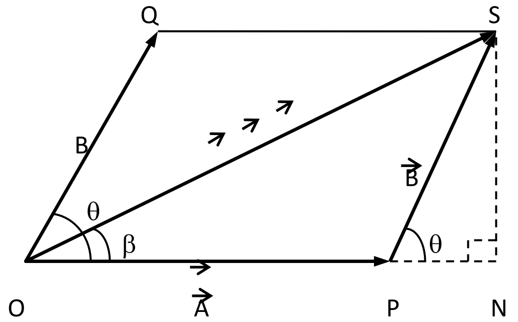
$\vec{OS} = \vec{OP} + \vec{OQ}$

or $\vec{R} = \vec{A} + \vec{B}$

ANALYTICAL METHOD OF parallelogram law of VECTOR - ADDITION:

Let the two vectors A and B inclined to each other at an angle θ be represented both in magnitude and direction by the adjacent sides OP and OQ of the parallelogram OPSQ.

Then according to the parallelogram law of vector addition, the resultant of A and B is represented both in magnitude and direction by the diagonal OS of the parallelogram.



[Analytical treatment of parallelogram law]

Magnitude of resultant R: Draw SN perpendicular to OP produced.

Then ∠SPN = ∠QOP = θ, OP = A, PS = OQ = B, OS = R

From right angle Δ SNP, we have

$$\frac{SN}{PS} = \sin\theta \text{ or } SN = PS\sin\theta = B\sin\theta$$

$$\frac{PN}{PS} = \cos\theta \text{ or } PN = PS\cos\theta = B\cos\theta$$

Using Pythagoras theorem is right-angled Δ ONS, we get

$$OS^2 = ON^2 + SN^2 = (OP + PN)^2 + SN^2$$

or $R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$

$= A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$

$= A^2 + B^2 + 2AB\cos\theta$

or $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

Direction of resultant R: Let the resultant $\vec{R}$ make angle β with the direction of $\vec{A}$. Then from right angled Δ ONS, we get

$$\tan\beta = \frac{SN}{ON} = \frac{SN}{OP + PN} \quad \text{or} \quad \tan\beta = \frac{B\sin\theta}{A + B\cos\theta}$$

♦♦ **Special cases:** (i) ***If the two vectors $\vec{A}$ and $\vec{B}$ are acting along the same direction, θ = 0°. Therefore, the magnitude of the resultant is given by***

$R = \sqrt{A^2 + B^2\cos^2\theta + 2AB\cdot\cos 0°}$

$= \sqrt{A^2 + B^2 + 2AB}$ [∵ cos 0° = 1]

$= \sqrt{(A + B)^2}$

or $R = A + B$

∴ **Magnitude of the resultant vector is equal to the sum of magnitude of two vectors acting along the same direction and the resultant vector also acts along that direction.**

(ii) ***If the vectors $\vec{A}$ and $\vec{B}$ are acting along mutually opposite directions, θ = 180° and therefore,***

$R = \sqrt{A^2 + B^2 + 2AB\cos 180°}$

$= \sqrt{A^2 + B^2 - 2AB}$ [∵ cos 180° = – 1]

or $R = (A - B)$

∴ **Resultant is equal to the positive difference between magnitudes of two vectors and acts along the direction of bigger vector.**

(iii) ***When the two vectors are acting at right angle to each other, θ = 90° and therefore,***

$R = \sqrt{A^2 + B^2 + 2AB\cos 90°}$

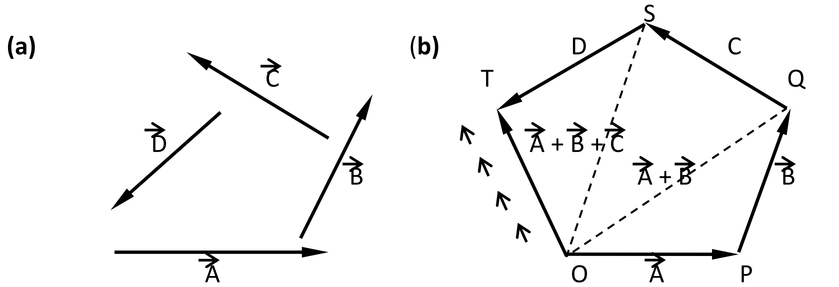
$= \sqrt{A^2 + B^2}$ [∵ cos 90° = 0]

Also, $\tan\beta = \frac{B\sin 90°}{A + B\cos 90°} = \frac{B}{A}$

∴ $\beta = \tan^{-1}\left(\frac{B}{A}\right)$

◙ **Polygon law of vector addition: If a number of vectors are represented both in magnitude and direction by the sides of an open polygon taken in the same order, then their resultant is represented both in magnitude and direction by the closing side of the polygon taken in opposite order.**

Illustration: Suppose we wish to add four vectors A, B, C and D, as shown in Fig. (a). Draw vector OP = A. Move vectors B, C and D parallel to themselves so that the tail of B touches the head of A, the tail of C touches the head of B and the tail of D touches the head of C, as shown in Fig. (b). According to the polygon law, the closing side OT of the polygon taken in the reverse order represents the resultant R. Thus $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$



[Polygon law of vector addition]

Proof: We apply triangle law of vector addition to different triangles of the polygon shown in Fig. (b).

In Δ OPQ, $\vec{OQ} = \vec{OP} + \vec{PQ} = \vec{A} + \vec{B}$

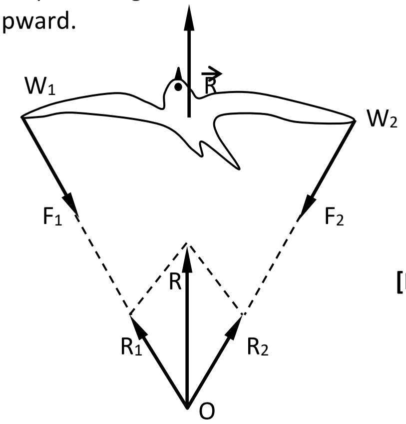
In Δ OQS, $\vec{OS} = \vec{OQ} + \vec{QS} = \vec{A} + \vec{B} + \vec{C}$

In Δ OST, $\vec{OT} = \vec{OS} + \vec{ST} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$

or $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$ This proves the polygon law of vector addition.

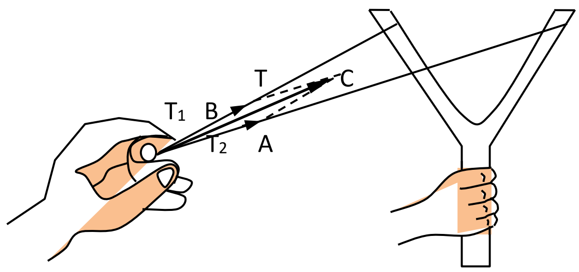
► **Examples of composition of vectors:**

Flight of bird: When a bird flies, it pushes the air with forces F_1 and F_2 in the downward direction with its wings W_1 and W_2. The lines of action of these two forces meet at point O. In accordance with Newton's third law of motion, the air exists equal and opposite reactions R_1 and R_2. According to the parallelogram law, the resultant R of the reaction R_1 and R_2 acts on the bird in the upward direction and helps the bird to fly upward.



[Flight of bird]

Working a sling: A sling consists of a Y-shaped wooden or metallic frame, to which a rubber band is attached, as shown in Fig. When a stone held at the point O on the rubber band is pulled, the tensions T_1 and T_2 and produced along OA and OB in the two segments of the rubber band. According to the parallelogram law of forces, the resultant T of the tensions T_1 and T_2 acts on the stone along OC. As the stone is released, it moves under the action of the resultant tension T in forward direction with a high speed.



Properties of Vector -Addition:-

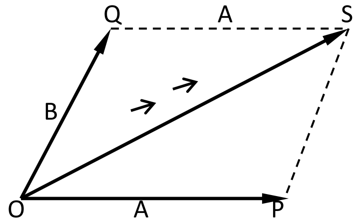
► **Vector addition is commutative:** In Fig., the sides OP and OQ of a parallelogram OPSQ represent vectors A and B respectively. According to parallelogram law of vector addition, diagonal OS gives the resultant. Thus

$A + B = OS$

or $OP + OQ = OS$ ... (1)

As $OP = QS$ and $OP \parallel QS$, therefore

$QS = OP = A$



[Vector addition is commutative]

Using triangle law of vector addition in ΔOQS, we get

$\vec{OQ} + \vec{QS} = \vec{OS}$

or $\vec{B} + \vec{A} = \vec{OS}$ ... (2)

From equations (1) and (2), we find that

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ This proves that the vector addition is commutative.

► **Vector addition is associative**: Suppose three vectors A, B and C are represented by the sides OP, PQ and QS of a polygon OPQS. According to polygon law, OS represents the resultant R both in magnitude and direction. Join O to Q and P to S.

Using triangle law of vector addition,

In Δ OPQ, $OQ = OP + \vec{PQ} = \vec{A} + \vec{B}$

In Δ OQS, $\vec{OS} = \vec{OQ} + \vec{QS} = (\vec{A} + \vec{B}) + \vec{C}$ ... (1)

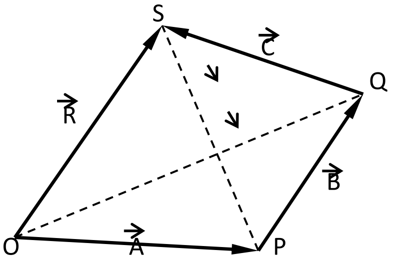
In Δ PQS, $\vec{PS} = \vec{PQ} + \vec{QS} = \vec{B} + \vec{C}$

In Δ OPS, $\vec{OS} = \vec{OP} + \vec{PS} = \vec{A} + (\vec{B} + \vec{C})$ ... (2)

From equations (1) and (2), we find that

$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

This proves that vector addition is associative.



SUBTRACTION OF VECTOR:

Let us subtract a vector $\vec{B}$ from $\vec{A}$.

i.e., $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Thus, "**subs traction of a vector B from vector by adding A is same as adding (-B) in vector A**

Consider two vector $\vec{A}$ & $\vec{B}$ represents both in magnitude and direction by $\vec{OP}$ and $\vec{PQ}$ respectively.

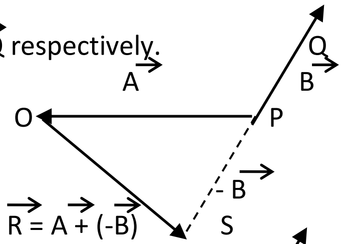
Now, we have to draw a vector PS = (- B) (Equal & opp. to B)

Using △ law of vector addition to△ OPS.

$\vec{OS} = \vec{OP} + \vec{PS}$

$\vec{R} = \vec{A} + (-\vec{B})$

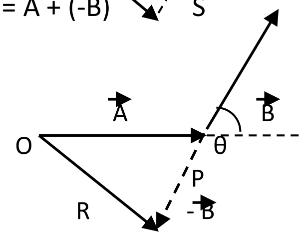
$\vec{R} = \vec{A} - \vec{B}$



Direction and Mag:

If θ is the angle between $\vec{A}$ & $\vec{B}$

Then angle between A and - $\vec{B}$ is (180 – θ)



∴ Mag. R = $\sqrt{A^2 + B^2 + 2AB\cos(180 - \theta)} = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

Direction, $\tan\alpha = \dfrac{B\sin(180 - \theta)}{A + B\cos(180 - \theta)} = \dfrac{B\sin\theta}{A - B\cos\theta}$

▪ **Vector subs traction does not follow associative law:-**

i.e., $\vec{A} - (\vec{B} - \vec{C}) \neq (\vec{A} - \vec{B}) - \vec{C}$

▪ **Vector subs traction does not follow communicative law**

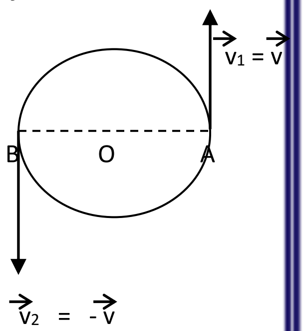
► **Illustration of vector subtraction: -**

i) ***A particle is moving with constant speed in a circular orbit. Find the change in its velocity when it completes half the revolution***.

Sol: - When a particle moves in a circular path with constant speed then its velocity changes continuously due to change in direction. However, the magnitude of velocity remains constant.

Let $\vec{v}_1 = v$ (at $\vec{A}$) then $v_2 \vec{=}$ -v (at B)

after travelling half revolution change in velocity $\triangle\vec{v} = \vec{v}_1 - \vec{v}_2 = v - (-v) = 2v$

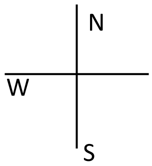


(ii) **A car moving towards south changes its direction and moves towards west with same speed. Find the change in the velocity. of the car.**

Sol. Let $\vec{v}_1$ be the velocity of the car towards the south direction and $\vec{v}_2$ be the velocity of the car towards the west.

Here, $|\vec{v}_1| = |\vec{v}_2| = v$ (say].

∴ Changes in velocity of the car, $\triangle \vec{v} = \vec{v}_2 - \vec{v}_1$

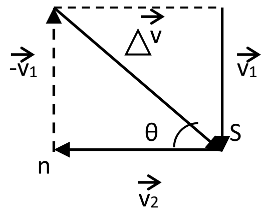


Mag. of change in velocity $|\triangle \vec{v}| = \sqrt{v_1^2 + v_2^2 + 2 v_1 v_2 \cos 90^0} = \sqrt{v_1^2 + v_2^2 + 0}$

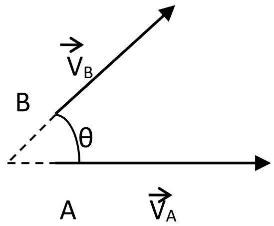
$= \sqrt{2 v^2} = \sqrt{2}\, v$

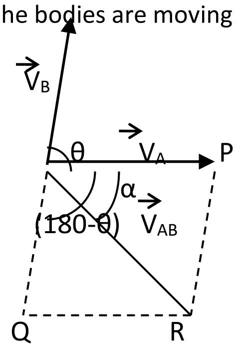
Direction: - $\tan \theta = \dfrac{|\vec{v}_1|}{|\vec{v}_2|} = \dfrac{v}{v} = 1 = \tan 45^0$

$\theta = 45^0$



Relative velocity (Rel. velo. of one body w. r. t. another body when both the bodies are moving in the direction inclined to each other)





Consider two bodies A & B moving with velo. $\vec{V}_A$ & $\vec{V}_B$ resp. making an. angle θ with each other.

Rel. velocity of A w. r. t. B, $\vec{V}_{AB} = \vec{V}_A - \vec{V}_B = \vec{V}_A + (-\vec{V}_B)$

∴ Mag. of $\vec{V}_{AB} = V_{AB} = \sqrt{V_A^2 + V_B^2 + 2 V_A V_B \cos(180 - \theta)}$

∴ $V_{AB} = \sqrt{V_A^2 + V_B^2 - 2 V_A V_B \cos\theta}$

[∴ $\cos(180 - \theta) = -\cos\theta$]

Direction:-

Let, $\vec{V}_{AB}$ makes an angle α with $\vec{V}_A$, then

$\tan\alpha = \dfrac{V_B \sin(180 - \theta)}{V_A + V_B \cos(180 - \theta)}$

$\tan\alpha = \dfrac{V_B \sin\theta}{V_A - V_B \cos\theta}$

If $\theta = 90^0$ then $|\vec{V}_{AB}| = \sqrt{V_A^2 + V_B^2}$ (mag.)

(direction) $\tan\alpha = V_B / V_A$

Examples based on Composition of Vectors

- **Formulae Used**

1. By triangle law or parallelogram law of vector addition, the magnitude of resultant $\vec{R}$ of two vectors $\vec{P}$ and $\vec{Q}$ inclined to each other at angle θ, is given by

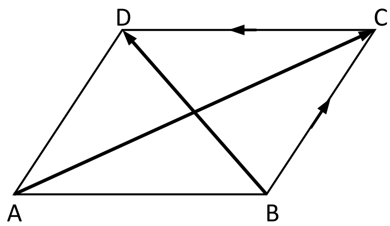
$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

2. If resultant $\vec{R}$ makes an angle β with $\vec{P}$, then

$\tan\beta = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

Q. 1. ABCD is a parallelogram and $\vec{AC}$ and $\vec{BD}$ are its diagonals. Prove that (i) $\vec{AC} + \vec{BD} = 2\vec{BC}$ and (ii) $\vec{AC} - \vec{BD} = 2\vec{AB}$.

Sol. Using triangle law of vector addition in Fig.



$\vec{AC} = \vec{AB} + \vec{BC}$

$\vec{BD} = \vec{BC} + \vec{CD}$

(i) $\vec{AC} + \vec{BD} = \vec{AB} + \vec{BC} + \vec{BC} + \vec{CD}$

But $\vec{AB} = -\vec{CD}$

∴ $\vec{AC} + \vec{BD} = -\vec{CD} + 2\vec{BC} + \vec{CD} = 2\vec{BC}$

(ii) $\vec{AC} - \vec{BD} = \vec{AB} + \vec{BC} - \vec{BC} - \vec{CD}$

$= \vec{AB} - \vec{CD} = \vec{AB} - (-\vec{AB})$

$= 2\vec{AB}$

Q. 2. ***A body is simultaneously given two velocities, one 30 ms^{-1} due east and other 40 ms^{-1} due north. Find the resultant velocity.***

Sol. Let the body start moving from O, as shown in Fig.

$\vec{v}_A = \vec{OA}$ = 30 ms^{-1}, due east

$\vec{v}_B = \vec{OB}$ = 40 ms^{-1}, due north

According to parallelogram law, OC is the resultant velocity.

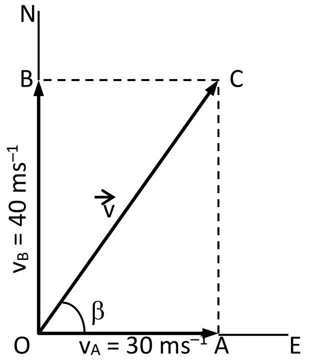
Its magnitude is

$v = \sqrt{v_A^2 + v_B^2} = \sqrt{30^2 + 40^2} = 50$ m

Suppose the resultant velocity $\vec{v}$ makes angle β with the east direction. Then

$\tan\beta = \frac{CA}{OA} = \frac{40}{30} = 1.3333$

∴ $\beta = \tan^{-1}(1.3333) = 53°8'$.



Q. 3. ***A particle has a displacement of 12 m towards east and 5 m towards the north and then 6 m vertically upward. Find the magnitude of the sum of these displacements.***

Sol. As shown in Fig., suppose initially the particle is at origin O.

Then its displacement vectors are

$\vec{OA}$ = 12 m, due east

$\vec{AB}$ = 5 m, due north

$\vec{BC}$ = 6 m, vertically upwards.

According to polygon law, $\vec{OC}$ is the resultant displacement.

From right ΔOAB,

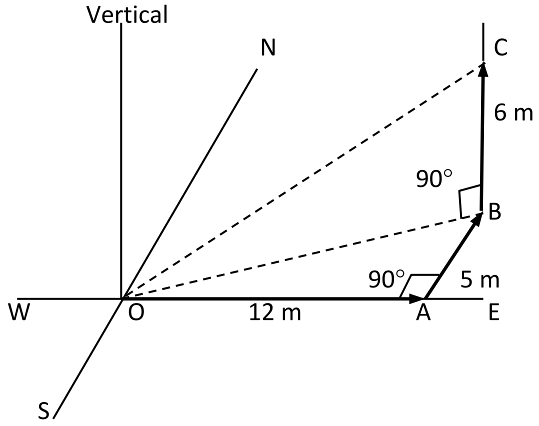
$OB = \sqrt{OA^2 + AB^2} = \sqrt{12^2 + 5^2}$

$= \sqrt{144 + 25} = 13$ m

From right ΔOBC,

$OC = \sqrt{OB^2 + BC^2} = \sqrt{13^2 + 6^2}$

$= \sqrt{169 + 36} = \sqrt{205} = 14.32$ m



Q. 4. ***Two forces of 5 N and 7 N act on a particle with an angle of 60° between them. Find the resultant force.***

Sol. Here P = 5 N, Q = 7N, θ = 60°

The magnitude of the resultant force is

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

$= \sqrt{5^2 + 7^2 + 2 \times 5 \times 7 \times \cos 60°}$

$= \sqrt{109} = 10.44$ N

If $\vec{R}$ makes angle β with the force $\vec{P}$, then

$\tan\beta = \frac{Q\sin\theta}{P + Q\cos\theta} = \frac{7\sin 60°}{5 + 7\cos 60°} = 0.7132$

or $\beta = \tan^{-1} 0.7132 = 35°\ 29'$.

Q. 5. ***Two vectors, both equal in magnitude, have their resultant equal in magnitude of the either. Find the angle between the two vectors.*** ***or*** ***Two equal forces have their resultant equal to either. What is the inclination between them?***

Sol. Here P = Q = R?

As $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

∴ $P = \sqrt{P^2 + P^2 + 2P \cdot P\cos\theta}$

or $P^2 = 2P^2(1 + \cos\theta)$ or $1 + \cos\theta = ½$

or $\cos\theta = -½ = \cos 120°$ or $\theta = 120°$

Q. 6. ***Two forces whose magnitudes are in the ratio of 3:5 give a resultant of 35 N. If the angle of inclination be 60°, calculate the magnitude of each force.***

Sol. Let P = 3x newton, Q = 5x newton, R = 35 n, θ = 60°

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

or $35 = \sqrt{(3x)^2 + (5x)^2 + 2 \times 3x + 5x\cos 60°}$

or 35 = 7x or $x = \frac{35}{7} = 5$

∴ 35 = 7x or $x = \frac{35}{7} = 5$

∴ P = 3 × 5 = 15 N and Q = 5 × 5 = 25 N

Q. 7. ***Two forces equal to P and 2 P newton act on a particle. If the first be doubled and the second be increased by 20 newtons, the direction of the resultant is unaltered. Find the value of P.***

Sol. Let the resultant make angle β with the force P.

∴ In first case, $\tan\beta = \frac{2P\sin\theta}{P + 2P\cos\theta}$

In second case, $\tan\beta = \dfrac{(2P + 20)\sin\theta}{2P + (2P + 20)\cos\theta}$

Hence $\dfrac{(2P + 20)\sin\theta}{2P + (2P + 20)\cos\theta} = \dfrac{2P\sin\theta}{P + 2P\cos\theta}$

or $\dfrac{2P\sin\theta}{P + 2P\cos\theta} = \dfrac{20\sin\theta}{P + 20\cos\theta}$ $\left(\because \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{a-b}{c-d}\right)$

From the above equation, 2P = 20

or P = 10 N.

Q. 8. ***The greatest and the least resultant of two forces acting at a point are 29 N and 5 N respectively. If each force is increased by 3 N, find the resultant of two new forces acting at right angle to each other.***

Sol. Let P and Q be the two forces. Then

Greatest resultant = P + Q = 29 N ... (i)

Least resultant = P – Q = 5 N ... (ii)

On solving (i) and (ii), we get

P = 17 N, Q = 12 N

When each force is increased by 3 N, new forces are

p = P + 3 = 17 + 3 = 20 N

q = Q + 3 = 12 + 3 = 15 N

As the new forces act at right angle to each other, their resultant is

$R = \sqrt{p^2 + q^2} = \sqrt{20^2 + 15^2} = \sqrt{625} = 25$ N

If the resultant R makes angle β with the force p, then

$\tan\beta = \dfrac{p}{q} = \dfrac{15}{20} = 0.75$

or $\beta = \tan^{-1}(0.75) = 36° 52'$.

Q. 9. ***The sum of the magnitude of two forces acting on a point is 18 N and the magnitude of their resultant is 12 N. If the resultant makes an angle of 90° with the force of smaller magnitude, what are the magnitudes of the two forces?***

Sol. Let the two individual forces be $\vec{P}$ and $\vec{Q}$ and θ be the angle between them. Let P < Q. If the resultant $\vec{R}$ makes angle β with the force $\vec{P}$, then

$\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

But $\alpha = 90°$ $\therefore$ $\dfrac{Q\sin\theta}{P + Q\cos\theta} = \tan 90° = \infty$

or $P + Q\cos\theta = 0$

Also $P + Q = 18$ N

As $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = 12$

$\therefore$ $P^2 + Q^2 + 2PQ\cos\theta = 144$ $[\because Q = 18 - P,\ Q\cos\theta = -P]$

or $P^2 + (324 + P^2 - 36P) - 2P^2 = 144$ or $6P = 180$

or P = 5 N and Q = 18 – 5 = 13 N

Q. 10. ***Establish the following vector inequalities:***

(i) $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ ***(ii)*** $|\vec{a}+\vec{b}| \ge |\vec{a}| - |\vec{b}|$ ***When does the equality sign apply?***

Sol. (i) If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$|\vec{a}+\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$

Now $|\vec{a}+\vec{b}|$ will be maximum when

$\cos\theta = 1$ or $\theta = 0°$

$\therefore$ $|\vec{a}+\vec{b}|_{max} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$

$= \sqrt{|\vec{a}|^2 + |\vec{b}|^2} = |\vec{a}| + |\vec{b}|$

Hence $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$

The equality sign is applicable when $\theta = 0°$ i.e., when $\vec{a}$ and $\vec{b}$ are in the same direction.

(ii) Again

$|\vec{a}+\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$

The value of $|\vec{a}+\vec{b}|$ will be minimum when $\cos\theta = -1$

or $\theta = 180°$

$\therefore$ $|\vec{a}+\vec{b}|_{min} = \sqrt{(\vec{a})^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos 180°}$

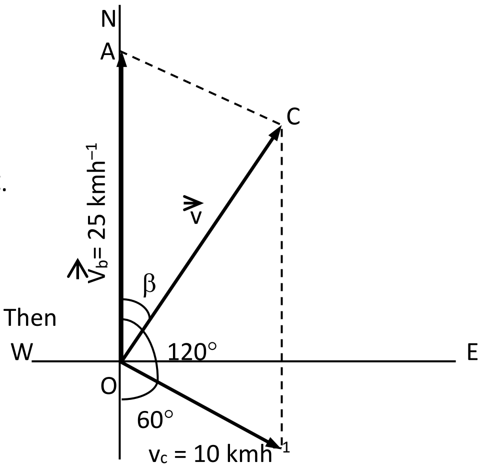
$= \sqrt{(|\vec{a}| - |\vec{b}|)^2} = |\vec{a}| - |\vec{b}|$

Hence $|\vec{a}+\vec{b}| \ge |\vec{a}| - |\vec{b}|$

The equality sign is applicable when $\theta = 180°$ i.e., when $\vec{a}$ and $\vec{b}$ are in opposite directions.

Q. 11. ***A motorboat is racing towards north at 25 km h^{-1} and the water current in that region is 10 km h^{-1} in the direction of 60° east of south. Find the resultant velocity of the boat.***

Sol. Let the motorboat start moving from O, as shown in Fig.



v_b = velocity of motorboat

= 25 km h^{-1}, due north

v_c = velocity of water current

= 10 km h^{-1}, 60° east of south

By parallelogram law, the resultant velocity v is equal to the diagonal OC.

Its magnitude is

$$v = \sqrt{v_b^2 + v_c^2 + 2v_b v_c \cos 120°}$$

$$= \sqrt{25^2 + 10^2 + 2 \times 25 \times 10 (-½)} = 21.8 \text{ km h}^{-1}$$

Suppose the resultant velocity v makes angle β with the north direction. Then

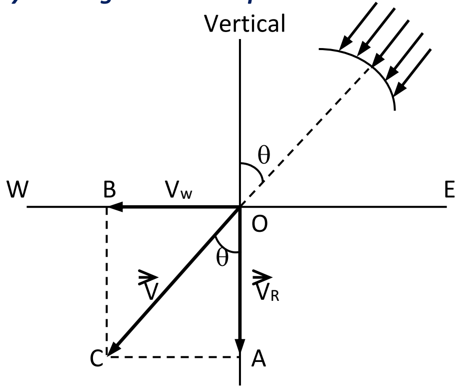
$$\tan\beta = \frac{v_c \sin 120°}{v_b + v_c \cos 120°} = \frac{10 \times (\sqrt{3}/2)}{25 + 10 \times (-½)}$$

$$= \sqrt{3}/4 = 0.433$$

∴ $\beta = \tan^{-1}(0.433) = 23.4°$

Q. 12. ***On a certain day, rain was falling vertically with a speed of 35 ms^{-1}. A wind started blowing after some time with a speed of 12 ms^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella?***

Sol. In Fig.



Velocity of rain, v_R = OA = 35 ms^{-1}, vertically downward

Velocity of wind, v_W = OB = 12 ms^{-1}, east to west.

The magnitude of the resultant velocity is

$$v = \sqrt{v_R^2 + v_w^2} = \sqrt{35^2 + 12^2} = 37 \text{ ms}^{-1}$$

Let the resultant velocity v (= OC) make an angle θ with the vertical. Then

$$\tan\theta = \frac{AC}{OA} = \frac{v_W}{v_R} = \frac{12}{35} = 0.343$$

∴ $\theta = \tan^{-1}(0.343) \simeq 19°$

Hence the body should hold umbrella bending it towards east making an angle of about 19° with the vertical.

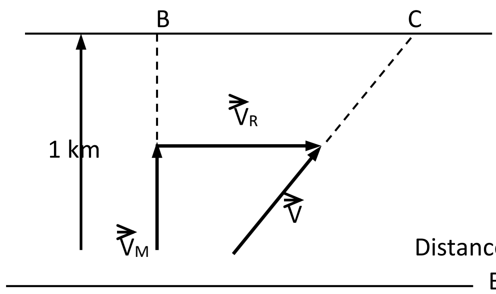
Q. 13. ***A man can swim with a speed of 4 kmh^{-1} in still water. How long does he take a cross the river 1 km wide, if the river flows steadily at 3 kmh^{-1} and he makes his strokes normal to the river current? How far from the river does he go, when he reaches the other bank?***

Sol. In Fig. v_M and v_R represent the velocities of man and river. Clearly v is the resultant of these velocities. If the man begins to swim along AB, he will be deflected to the path AC by the flowing river.

Time taken to cover distance AC with velocity v will be same as the time taken to cover distance AB with velocity v_M.

∴ Time taken by the man to cross the river is

$$t = \frac{AB}{v_M} = \frac{1 \text{ km}}{4 \text{ km h}^{-1}} = \frac{1}{4} \text{ h} = 15 \text{ min}$$



Distance through which the man goes down the river is

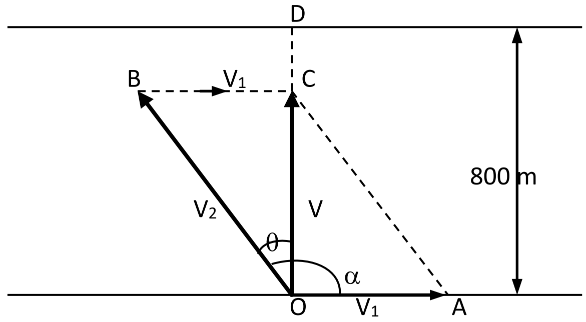
$$BC = v_R \times t = 3 \text{ kmh}^{-1} \times ¼ \text{ h} = 0.75 \text{ km}$$

Q. 14. ***A river 800 m wide flows at the rate of 5 kmh^{-1}. A swimmer who can swim at 10 kmh^{-1} in still water, wishes to cross the river straight. (i) Along what direction must be strike? (ii) What should be his resultant velocity? (iii) How much time he would take?***

Sol. (i) The situation is shown in Fig.

OA = v_1 = Velocity of river = 5 kmh^{-1}

OB = v_2 = Velocity of swimmer in still water.



The swimmer will cross the river straight if the resultant velocity v is perpendicular to the bank of the river. This will be possible if the swimmer moves making an angle θ with the upstream of the river.

In right ΔOCB, $\sin\theta = \frac{BC}{OB} = \frac{v_1}{v_2} = \frac{5}{10} = 0.5$

$\theta = \sin^{-1}(0.5) = 30°$

(ii) Resultant velocity of the swimmer,

$v = \sqrt{v_2^2 - v_1^2} = \sqrt{10^2 - 5^2} = \sqrt{75} = 8.66\ kmh^{-1}$

$= \frac{8.66 \times 5}{18} = 2.4\ ms^{-1}$

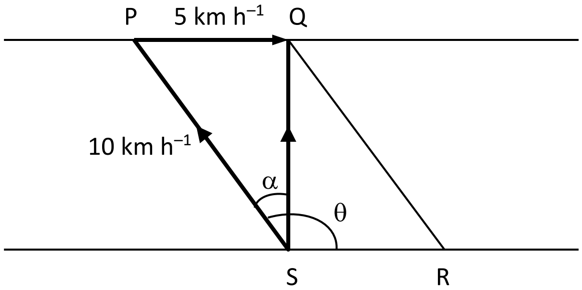
(iii) Time taken to cross the river,

$t = \frac{\text{width of river}}{v} = \frac{800\ m}{2.4\ ms^{-1}} = 333.3\ s$

Q. 16. A boatman can row with a speed for 10 kmh^{-1} in still water. If the river flows steadily at 5 kmh^{-1}, in which direction should the boatman row in order to reach a point on the other bank directly opposite to the point from where he started? The width of the river is 2 km.

Sol. As shown in Fig. the boatman starts from S. He should reach Q. Since the river flows along PQ with a velocity 5 kmh^{-1}, he should travel along SP.

$\vec{PQ} = 5\ kmh^{-1}$



Speed of boatman is shown by vector SP.

SP = 10 km h^{-1}

$\vec{SQ}$ is the resultant of SP and PQ

∠QSP = α, ∠PQS = 90°

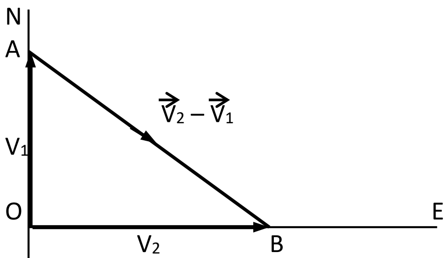
Since Q and S are directly opposite.

$\sin\alpha = \frac{PQ}{SP} = \frac{5}{10} = ½$ i.e., α = 30° ∴ θ = 90° + α = 120°

Thus the boatman must row the boat in a direction at an angle of 120° with the direction of river flow. The direction does not depend on width of the river.

Q. 17. A car travelling at a speed of 20 ms^{-1} due north along the highway makes a right turn on to a side road that heads due east. It takes 50 s for the car to complete the turn. At the end of 50 seconds, the car has a speed of 15 ms^{-1} along the side road. Determine the magnitude of average acceleration over the 50 second interval.

Sol. In Fig.



Initial velocity $\vec{v}_1 = \vec{OA} = 20\ ms^{-1}$, due north

Final velocity $\vec{v}_2 = \vec{OB} = 15\ ms^{-1}$, due east

As $\vec{OA} + \vec{AB} = \vec{OB}$

∴ $\vec{AB} = OB - \vec{OA} = \vec{v}_2 - \vec{v}_1$ = Change in velocity

$\therefore \quad |\vec{v}_2 - \vec{v}_1| = AB = \sqrt{OA^2 + OB^2}$

$= \sqrt{20^2 + 15^2} = \sqrt{625} = 15\ ms^{-1}$

Average acceleration $= \dfrac{|v_2 - v_1|}{t} = \dfrac{15}{50} = 0.3\ ms^{-2}$

Problems For Practice

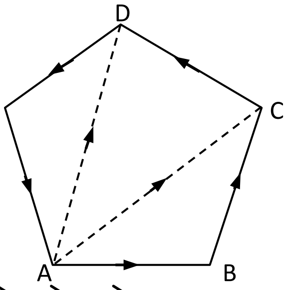
Q. 1. ***ABCDE is a pentagon. Prove that***

$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EA} = \vec{0}$$

Sol. Applying triangle law of vector addition,

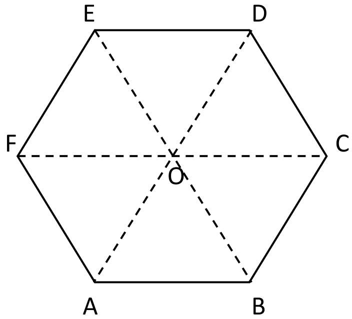
LHS $= (\vec{AB} + \vec{BC}) + \vec{CD} + \vec{DE} + \vec{EA}$

$= \vec{AC} + \vec{CD} + \vec{DE} + \vec{EA} = (\vec{AC} + \vec{CD}) + \vec{DE} + \vec{EA}$



$= \vec{AD} + \vec{DE} + \vec{EA} = (\vec{AD} + \vec{DE}) + \vec{EA}$ $\quad = \vec{AE} + \vec{EA} = -\vec{EA} + \vec{EA} = 0 =$ RHS

Q. 2. ***In Fig. ABCDEF is a regular hexagon. Prove that AB + AC + AD + AE + AF = 6 AO.***



Sol. We use triangle law of vector addition.

AB + AC + AD + AE + AF = AB + (AD + DC) + AD + (AD + DE) + AF = 3 AD + (AB + DE) + (DC + AF)

$= 3 \times 2$ AO + O + O = 6 AO $\qquad$ [$\because$ AD = 2 AO, DE = – AB, AF = – DC]

Q. 3. ***A boy travels 10 m due north and then 7 m due east. Find the displacement of the boy.***

Sol. Let the boy start moving from point O as shown in Fig.

OA = 10 m, due north; $\quad$ AB = 7 m, due east

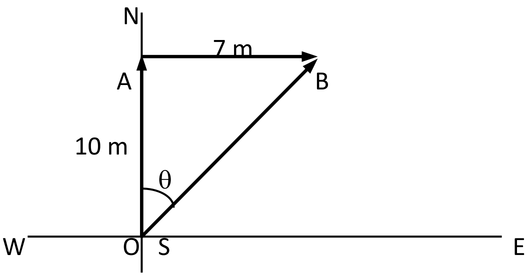
By triangle law of vector addition, the resultant displacement is

OA + AB = OB

$(OB) = OB = \sqrt{OA^2 + AB}$

$= \sqrt{10^2 + 7^2} = 12.21$ m

$\tan\theta = \dfrac{AB}{OA} = \dfrac{7}{10} = 0.7$

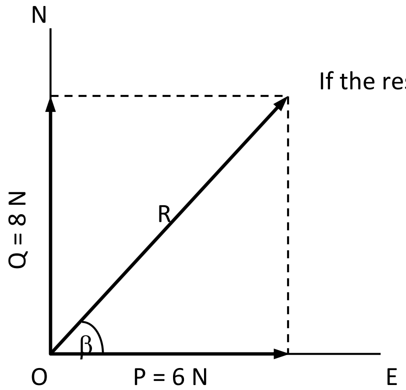


$\theta = \tan^{-1}(0.7) \simeq 35°$ $\qquad \therefore \qquad$ Displacement = 12.21 m, along N 35° E

Q. 4. ***Find the resultant of two forces, one 6N due east and other 8 N due north.***

Sol. As shown in Fig. $\quad$ P = 6N, $\quad$ Q = 8 N, $\quad$ R = ?

$R = \sqrt{P^2 + Q^2} = \sqrt{6^2 + 8^2} = 10$ N



If the resultant R makes angle β with the force of 6 N, than

$\tan\beta = \dfrac{Q}{P} = \dfrac{8}{6} = 1.3333$

$\therefore \qquad \beta = \tan^{-1}(1.3333) = 53°\ 8'$

Q. 5. ***Calculate the angle between a 2N force and a 3N force so that their resultant is 4 N.***

Sol. Here P = 2N, Q = 3 N, R = 4 N, θ = ?

As $R^2 = P^2 + Q^2 + 2\,PQ\cos\theta$

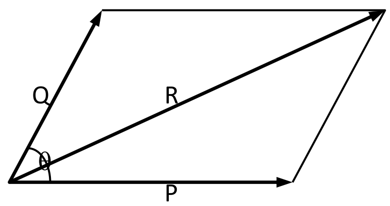
$\therefore$ $4^2 = 2^2 + 3^2 + 2 \times 2 \times 3\cos\theta$

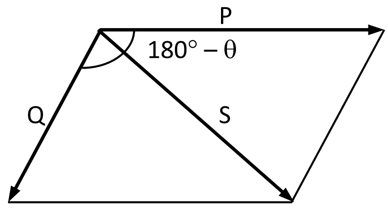
or $16 = 4 + 9 + 12\cos\theta$

or $\cos\theta = \frac{3}{12} = \frac{1}{4} = 0.25$ $\theta = \cos^{-1}(0.25) = 75^\circ\,31'$.

Q. 6. ***The resultant vector of P and Q is R. On reversing the direction of Q, the resultant vector becomes S. Show that: $R^2 + S^2 = 2(P^2 + Q^2)$***

Sol. Let θ be the angle between P and Q.





As the resultant of P and Q is R, therefore

$R^2 = P^2 + Q^2 + 2PQ\cos(180^\circ - \theta)$ ... (i)

When the direction of Q is reversed, the resultant becomes S, therefore

$S^2 = P^2 + Q^2 + 2PQ\cos(180^\circ - \theta)$

$= P^2 + Q^2 - 2PQ\cos\theta$ ... (ii)

Adding (i) and (ii), we get: $R^2 + S^2 = 2P^2 + 2Q^2 = 2(P^2 + Q^2)$

Q. 7. ***When the angle between two vectors of equal magnitude is $2\pi/3$, prove that the magnitude of the resultant is equal to either.***

Sol. Here P = Q, $\theta = \frac{2\pi}{3} = 120^\circ$

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

or $R = \sqrt{P^2 + P^2 + 2PP\cos 120^\circ} = \sqrt{2P^2 + 2P^2 \times (-\frac{1}{2})} = P$

Q. 8. ***At what angle do the two forces (P + Q) and (P – Q) act so that the resultant is $\sqrt{3P^2 + Q^2}$.***

Sol. Here $F_1 = P + Q$, $F_2 = P - Q$, $R = \sqrt{3P^2 + Q^2}$

As $R^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\theta$

$\therefore$ $3P^2 + Q^2 = (P + Q)^2 + (P - Q)^2 + 2(P + Q)(P - Q)\cos\theta$

or $P^2 - Q^2 = 2(P^2 - Q^2)\cos\theta$

or $\cos\theta = \frac{1}{2}$ $\therefore$ $\theta = 60^\circ$

Q. 9. ***The resultant of two equal forces acting at right angles to each other is 1414 dyne. Find the magnitude of either force.***

Sol. Here P = Q, $\theta = 90^\circ$, R = 1414 dyne

As $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

$\therefore$ $1411 = \sqrt{P^2 + P^2 + 2P^2\cos 90^\circ} = \sqrt{2}\,P$

or $P = \frac{1414}{\sqrt{2}} = \frac{1414}{1.414} = 1000$ dyne

Q. 10. ***A particle is acted upon by four forces simultaneously: (i) 30 N due east (ii) 20 N due north (ii) 50 N due west and (iv) 40 N due south. Find the resultant force on the particle.***

Sol. Net force due west, P = 50 – 30 = 20 N

Net force due south, Q = 40 – 20 = 20 N

$\therefore$ $R = \sqrt{P^2 + Q^2} = \sqrt{20^2 + 20^2} = 20\sqrt{2}$ N

$\tan\beta = \frac{Q}{P} = \frac{20}{20} = 1$ or $\beta = 45^\circ$

Q. 11. ***Two boys raising a load pull at an angle to each other. If they exert forces of 30 N and 60 N respectively and their effective pull is at right angles to the direction of the pull of the first boy, what is the angle between their arms? What is the effective pull?***

Sol. Here P = 30 N, Q = 60°, $\beta = 90^\circ$, R =? θ = ?

As $\tan\beta = \frac{Q\sin\theta}{P + Q\cos\theta}$

$\therefore$ $\tan 90^\circ = \frac{60\sin\theta}{30 + 60\cos\theta} = \infty$

or $30 + 60\cos\theta = 0$ or $\cos\theta = -\frac{1}{2}$ $\therefore$ $\theta = 120^\circ$

$R = \sqrt{30^2 + 60^2 + 2 \times 30 \times 60 \times (-\frac{1}{2})} = 30\sqrt{3}$ N

Q. 12. A ship is steaming due east at 12 ms^{-1}. A woman runs across the deck at 5 ms^{-1} in a direction at right angles to the direction of motion of the ship and then towards north. Calculate the velocity of the woman relative to sea.

Sol. In Fig. v_S = 12 ms^{-1}, due east

v_W = 5 ms^{-1}, due north.

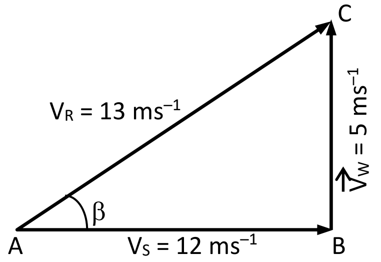
Resultant velocity of woman is

$$v_R = v_S + v_W$$

∴ $v_R = \sqrt{12^2 + 5^2} = 13$ ms^{-1}

Also, $\tan\beta = \frac{5}{12} = 0.4167$

∴ $\beta = 22° \ 37'$ north of east.

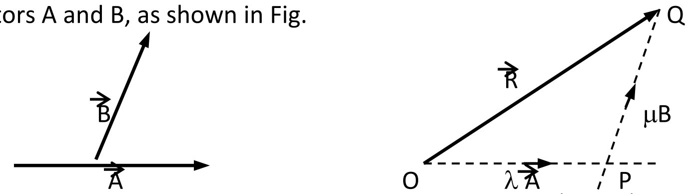


RESOLUTION OF A VECTOR

It is the process of splitting a vector into two or more vectors in such a way that their combined effect is same as that of the given vector.

► The vector into which the given vector is splited are called component vectors.
► A component of a vector in any direction gives a measure of the effect of the given vector in that direction.
► The resolution of a vector is just opposite to the process of vector addition.

Resolution of a vector along two given directions: Suppose wish to resolve a vector R in the direction of two coplanar and non-parallel vectors A and B, as shown in Fig.



[Resolving a vector R in the directions of $\vec{A}$ and $\vec{B}$]

Suppose $\vec{OQ}$ represents vector $\vec{R}$. Through O and Q, draw lines parallel to vectors $\vec{A}$ and $\vec{B}$ respectively to meet at point P. Form triangle law of vector addition,

$$\vec{OQ} = \vec{OP} + \vec{PQ}$$

As $\vec{OP} \parallel \vec{A}$, therefore, $\vec{OP} = \lambda\vec{A}$

As $\vec{PQ} \parallel \vec{B}$, therefore, $\vec{PQ} = \mu\vec{B}$

Here λ and μ are scalars. Hence

$$\vec{R} = \lambda\vec{A} + \mu\vec{B} \quad \text{... (1)}$$

Thus, the vector $\vec{R}$ has been resolved in the directions of $\vec{A}$ and $\vec{B}$. Here $\lambda\vec{A}$ is the component of $\vec{R}$ in the direction $\vec{A}$ and $\mu\vec{B}$ is the component in the direction of $\vec{B}$.

Uniqueness of resolution: Let us assume that R can be resolved in the directions of $\vec{A}$ and $\vec{B}$ in another way. Then

$$\vec{R} = \lambda'\vec{A} + \mu'\vec{B} \quad \text{... (2)}$$

From equations (1) and (2), we have

$$\lambda\vec{A} + \mu\vec{B} = \lambda'\vec{A} + \mu'\vec{B}$$

or $(\lambda - \lambda')\vec{A} = (\mu - \mu)\vec{B}$

But $\vec{A}$ and $\vec{B}$ are non-zero vectors acting along different directions. The above equation is possible only if

$\lambda - \lambda' = 0$ and $\mu' - \mu = 0$

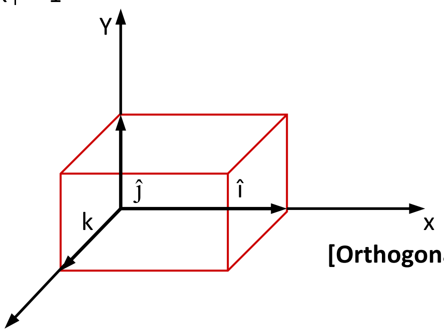
or $\lambda' = \lambda$ and $\mu' = \mu$

Hence there is one and only one way in which a vector $\vec{R}$ can be resolved in the directions of vectors $\vec{A}$ and $\vec{B}$.

ORTHOGONAL TRIAD OF UNIT VECTORS: BASE VECTORS

Orthogonal triad of unit vectors or base vectors: In a right-handed Cartesian coordinate system, three-unit vectors î, ĵ, k are used to represent the positive directions of X-axis, Y-axis and Z-axis respectively. These three mutually perpendicular unit vectors î, ĵ, k are collectively known as orthogonal triad of unit vectors or base vectors. Thus

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$



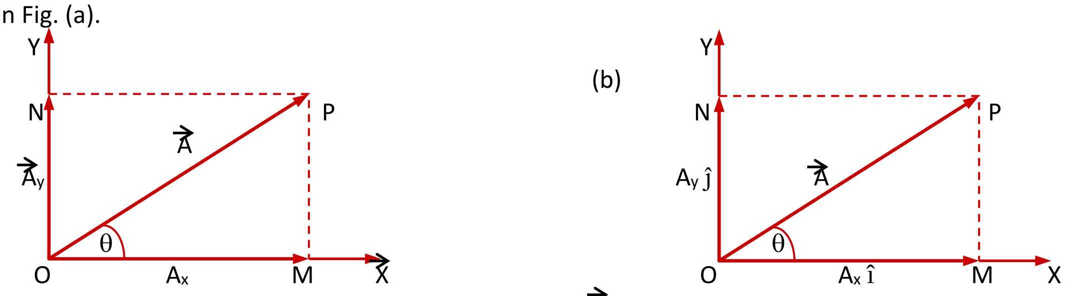
[Orthogonal triad of unit vectors]

RECTANGULAR COMPONENTS OF A VECTOR [Resolution of a Vector]

Rectangular components: When a vector is resolved along two mutually perpendicular directions, the components so obtained are called rectangular components of the given vector.

Rectangular components of a vector in a plane: Suppose we wish to resolve vector A along X- and Y-axes. Taking the initial point of $\vec{A}$ as the origin O, draw two axes OX and OY perpendicular to each other. From the head P of $\vec{A}$, draw PM ⊥ OX and PN ⊥ OY, as shown in Fig. (a).

(a) (b)



[Rectangular components of $\vec{A}$]

According to parallelogram law of vector addition,

$\vec{OP} = \vec{OM} + \vec{ON}$

or $\vec{A} = \vec{A}_x + \vec{A}_y$

Thus $\vec{A}_x$ is the horizontal or X-component of $\vec{A}$ and $\vec{A}_y$ is the vertical or Y-component of $\vec{A}$.

Now, let $\hat{i}$ and $\hat{j}$ be the unit vectors along X- and Y- axes respectively, and A_x and A_y be the scalar magnitudes of $\vec{A}_x$ and A_y respectively. Then, we can write

$\vec{A}_x = A_x\,\hat{i}$ and $\vec{A}_y = A_y\,\hat{j}$

∴ $\vec{A} = A_x\,\hat{i} + A_y\,\hat{y}$

◙*This is the equation for vector $\vec{A}$ in terms of its rectangular components*.

If vector $\vec{A}$ makes an angle θ with X-axis, then

$A_x = A\cos\theta$

and $A_y = A\sin\theta$

Conversely, if A_x and A_y are given, we can find A and θ as :

$Ax^2 + A_y^2 = A^2(\cos^2\theta + \sin^2\theta) = A^2$

or $A = \sqrt{A_x^2 + A_y^2}$

and $\tan\theta = \frac{A_y}{A_x}$ or $\theta = \tan^{-1}\frac{A_y}{A_x}$

Resolution of a position vector into rectangular components:

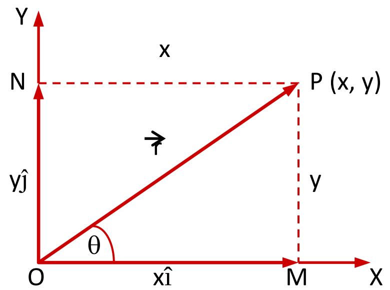
Draw PM ⊥ X-axis and PN ⊥ Y-axis. Then OM = x and ON = y. According to parallelogram law of vector addition,

$\vec{OP} = \vec{OM} + \vec{ON}$

or $\vec{r} = x\,\hat{i} + y\,\hat{j}$

This equation expresses position vector $\vec{r}$ in terms of its rectangular components along X-and Y-axes. Clearly,

$|\vec{r}| = \sqrt{OM^2 + ON^2} = \sqrt{x^2 + y^2}$



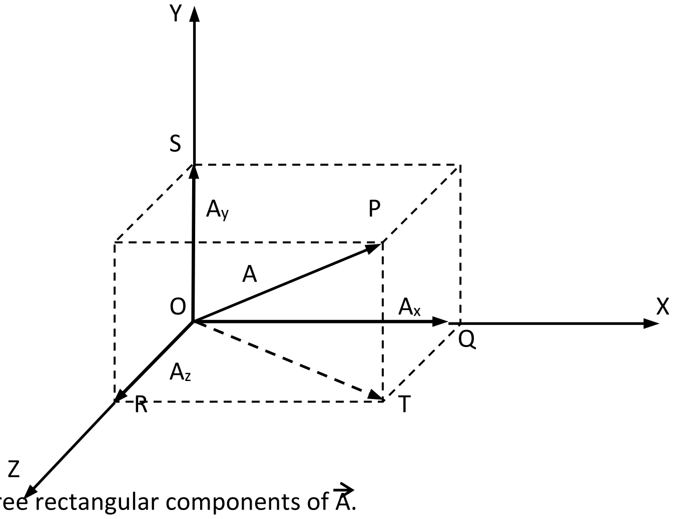
[Rectangular resolution of a position vector]

If r makes angle θ with X-axis, then

$x = r\cos\theta$ and $y = r\sin\theta$

Rectangular components a vector in three dimensions: Suppose vector $\vec{A}$ is represented by $\vec{OP}$, Taking O as original construct a rectangular parallelopiped with its three edges along the three rectangular axes i.e., X-, Y- and Z-axes. Clearly,

$\vec{A}$ represents the diagonal of the parallelepiped whose intercepts along X-, Y- and Z-axes are $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ respectively.



Thus $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ are the three rectangular components of $\vec{A}$.

Applying triangle law of vectors,

$$\vec{OP} = \vec{OT} + \vec{TP}$$

Applying parallelogram law of vectors,

$$\vec{OT} = \vec{OR} + \vec{OQ}$$

$\therefore$ $\vec{OP} = \vec{OR} + \vec{OQ} + \vec{TP}$

But $\vec{TP} = \vec{OS}$, hence $\vec{OP} = \vec{OR} + \vec{OQ} + \vec{OS}$

or $\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = A_x\,\hat{\imath}, A_y\,\hat{\jmath}, A_z\,k$

Magnitude of $\vec{A}$: We note that

$$OP^2 + OT^2 + TP^2 = OQ^2 + QT^2 + TP^2$$

or $A^2 = A_x^2 + A_y^2 + A_z^2$

or $A = \sqrt{A_x^2 + A_y^2 + Az^2}$

□□□□: A position vector $\vec{r}$ in three dimensions can be expressed as $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,k$

where x, y and z are the components of r along X-, Y-, and Z-axes respectively.

Direction cosine of a vectors: -

If α , β and γ are the angles which $\vec{A}$ makes with X , Y & Z axis resp.

$\therefore$ $\cos\alpha = Ax / A$

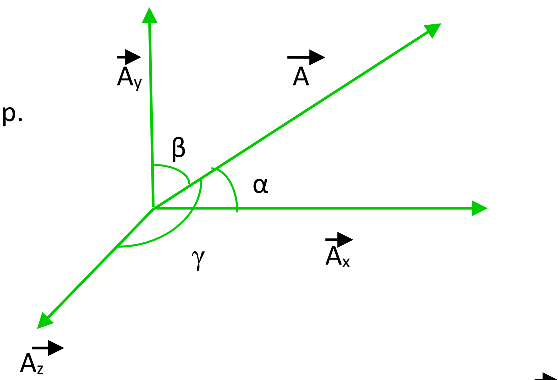
$\therefore$ **$A_x = A\cos\alpha$** ....(1)

$\cos\beta = Ay / A$

$\therefore$ **$A_y = A\cos\beta$** .... (2)

& **$\cos\gamma = Az / A$**

$\therefore$ **$A_z = A\cos\gamma$** .... (3)



Where cos α, cos β, cos γ are called direction cosine of vector $\vec{A}$.

Putting Ax, A_y & A_z in Eq - ---- $A^2 = A^2x + A^2y + A^2z$

Then $A^2 = A^2\cos^2\alpha + A^2\cos^2\beta + A^2\cos^2\gamma$

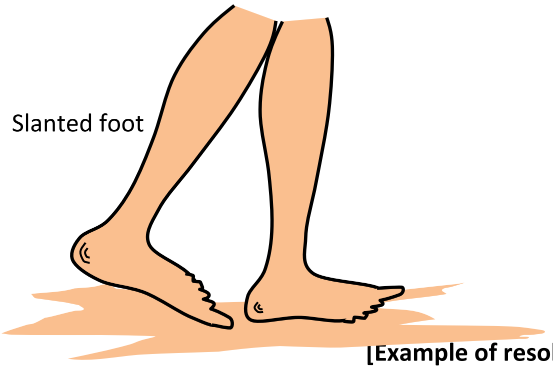
1 $A^2 = A^2 (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$

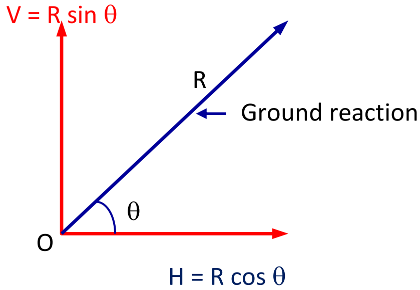
$\therefore$ **$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$**

Since, maximum cosine of an angle is unity so from (1), (2) & (3), it is clear that the magnitude of the rectangular component Is always less than the mag. of vector itself.

* *A vector can have infinite components, but the maximum no. of rectangular components vectors is three.*

* **Walking of a man is an example of resolution of forces:** While walking, a person presses the ground with his feet slightly slanted in the backward direction, as shown in Fig. The ground exerts upon him an equal and opposite reaction R. Its horizontal component H = R cos θ enables the person to move forward while the vertical component V = R sin θ balances his weight.





[Example of resolution of a vector]

Examples based on *Expressing the Vectors in terms of Base Vectors and Rectangular Components of Vectors*

♦ **Formulae Used**

1. If A_x, A_y, A_z are the rectangular components of $\vec{A}$ and $\hat{i}, \hat{j}, k$ are the unit vector along X-, Y- and Z-axes respectively, then $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z k$

2. $|\vec{A}| = \sqrt{A^2_x + A^2_y + A^2_z}$

3. $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{A_x \hat{i} + A_y \hat{j} + A_z k}{\sqrt{A^2_x + A^2_y + A^2_z}}$

4. If vector $\vec{A}$ makes angle θ with the horizontal, then horizontal component of $\vec{A} = A_x = A \cos\theta$ vertical component of $\vec{A} = A_y = A \sin\theta$ and $\vec{A} = \sqrt{A^2_x + A^2_y}$

♦ **Units Used**

Units of A_x, A_y and A_z are same as that of A and angle θ is in radians.

Q. 1. ***Find the vector $\vec{AB}$ and its magnitude if it has initial point A (1, 2, – 1) and final point B (3, 2, 2).***

Sol. Here $\vec{OA} = \hat{i} + 2\hat{j} + 2k$

$\vec{OB} = 3\hat{i} + 2\hat{j} + 2k$

∴ $\vec{AB} = \vec{OB} - \vec{OA} = (3 - 1)\hat{i} + (2 - 2)\hat{j} + [2 - (-1)]k = 2\hat{i} + 3k$

Q. 2. ***Find unit vector parallel to the resultant of the vectors $\vec{A} = \hat{i} + 4\hat{j} - 2k$ and $\vec{B} = 3\hat{i} - 5\hat{j} + k$.***

Sol. The resultant of $\vec{A}$ and $\vec{B}$ is

$\vec{R} = \vec{A} + \vec{B} = (\hat{i} + 4\hat{j} - 2k) + (3\hat{i} - 5\hat{j} + k)$

$= (1 + 3)\hat{i} + (4 - 5)\hat{j} + (-2 + 1)k = 4\hat{i} + \hat{j} + k$

$|R| = \sqrt{4^2 + (-1)^2 + (-1)^2} = \sqrt{16 + 1 + 1} = 3\sqrt{2}$

The unit vector parallel to $\vec{R}$ is

$R = \dfrac{\vec{R}}{|\vec{R}|} = \dfrac{1}{3\sqrt{2}}(4\hat{i} - \hat{j} - k)$

Q. 3. ***A vector $\vec{X}$, when added to the resultant of the vectors $\vec{A} = 3\hat{i} + 5\hat{j} + 7k$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 3k$ gives a unit vector along Y-axis. Find the vector $\vec{X}$.***

Sol. Resultant of $\vec{A}$ and $\vec{B}$ is

$R = \vec{A} + \vec{B} = (3\hat{i} - 5\hat{j} + 7k) + (2\hat{i} + 4\hat{j} - 3k) = 5\hat{i} - \hat{j} + 4k$

Unit vector along Y-axis = $\hat{j}$

∴ Required vector $\vec{X} = \hat{j} - R = \hat{j} - (5\hat{i} - \hat{j} + 4k) = -5\hat{i} + 2\hat{j} - 4k$

Q. 4. ***Two forces $\vec{F_1} = 3\hat{i} + 4\hat{j}$ and $\vec{F_2} = 3\hat{j} + 4k$ are acting simultaneously at a point. What is the magnitude of the resultant force?***

Sol. The resultant force is

$R = F_1 + F_2 = (3\hat{i} + 4\hat{j}) + (3\hat{j} + 4k) = 3\hat{i} + 7\hat{j} + 4k$

Magnitude of the resultant force is

$|R| = \sqrt{3^2 + 7^2 + 4^2} = \sqrt{9 + 49 + 16} = \sqrt{74}$ units of force

Q. 5. ***If A = 3î + 4ĵ and B = 7u + 24 ĵ, find a vector having the same magnitude as $\vec{B}$ and parallel to $\vec{A}$.***

Sol. $|\vec{A}| = \sqrt{3^2 + 4^2} = 5$

Unit vector in the direction of $\vec{A}$ is

$\hat{A} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{1}{5}(3\hat{i} + 4\hat{j})$

Also, $|\vec{B}| = \sqrt{7^2 + 24^2} = 25$

The vector having the same magnitude as B and parallel to A

$= |\vec{B}| = A = 25 \times \dfrac{1}{5}(3\hat{i} + 4\hat{j}) = 15\hat{i} + 20\hat{j}$

Q. 6. ***A bird moves with velocity 20 ms^{-1} in a direction making an angle of 60° with the eastern line and 60° with vertical upward. Represent the velocity vector in rectangular form.***

Sol. Let eastern line be taken as X-axis, northern as Y-axis and vertical upward as Z-axis. Let the velocity vector $\vec{v}$ make angles α, β and γ with X-, Y- and Z-axis respectively. Then α = 60°, γ = 60°.

As $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

∴ $\cos^2 60° + \cos^2 \beta + \cos^2 60° = 1$

or $\left(\frac{1}{2}\right)^2 + \cos^2 \beta + \left(\frac{1}{2}\right)^2 = 1$

or $\cos^2 \beta = 1 - ½ = ½$ or $\cos \beta = 1/\sqrt{2}$

∴ $v = v \cos \alpha\, \hat{i} + v \cos \beta\, \hat{j} + v \cos \gamma\, k$

$= 20 \times ½\, \hat{i} + 20 \times 1/\sqrt{2}\, \hat{j} + 20 \times ½\, k = 10\, \hat{i} + 10\sqrt{2}\, \hat{j} + 10\, k$

Q. 7. ***One of the rectangular components of a velocity of 80 kmh^{-1} is 40 kmh^{-1}. Find the other component.***

Sol. Let v = 80 kmh^{-1}, v_x = 40 kmh^{-1}, then v_y = ?

As $v = \sqrt{v_x^2 + v_y^2}$ ∴ $v_y = \sqrt{v^2 - v_x^2} = \sqrt{80^2 - 40^2} = \sqrt{6400 - 1600} = \sqrt{4800} = 69.28\ kmh^{-1}$

Q. 8. ***A force is inclined at 50° to the horizontal. If its rectangular component in the horizontal direction be 50 N, find the magnitude of the force and its vertical component.***

Sol. Here F_x = 50 N, θ = 50° But $F_x = F \cos \theta$

∴ $F = \frac{F_x}{\cos \theta} = \frac{50}{\cos 50°} = \frac{50}{0.6428} = 77.78$ N

Also, $F_y = F \sin \theta = 50 \times \sin 50° = 77.75 \times 0.7660 = 59.58$ N.

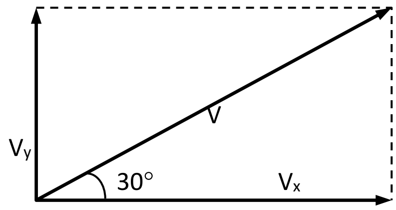
Q. 9. ***An aeroplane takes off at angle of 30° to the horizontal. If the component of its velocity along the horizontal. If the component of its velocity along the horizontal is 250 kmh^{-1}, what is the actual velocity? Find also the vertical component of the velocity.***

Sol. Let v_x and v_y be the horizontal and vertical components of actual velocity v (Fig.) Then

$v_x = v \cos 30° = 250\ kmh^{-1}$

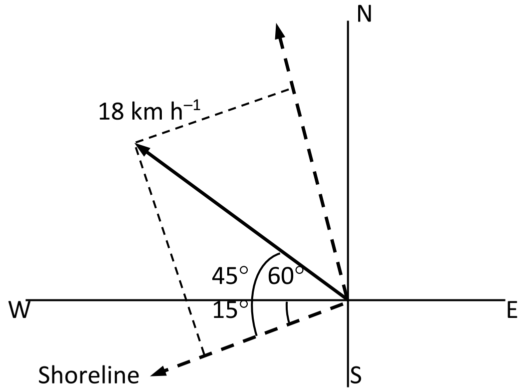
∴ $v = \frac{250}{\cos 30°} = \frac{250 \times 2}{\sqrt{3}} = 288.67\ kmh^{-1}$

$v_y = v \sin 30° = 288.67 \times 0.5 = 144.33\ kmh^{-1}$



Q. 10. ***A man rows a boat with a speed of 18 kmh^{-1} in the north-west direction. The shoreline makes an angle of 15° south of west. Obtain the component of the velocity of the boat along the shoreline and perpendicular to the shoreline.***

Sol. As shown in Fig., the boat makes an angle of 45° with the west direction while the shoreline makes an angle of 15° with it. Thus the boat makes angle of 45° + 15° = 60° with the shoreline.



∴ Component of the velocity of the boat along the shoreline= 18 cos 60° = 18 × ½ = 9 kmh^{-1}

Component of the velocity of the boat perpendicular to the shoreline= 18 sin 60° = $\frac{18 \times \sqrt{3}}{2}$ = 15.6 kmh^{-1}

Q. 11. ***Two billiard balls are rolling on a flat table. One has the velocity components $v_x = 1\ ms^{-1}$, $v_y = \sqrt{3}\ ms^{-1}$ and the other has component $v'_x = 2\ ms^{-1}$ and $v'_y = 2\ ms^{-1}$. If both the balls start moving from the same point, what is the angle between their paths?***

Sol. **For first ball:** OC = v_x = 1 ms^{-1}, OE = v_y = $\sqrt{3}$ ms^{-1}

Let θ be the angle which the resultant OA of v_x and v_y makes with the X-axis. Then

$$\tan\theta = \frac{v_y}{v_x} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

∴ $\theta = 60°$

For second ball:

$OD = v'_x = 2\ ms^{-1}$, $OF = v_y' = 2\ ms^{-1}$

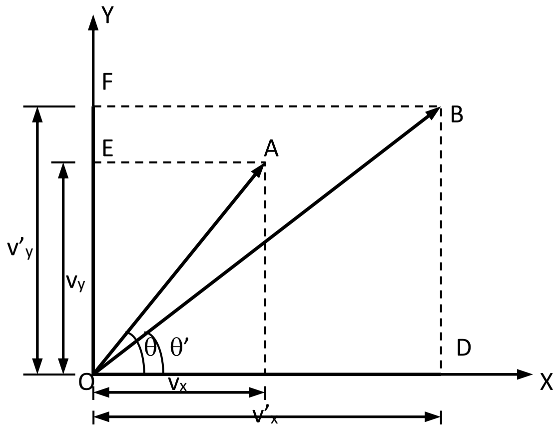
Let θ′ be the angle which the resultant OB of v'_x and v_y' makes with the X-axis. Then

$$\tan\theta' = \frac{v'_y}{v'^x} = \frac{2}{2} = 1$$

∴ $\theta' = 45°$

∴ Angles between the paths of two balls

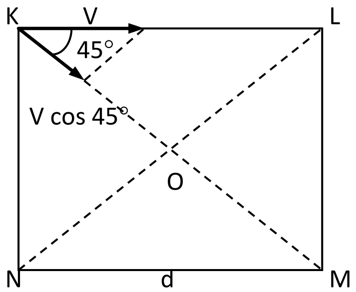
$= \theta = \theta' = 60° - 45° = 15°$



Q. 12. ***Four persons K, L, M and N are initially at rest at the four corners of a square of side d. Each person now moves with a uniform speed v in such a way that K always moves directly towards L, L directly towards M, M directly towards N and N directly towards K. Show that the four persons meet at a time d/v.***

Sol. All the four persons will meet at the centre O of the square. Each person covers a displacement,

$$s = ½ . \sqrt{d^2 + d^2} = \frac{d}{\sqrt{2}}$$



Component of velocity towards O,

$$v' = v\cos 45° = v/\sqrt{2}$$

∴ Required time, $t = \frac{s}{v'} = \frac{d/\sqrt{2}}{v/\sqrt{2}} = \frac{d}{v}$

Q. 13. ***The x- and y-components of A are 4 m and 6 m respectively. The x- and y- components of vector (A + B) are 10 m and 9 m respectively. Calculate for the vector B (i) its x- and y- components (ii) its length and (iii) the angle it makes with the x-axis.***

Sol. Here $A_x = 4$ m, $A_y = 6$ m, $A_x + B_x = 10$ m, $A_y + B_y = 9$ m

(i) $B_x = 10 - 4 = 6$ m, $B_y = 9 - 6 = 3$ m

(ii) $B = \sqrt{B_x^2 + B_y^2} = \sqrt{36 + 9} = \sqrt{45}$ m

(iii) $\theta = \tan^{-1}\frac{B_y}{B_x} = \tan^{-1}\frac{3}{6} = 26.6°$

Problems For Practice

Q. 1. ***Find the value of λ in the unit vector 0.4 î + 0.8 ĵ + λ k.***

Sol. As the given vector is a unit vector, so

$$|0.4\hat{i} + 0.8\hat{j} + \lambda k| = 1$$

or $\sqrt{(0.4)^2 + (0.8)^2 + \lambda^2} = 1$

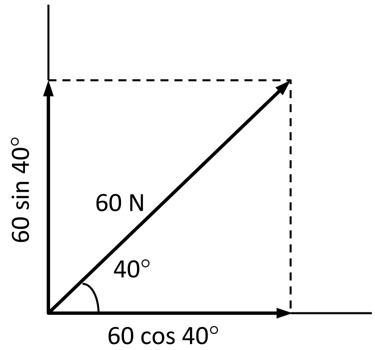
or $\lambda^2 = 1 - (0.4)^2 - (0.8)^2$

$= 1 - 0.16 - 0.64 = 0.2$

or $\lambda = \sqrt{0.2}$

Q. 2. ***A child pulls a rope attached to a stone with a force of 60 N. The rope makes an angle of 40° to the ground. (i) Calculate the effective value of the pull tending to move the stone along the ground. (ii) Calculate the force tending to lift the stone.***

Sol. From Fig.



(i) Pull along the ground = 60 cos 40° = 45.96 N

(ii) Force tending to lift the stone vertically up = 60 sin 40° = 38.57 N

Product of vectors: -

Two vectors can be multiplied in two ways:

.... **(i) Scalarly** .... **(ii) Vectorially**

These are two ways in which a vector can be multiplied by another vector:

------------------(i) One **way produces a scalar** and is known as scalar product.

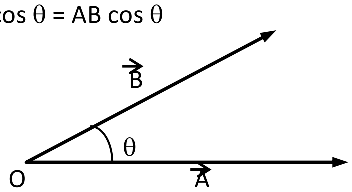
------------------(ii) Another **way produces a new vector** and is known as vector product.

SCALAR PRODUCT OF TWO VECTORS (or *Dot product of two vector*)

"The scalar product of two vector A & B is defined as the product of magnitude of $\vec{A}$ & $\vec{B}$ multiplied by the cosine of smaller angle between them".

------ Dot product of $\vec{A}$ & $\vec{B}$ represented by **A. B** & is read as A **dot B.**

$\vec{A} \cdot \vec{B} = |\vec{A}|\ |\vec{B}|\ \cos\theta = AB\cos\theta$



[Scalar product of $\vec{A}$ and $\vec{B}$ is a scalar: $\vec{A}.\ \vec{B}$ = AB cos θ]

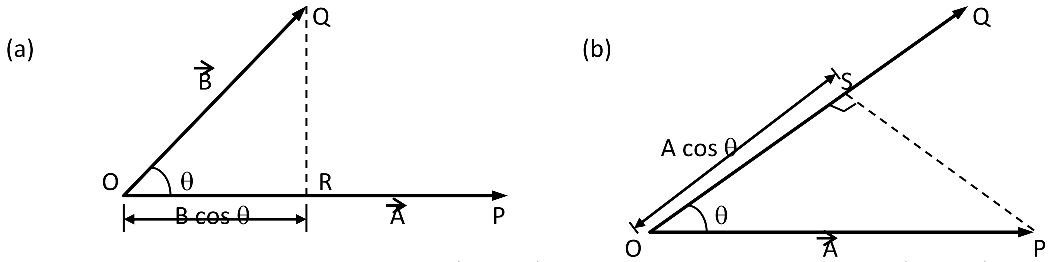
As A, B and cos θ are all scalars, so the dot product of $\vec{A}$ and $\vec{B}$ is a scalar quantity. Both $\vec{A}$ and $\vec{B}$ have directions, but their dot product has no directions.

Geometrical interpretation of scalar product:

As shown in Fig (a), suppose two vectors A and B are represented by $\vec{OP}$ and $\vec{OQ}$ and ∠POQ = θ

$\vec{A}.\ \vec{B}$ = AB cos θ = A (B cos θ) = A (OR)

= A × Magnitude of component of $\vec{B}$ in the direction of $\vec{A}$



[(a) B cos θ is the projection of $\vec{B}$ onto $\vec{A}$. (b) A cos θ is the projection of $\vec{A}$ onto $\vec{B}$]

From Fig. (b), we have

A. $\vec{B}$ = AB cos θ = B (A cos θ) = B cos θ = B (OS)

= B × Magnitude of component of $\vec{A}$ in the direction of $\vec{B}$

Thus, the scalar product of two vectors is equal to the product of magnitude of one vector and the magnitude of component of other vector in the direction of first vector.

Physical examples of scalar product of two vectors:

(i) Work done (W): It is defined on the scalar product of the force ($\vec{F}$), acting on the body and the displacement ($\vec{s}$) produced. Thus $W = \vec{F}.\vec{s}$

(ii) Instantaneous power (P): It is defined as the scalar product of force ($\vec{F}$) and the instantaneous velocity ($\vec{v}$) of the body. Thus $P = \vec{F}.\vec{v}$

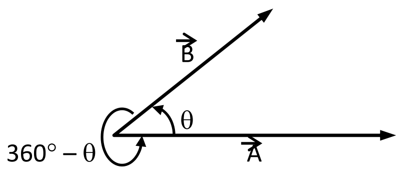
(iii) Magnetic flux (ϕ): The magnetic flux linked with a surface is defined as the scalar product of magnetic induction ($\vec{B}$) and the area vector ($\vec{A}$). Thus $\phi = \vec{B}.\vec{A}$

As the scalar product of two vectors is a scalar quantity, so work, power and magnetic flux are all scalar quantities.

PROPERTIES OF SCALAR PRODUCT OF TWO VECTORS

- (i) The scalar product is commutative i.e.,
$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$
- (ii) The scalar product is distributive over addition i.e.,
$$\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$$
- (iii) If $\vec{A}$ and $\vec{B}$ are two vectors perpendicular to each other, then their scalar product is zero.
$$\vec{A}.\vec{B} = A\,B\cos 90^\circ = 0$$
- (iv) If $\vec{A}$ and $\vec{B}$ are two parallel vectors having same direction, then their scalar product has the maximum positive magnitude.
$$\vec{A}.\vec{B} = AB\cos 0^\circ = A\,B$$
- (v) If $\vec{A}$ and $\vec{B}$ are two parallel vectors having opposite directions, then their scalar product has the maximum negative magnitude.
$$\vec{A}.\vec{B} = A\,B\cos 180^\circ = -A\,B.$$
- (vi) The scalar product of a vector with itself is equal to the square of its magnitude.
$$\vec{A}.\vec{A} = A\,.\,A\cos 0^\circ = A\,.\,A = A^2$$
- (vii) Scalar product of two similar base vectors is unity and that of two different base vectors is zero.
$$\hat{i}.\hat{i} = (1)(1)\cos 0^\circ = 1$$
$$\therefore \quad \boxed{\hat{i}.\hat{i} = \hat{j}.\hat{j} = k.k = 1}$$
$$\hat{i}.\hat{j} = (1)(1)\cos 90^\circ = 0$$
$$\therefore \quad \boxed{\hat{i}.\hat{j} = \hat{j}.k = k.\hat{i} = 0}$$
- (viii) Scalar product of two vectors $\vec{A}$ and $\vec{B}$ is equal to the sum of the products of their corresponding rectangular components.
$$\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z$$
- (ix) The cosine of the angle θ between A and B is given by
$$\text{Cos}\,\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}|\,|\vec{B}|}$$
$$= \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

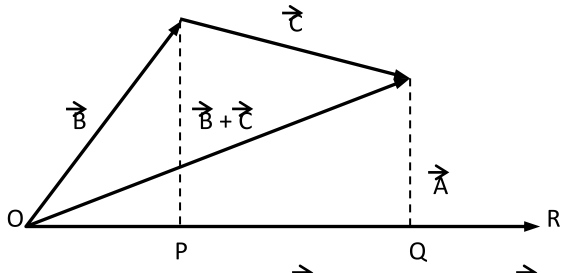
Scalar product of two vectors is commutative: We know that
$$\vec{A}.\vec{B} = AB\cos\theta \qquad \text{... (i)}$$
where θ is the angle between $\vec{A}$ and $\vec{B}$ measured anticlockwise, as shown in Fig.



Also, $\vec{B}.\vec{A} = BA\cos(360^\circ - \theta)$
$$= BA\cos\theta = AB\cos\theta \qquad \text{... (ii)}$$
Where $(360^\circ - \theta)$ is the angle between $\vec{B}$ and $\vec{A}$ measured anticlockwise.
From equations (i) and (ii), we get
$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$
Hence the scalar product of two vectors is commutative.

Scalar product of vectors is distributive over addition: In Fig. OP, PQ, and OQ are the projection or components of $\vec{B}$, $\vec{C}$ and $(\vec{B}+\vec{C})$ in the direction of vector A. By definition of scalar product,
$$\vec{A}.(\vec{B}+\vec{C})$$
$$= \text{Magnitude of } \vec{A} \times \text{Magnitude of component of } (\vec{B}+\vec{C}) \text{ in the direction of } \vec{A}$$

= (OR) (OQ) = (OR) (OP + PQ)
= (OR) (PO) + (OR) (PQ)



= Magnitude of $\vec{A}$ × Magnitude of component of $\vec{B}$ in the direction of $\vec{A}$ + Magnitude of A × Magnitude of component of $\vec{C}$ in the direction of A

$= \vec{A} . \vec{B} + \vec{A} . \vec{C}$ [By definition of scalar product]

Hence the scalar product is distributive over addition.

Scalar product in terms of rectangular components: We can express A and B in terms of their rectangular components as

$A = A_x \hat{i} + A_y \hat{j} + A_x k$

And $B = B_x \hat{i} + B_y \hat{i} + B_z k$

$\vec{A} . \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z k) . (B_x \hat{i} + B_y \hat{j} + B_z k)$

$= A_x \hat{i} . (B_x \hat{i} + B_y \hat{j} + B_z k) + A_y \hat{j} . (B_x \hat{i} + B_y \hat{j} + B_z k) + A_z k + B_x \hat{i} + By \hat{j} + Bz k)$

$= A_x B_x \hat{i} . \hat{i} + A_x B_y \hat{i} . \hat{j} + A_x B_z \hat{i} . k + A_y B_x \hat{j} . \hat{i} + A_y B_y \hat{j} . \hat{j} + A_y B_z \hat{j} . k + A_z B_x k + \hat{i} + A_z B_y k . \hat{j} + A_z B_z k . k$

$= A_x B_x (1) + A_x B_y (0) + A_x B_z (0) + A_y B_x (0) + A_y B_y (1) + A_y B_z (0) + A_z B_x (0) + A_z B_y (0) + A_z B_z (1)$

or $\vec{A} . \vec{B} = A_x B_x + A_y B_y + A_z B_z$

Examples based on *Scalar or Dot Product of two Vectors*

♦ **Formulae Used**

1. $\vec{A} . \vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta = AB \cos\theta$
2. If $A \perp B$, $\theta = 90°$ and A . B = 0
3. Angle θ between A and B is given by

$$\cos\theta = \frac{\vec{A} . \vec{B}}{|\vec{A}|\ |\vec{B}|}$$

4. In terms of rectangular components,

$\vec{A} . \vec{B} = A_x B_x + A_y B_y + A_z B_z$

5. Work done, $W = \vec{F} . \vec{S}$
6. Power, $P = \vec{F} . \vec{v}$

Q. 1. ***Find the angle between the vectors $A = \hat{i} + 2\hat{j} - k$ and $B = -\hat{i} + \hat{j} - 2k$,***

Sol. $|\vec{A}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

$|\vec{B}| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}$

$\vec{A} . \vec{B} = 1 \times (-1) + 2 \times 1 + (-1) \times (-2)$

$= -1 + 2 + 2 = 3$

$\therefore \cos\theta = \frac{\vec{A} . \vec{B}}{|\vec{A}|\ |\vec{B}|} = \frac{3}{\sqrt{6} \times \sqrt{6}} = \frac{3}{6} = \frac{1}{2}$ Hence $\theta = 60°$

Q. 2. ***Prove that the vectors $\vec{A} = \hat{i} + 2\hat{j} + 3k$ and $\vec{B} = 2\hat{i} - \hat{j}$ are perpendicular to each other.***

Sol. $\vec{A} . \vec{B} = (\hat{i} + 2\hat{j} + 3k) . (2\hat{i} - \hat{j} + 0 k)$

$= 1 \times 2 + 2 \times (-1) + 3 \times 0 = 0$

Hence $\vec{A} \perp \vec{B}$

Q. 3. ***Find the value of λ so that the vectors $\vec{A} = 2\hat{i} + \lambda\hat{j} + k$ and $\vec{B} = 4\hat{i} - 2\hat{j} - 2k$ are perpendicular to each other.***

Sol. As $\vec{A} \perp \vec{B}$, so $\vec{A} . \vec{B} = 0$

or $(2\hat{i} + \lambda\hat{j} + k) . (4\hat{i} - 2\hat{j} - 2k) = 0$

or $2 \times 4 + \lambda \times (-2) + 1 \times (-2) = 0$

or $\lambda = 3$

Q. 4. ***If the magnitudes of two vectors are 3 and 4 and their scalar product is 6, then find the angle between the two vectors.***

Sol. Here $|\vec{A}| = 3$, $|\vec{B}| = 4$, $\vec{A} . \vec{B} = 6$

$\therefore \quad \cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \dfrac{6}{3\times 4} = \dfrac{1}{2}$ Hence $\theta = 60°$

Q. 5. ***A body constrained to move along the z-axis of a co-ordinate system is subjected to a constant force $\vec{F}$ given by $\vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$ newton where $\hat{i}, \hat{j}$ and k represent unit vectors along x-, y- and z-axis of the system respectively. Calculate the work done by the force in displacing the body through a distance of 4 m along the z-axis.***

Sol. As the body moves 4 m along the z-axis, so the displacement vector is

$\vec{S} = 4\hat{k} = 0\hat{i} + 0\hat{j} + 4\hat{k}$ metre

Also $\vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$ newton

$\therefore \quad W = \vec{F}\cdot\vec{S} = (-\hat{i} + 2\hat{j} + 3\hat{k})\cdot(0\hat{i} + 0\hat{j} + 4\hat{k}) = -1\times 0 + 2\times 0 + 3\times 4 = 12$ joule.

Q. 6. ***A force of $7\hat{i} + 6\hat{k}$ newton makes a body move on a rough plane with a velocity of $3\hat{j} + 4\hat{k}$ ms^{-1} Calculate the power in watt.***

Sol. Power

$P = \vec{F}\cdot\vec{v} = (7\hat{i} + 0\hat{j} + 6\hat{k})\cdot(0\hat{i} + 3\hat{j} + 4\hat{k}) = 7\times 0 + 0\times 3 + 6\times 4 = 24$ W

Q. 7. ***Three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are such that $\vec{A} = \vec{B} + \vec{C}$ and their magnitudes are 5, 4 and 3 respectively. Find the angle between $\vec{A}$ and $\vec{C}$.***

Sol. Given $\vec{A} = \vec{B} + \vec{C}$ or $\vec{B} = \vec{A} - \vec{C}$

Now,

$\vec{B}\cdot\vec{B} = (\vec{A}-\vec{C})\cdot(\vec{A}-\vec{C}) = \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{C} - \vec{C}\cdot\vec{A} + \vec{C}\cdot\vec{C}$

$= \vec{A}\cdot\vec{A} + \vec{C}\cdot\vec{C} - 2\vec{A}\cdot\vec{C}$ $\quad [\because \vec{C}\cdot\vec{A} = \vec{A}\cdot\vec{C}]$

or $\quad B^2 = A^2 + C^2 - 2AC\cos\theta$

where θ is the angle between A and C. Thus

$\cos\theta = \dfrac{A^2 + C^2 - B^2}{2AC} = \dfrac{5^2 + 3^2 - 4^2}{2\times 5\times 3} = \dfrac{18}{30} = 0.6$

$\theta = \cos^{-1}(0.6) = 53°$

Q. 8. ***If $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$, find the angle between A and B.***

Sol. Given $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$

or $\quad |\vec{A}+\vec{B}|^2 = |\vec{A}-\vec{B}|^2$

or $\quad (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$ $\quad [\because |\vec{A}|^2 = \vec{A}\cdot\vec{A}]$

or $\quad \vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B} = \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} - \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B}$

or $\quad A^2 + 2\vec{A}\cdot\vec{B} + B^2 = A^2 - 2\vec{A}\cdot\vec{B} + B^2$ $\quad [\because \vec{B}\cdot\vec{A} = \vec{A}\cdot\vec{B}]$

or $\quad 4\vec{A}\cdot\vec{B} = 0$ or $4AB\cos\theta = 0$

As $\vec{A}$ and $\vec{B}$ are non-zero vectors, so

Cos $\theta = 0°$ or $\theta = 90°$

Q. 9. ***If vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$ have magnitudes 5, 12 and 13 units and $\vec{P} + \vec{Q} = \vec{R}$, find the angle between Q and R.***

Sol. As $\quad \vec{P} + \vec{Q} = \vec{R} \quad \therefore \quad \vec{R} - \vec{Q} = \vec{P}$

and $\quad (\vec{R}-\vec{Q})\cdot(\vec{R}-\vec{Q}) = \vec{P}\cdot\vec{P}$

or $\quad \vec{R}\cdot\vec{R} - \vec{R}\cdot\vec{O} - \vec{Q}\cdot\vec{R} + \vec{Q}\cdot\vec{Q} = \vec{P}\cdot\vec{P}$

or $\quad R^2 - 2\vec{R}\cdot\vec{Q} + Q^2 = P^2$

or $\quad \cos\theta = \dfrac{R^2 + Q^2 - P^2}{2RQ}$

$= \dfrac{13^2 + 12^2 - 5^2}{2\times 13\times 12} = \dfrac{288}{2\times 13\times 12} = \dfrac{12}{13}$

$\therefore \quad \theta = \cos^{-1} 12/13$

Q. 10. ***Determine the angles which the vector $\vec{A} = 5\hat{i} + 0\hat{j} + 5\hat{k}$ makes with X-, Y- and Z-axes.***

Sol. Here $\quad A = |\vec{A}|$

$= \sqrt{A^2x + A^2y + A^2z} = \sqrt{5^2 + 0^2 + 5^2} = 5\sqrt{2}$

If vector $\vec{OA}$ makes angles α, β and γ with X-, Y- and Z-axis respectively, then

$\cos\alpha = \dfrac{A_x}{A} = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}} \quad \therefore \quad \alpha = 45°$

$\cos\beta = \dfrac{A_y}{A} = \dfrac{0}{5\sqrt{2}} = 0 \quad \therefore \quad \beta = 90°$

$\cos\gamma = \dfrac{A_z}{A} = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}} \quad \therefore \quad \gamma = 45°$

Q. 11. ***If unit vectors $\hat{a}$ and $\hat{b}$ are inclined at angle θ, then prove that***
$|\hat{a} - \hat{b}|^2 = 2 \sin \theta/2$

Sol. For any vector $\vec{a}$, $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$

$\therefore \quad |\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b})$

$= \hat{a} \cdot \hat{a} - \hat{a} \cdot \hat{b} - \hat{b} \cdot \hat{a} + \hat{b} \cdot \hat{b}$

$= 1 - 2\hat{a} \cdot \hat{b} + 1 \qquad [\because \hat{a} \cdot \hat{a} = 1 \times 1 \times \cos 0° = 1]$

$= 2 - 2 \times 1 \times 1 \times \cos \theta = 2(1 - \cos \theta)$

$= 2 \cdot 2 \sin^2 \frac{\theta}{2} = 4 \sin^2 \frac{\theta}{2} \qquad [\because 1 - \cos 2\theta = 2 \sin^2 \theta]$

Hence $\quad |\hat{a} - \hat{b}| = 2 \sin \theta/2$

Q. 12. ***Find the components of $a = 2\hat{i} + 3\hat{j}$ along the directions of vectors $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$.***

Sol. Given $a = 2\hat{i} + 3\hat{j}$ and let $b = \hat{i} + \hat{j}$ and $c = \hat{i} - \hat{j}$. Then component of a in the direction b

$= (a \cos \theta)\, b$

$= \frac{ab \cos \theta}{b} \frac{b}{b} = \frac{a \cdot b}{|b|^2}\, b$

$= \frac{(2\hat{i} + 3\hat{j}) \cdot (\hat{i} + \hat{j})}{[\sqrt{1^2 + 1^2}]^2} (\hat{i} + \hat{j})$

$= \frac{2 \times 1 + 3 \times 1}{2} (\hat{i} + \hat{j}) = \frac{5}{2} (\hat{i} + \hat{j})$

Component of a in the direction of c

$= \frac{a \cdot c}{|c|^2}\, c = \frac{(2\hat{i} + 3\hat{j}) \cdot (\hat{i} - \hat{j})}{[\sqrt{1^2 + 1^2}]^2}$

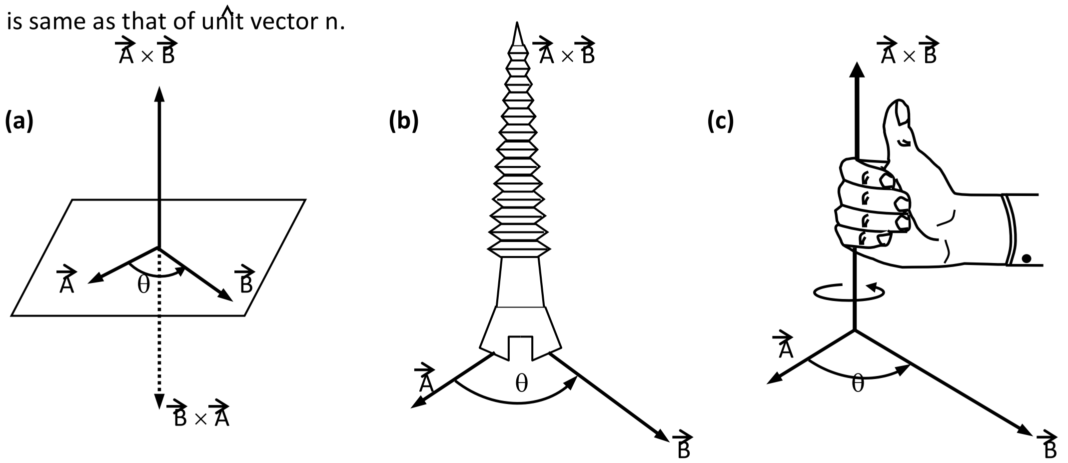
$= \frac{2 \times 1 + 3 \times (-1)}{2} (\hat{i} - \hat{j}) = -½ (\hat{i} - \hat{j})$

VECTOR PRODUCT OF TWO VECTORS

: The vector or cross product of two vectors is defined as the vector whose magnitude is equal to the product of the magnitudes of two vectors and sine of the angle between them and whose direction is perpendicular to the plane of the two vectors and is given by right hand rule Mathematically, if θ is the angle between vectors A and B, then

$$\vec{A} \times \vec{B} = AB \sin \theta\, \hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and its direction is given by right hand rule. Thus, the direction of $\vec{A} \times \vec{B}$ is same as that of unit vector $\hat{n}$.



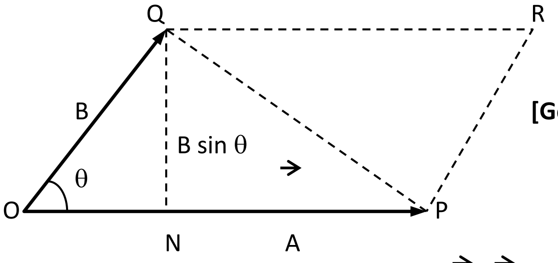
[Right hand rules for direction of vector product]

Rules for determining the direction of $\vec{A} \times \vec{B}$:

(i) Right-handed screw rule: As shown in Fig. (b), if a right-handed screw is placed with its axis perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$ and is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle, then the direction in which the screw advances gives the direction of $\vec{A} \times \vec{B}$.

(ii) Right hand thumb rule: As shown in Fig.(c), curl the fingers of the right hand in such a way that they point in the direction of rotation from vector $\vec{A}$ to $\vec{B}$ through the smaller angle, then the stretched thumb points in the direction of $\vec{A} \times \vec{B}$.

Geometrical interpretation of vector product: Suppose two vectors $\vec{A}$ and $\vec{B}$ are the represented by the sides OP and OQ of a parallelogram OPRQ, as shown in Fig.



[Geometrical significance of vector product]

Let ∠POQ = θ. Draw QN ⊥ OP. The magnitude of vector product $\vec{A} \times \vec{B}$ is

$$|\vec{A} \times \vec{B}| = AB \sin\theta$$
$$= (OP)(OQ) \sin\theta$$
$$= (OP)(QN) \quad [\because QN = OQ \sin\theta]$$
$$= \text{Area of parallelogram OPQR}$$

Thus the magnitude of the vector product of two vectors is equal to area of the parallelogram formed by the two vectors as its adjacent sides.

Moreover,

$$|\vec{A} \times \vec{B}| = 2 \times \tfrac{1}{2}(OP)(QN) = 2 \times \text{Area of } \Delta POQ$$

Thus, **the magnitude of the vector product of two vectors is equal to twice the area of the triangle formed by the two vectors its adjacent sides.**

Physical examples of vector product:

(i) Torque τ: The torque acting on a particle is equal to the vector product of its position vector ($\vec{r}$) and force vector ($\vec{F}$).

Thus $$\vec{\tau} = \vec{r} \times \vec{F}$$

(ii) Angular momentum $\vec{L}$: The angular momentum of a particle is equal to the cross product of its position. vector ($\vec{r}$) and linear momentum ($\vec{p}$). Thus

$$\vec{L} = \vec{r} \times \vec{p}$$

(iii) Instantaneous velocity v: The instantaneous velocity of a particle is equal to the cross product of its angular velocity ($\vec{\omega}$) and the position vector (r). Thus

$$\vec{v} = \vec{\omega} \times \vec{r}$$

PROPERTIES OF VECTOR PRODUCT

- (i) Vector product is anti-commutative i.e.,

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

- (ii) Vector product is distributive over addition i.e.,

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

- (iii) Vector product of two parallel or antiparallel vectors is a null vector. Thus

$$\vec{A} \times \vec{B} = AB \sin(0° \text{ or } 180°)\,\hat{n} = \vec{0}$$

- (iv) Vector product of a vector with itself is a null vector.

$$\vec{A} \times \vec{A} = AA \sin 0°\,\hat{n} = \vec{0}$$

- (v) The magnitude of the vector product of two mutually perpendicular vectors is equal to the product of their magnitudes.

$$|\vec{A} \times \vec{B}| = AB \sin 90° = AB$$

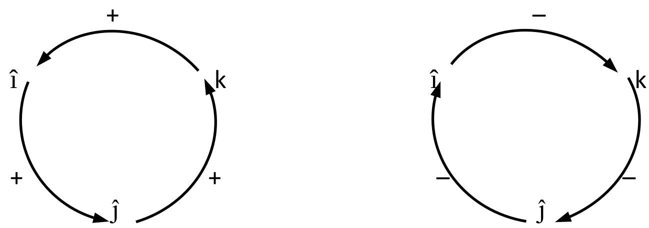
- (vi) Vector product of orthogonal unit vectors: The magnitude of each of the vectors $\hat{i}$, $\hat{j}$ and k is 1. and the angle between any of two of them is 90°.

∴ $\hat{i} \times \hat{j} = (1)(1) \sin 90°\ n = n \times n$

As n is a unit vector perpendicular to the plane of $\hat{i}$ and $\hat{j}$, so it is just the third vector k.

∴ $\hat{i} \times \hat{j} = k$

$$\hat{i} \times \hat{j} = k,\ \hat{j} \times k = \hat{i},\ k \times \hat{i} = \hat{j}$$
$$\hat{j} \times \hat{i} = -k,\ k \times \hat{j} = -\hat{i},\ \hat{i} \times k = -\hat{j}$$



[Vector product of base vectors is cyclic (i) Anticlock-wise positive, (b) Clockwise negative]

◘Aid to memory: Write $\hat{i}$, $\hat{j}$ and k cyclically round a circle, as shown in Fig. Multiplying two-unit vectors anticlockwise, we get positive value of third unit vector (e.g., $\hat{i} \times \hat{j} = +k$); and multiplying two-unit vectors clockwise, we get negative value of third unit vector (e.g., $\hat{j} \times \hat{i} = k$).

Also, $\hat{i} \times \hat{i} = (1)(1) \sin 0° \hat{n} = \vec{0}$

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = k \times k = 0$

◘ (vii) The vector product of two vectors can be expressed in terms of their rectangular components as a determinant.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

◘ **(viii) Sine of the angle between two vectors:** If θ is the angle between two vectors $\vec{A}$ and $\vec{B}$, then

$$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$$

or $$\sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}|\,|\vec{B}|}$$

◘ **(ix) Unit vector perpendicular to the plane of two vectors:** If n is a unit vector perpendicular to the plane of vectors A and B, then $A \times B = AB \sin\theta\, \vec{n} = |A \times B|\, \hat{n}$

or $$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

Cross product is anticommutative: As shown in Fig. (a) and (b), consider two vectors A and B such that the small angle between them is θ.

$\vec{A} \times \vec{B}$

Plane of $\vec{A}$ and $\vec{B}$ Plane of $\vec{A}$ and $\vec{B}$

$\vec{B}$

n   (– θ)

θ $\vec{A}$ $\vec{A}$

[Direction of $\vec{B} \times \vec{A}$ is opposite to that of $\vec{A} \times \vec{B}$] $\vec{B} \times \vec{A}$

Now, $\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$

and $\vec{B} \times \vec{A} = AB \sin(-\theta)\, \hat{n} = -AB \sin\theta\, \hat{n}$

Clearly, both $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ have equal magnitudes (AB sin θ) but they have opposite directions. Thus

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A} \quad \text{and} \quad \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Hence cross product of two vectors is not commutative, instead it is anticommutative.

Vector product in terms of rectangular components: We can express $\vec{A}$ and $\vec{B}$ in terms of their rectangular components as

$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z k$

$\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z k) \times (B_x \hat{i} + B_y \hat{j} + B_z k)$

$= A_x \hat{i} \times (B_x \hat{i} + B_y \hat{j} + B_z k) + A_y \hat{j} \times (B_x \hat{i} + B_y \hat{j} + B_z k) + A_z k \times (B_x \hat{i} + B_y \hat{j} + B_z k)$

$= A_x B_x (\hat{i} \times \hat{i}) + A_x B_y (\hat{j} \times \hat{j}) + A_y B_z (\hat{j} \times k) + A_y B_x (\hat{j} \times \hat{i}) + A_y B_y (\hat{j} \times \hat{j}) + A_y B_z (\hat{j} \times k)$

$+ A_z B_x (k \times \hat{i}) + A_z B_y (k \times \hat{j}) + A_z B_z (k \times k)$

or $= \hat{i}(A_y B_z - A_z B_y) - \hat{j}(A_z B_x - A_x B_z) + k(A_x B_y - A_y B_x)$

or $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

This equation expresses $\vec{A} \times \vec{B}$ in a determinant form.

Examples based on *Vector or Cross Product of two Vectors*

♦ **Formulae Used**

1. $\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$

2. Unit vector n perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$ is given by $n = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

3. Angle θ between vectors $\vec{A}$ and $\vec{B}$ is given by

$$\sin\theta = \frac{|\vec{A}\times\vec{B}|}{|\vec{A}|\,|\vec{B}|}$$

4. In terms of rectangular components, we have

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

or $\vec{A}\times\vec{B} = (A_y B_z - A_z B_y)\,\hat{i} + (A_z B_x - A_x B_z)\,\hat{j} + (A_x B_y - A_y B_x)\,k$

5. For parallel vectors, $\vec{A}\times\vec{B} = 0$

6. Moment of a force or torque, $\vec{\tau} = \vec{r}\times\vec{F}$.

Q. 1. ***Prove that the vectors $\vec{A} = 2\hat{i} - 3\hat{j} - k$ and $\vec{B} = -6\hat{i} + 9\hat{j} + 2k$ are parallel.***

Sol.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ A_x & -3 & -1 \\ -6 & 9 & 3 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -3 & -1 \\ 9 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} + k\begin{vmatrix} 2 & -3 \\ -6 & 9 \end{vmatrix}$$

$= \hat{i}(-9+9) - \hat{j}(6-6) + k(18-18) = 0$

Hence $\vec{A} \parallel \vec{B}$

Q. 2. ***Calculate the area of the parallelogram whose two adjacent sides are formed by the vectors $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = -3\hat{i} + 7\hat{j}$.***

Sol.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 3 & 4 & 0 \\ -3 & 7 & 0 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -4 & 0 \\ 7 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} + k\begin{vmatrix} 3 & 4 \\ -3 & 7 \end{vmatrix}$$

$= \hat{i}(0+0) - \hat{j}(0-0) + k(21-12) = 33\,k$

Area of parallelogram

$|\vec{A}\times\vec{B}| = \sqrt{0^2 + 0^2 + 33^2} = 33$ sq. Units.

Q. 3. ***If $\vec{A}$ and $\vec{B}$ denote the sides of a parallelogram and its area is AB/2, find the angle between $\vec{A}$ and $\vec{B}$.***

Sol. Area of parallelogram

$= (\vec{A}\times\vec{B}) = AB\sin\theta = AB/2$

$\therefore \quad \sin\theta = ½ \quad$ or $\quad \theta = 30°$

Q. 4. ***Determine a unit vector perpendicular to both $\vec{A} = 2\hat{i} + \hat{j} + k$ and $\vec{B} = \hat{i} - \hat{j} + 2k$.***

Sol. The perpendicular unit vector n is given by

$A\times B = AB\sin\theta\, n = |\vec{A}\times\vec{B}|\, n$

$$\therefore \quad n = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

Now

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$= \hat{i}(2+1) - \hat{j}(4-1) + k(-2-1)$

$= 3\hat{i} - 3\hat{j} - 3k$

$\therefore \quad |\vec{A}\times\vec{B}| = \sqrt{3^2 + (-3)^2 + (-3)^2} = \sqrt{27} = 3\sqrt{3}$

Hence $\quad n = \dfrac{3\hat{i} - 3\hat{j} - 3k}{3\sqrt{3}} = \dfrac{1}{\sqrt{3}}(\hat{i} - \hat{j} - k)$

Q. 5. ***Find a vector whose length is 7 and which is perpendicular to each of the vectors.***
$\vec{A} = 2\hat{i} - 3\hat{j} + 6k$ and $\vec{B} = \hat{i} + \hat{j} - k$

Sol. Here

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 2 & -3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

$= \hat{i}(3-6) - \hat{j}(-2-6) + k(2+3)$

$= -3\hat{i} + 8\hat{j} + 5k$

$|\vec{A}\times\vec{B}| = \sqrt{(-3)^2 + 8^2 + 5^2} = \sqrt{98} = 7\sqrt{2}$

Unit vector perpendicular to A and B is

$$n = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{-3\hat{i} + 8\hat{j} + 5k}{7\sqrt{2}}$$

Required vector

$$= 7n = 7\;\frac{-3\hat{i} + 8\hat{j} + 5k}{7\sqrt{2}}$$

$= \frac{1}{\sqrt{2}} (-3\hat{i} + 8\hat{j} + 5\hat{k})$

Q. 6. ***Determine the sine of the angle between the vectors $3\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} - 2\hat{j} + 4\hat{k}$.***

Sol. Let $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$. Then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$$

$= \hat{i}(4 + 4) - \hat{j}(12 - 4) + \hat{k}(-6 - 2)$

$= 8\hat{i} - 8\hat{j} - 8\hat{k}$

$|\vec{A} \times \vec{B}| = \sqrt{8^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$

$|\vec{A}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$

$|\vec{B}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24}$

$\therefore \quad \sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}|\,|\vec{B}|} = \frac{8\sqrt{3}}{\sqrt{14} \times \sqrt{24}} = \frac{2}{\sqrt{7}}$

Q. 7. **Show that: $(A - B) \times (A + B) = 2(\vec{A} \times \vec{B})$**

Sol. LHS $= (\vec{A} - \vec{B}) \times (\vec{A} + \vec{B})$

$= \vec{A} \times \vec{A} + \vec{A} \times \vec{B} - \vec{B} \times \vec{A} - \vec{B} \times \vec{B}$

$= 0 + \vec{A} \times \vec{B} + \vec{A} \times \vec{B} + 0 \qquad [\because \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}]$

$= 2(\vec{A} \times \vec{B})$ = RHS

Q. 8. ***For any three vector A, $\vec{B}$ and $\vec{C}$, prove that $\vec{A} \times (\vec{B} + \vec{C}) + \vec{B} \times (\vec{C} + \vec{A}) + \vec{C} \times (\vec{A} + \vec{B}) = 0$***

Sol. LHS

$= \vec{A} \times (\vec{B} + \vec{C}) + \vec{B}(\vec{C} + \vec{A}) + \vec{C}(\vec{A} + \vec{B})$

$= \vec{A} \times \vec{B} + \vec{A} \times \vec{C} + \vec{B} \times \vec{C} + \vec{B} \times \vec{A} + \vec{C} \times \vec{A} + \vec{C} \times \vec{B}$

$= \vec{A} \times \vec{B} + \vec{A} \times \vec{C} + \vec{B} \times \vec{C} - \vec{A} \times \vec{B} - \vec{A} \times \vec{C} - \vec{B} \times \vec{C} \qquad [\because \vec{B} \times \vec{A} = -\vec{A} \times \vec{B}]$

$= \vec{0}$ = RHS

Q. 9. ***For any two vectors $\vec{A}$ and $\vec{B}$, prove that $(\vec{A} \times \vec{B})^2 = A^2B^2 - (\vec{A} . \vec{B})^2$***

Sol. LHS $= (\vec{A} \times \vec{B})^2$

$= |\vec{A} \times \vec{B}|^2 = (AB\sin\theta)^2$

$= A^2B^2(1 - \cos^2\theta) = A^2B^2 - (AB\cos\theta)^2$

$= A^2B^2 - (\vec{A} . \vec{B})^2$ = RHS

Q. 10. ***Find $A.\vec{B}$ if $|\vec{A}| = 2$, $|\vec{B}| = 5$ and $|\vec{A} \times \vec{B}| = 8$***

Sol. As $|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 - (\vec{A} . \vec{B})^2$

$\therefore \quad 8^2 = 2^2 \times 5^2 - (\vec{A} . \vec{B})^2$

or $\quad (\vec{A} . \vec{B})^2 = 100 - 64 = 36$

$\therefore \quad \vec{A} . \vec{B} = \pm 6$

Q. 11. ***Find the area of the triangle formed by the tips of the vectors $\vec{a} = \hat{i} - \hat{j} - 3\hat{k}$, $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$.***

Sol. Let ABC the triangle formed by the tips of the given vectors. Then

$BA = \vec{a} - \vec{b} = (\hat{i} - \hat{j} - 3\hat{k}) - (4\hat{i} - 3\hat{j} + \hat{k})$

$= -3\hat{i} + 2\hat{j} - 4\hat{k}$

$BC = \vec{c} - \vec{b} = (3\hat{i} - \hat{j} + 2\hat{k}) - (4\hat{i} - 3\hat{j} + \hat{k})$

$= -\hat{i} + 2\hat{j} + \hat{k}$

Now $\vec{BA} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & -4 \\ -1 & 2 & 1 \end{vmatrix}$

$= \hat{i}(2 + 8) - \hat{j}(-3 - 4) + \hat{k}(-6 + 2) = 10\hat{i} + 7\hat{j} - 4\hat{k}$

$|\vec{BA} \times \vec{BC}| = \sqrt{(10)^2 + 7^2 + (-4)^2} = \sqrt{165} = 12.8$

Area of $\Delta ABC = ½\,|\vec{BA} \times \vec{BC}| = ½ \times 12.8 = 6.4$ sq. Units.

Q. 12. ***Find the moment about the point (1, – 1, – 1) of the force $3\hat{i} + 4\hat{j} - \hat{k}$ acting at the point (1, 0, – 2).***

Sol. Here $\quad \vec{F} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

Let P be the point about which moment is to be obtained and A be the point at which force is applied. If O is the origin, then

$\vec{OP} = \hat{i} - \hat{j} - \hat{k}, \qquad \vec{OA} = \hat{i} + 0\hat{j} - 2\hat{k}$

$\therefore \quad \vec{PA} = \vec{OA} - \vec{OP} = (\hat{i} - 0\,h - 2\hat{k}) - (\hat{i} - \hat{j} - \hat{k}) = \hat{j} - \hat{k}$

Moment of force $\vec{F}$ about the point P is

$\vec{\tau} = \vec{PA} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 3 & 4 & -5 \end{vmatrix}$

$= \hat{i}(-5 + 4) - \hat{j}(0 + 3) + \hat{k}(0 - 3) = -\hat{i} - 3\hat{j} - 3\hat{k}$

Q. 13. ***The diagonals of a parallelogram are given by the vectors $3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$. Find the area of the parallelogram.***

Sol. If A and B are the adjacent sides of the parallelogram, then its diagonals will be

$\vec{A} + \vec{B} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $A - B = \hat{i} - 3\hat{j} + 4\hat{k}$

Now $\vec{A} \times \vec{B} = ½ (A - B) \times (A + B)$

$$= ½ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 4 \\ 3 & 1 & 2 \end{vmatrix}$$

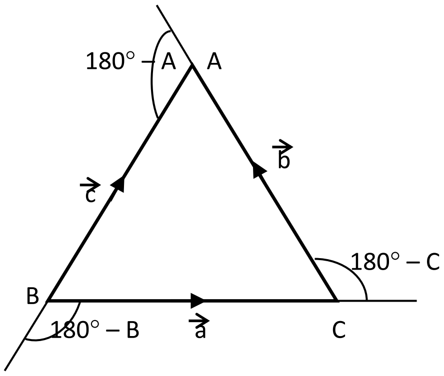
$= ½ [\hat{i}(-6 - 4) - \hat{j}(2 - 12) + \hat{k}(1 + 9)] = ½ (-10\hat{i} + 10\hat{j} + 10\hat{k})$

Area of parallelogram

$= |\vec{A} \times \vec{B}| = ½ \sqrt{(-10)^2 + 10^2 + 10^2} = ½ \times 17.32 = 8.66$ sq. Units

Q. 14. ***In any ΔABC, prove that***

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Sol. As shown in Fig. the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are cyclic, therefore

$\vec{a} + \vec{b} + \vec{c} = 0$ or $\vec{a} + \vec{b} = -\vec{c}$

or $(\vec{a} + \vec{b}) \times \vec{c} = -\vec{c} \times \vec{c}$ or $\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = 0$

or $-\vec{c} \times \vec{a} = \vec{b} \times \vec{c} = 0$ or $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ ... (i)

Similarly, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ... (ii)

From (i) and (ii), we get

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

or $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

or ab sin (180° – C) = bc sin (180° – A) = ca sin (180° – B)

or ab sin C = bc sin A = ca sin B

Dividing throughout by abc, we get

$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$ or $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Q. 15. ***If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} . \vec{b} = \vec{a} . \vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{a} \neq \vec{0}$ then prove that $\vec{b} = \vec{c}$.***

Sol. Given $\vec{a} . \vec{b} = \vec{a} . \vec{c}$ or $\vec{a} . \vec{b} - \vec{a} . \vec{c} = 0$

or $\vec{a} . (\vec{b} - \vec{c}) = 0$

But $\vec{a} \neq 0$ ∴ Either $\vec{b} - \vec{c} = 0$ or $\vec{a} \perp (\vec{b} - \vec{c})$

=> Either $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$ ... (i)

Also, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ or $\vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0$

or $\vec{a} \times (\vec{b} - \vec{c}) = 0$

But $\vec{a} \neq 0$ ∴ Either $\vec{b} - \vec{c} = 0$ or $\vec{a} \parallel (\vec{b} - \vec{c})$

=> Either $\vec{b} = \vec{c}$ or $\vec{a} \parallel (\vec{b} - \vec{c})$ ... (ii)

But a cannot be simultaneously perpendicular and parallel to (b – c), so equation (i) and (ii) will hold simultaneously if b = c.

Q. 16. ***If $\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - 2\hat{k}$, then find $\vec{a} \times (\vec{b} \times \vec{c})$.***

Sol. $$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix}$$

$= \hat{i}(-2 + 3) - \hat{j}(-4 + 1) + \hat{k}(6 - 1) = \hat{i} + 3\hat{j} + 5\hat{k}$

Now, $$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 1 & 3 & 5 \end{vmatrix}$$

$= \hat{i}(-10 + 9) - \hat{j}(5 + 3) + \hat{k}(3 + 2) = -\hat{i} - 8\hat{j} + 5\hat{k}$

VERY SHORT ANSWER CONCEPTUAL PROBLEMS

Q. 1. Pick out the two scalar quantities in the following list: force, angular momentum, work, current, linear momentum, electric field, average velocity, magnetic moment, reaction as per Newton's third law, relative velocity.

Sol. Work and current

Q. 2. Pick out the only vector quantity in the following list: Temperature, pressure, impulse, time, power, total path length, energy, gravitational potential, coefficient of friction, charge.

Sol. Impulse

Q. 3. Why cannot be vectors added algebraically?

Sol. Apart from magnitude, the vectors also have directions, so they cannot be added algebraically.

Q. 4. State the essential condition for the addition of vectors.

Sol. The essential condition for the addition of vectors is that they must represent the physical quantities of same nature.

Q. 5. Is time a vector quantity? Give reason.

Sol. Time flows from past to present and present to future. Thus, the direction of time flow is unique and does not need to be specified. Hence time is not a vector, though it has a direction.

Q. 6. Is pressure a vector? Given reason

Sol. Pressure is always taken to be normal to the plane of the area on which it is acting. As this direction is unique, it does need any specification. So pressure is not a vector.

Q. 7. Can two vectors of different magnitudes be combined to given zero resultant?

Sol. No, two vectors of different magnitudes cannot be combined to given zero resultant.

Q. 8. When is the magnitude of the resultant of two vectors equal to either of them?

Sol. When two vectors of equal magnitude are inclined to each other at an angle of 120°, the magnitude of their resultant is equal to that of the either vector.

$a^2 = a^2 + a^2 + 2a^2 \cos^2\theta$

=> $\cos\theta = -\frac{1}{2}$ => $\theta = 120°$

Q. 9. What is the difference between $(\vec{A} - \vec{B})$ and $(\vec{B} - \vec{A})$?

Sol. The two vectors have equal magnitude but opposite directions.

Q. 10. Can $\vec{A} + \vec{B} = \vec{A} - \vec{B}$?

Sol. Yes, the equality holds when B is a null vector.

Q. 11. When is the magnitude of (A + B) equal to the magnitude of (A – B)?

Sol. When the vectors A and B are perpendicular to each other, $|A + B| = |A - B|$.

Q. 12. Under what condition will the directions of sum and difference of two vectors be same?

Sol. When the two vectors are unequal in magnitude and are in the same direction.

Q. 13. If A + B = B + A, what can you about the angle between A and B?

Sol. A and B can have any angle between them because commutative law holds good for any two coplanar vectors.

Q. 14. Can we add a velocity vector to a displacement vector?

Sol. No, only vectors repressing physical quantites of same nature can be added together.

Q. 15. What is the minimum number of coplanar vectors of different magnitudes which can give zero resultant?

Sol. Three. If three vectors can be represented by the three sides of a triangle taken in the same order, then their resultant is zero vector.

Q. 16. If a + b = c and $|a| + |b| = |c|$, what can we say about the direction of these vectors?

Sol. The three vectors have the same direction.

Q. 17. What is the resultant of vector $\vec{A}$ multiplied by real number m?

Sol. The resultant vector $m\vec{A}$ has magnitude m times that of $\vec{A}$. It has same direction as that of $\vec{A}$ if m is positive. It has direction opposite to that of A if m is negative.

Q. 18. Can a vector be multiplied by both dimensional and non-dimensional scalars?

Sol. Yes, When a vector is multiplied by a dimensional scalar, the resultant has different dimensions. When the vector is multiplied by a non-dimensional scalar, its dimensional remain unchanged.

Q. 19. What is the maximum number of components into which a vector can resolved?

Sol. Infinite

Q. 20. Can the magnitude of the rectangular component of a vector be greater than the magnitude of that vector?

Sol. No, For example, the rectangular components of vector A are $A_x = A\cos\theta$ and $A_y = A\sin\theta$. As both $\sin\theta$ and $\cos\theta$ can take values between – 1 and + 1, so the magnitude of both A_x and A_y cannot be greater than A.

Q. 21. Can a vector be zero when one of the components is not zero while all the other components are zero?

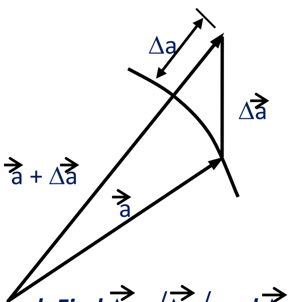
Sol. No, Any vector $\vec{A}$ in three dimensions can be written as

$A = A_x\hat{i} + A_y\hat{j} + A_z k$, where $A = \sqrt{A^2_x + A^2_y + A^2_z}$

Clearly, if any of the components A_x, A_y or A_z is not zero, the vector $\vec{A}$ will not be a zero vector.

Q. 22. Can the increment $\vec{\Delta}a$ in the magnitude of vector a be greater than the modulus of the increment of the vector, that is $/\vec{\Delta a}/$? Can they be equal?

Sol. No, the increment Δ a cannot be greater than the increment $|\vec{\Delta a}|$, as shown in Fig. The two will be equal if $\vec{a}$ and $\vec{\Delta a}$ have the same direction.



Q. 23. The direction of vector $\vec{a}$ is reversed. Find $\vec{\Delta a}$, $/\vec{\Delta a}/$ and $\vec{\Delta}a$.

Sol. $\vec{\Delta a} = -\vec{a} - \vec{a} = -\vec{2a}$

$|\vec{\Delta a}| = 2a$

Q. 24. ***If $\vec{a}$ and $\Delta\vec{a}$ are directed opposite each other, what is the relation between Δa and $|\Delta\vec{a}|$?***

Sol. $\Delta a = -|\Delta\vec{a}|$.

Q. 25. ***A vector $\vec{a}$ is turned through a small angle $d\theta$ without a change in its length. What are $|\Delta\vec{a}|$ and Δa?***

Sol. $|\Delta\vec{a}| = a\,d\theta$ and $\Delta a = 0$

Q. 26. ***Give two conditions necessary for a given quantity to be a vector.***

Sol. (i) The quantity must have both magnitude and direction.

(ii) It must obey the laws of vector addition.

Q. 27. ***Is finite rotation a vector?***

Sol. No, This is because the addition of two finite rotations about different axes does not obey commutative law.

Q. 28. ***Can the scalar product of two vectors be negative?***

Sol. Yes, the scalar product is negative when the angle between two vectors lies between 90° and 270°.

Q. 29. ***What is the dot product of two perpendicular vectors $\vec{A}$ and $\vec{B}$?***

Sol. Zero, $\vec{A}\cdot\vec{B} = AB\cos 90° = 0$

Q. 30. ***What is the dot product of two similar unit vectors?***

Sol. Unity. For example $\hat{\imath}\cdot\hat{\jmath} = (1)(1)\cos 0° = 1$

Q. 31. ***What is the dot product of two dissimilar unit vectors? Or Calculate the value of $\hat{\imath}\cdot\hat{\jmath}$.***

Sol. Zero

For example, $\hat{\imath}\cdot\hat{\jmath} = (1)(1)\cos 90° = 0$

Q. 32. ***If A . B = A . C, is it correct to conclude that B = C.***

Sol. No, we can write A : (B – C) = 0 which implies that A may be perpendicular to (B – C).

Q. 33. ***If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are non-zero vector and A . B = 0 and B . C = 0, then find out the value of A . C.***

Sol. $A\cdot B = 0 \Rightarrow A \perp B$

$B\cdot C = 0 \Rightarrow B \perp C$

$\therefore$ $A \parallel C$ and $A\cdot C = AC\cos 0° = AC$

Q. 34. ***What is the magnitude and direction of i + j?***

Sol. $|\hat{\imath}+\hat{\jmath}| = \sqrt{1^2+1^2} = \sqrt{2}$

If the vector $\hat{\imath}+\hat{\jmath}$ makes angle β with x-axis, then

$$\tan\beta = \frac{\text{coeff. of } \hat{\imath}}{\text{coeff. of } \hat{\jmath}} = \frac{1}{1} = 1 \quad \therefore \quad \beta = 45°$$

Q. 35. ***What is the angle made by vector A = 2 $\hat{\imath}$ + 2 $\hat{\jmath}$ with x-axis?***

Sol. The angle θ between A and x-axis is given by

$$\cos\theta = \frac{A\cdot\hat{\imath}}{|A|\,|\hat{\imath}|} = \frac{(2\hat{\imath}+2\hat{\jmath})\cdot\hat{\imath}}{|2\hat{\imath}+2\hat{\jmath}|\,|\hat{\imath}|} = \frac{2\times1+2\times0}{\sqrt{2^2+2^2}\sqrt{1^2}} = \frac{2}{2\sqrt{2}}$$

or $\cos\theta = 1/\sqrt{2}$ $\therefore$ $\theta = 45°$

Q. 36. ***What should be the angle θ between two vectors A and B for their resultant R to be maximum?***

Sol. $R = \sqrt{A^2+B^2+2AB\cos\theta}$

R will be maximum when $\cos\theta = +1$ or $\theta = 0°$

$R_{max} = A + B$

Q. 37. ***What should be the angle θ between tow vectors A and B for their resultant R to be minimum?***

Sol. $R = \sqrt{A^2+B^2+2AB\cos\theta}$

R will be minimum when $\cos\theta = -1$ or $\theta = 180°$ $R_{min} = A \sim B$

Q. 38. ***What is the effect on the magnitude of the resultant of two vectors when the angle θ between them is increased from 0° to 180°?***

Sol. $R = \sqrt{A^2+B^2+2AB\cos\theta}$

As the angle θ increases from 0° to 180°, the value of $\cos\theta$ decreases, so the magnitude R of the resultant also decreases

Q. 39. ***Two persons are pulling the ends of a string in such a way that the string is stretched horizontally. When a weight of 10 kg is suspended in the middle of the string, the string does not remain horizontal. Can the persons make it horizontal again by putting it with a greater force?***

Sol. No, the vertical weight cannot be balanced by the horizontal force, however larger the two forces may be.

Q. 40. ***What is the vector sum of n coplanar forces, each of magnitude F, if each force makes an angle of $2\pi/n$ with the proceeding force?***

Sol. Total angle between n coplanar force = $(2\pi/n)\times n = 2\pi$. This shows that the n forces can be represented by the n sides of a closed polygon taken in the same order. So, the resultant force is zero.

Q. 41. ***Two vectors A and B are in the same plane and angle between them is θ. What is the magnitude and direction of $A\times B$?***

Sol. $|A\times B| = AB\sin\theta$

The direction of $A\times B$ is perpendicular to the plane of A and B and points in the same direction in which a right-handed screw would advance when rotated from A to B.

Q. 42. ***If A and B are two length vectors, then what is the geometrical significance of $|A\times B|$?***

Sol. $|A \times B|$ = AB sin θ, this gives area of the parallelogram with adjacent sides A and B.

Q. 43. What is the unit vector perpendicular to the plane of vectors A and B?

Sol. $n = \dfrac{A \times B}{|A \times B|} = \dfrac{A \times B}{AB \sin \theta}$

Q. 44. What is the value of A × A?

Sol. Zero, because A × A = AA sin 0° = 0

Q. 45. What is the condition for two vectors to be collinear?

Sol. For two given vectors to be collinear, their cross-product must be zero.

Q. 46. If A × B = C × B, show that C need not be equal to A.

Sol. As A × B = C × B

A × B – C × B = 0

or (A – C) × B = 0

This implies that if C ≠ A, then either B = 0 or (A – C) ‖ B

Q. 47. Find the value of $\hat{i} \times \hat{j}$.

Sol. $\hat{i} \times \hat{j}$ = (1) (1) sin 90° k = k

Q. 48. What is i . (j × k)?

Sol. $\hat{i} \cdot (\hat{j} \times k) = \hat{i} \cdot \hat{i}$ = (1) (1) cos 0° = 1

Q. 49. Under what condition will the equality: |A × B| = A . B hold good?

Sol. As |A × B| = A . B

∴ AB sin θ = AB cos θ or tan θ = 1 => θ = 45°

Q. 50. What is the angle between (A + B) and (A × B)?

Sol. The resultant vector (A + B) lies in the plane of A and B, while the cross product (A × B) is perpendicular to this plane. So the angle between (A + B) and (A × B) is 90°.

SHORT ANSWER CONCEPTUAL PROBLEMS

Q. 1. State for each of the following physical quantities, if its is a scalar or a vector: volume, mass, speed, acceleration, density, number of moles, velocity, angular frequency, displacement, angular velocity.

Sol. **Scalars:** volume, mass, speed, density, number of moles and angular frequency.

Vectors: Acceleration, velocity, displacement and angular velocity.

Q. 2. State with reasons, whether the following algebraic operations with scalar and vector physical quantities are meaningful:

(a) Adding any two scalars. ***(b) Adding a scalar to a vector of the same dimensions.***

(c) Multiplying any vector by any scalar. ***(d) Multiplying any two scalars.***

(e) Adding any two vectors. ***(f) Adding a component of a vector to the same vector.***

Sol. (a) No, Only two such scalars can be added which represent the same physical quantity.

(b) No, A scalar cannot be added to a vector even of same dimensions because a vector has a direction while a scalar has no direction e.g., speed cannot be added to velocity.

(c) Yes, We can multiply any vector by a scalar. For example, when mass (scalar) is multiplied with acceleration (vector), we get force (vector) i.e., F = ma

(d) Yes, We can multiply any two scalars. When we multiply power (scalar) with time (scalar), we get work done (scalar) i.e., W = Pt.

(e) No, Only two vectors of same nature can be added by using the law of vector addition.

(f) No, A component of a vector can be added to the same vector only by using the law of vector addition. So the addition of a component of a vector to the same vector is not a meaningful algebraic operation.

Q. 3. Read each statement below carefully and state, with reasons and examples, if it is true or false:

A scalar quantity is one that

(a) is conserved in a process ***(b) can never take negative values***

(c) must be dimensionless ***(d) does not vary from one point to another in space***

(e) has the same value for observers with different orientations of axes.

Sol. (a) False, Kinetic energy (scalar) is not conserved in an inelastic collision. Moreover, vector quantities like linear momentum, angular momentum, etc., are also conserved.

(b) False, Scalar quantities such as electric potential, temperature, etc., can take negative values.

(c) False, Scalar quantities like mass, density, energy etc., are not dimensionless.

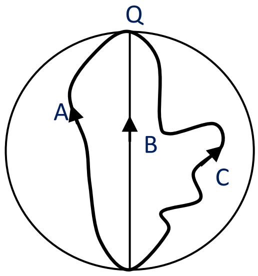
(d) False, Density (scalar) varies from point to point in the atmosphere.

(e) True, the mass (scalar) of a body as measured by different observes with different orientations of axes has the same value.

Q. 4. ***Read each statement below carefully and state with reasons, if it is true or false:***
(a) The magnitude of a vector is always a scalar.
(b) Each component of a vector is always a scalar.
(c) The total path length is always equal to the magnitude of the displacement vector of a particle.
(d) The average speed of a particle (defined as total path length divided by the time taken to cover the path) is either greater or equal to the magnitude of average velocity of the particular over the same interval of time.
(e) Three vectors not lying in a plane can never add up to give a null vector.

Sol. (a) True, the magnitude of a vector is a pure number and has no direction.
(b) False, Each component of a vector is also a vector.
(c) False, The displacement depends only on the end points while the path length depends on the actual path. The two quantities are equal only if the direction of motion of the object does not change. In all other cases, path length is greater than the magnitude of displacement.
(d) True, This is because the total path length is either greater than or equal to the magnitude of displacement over the same interval ot time.
(e) True, This is because the resultant of two vectors will not lie in the plane of third vector and hence cannot cancel its effect to give null vector.

Q. 5. ***Three girsl skating on a circular ice ground of radius 200 m start from a point P on the edge of the ground and reach a point Q diametrically opposite to P following different paths as shown in Fig. What is the magnitude of the displacement vector for each? For which girl is this equal to the actual length of path skated?***



Sol. Displacement of each girl = $\vec{PQ}$
Magnitude of displacement vector for each girl = $|\vec{PQ}|$ = 2 × radius = 2 × 200 = 400 m
For girl B, the magnitude of displacement vector = actual length of path.

Q. 6. ***A vector has magnitude and direction. Does it have a location in space? Can it vary with time? Will two equal vector $\vec{a}$ and b at different locations in space necessarily have identical physical effects? Give examples in support of your answer.***

Sol. In addition to magnitude and direction, each vector also has a definite location in space. For example, a velocity vector has definite location at every point of uniform circular motion.
A vector can vary with time. For example, increase in velocity produces acceleration.
Two equal vectors a and b having different locations may not produce identical physical effects. For example two equal forces (vectors) acting at two different points may not produce equal turning effects.

Q. 7. ***A vector has both magnitude and direction. Does it mean that anything that has magnitude and direction is necessarity a vector? The rotation of a body can be specified by the direction of the axis of rotation, and the angle of rotation about the axis. Does that make any rotation a vector?***

Sol. No, anything that has both magnitude and direction is not necessarily a vector. It must obey the laws of vector addition. Rotation is not generally considered a vector even though it has magnitude and direction because the commutative law, However, infinitesimally small rotations obey commutative law and hence an infinitesimally small rotation is considered a vector.

Q. 8. ***Can you associate vectors with (a) the length of a wire bent into a loop, (b) a plane area, (c) a sphere? Explain.***

Sol. Out of these, only a plane area can be associated with a vector. The direction of this area vector is taken normal to the plane.

Q. 9. ***Which of the following quantities are independent of the choice of orientation of the coordinate axes: $\vec{a}+\vec{b}$, $3\vec{a}_x + 2b_y$, $|\vec{a}+\vec{b}-\vec{c}|$, angle between b and c, λa, where λ is a scalar?***

Sol. A vector, its magnitude and the angle between two vectors do not depend on the choice of the orientation of the coordinate axes, so a + b, $|a + b - c|$, angle between b and c and λa are independent of the orientation of the coordinate axes.
But the quantity $3a_x + 2b_y$ depends upon the magnitude of the components along x- and y-axes, so it will change with the change in coordinate axes.

Q. 10. **Given a + b + c + d = 0, which of the following statements are correct:**
(a) a, b, c and d must each be a null vector,
(b) The magnitude of (a + c) equals the magnitude of (b + d),
(c) The magnitude of a can never be greater than the sum of the magnitudes of b, c, and d,
(d) b + c must lie in the plane of a and d if a and d are not collinear, and in the line of a and d, if they are collinear?

Sol. (a) a, b, c and d need not each be a null vector. The resultant of four non-zero vectors can be a null vector in many ways e.g., the resultant of any three vectors may be equal to the magnitude of fourth vector but has the opposite direction. Hence the statement a, b, c and d must each be a null vector, is not correct.

(b) Because, a + b + c + d = 0 or a = – (b + c + d). Hence, magnitude of vector a is equal to magnitude of vector (b + c + d). The sum of the magnitude of vectors b, c and d may be greater than or equal to that of vector a (or vector b + c + d). Hence the statement that the magnitude of a can never be greater than the sum of the magnitudes of b, c and d is correct.

(d) Because a + b + c + d = 0, hence (b + c) + a + d = 0. The resultant sum of three vectors (b + c, a and d can be zero only if b + c is in plane of a and d. In case a and d are collinear, b + c must be in line of a and d. Hence the given statement is correct.

Q. 11. **Do a + b and a – b lie in the same plane. Give reason.**

Sol. Yes, because a + b is represented by the diagonal of the parallelogram drawn with a and b as adjacent sides. The diagonal passes through the common tail of a and b. However, a – b is represented by the other diagonal of the same parallelogram not passing through the common tail of a and b. Thus both a + b and a – b lie in plane of the same parallelogram.

Q. 12. **Can we apply the commutative and associative laws to vector subtraction also?**

Sol. (i) No, we cannot apply commutative law to vector subtraction because $a - b \neq b - a$.

(ii) Yes, association law can be applied to vector subtraction because (a + b) – c = a + (b – c).

Q. 13. **Can three vectors not in one plane give a zero resultant? Can four vectors do so?**

Sol. No, three vectors not in one plane cannot give a zero resultant because the resultant of two vectors will not lie in the plane of the third vector and hence cannot cancel its effect.

The resultant of four vectors not in one plane may be a zero vector.

Q. 14. **What is the difference between the following two data?**

(i) 8 (5 km/hr, east) (ii) (8 hr) (5 km/hr, east)

Sol. (i) It is the product of a pure number and a vector (velocity), hence the unit of product is the same as that of vector i.e., the product is a velocity of 40 km/hr, towards east.

(ii) It is the product of a scalar (time) and a vector (velocity). Hence the unit of the product will be hr × (km/hr). Thus the product is a displacement of magnitude 40 km, towards east.

Q. 15. **Is $|a + b|$ greater than or less than $|a|+|b|$? Give reason.**

Sol. $|a+b|^2 - (|a| + |b|)^2 = |a|^2 + |b|^2 + 2|a||b| \cos\theta - |a|^2 - |b|^2 - 2|a||b|$

$= -2|a||b|(1 - \cos\theta)$

$= -2|a||b| \sin^2\theta/2$ = a negative quantity

Hence $|a+b| \leq |a| + |b|$.

Q. 16. **Is $|a - b|$ greater than or less than $|a| + |b|$?**

Sol. $|a-b|^2 - (|a| + |b|)^2$

$= |a|^2 + |b|^2 - 2|a||b| \cos\theta - |a|^2 - |b|^2 - |a||b|$

$= -4|a||b| \cos^2\theta/2$ = a negative quantity

Hence $|a-b| \geq |a| + |b|$

Q.17 **Two vectors of magnitudes 3 and 4 give a resultant of magnitude 5. What is the dot product of the two vectors ?**

Ans. $5^2 = 3^2 + 4^2$. So, the given vectors are perpendicular. Thus, their dot product is zero.

Q.18 **Two vectors of magnitudes 2 and 3 give a resultant of magnitude 5. What is the cross product of the given vectors**

Ans. 2 + 3 = 5. So, the given vectors are parallel. Thus, their cross product is zero.

Q.19 **What is the basic condition for the composition of vectors?**

Ans. The vectors to be added must represent quantities of same physical nature. As an example, we cannot add displacement and momentum vectors.