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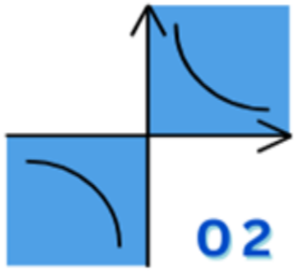


HYPERBOLA

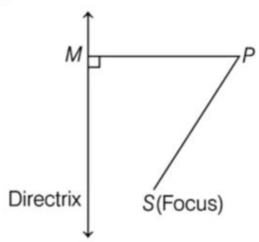
02

IN THIS CHAPTER ....

A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have constant difference. The two fixed points are the foci of the hyperbola.

OR

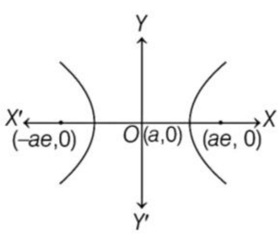
A hyperbola is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (i.e. focus) in the same plane to its distance from a fixed line (i.e. directrix) is always constant which is always greater than unity.



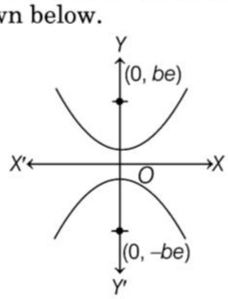
Mathematically, $\dfrac{SP}{PM} = e$, where $e > 1$ is eccentricity of the hyperbola.

Standard Equation of Hyperbola

If the centre of the hyperbola is at the origin and foci are on the X-axis or Y-axis, then that types of equation are called standard equation of an ellipse. The two such possible orientations are shown below.

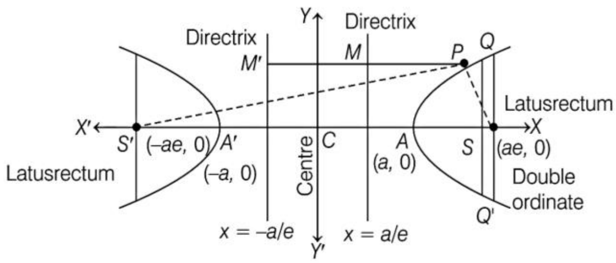


(i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



(ii) $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Terms Related to Hyperbola



Vertices

The point A and A', where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola.

Transverse and Conjugate Axes

Transverse axis is the one which lie along the line passing through the foci and perpendicular to the directrices and conjugate axis is the one which is perpendicular to the transverse axis and passes through the mid-point of the foci (i.e. centre).

Centre

The middle point C of AA' bisects every chord of the hyperbola passing through it and is called the centre of the hyperbola.

Focal Chord

A chord of a hyperbola which is passes through the focus is called a focal chord of the hyperbola.

Directries

A line which is perpendicular to the axis and it lies between centre and vertex. The equations of directries are $x = \pm \frac{a}{e}$.

Double Ordinates

If Q be a point on the hyperbola draw QN perpendicular to the axis of the hyperbola and produced to meet the curve again at Q'. Then, QQ' is called a double ordinate of Q.

Latusrectum

The double ordinate passing through focus is called latusrectum.

Note

- The vertex divides the join of focus and the point of intersection of directrix with axis internally and externally in the ratio $e : 1$.
- Domain and range of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $x \le -a$ or $x \ge a$ and $y \in R$ respectively.

Example 1. *The equation of the hyperbola whose directrix is $2x + y = 1$, focus is $(1, 2)$ and eccentricity is $\sqrt{3}$, is $7x^2 - 2y^2 + 12xy - 2x + 14y + k = 0$, then k is equal to*

(a) 0
(b) 22
(c) –22
(d) None of the above

Sol. (c) We know that, $SP^2 = e^2 PM^2$

$\Rightarrow \quad 5[(x-1)^2 + (y-2)^2] = 3(2x + y - 1)^2$

$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 20y + 25$
$$= 3(4x^2 + y^2 + 1 + 4xy - 4x - 2y)$$

$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

Thus, $\quad k = -22$

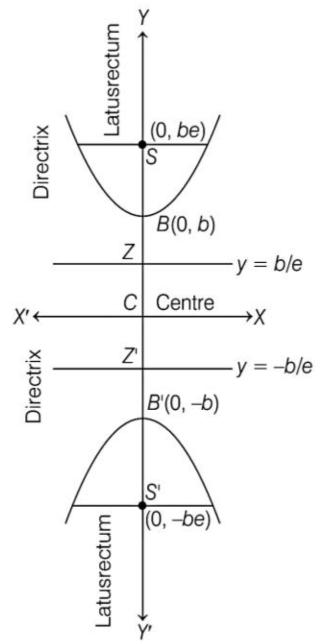
Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

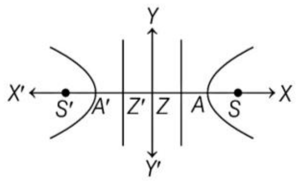
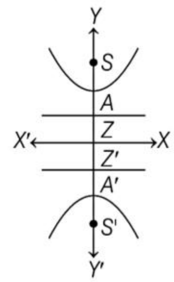
The hyperbola conjugate of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is
$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Detailed Description of Hyperbola

	Fundamental Terms	Hyperbola	Conjugate Hyperbola
(a)	Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
(b)	Graph		
(c)	Centre	$C(0, 0)$	$C(0, 0)$
(d)	Vertices	$(\pm a, 0)$	$(0, \pm b)$
(e)	Length of transverse axis	$2a$	$2b$
(f)	Length of conjugate axis	$2b$	$2a$
(g)	Foci	$(\pm ae, 0)$	$(0, \pm be)$
(h)	Equation of directrices	$x=\pm\left(\frac{a}{e}\right)$	$y=\pm\left(\frac{b}{e}\right)$
(i)	Eccentricity	$e=\sqrt{1+\frac{b^2}{a^2}}$	$e=\sqrt{1+\frac{a^2}{b^2}}$
(j)	Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
(k)	Ends of latusrectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
(l)	Parametric equations	$\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$	$\begin{cases} x = b\sec\theta \\ y = a\tan\theta \end{cases}$
(m)	Parametric coordinates	$(a\sec\theta, b\tan\theta)$, $0\le\theta\le 2\pi$	$(b\sec\theta, a\tan\theta)$, $0\le\theta\le 2\pi$
(n)	Focal radi	$\lvert SP\rvert=(ex_1-a)$ and $\lvert S'P\rvert=(ex_1+a)$	$\lvert SP\rvert=(ey_1-b)$ and $\lvert S'P\rvert=(ey_1+b)$
(o)	Difference of focal radi $=\lvert SP\rvert-\lvert S'P\rvert$	$2a$	$2b$
(p)	Distance between foci	$2ae$	$2be$
(q)	Tangents at vertices	$x=a$ and $x=-a$	$x=b$ and $y=-b$

Example 2. *A hyperbola has its centre at the origin, passes through the point (4, 2) and has transverse axis of length 4 along the X-axis. Then the eccentricity of the hyperbola is* **(JEE Main 2019)**

(a) 2 (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{3}{2}$ (d) $\sqrt{3}$

Sol. (*b*) Equation of hyperbola is given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\because$ Length of transverse axis $=2a=4$

$\therefore$ $a=2$

Thus, $\frac{x^2}{4}-\frac{y^2}{b^2}=1$ is the equation of hyperbola

$\because$ It passes through $(4, 2)$.

$$\therefore \quad \frac{16}{4}-\frac{4}{b^2}=1 \Rightarrow 4-\frac{4}{b^2}=1$$

$$\Rightarrow \quad b^2=\frac{4}{3}\Rightarrow b=\frac{2}{\sqrt{3}}$$

Now, eccentricity,

$$e=\sqrt{1+\frac{b^2}{a^2}}$$

$$=\sqrt{1+\frac{\frac{4}{3}}{4}}=\sqrt{1+\frac{1}{3}}=\frac{2}{\sqrt{3}}$$

Example 3. *If the vertices of a hyperbola be at $(-2, 0)$ and $(2, 0)$ and one of its foci be at $(-3, 0)$, then which one of the following points does not lie on this hyperbola?* **(JEE Main 2019)**

(a) $(2\sqrt{6}, 5)$ (b) $(6, 5\sqrt{2})$

(c) $(4, \sqrt{15})$ (d) $(-6, 2\sqrt{10})$

Sol. (*b*) The vertices of hyperbola are given as $(\pm 2, 0)$ and one of its foci is at $(-3, 0)$.

$\therefore (a, 0) = (2, 0)$ and $(-ae, 0), = (-3, 0)$

On comparing *x*-coordinates both sides, we get

$\Rightarrow \quad a = 2$ and $-ae = -3$

$\Rightarrow \quad 2e = 3 \Rightarrow e = \frac{3}{2}$

Also, $\frac{9}{4} = 1 + \frac{b^2}{4} \Rightarrow b^2 = 5 \quad \left[\because e^2 = 1 + \frac{b^2}{a^2}\right]$

So, equation of the hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{5} = 1 \quad \text{... (i)}$$

The point $(6, 5\sqrt{2})$ from the given options does not satisfy the above equation of hyperbola.

Example 4. *Let e_1 and e_2 be the eccentricities of the ellipse, $\frac{x^2}{25} + \frac{y^2}{b^2} = 1 (b < 5)$ and the hyperbola, $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$ respectively satisfying $e_1 e_2 = 1$. If α and β are the distances between the foci of the ellipse and the foci of the hyperbola respectively, then the ordered pair (α, β) is equal to* **(JEE Main 2020)**

(a) (8, 12) (b) (8, 10) (c) $\left(\frac{20}{3}, 12\right)$ (d) $\left(\frac{24}{5}, 10\right)$

Sol. (*b*) For the given ellipse, $\frac{x^2}{25} + \frac{y^2}{b^2} = 1, (b < 5)$,

the eccentricity $e_1 = \sqrt{1 - \frac{b^2}{25}}$

and for the given hyperbola $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$,

the eccentricity $e_2 = \sqrt{1 + \frac{b^2}{16}}$

Since, $e_1 e_2 = 1 \Rightarrow \sqrt{1 - \frac{b^2}{25}}\sqrt{1 + \frac{b^2}{16}} = 1$

$\Rightarrow \left(1 - \frac{b^2}{25}\right)\left(1 + \frac{b^2}{16}\right) = 1 \Rightarrow 1 - \frac{b^2}{25} + \frac{b^2}{16} - \frac{b^4}{400} = 1$

$\Rightarrow \quad b^2 = 25 - 16 \Rightarrow b^2 = 9$

$\therefore \quad e_1 = \frac{4}{5}$ and $e_2 = \frac{5}{4}$

Now, as α = distance between the foci of the ellipse

$\Rightarrow \quad \alpha = 2 \times 5 \times \frac{4}{5} \Rightarrow \alpha = 8$

and β = distance between the foci of the hyperbola

$\Rightarrow \quad \beta = 2 \times 4 \times \frac{5}{4} = 10$

$\therefore$ The order pair $(\alpha, \beta) = (8, 10)$

Hence, option (b) is correct.

Example 5. *If $5x + 9 = 0$ is the directrix of the hyperbola $16x^2 - 9y^2 = 144$, then its corresponding focus is* **(JEE Main 2019)**

(a) $\left(-\frac{5}{3}, 0\right)$ (b) $(-5, 0)$ (c) $\left(\frac{5}{3}, 0\right)$ (d) (5, 0)

Sol. (*b*) Equation of given hyperbola is $16x^2 - 9y^2 = 144$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \text{...(i)}$$

So, the eccentricity of Eq. (i)

$$e = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$$

$\left[\because \text{the eccentricity (e) of the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \sqrt{1 + (b/a)^2}\right]$

and given directrix is $5x + 9 = 0$

$\Rightarrow \quad x = -9/5$

So, corresponding focus is $\left(-3\left(\frac{5}{3}\right), 0\right) = (-5, 0)$

Example 6. *A hyperbola having the transverse axis of length $\sqrt{2}$ has the same foci as that of the ellipse $3x^2 + 4y^2 = 12$, then this hyperbola does not pass through which of the following points?* **(JEE Main 2020)**

(a) $\left(\frac{1}{\sqrt{2}}, 0\right)$ (b) $\left(-\sqrt{\frac{3}{2}}, 1\right)$

(c) $\left(1, -\frac{1}{\sqrt{2}}\right)$ (d) $\left(\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}\right)$

Sol. (*d*) Equation of given ellipse is

$$3x^2 + 4y^2 = 12 \Rightarrow \frac{x^2}{4} + \frac{y^2}{3} = 1 \quad \text{... (i)}$$

$\therefore$ Eccentricity of ellipse (i) is $e_1 = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

$\therefore$ Coordinate of foci is $(\pm 1, 0)$.

Now, it is given that length of the transverse axis of hyperbola is $\sqrt{2}$, so $\frac{\sqrt{2}}{2} e_2 = 1$,

where e_2 is the eccentricity of the hyperbola.

So, $\quad e_2 = \sqrt{2}$

$\therefore$ Given hyperbola is rectangular hyperbola and it's equation is

$$\frac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1 \Rightarrow x^2 - y^2 = \frac{1}{2} \quad \text{... (ii)}$$

So, from the given options point $\left(\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}\right)$ does not pass through the hyperbola (ii). Hence, option (d) is correct.

Auxiliary Circle

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola, then equation of the auxiliary circle is $x^2 + y^2 = a^2$

Example 8. *The line $x \cos\alpha + y \sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $a^2\cos^2\alpha - b^2\sin^2\alpha$ is equal to*

(a) p (b) p^2 (c) $-p^2$ (d) $2p^2$

Sol. (b) The given line is $x\cos\alpha + y\sin\alpha = p$

or $y = -x\cot\alpha + p\operatorname{cosec}\alpha$

On comparing with $y = mx + c$, we get

$$m = -\cot\alpha,\ x = p\operatorname{cosec}\alpha$$

Since, the given line touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ then } c^2 = a^2m^2 - b^2$$

$\therefore$ $p^2\operatorname{cosec}^2\alpha = a^2\cot^2\alpha - b^2$

$\Rightarrow$ $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$

Tangent to a Hyperbola

A line which intersect the hyperbola at only one point is called the tangent to a hyperbola.

Equation of Tangent in Different Forms

Point Form

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Slope Form

The equations of tangents of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $y = mx \pm \sqrt{a^2m^2 - b^2}$.

Parametric Form

The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Note

- The tangents at the points $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ intersect at the point

$$R\left\{\frac{a\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}\right\}.$$

- To a hyperbola, two tangents can be drawn from a point outside the hyperbola.

Example 9. *If the line $y = mx + \sqrt{a^2m^2 - b^2}$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$, then θ is equal to*

(a) $\sin^{-1}\left(\frac{b}{m}\right)$ (b) $\sin^{-1}\left(\frac{b}{a}\right)$

(c) $\sin^{-1}\left(\frac{b}{ma}\right)$ (d) None of these

Sol. (c) Since, the point $(a\sec\theta, b\tan\theta)$ lies on the line

$$y = mx + \sqrt{a^2m^2 - b^2}$$

$\therefore$ $b\tan\theta = am\sec\theta + \sqrt{a^2m^2 - b^2}$

$\Rightarrow$ $(b\tan\theta - am\sec\theta)^2 = a^2m^2 - b^2$

$\Rightarrow$ $b^2\tan^2\theta + a^2m^2\sec^2\theta - 2abm\tan\theta\sec\theta = a^2m^2 - b^2$

$\Rightarrow$ $a^2m^2\tan^2\theta - 2abm\tan\theta\sec\theta + b^2\sec^2\theta = 0$

$\Rightarrow$ $a^2m^2\sin^2\theta - 2abm\sin\theta + b^2 = 0$ $[\because \cos\theta \neq 0]$

$\therefore$ $\sin\theta = \frac{2abm \pm \sqrt{4a^2b^2m^2 - 4a^2b^2m^2}}{2a^2m^2} = \left(\frac{b}{am}\right)$

$\therefore$ $\theta = \sin^{-1}\left(\frac{b}{am}\right)$

Example 10. *If the eccentricity of the standard hyperbola passing through the point (4, 6) is 2, then the equation of the tangent to the hyperbola at (4, 6) is* **(JEE Main 2019)**

(a) $3x - 2y = 0$ (b) $x - 2y + 8 = 0$

(c) $2x - y - 2 = 0$ (d) $2x - 3y + 10 = 0$

Sol. (c) Let the equation of standard hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Now, eccentricity of hyperbola is

$$\sqrt{1 + \frac{b^2}{a^2}} = 2 \quad \text{(given)}$$

$\Rightarrow$ $a^2 + b^2 = 4a^2$

$\Rightarrow$ $b^2 = 3a^2$ ...(ii)

Since, hyperbola (i) passes through the point (4, 6)

$\therefore$ $\frac{16}{a^2} - \frac{36}{b^2} = 1$ ...(iii)

On solving Eqs. (ii) and (iii), we get

$$a^2 = 4 \text{ and } b^2 = 12 \quad \text{...(iv)}$$

Now, equation of tangent to hyperbola (i) at point (4, 6), is

$$\frac{4x}{a^2} - \frac{6y}{b^2} = 1$$

$\Rightarrow$ $\frac{4x}{4} - \frac{6y}{12} = 1$ [from Eq. (iv)]

$\Rightarrow$ $x - \frac{y}{2} = 1$

$\Rightarrow$ $2x - y - 2 = 0$

Example 11. *The equation of a tangent to the hyperbola $4x^2 - 5y^2 = 20$ parallel to the line $x - y = 2$ is* **(JEE Main 2019)**

(a) $x - y - 3 = 0$ (b) $x - y + 9 = 0$

(c) $x - y + 1 = 0$ (d) $x - y + 7 = 0$

Sol. (c) Given equation of hyperbola is $4x^2 - 5y^2 = 20$

which can be rewritten as $\frac{x^2}{5} - \frac{y^2}{4} = 1$

The line $x - y = 2$ has slope, $m = 1$

$\therefore$ Slope of tangent parallel to this line = 1

We know equation of tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

having slope m is given by $y = mx \pm \sqrt{a^2m^2 - b^2}$

Here, $a^2 = 5, b^2 = 4$ and $m = 1$

$\therefore$ Required equation of tangent is

$\Rightarrow \quad y = x \pm \sqrt{5-4} \Rightarrow y = x \pm 1 \Rightarrow x - y \pm 1 = 0$

Normal to a Hyperbola

A line which is perpendicular to the tangent of the hyperbola is called the normal to a hyperbola.

Equation of Normal in Different Forms

Point Form

The equation of normal at the point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

Slope Form

The equation of the normals of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2m^2}}$ at the points

$$\left(\pm \frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp \frac{b^2m}{\sqrt{a^2 - b^2m^2}}\right)$$

Note The line $y = mx + c$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c^2 = \frac{m^2(a^2+b^2)^2}{a^2 - b^2m^2}$

Parametric Form

The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$ is $ax\cos\theta + by\cot\theta = a^2 + b^2$

Note Four normals can be drawn to the hyperbola from any point outside to the hyperbola.

Example 12. *If the line $lx + my - n = 0$ will be a normal to the hyperbola, then $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{k}$, where k is equal to*

(a) n (b) n^2
(c) n^3 (d) None of these

Sol. (b) The equation of any normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$ax\cos\phi + by\cot\phi = a^2 + b^2$$

$$\Rightarrow \quad ax\cos\phi + by\cot\phi - (a^2+b^2) = 0 \qquad ...(i)$$

The straight line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then Eq. (i) and $lx + my - n = 0$ represent the same line.

$$\therefore \quad \frac{a\cos\phi}{l} = \frac{b\cot\phi}{m} = \frac{(a^2+b^2)}{n}$$

$$\Rightarrow \quad \sec\phi = \frac{na}{l(a^2+b^2)}$$

and $\quad \tan\phi = \frac{nb}{m(a^2+b^2)}$

$$\therefore \quad \sec^2\phi - \tan^2\phi = 1$$

$$\Rightarrow \frac{n^2a^2}{l^2(a^2+b^2)^2} - \frac{n^2b^2}{m^2(a^2+b^2)^2} = 1 \Rightarrow \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$$

Thus, $\quad k = n^2$

Example 13. *If a hyperbola passes through the point $P(10, 16)$ and it has vertices at $(\pm 6, 0)$, then the equation of the normal to it at P is* **(JEE Main 2020)**

(a) $3x + 4y = 94$ (b) $x + 2y = 42$
(c) $2x + 5y = 100$ (d) $x + 3y = 58$

Sol. (c) Since, the vertices of hyperbola on X-axis at $(\pm 6, 0)$, so equation of hyperbola we can assume as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } |a| = 6$$

and hyperbola passes through point $P(10, 16)$ so

$$\frac{(10)^2}{6^2} - \frac{(16)^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{(16)^2}{b^2} = \frac{100-36}{36} = \frac{64}{36}$$

$$\Rightarrow \quad b^2 = \frac{36 \times 256}{64} = 144$$

So, equation of hyperbola is $\frac{x^2}{36} - \frac{y^2}{144} = 0$, and the equation of normal to hyperbola at point P is

$$\frac{36x}{10} + \frac{144y}{16} = 36 + 144$$

$$\Rightarrow \quad \frac{18}{5}x + 9y = 180$$

$$\Rightarrow \quad 2x + 5y = 100$$

Hence, option (c) is correct.

Example 14. *If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$, then a value of m is* **(JEE Main 2019)**

(a) $\frac{3}{\sqrt{5}}$ (b) $\frac{\sqrt{15}}{2}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{2}$

Sol. (c) Given equation of hyperbola, is

$$\frac{x^2}{24} - \frac{y^2}{18} = 1 \qquad ...(i)$$

Since, the equation of the normals of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, are given by

$$y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

$\therefore$ Equation of normals of slope m, to the hyperbola (i), are

$$y = mx \pm \frac{m(24+18)}{\sqrt{24-m^2(18)}} \quad \text{...(ii)}$$

$\because$ Line $y = mx + 7\sqrt{3}$ is normal to hyperbola (i).

$\therefore$ On comparing with Eq. (ii), we get

$$\pm\frac{m(42)}{\sqrt{24-18m^2}} = 7\sqrt{3} \Rightarrow \pm\frac{6m}{\sqrt{24-18m^2}} = \sqrt{3}$$

$$\Rightarrow \frac{36m^2}{24-18m^2} = 3 \quad \text{[squaring both sides]}$$

$$\Rightarrow 12m^2 = 24-18m^2 \Rightarrow 30m^2 = 24$$

$$\Rightarrow 5m^2 = 4 \Rightarrow m = \pm\frac{2}{\sqrt{5}}$$

Number of Normals and Conormal Points

There are exactly four lines passing through a given point such that they are normals to the hyperbola at the points where they intersect the hyperbola. Such points on the hyperbola are known as the **conormal points**.

Properties of Eccentric Angles of Conormal Points

(i) The sum of the eccentric angles of conormal points is an odd multiple of π.

(ii) If $\theta_1, \theta_2, \theta_3$ and θ_4 are eccentric angles of four points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the normals at which are concurrent, then

(a) $\Sigma \cos(\theta_1+\theta_2) = 0$

(b) $\Sigma \sin(\theta_1+\theta_2) = 0$

(iii) If θ_1, θ_2 and θ_3 are the eccentric angle of three points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that

$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$,

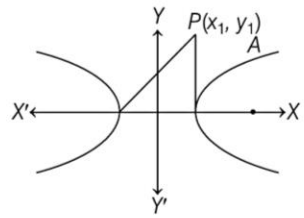
then the normals at these points are concurrent.

(iv) If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are concurrent, then

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right) = 4$$

Equation of Pair of Tangents

Let $P(x_1, y_1)$ be a point outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.



Then, the equation of pair of tangents drawn from external point P to the hyperbola is given by

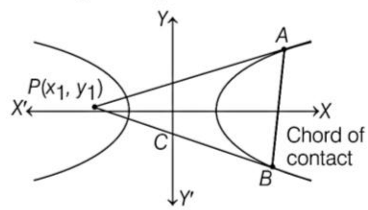
$$SS_1 = T^2$$

where, $S = \frac{x^2}{a^2}-\frac{y^2}{b^2}-1$, $S_1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

and $T = \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

Chord of Contact

Let PA and PB be any two tangents to the hyperbola from a point $P(x_1, y_1)$, then AB is known as chord of contact and its equation is given by



$$T = 0 \Rightarrow \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1 = 0$$

Equation of the Chord Bisected at a Given Point

The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the point (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1 \quad \text{or} \quad T = S_1$$

where T and S_1 have their usual meanings.

Example 15. *From the points on the circle $x^2+y^2=a^2$ tangents are drawn to the hyperbola $x^2-y^2=a^2$, then the locus of the middle points of the chords of contact is the curve $(x^2-y^2)^2 = k(x^2+y^2)$, where k is equal to*

(a) a
(b) a^3
(c) a^2
(d) None of these

Sol. (c) Since, any point of the circle $x^2+y^2=a^2$ is $(a\cos\theta, a\sin\theta)$, chord of contact of this point w.r.t. hyperbola $x^2-y^2=a^2$ is

$$x(a\cos\theta) - y(a\sin\theta) = a^2$$

or $$x\cos\theta - y\sin\theta = a \quad \text{...(i)}$$

If its mid-point be (h, k), then it is same as

$$T = S_1$$

$$\Rightarrow \quad hx - ky - a^2 = h^2 - k^2 - a^2$$

$$\Rightarrow \quad hx - ky = h^2 - k^2 \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{(h^2 - k^2)}$$

$$\therefore \quad (h^2 - k^2)\cos\theta = ah \quad \text{...(iii)}$$

and $$(h^2 - k^2)\sin\theta = ak \quad \text{...(iv)}$$

On squaring and adding Eqs. (iii) and (iv), we get

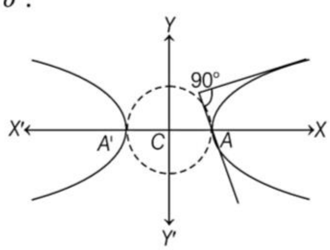
$$(h^2 - k^2)^2 = a^2h^2 + a^2k^2 \Rightarrow (h^2 - k^2)^2 = a^2(h^2 + k^2)$$

Hence, the required locus is

$$(x^2 - y^2)^2 = a^2(x^2 + y^2)$$

Director Circle

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which are perpendicular to each other is called director circle. The equation of the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$.



Note For director circle of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, a must be greater than b. If $a < b$, then director circle $x^2 + y^2 = a^2 - b^2$ does not exist.

Example 16. *The equation of the director circle of the hyperbola* $9x^2 - 16y^2 = 144$ *is*

(a) $x^2 + y^2 = 7$ (b) $x^2 + y^2 = 9$
(c) $x^2 + y^2 = 16$ (d) $x^2 + y^2 = 25$

Sol. (b) Given equation can be rewritten as $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$\therefore$ Required equation of director circle is

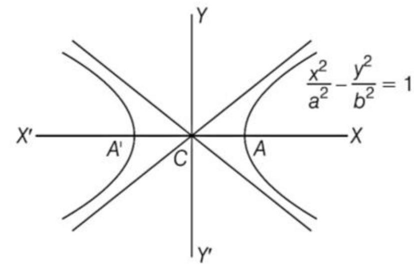
$$x^2 + y^2 = 16 - 9$$

$$\therefore \quad x^2 + y^2 = 9$$

Asymptotes

An asymptote of any hyperbola is a straight line which touches in it two points at infinity. In other words asymptotes are the lines which are tangents to the curve at infinity.

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\frac{b}{a}x$.



Note

- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.
- The asymptotes pass through the centre of the hyperbola.
- The bisectors of the angles between the asymptotes are the coordinate axes.
- The product of the perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is equal to $\frac{a^2b^2}{a^2 + b^2}$.

Example 17. *The asymptotes of a hyperbola having centre at the point* (1, 2) *are parallel to the lines* $2x + 3y = 0$ *and* $3x + 2y = 0$. *If the hyperbola passes through the point* (5, 3), *its equation is* $(2x + 3y - 8)(3x + 2y - 7) = k$, *where k is equal to*

(a) 15 (b) 154
(c) 0 (d) None of these

Sol. (b) Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$.

Since, asymptotes passes through (1, 2), then

$$\lambda = -8 \quad \text{and} \quad \mu = -7$$

Thus, the equations of asymptotes are

$$2x + 3y - 8 = 0 \quad \text{and} \quad 3x + 2y - 7 = 0$$

Let the equation of hyperbola be

$$(2x + 3y - 8)(3x + 2y - 7) + v = 0 \quad \text{...(i)}$$

It passes through (5, 3), then

$$(10 + 9 - 8)(15 + 6 - 7) + v = 0 \Rightarrow 11 \times 14 + v = 0$$

$$\therefore \quad v = -154$$

On putting the value of v in Eq. (i), we get

$$(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$$

This is the equation of required hyperbola.

Rectangular Hyperbola

A hyperbola whose asymptotes are at right angles to each other, is said to be a rectangular hyperbola.

Or

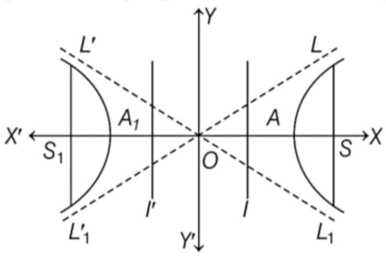
If the length of transverse and conjugate axes of any hyperbola are equal, then hyperbola is known as rectangular hyperbola.

The equation of rectangular hyperbola to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 - y^2 = a^2$ and equation of the asymptotes are $y = \pm x$ i.e. $y = x$ and $y = -x$. Clearly, each of these two asymptotes is inclined at 45° to the transverse axis.

When the centre of any rectangular hyperbola is at the origin and its asymptotes coincide with the coordinate axes, then the equation of rectangular hyperbola is $xy = c^2$. Its eccentricity is $\sqrt{2}$.

Rectangular Hyperbola of the Form $x^2 - y^2 = a^2$

(a) Asymptotes are perpendicular lines *i.e.*, $x \pm y = 0$



(b) Eccentricity, $e = \sqrt{2}$. (c) Centre, (0, 0)

(d) Foci $(\pm \sqrt{2}\, a, 0)$ (e) Directrices, $x = \pm \frac{a}{\sqrt{2}}$

(f) Latusrectum = $2a$

(g) Point form, $x = x_1, y = y_1$

Equation of tangent, $xx_1 - yy_1 = a^2$

Equation of normal, $\frac{x_1}{x} + \frac{y_1}{y} = 2$.

(h) Parametric form, $x = a\sec\theta, y = a\tan\theta$

Equation of tangent, $x\sec\theta - y\tan\theta = a$

Equation of normal, $\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$

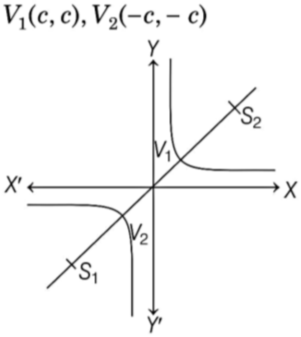
Rectangular Hyperbola of the Form $xy = c^2$

(a) Asymptotes are perpendicular lines *i.e.*, $x = 0$ and $y = 0$

(b) Eccentricity, $e = \sqrt{2}$ (c) Centre,(0, 0)

(d) Foci are $S(\sqrt{2}c, \sqrt{2}c), S_1(-\sqrt{2}c, -\sqrt{2}c)$

(e) Vertices, $V_1(c, c), V_2(-c, -c)$



(f) Directrices, $x + y = \pm\sqrt{2}c$

(g) Latusrectum = $2\sqrt{2}c$

(h) Point form, $x = x_1, y = y_1$

Equation of tangent, $xy_1 + yx_1 = 2c^2 \Rightarrow \frac{x}{x_1} + \frac{y}{y_1} = 2$

Equation of normal, $xx_1 - yy_1 = x_1^2 - y_1^2$

(i) Parametric form : $x = ct, y = \frac{c}{t}$ where $t \in R - \{0\}$

Equation of tangent, $x + yt^2 = 2ct$

Equation of normal, $t^2x - y = c\left(t^3 - \frac{1}{t}\right)$

Properties of Rectangular Hyperbola $xy = c^2$

(a) Equation of the chord joining t_l and t_2 is $x + yt_1t_2 - c(t_1 + t_2) = 0$.

(b) Equation of tangent at (x_1, y_1) is $xy_1 + x_1y = 2c^2$.

(c) Equation of tangent at $\left(ct, \frac{c}{t}\right)$ is $\frac{x}{t} + yt = 2c$.

(d) Point of intersection of tangents at t_1 and t_2 is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

(e) Equation of normal at $\left(ct, \frac{c}{t}\right)$ is $xt^3 - yt - ct^4 + c = 0$.

(f) Equation of normal to rectangular hyperbola at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

Example 18. *Number of common tangent with finite slope to the curve $xy = c^2$ and $y^2 = 4ax$ is*

(a) 0 (b) 1 (c) 2 (d) 4

Sol. *(b)* Let the point on parabola $P(at_1^2, 2at_1)$ and on rectangular hyperbola $\theta\left(ct_2, \frac{c}{t_2}\right)$

Equation of tangent at $P(at_1^2, 2at_1)$ is

$$Yt_1 - x = at_1^2 \quad ...(i)$$

and equation of tangent at $Q\left(ct_2, \frac{c}{t_2}\right)$ is

$$\frac{x}{t_2} + yt_2 = 2c \quad ...(ii)$$

Now, Eqs. (i) and (ii) are identical.

$\therefore \quad -t_2 = \frac{t_1}{t_2} = \frac{at_1^2}{2c}$

$\Rightarrow \quad \frac{a(t_2)^4}{2c} = -t_2$

$\Rightarrow \quad (t_2)^3 = \frac{-2c}{a}$

$\Rightarrow$ only one real value of t_2 exists.

Hence, only one common tangent of the curve

$$xy = c^2 \text{ and } y^2 = 4ax$$

Practice Exercise

ROUND I Topically Divided Problems

Basic Terms of Hyperbola, Equation of Hyperbola and Intersection of Two Curves

1. If equation $(10x-5)^2+(10y-4)^2=\lambda^2(3x+4y-1)^2$ represents a hyperbola, then

(a) $-2<\lambda<2$ (b) $\lambda>2$
(c) $\lambda<-2$ or $\lambda>2$ (d) $0<\lambda<2$

2. If a directrix of a hyperbola centred at the origin and passing through the point $(4,-2\sqrt{3})$ is $5x=4\sqrt{5}$ and its eccentricity is e, then *(JEE Main 2019)*

(a) $4e^4-12e^2-27=0$ (b) $4e^4-24e^2+27=0$
(c) $4e^4+8e^2-35=0$ (d) $4e^4-24e^2+35=0$

3. If a hyperbola has length of its conjugate axis equal to 5 and the distance between its foci is 13, then the eccentricity of the hyperbola is *(JEE Main 2019)*

(a) $\frac{13}{12}$ (b) 2
(c) $\frac{13}{8}$ (d) $\frac{13}{6}$

4. A general point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

(a) $(a\sin\theta, b\cos\theta)$ (where, θ is parameter)
(b) $(a\tan\theta, b\sec\theta)$ (where, θ is parameter)
(c) $\left(a\frac{e^t+e^{-t}}{2}, b\frac{e^t-e^{-t}}{2}\right)$ (where, t is parameter)
(d) None of the above

5. If a point $(x,y)=(\tan\theta+\sin\theta, \tan\theta-\sin\theta)$, then locus of (x,y) is

(a) $(x^2y)^{2/3}+(xy^2)^{2/3}=1$
(b) $x^2-y^2=4xy$
(c) $(x^2-y^2)^2=16xy$
(d) $x^2-y^2=6xy$

6. The equation $16x^2-3y^2-32x+12y-44=0$ represents a hyperbola

(a) the length of whose transverse axis is $4\sqrt{3}$
(b) the length of whose conjugate axis is 4
(c) whose centre is $(-1,2)$
(d) whose eccentricity is $\sqrt{\frac{19}{3}}$

7. The length of transverse axis of the hyperbola $3x^2-4y^2=32$ is

(a) $\frac{8\sqrt{2}}{\sqrt{3}}$ (b) $\frac{16\sqrt{2}}{\sqrt{3}}$
(c) $\frac{3}{32}$ (d) $\frac{64}{3}$

8. The difference between the length $2a$ of the transverse axis of a hyperbola of eccentricity e and the length of its latusrectum is

(a) $2a(3-e^2)$ (b) $2a|2-e^2|$
(c) $2a(e^2-1)$ (d) $a(2e^2-1)$

9. If eccentricity of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is e and e' is the eccentricity of its conjugate hyperbola, then

(a) $e=e'$ (b) $ee'=1$
(c) $\frac{1}{e^2}+\frac{1}{(e')^2}=1$ (d) None of these

10. The equation of the transverse axis of the hyperbola $(x-3)^2+(y+1)^2=(4x+3y)^2$ is

(a) $x+3y=0$ (b) $4x+3y=9$
(c) $3x-4y=13$ (d) $4x+3y=0$

11. The equation of the hyperbola whose eccentricity is $\sqrt{2}$ and the distance between the foci is 16, taking transverse and conjugate axes of the hyperbola as x and y axes respectively, is

(a) $x^2-y^2=0$ (b) $x^2-y^2=32$
(c) $x^2-y^2=2$ (d) None of these

12. The equation of the hyperbola whose foci are (6,4) and (−4,4) and eccentricity 2, is

(a) $\frac{4(x-1)^2}{25}+\frac{4(y-4)^2}{25}=1$ (b) $\frac{4(x+1)^2}{25}+\frac{4(y+4)^2}{75}=1$
(c) $\frac{4(x-1)^2}{75}-\frac{4(y-4)^2}{25}=1$ (d) $\frac{4(x-1)^2}{25}-\frac{4(y-4)^2}{75}=1$

13. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Its equation is

(a) $x^2-y^2=32$ (b) $\frac{x^2}{4}-\frac{y^2}{9}=1$
(c) $2x-3y^2=7$ (d) None of these

14. Equation of the hyperbola with eccentricity $\frac{3}{2}$ and foci at $(\pm 2, 0)$ is

(a) $\frac{x^2}{4}-\frac{y^2}{5}=\frac{4}{9}$ (b) $\frac{x^2}{9}-\frac{y^2}{9}=\frac{4}{9}$

(c) $\frac{x^2}{4}-\frac{y^2}{9}=1$ (d) None of these

15. The vertices of a hyperbola are at (0, 0) and (10, 0) and one of its foci is at (18, 0). The equation of hyperbola is

(a) $\frac{x^2}{25}-\frac{y^2}{144}=1$ (b) $\frac{(x-5)^2}{25}-\frac{y^2}{144}=1$

(c) $\frac{x^2}{5}-\frac{(y-5)^2}{144}=1$ (d) $\frac{(x-5)^2}{25}-\frac{(y-5)^2}{144}=1$

16. The points of intersection of curves whose parametric equations are $x=t^2+1$, $y=2t$ and $x=2S$, $y=\frac{2}{S}$ is

(a) (1, –3) (b) (2, 2) (c) (–2,4) (d) (1, 2)

17. The equation $9x^2-16y^2-18x+32y-151=0$ represents a hyperbola

(a) the length of the transverse axes is 4

(b) the length of latusrectum is 9

(c) the equation of directrix is $x=\frac{21}{5}$ and $x=-\frac{11}{5}$

(d) None of the above

18. The equations of transverse and conjugate axes of a hyperbola are respectively $x+2y-3=0$, $2x-y+4=0$ and their respectively lengths are $\sqrt{2}$ and $2/\sqrt{3}$. The equation of the hyperbola is

(a) $\frac{2}{5}(x+2y-3)^2-\frac{3}{5}(2x-y+4)^2=1$

(b) $\frac{2}{5}(2x-y+4)^2-\frac{3}{5}(x+2y-3)^2=1$

(c) $2(2x-y+4)^2-3(x+2y-3)^2=1$

(d) $2(2x-y+4)^2-(x+2y-3)^2=1$

Tangent and Normal to the Hyperbola

19. The straight line $x+y=\sqrt{2}p$ will touch the hyperbola $4x^2-9y^2=36$, if

(a) $p^2=2$ (b) $p^2=5$ (c) $5p^2=2$ (d) $2p^2=5$

20. The product of the perpendicular from two foci on any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is

(a) a^2 (b) b^2 (c) $-a^2$ (d) $-b^2$

21. Let $P(3,3)$ be a point on the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal to it at P intersects the X-axis at (9, 0) and e is its eccentricity, then the ordered pair (a^2, e^2) is equal to *(JEE Main 2020)*

(a) $\left(\frac{9}{2},3\right)$ (b) $\left(\frac{3}{2},2\right)$ (c) $\left(\frac{9}{2},2\right)$ (d) $(9,3)$

22. If a hyperbola passes through the point $P(\sqrt{2},\sqrt{3})$ and has foci at $(\pm 2, 0)$, then the tangent to this hyperbola at P also passes through the point *(JEE Main 2017)*

(a) $(3\sqrt{2},2\sqrt{3})$ (b) $(2\sqrt{2},3\sqrt{3})$

(c) $(\sqrt{3},\sqrt{2})$ (d) $(-\sqrt{2},-\sqrt{3})$

23. The point of intersection of two tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the product of whose slope is c^2, lies on the curve

(a) $y^2-b^2=c^2(x^2+a^2)$ (b) $y^2+a^2=c^2(x^2-b^2)$

(c) $y^2+b^2=c^2(x^2-a^2)$ (d) $y^2-a^2=c^2(x^2+b^2)$

24. Equation of the normal to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ perpendicular to the line $2x+y=1$ is

(a) $\sqrt{21}(x-2y)=41$ (b) $x-2y=1$

(c) $\sqrt{41}(x-2y)=41$ (d) $\sqrt{21}(x-2y)=21$

25. The locus of point of intersection of two lengths of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if the product of their slopes is c^2, will be

(a) $y^2-b^2=c^2(x^2+a^2)$ (b) $y^2+b^2=c^2(x^2-a^2)$

(c) $y^2+a^2=c^2(x^2-b^2)$ (d) $y^2-a^2=c^2(x^2+b^2)$

26. The equation of common tangents to the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ are

(a) $y=\pm x\pm\sqrt{b^2-a^2}$ (b) $y=\pm x\pm\sqrt{a^2-b^2}$

(c) $y=\pm x\pm(a^2-b^2)$ (d) $y=\pm x\pm\sqrt{a^2+b^2}$

27. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$ be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h,k) is the point of intersection of normals at P and Q, then k is equal to

(a) $\frac{a^2+b^2}{a}$ (b) $-\left[\frac{a^2+b^2}{a}\right]$

(c) $\frac{a^2+b^2}{b}$ (d) $-\left[\frac{a^2+b^2}{b}\right]$

28. The value of m, for which the line $y=mx+\frac{25\sqrt{3}}{3}$ is a normal to the conic $\frac{x^2}{16}-\frac{y^2}{9}=1$, is

(a) $\pm\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$

(c) $-\frac{\sqrt{3}}{2}$ (d) None of these

Chord of Contact of a Hyperbola

29. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $4x^2 - 9y^2 - 36 = 0$ are at right angles, then $\frac{x_1 x_2}{y_1 y_2}$ is equal to

(a) $\frac{9}{4}$ (b) $-\frac{9}{4}$ (c) $\frac{81}{16}$ (d) $-\frac{81}{16}$

30. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$
(b) $9x^2 - 8y^2 - 18x + 9 = 0$
(c) $9x^2 - 8y^2 - 18x - 9 = 0$
(d) $9x^2 - 8y^2 + 18x + 9 = 0$

31. If chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Then, the locus of the middle points of these chords is

(a) $y^2 = (x - a)x^3$ (b) $y^2(x - a) = x^3$
(c) $x^2(x - a) = x^3$ (d) None of these

32. The locus of middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is

(a) $3x - 4y = 4$ (b) $3y - 4x + 4 = 0$
(c) $4x - 3y = 3$ (d) $3x - 4y = 2$

Rectangular Hyperbola

33. The equation of a line passing through the centre of a rectangular hyperbola is $x - y - 1 = 0$. If one of the asymptotes is $3x - 4y - 6 = 0$, the equation of other asymptote is

(a) $4x - 3y + 17 = 0$ (b) $-4x - 3y + 17 = 0$
(c) $-4x + 3y + 1 = 0$ (d) $4x + 3y + 17 = 0$

34. The asymptotes of the hyperbola $xy = hx + ky$ are

(a) $x = k, y = h$ (b) $x = h, y = k$
(c) $x = h, y = h$ (d) $x = k, y = k$

35. The product of the lengths of perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes, is

(a) 1/2 (b) 2/3 (c) 3/2 (d) 2

36. A hyperbola has the asymptotes $x + 2y = 3$ and $x - y = 0$ and passes through (2, 1). Its centre is

(a) (1, 2) (b) (2, 2)
(c) (1, 1) (d) (2, 1)

37. The angle between the asymptotes of the hyperbola $x^2 + 2xy - 3y^2 + x + 7y + 9 = 0$ is

(a) $\tan^{-1}(\pm 2)$ (b) $\tan^{-1}(\pm\sqrt{3})$
(c) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

38. The angle between the two asymptotes of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is

(a) $\pi - 2\tan^{-1}\left(\frac{3}{4}\right)$ (b) $\pi - 2\tan^{-1}\left(\frac{3}{2}\right)$
(c) $2\tan^{-1}\left(\frac{3}{4}\right)$ (d) $\pi - 2\tan^{-1}\left(\frac{4}{3}\right)$

39. If tangent and normal to a rectangular hyperbola $xy = c^2$ cut off intercepts a_1 and a_2 on one axis and b_1, b_2 on the other, then

(a) $a_1 = b_1$ (b) $a_2 = b_2$
(c) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ (d) $a_1 a_2 + b_1 b_2 = 0$

40. The length of the transverse axis of the rectangular hyperbola $xy = 18$ is

(a) 6 (b) 12
(c) 18 (d) 9

41. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$ (b) $y = 2x + 1$
(c) $2y = y + 8$ (d) $y = x + 2$

ROUND II Mixed Bag

Only One Correct Option

1. If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola. Then, the eccentricity e of the hyperbola satisfies

(a) $1 < e < \frac{2}{\sqrt{3}}$ (b) $e = \frac{2}{\sqrt{3}}$
(c) $e = \frac{\sqrt{3}}{2}$ (d) $e > \frac{2}{\sqrt{3}}$

2. The normal at P to a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M respectively. If locus of the mid-point of LM is hyperbola, then eccentricity of the hyperbola is

(a) $\left(\frac{e+1}{e-1}\right)$
(b) $\frac{e}{\sqrt{(e^2 - 1)}}$
(c) e
(d) None of the above

3. The eccentricity of the hyperbola whose length of the latusrectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is *(JEE Main 2016)*

(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{3}$

4. With one focus of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ as the centre, a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is

(a) less than 2 (b) 2
(c) 1/3 (d) None of these

5. Let $0<\theta<\frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2\theta}-\frac{y^2}{\sin^2\theta}=1$ is greater than 2, then the length of its latus rectum lies in the interval *(JEE Main 2019)*

(a) $(1,\frac{3}{2}]$ (b) $(3,\infty)$
(c) $(\frac{3}{2},2]$ (d) $(2, 3]$

6. The tangents to the hyperbola $x^2-y^2=3$ are parallel to the straight line $2x+y+8=0$, at which of the following points?

(a) (2, 1) (b) (2, –1)
(c) (–2, –1) (d) (–2, 1)

7. P is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT\times ON$ is equal to

(a) e^2 (b) a^2
(c) b^2 (d) $\frac{b^2}{a^2}$

8. Consider a hyperbola $H: x^2-2y^2=4$. Let the tangent at a point $P(4,\sqrt{6})$ meet the X-axis at Q and latus rectum at $R(x_1,y_1)$, $x_1>0$. If F is a focus of H which is nearer to the point P, then the area of ΔQFR is equal to *(JEE Main 2021)*

(a) $4\sqrt{6}$ (b) $\sqrt{6}-1$ (c) $\frac{7}{\sqrt{6}}-2$ (d) $4\sqrt{6}-1$

9. Number of points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=3$ from which mutually perpendicular tangents can be drawn to the circle $x^2+y^2=a^2$, is/are

(a) 0 (b) 2 (c) 3 (d) 4

10. If the tangent and normal to $x^2-y^2=4$ at a point cut off intercepts a_1, a_2 on the X-axis respectively and b_1, b_2 on Y-axis respectively, then the value of $a_1a_2+b_1b_2$ is

(a) 1 (b) –1 (c) 0 (d) 4

11. The locus of the mid-points of the chord of the circle, $x^2+y^2=25$ which is tangent to the hyperbola, $\frac{x^2}{9}-\frac{y^2}{16}=1$ is *(JEE Main 2021)*

(a) $(x^2+y^2)^2-16x^2+9y^2=0$
(b) $(x^2+y^2)^2-9x^2+144y^2=0$
(c) $(x^2+y^2)^2-9x^2-16y^2=0$
(d) $(x^2+y^2)^2-9x^2+16y^2=0$

12. Let P be the point of intersection of the common tangents to the parabola $y^2=12x$ and the hyperbola $8x^2-y^2=8$. If S and S' denotes the foci of the hyperbola where S lies on the positive X-axis then P divides SS' in a ratio *(JEE Main 2019)*

(a) 13 : 11 (b) 14 : 13 (c) 5 : 4 (d) 2 : 1

13. A straight line touches the rectangular hyperbola $9x^2-9y^2=8$ and the parabola $y^2=32x$. The equation of the line is

(a) $9x+3y-12=0$ (b) $9x-3y+11=0$
(c) $9x+3y+8=0$ (d) $9x-3y-7=0$

14. If the normal at θ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the transverse axis at G, then $AG\cdot A'G$ is equal to (where, A and A' are the vertices of hyperbola)

(a) $a^2(e^4\sec^2\theta-1)$ (b) $a^2(e^4\tan^2\theta-1)$
(c) $b^2(e^4\sec^2\theta-1)$ (d) $b^2(e^4\sec^2\theta+1)$

15. Equation of a common tangent to the parabola $y^2=4x$ and the hyperbola $xy=2$ is *(JEE Main 2019)*

(a) $x+2y+4=0$ (b) $x-2y+4=0$
(c) $4x+2y+1=0$ (d) $x+y+1=0$

16. If normal to the rectangular hyperbola $xy=c^2$ at the point t on it intersects the hyperbola at t_1, then t^3t_1 is equal to

(a) 1 (b) 2 (c) –1 (d) –2

17. The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by

(a) $-3x^2+y^2=3$ (b) $x^2-3y^2=3$
(c) $3x^2-y^2=3$ (d) $-x^2+3y^2=3$

18. For some $\theta\in\left(0,\frac{\pi}{2}\right)$, if the eccentric of the hyperbola, $x^2-y^2\sec^2\theta=10$ is $\sqrt{5}$ times the eccentricity of the ellipse, $x^2 \sec^2\theta + y^2 = 5$, then the length of the latus rectum of the ellipse, is *(JEE Main 2020)*

(a) $2\sqrt{6}$ (b) $\sqrt{30}$ (c) $\frac{2\sqrt{5}}{3}$ (d) $\frac{4\sqrt{5}}{3}$

19. If e_1 and e_2 are the eccentricities of the ellipse, $\frac{x^2}{18}+\frac{y^2}{4}=1$ and the hyperbola, $\frac{x^2}{9}-\frac{y^2}{4}=1$ respectively and (e_1, e_2) is a point on the ellipse, $15x^2+3y^2=k$, then k is equal to *(JEE Main 2020)*

(a) 14 (b) 15 (c) 17 (d) 16

20. Tangents at any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cut the axes at A and B respectively. If the rectangle $OAPB$, where O is the origin is completed, then locus of point P is given by

(a) $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$ (b) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$

(c) $\frac{a^2}{y^2}-\frac{b^2}{x^2}=1$ (d) None of these

Numerical Value Type Questions

21. The equation of one of the latusrectum of the hyperbola $(10x-5)^2+(10y-2)^2=9(3x+4y-7)^2$ is $ax+by+c=0$, then the value of $|b+2c|$ is ....

22. If (5, 12) and (24, 7) are foci of a hyperbola passing through the origin, then the value of $\frac{12e}{\sqrt{386}}$ is .... (where, e is the eccentricity of hyperbola)

23. PQ is double ordinate of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that OPQ is an equilateral triangle, O being the centre of hyperbola, where the eccentricity of hyperbola e satisfy, $\sqrt{3}\,e>k$, then the value of k is ....

24. If the tangent at point $P(h, k)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the circle $x^2+y^2=a^2$ at the point $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, where $\frac{1}{y_1}+\frac{1}{y_2}=\frac{\lambda}{k}$, then the value of λ is ....

25. If the tangent drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at any point P meets the coordinate axes at the points A and B respectively. If the rectangle $OACB$ (O being the origin) is completed, where C lies on $\frac{a^2}{x^2}-\frac{b^2}{y^2}=\lambda$. Then, the value of λ is

Answers

Round I

1. (c)	2. (d)	3. (a)	4. (c)	5. (c)	6. (d)	7. (a)	8. (b)	9. (c)	10. (c)
11. (b)	12. (d)	13. (a)	14. (a)	15. (b)	16. (b)	17. (c)	18. (b)	19. (d)	20. (b)
21. (a)	22. (b)	23. (c)	24. (a)	25. (b)	26. (b)	27. (d)	28. (a)	29. (d)	30. (b)
31. (d)	32. (a)	33. (d)	34. (a)	35. (b)	36. (c)	37. (a)	38. (c)	39. (d)	40. (b)
41. (d)									

Round II

1. (d)	2. (b)	3. (c)	4. (b)	5. (b)	6. (b)	7. (b)	8. (c)	9. (a)	10. (c)
11. (d)	12. (c)	13. (c)	14. (a)	15. (a)	16. (c)	17. (c)	18. (d)	19. (d)	20. (a)
21. (6)	22. (1)	23. (2)	24. (2)	25. (1)					

Solutions

Round I

1. Given equation of hyperbola is

$$(10x-5)^2+(10y-4)^2=\lambda^2(3x+4y-1)^2$$

can be rewritten as

$$\frac{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{2}{5}\right)^2}}{\left|\frac{3x+4y-1}{5}\right|}=\left|\frac{\lambda}{2}\right|$$

This is of the form of $\frac{PS}{PM}=e$

where, P is any point on the hyperbola and S is a focus and M is the point of directrix.

Here, $\left|\frac{\lambda}{2}\right|>1 \Rightarrow |\lambda|>2$ $\quad (\because e>1)$

$\Rightarrow \quad \lambda<-2$ or $\lambda>2$

2. Let the equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

Since, equation of given directrix is $5x=4\sqrt{5}$

So, $5\left(\frac{a}{e}\right)=4\sqrt{5}$ $\quad [\because$ equation of directrix is $x=\frac{a}{e}]$

$$\Rightarrow \frac{a}{e}=\frac{4}{\sqrt{5}} \quad \text{...(ii)}$$

and hyperbola (i) passes through point $(4,-2\sqrt{3})$

so, $$\frac{16}{a^2}-\frac{12}{b^2}=1 \quad \text{...(iii)}$$

The eccentricity $e=\sqrt{1+\frac{b^2}{a^2}}$

$$\Rightarrow e^2=1+\frac{b^2}{a^2} \Rightarrow a^2e^2-a^2=b^2 \quad \text{...(iv)}$$

From Eqs. (ii) and (iv), we get

$$\frac{16}{5}e^4-\frac{16}{5}e^2=b^2 \quad \text{...(v)}$$

From Eqs. (ii) and (iii), we get

$$\frac{16}{\frac{16}{5}e^2}-\frac{12}{b^2}=1 \Rightarrow \frac{5}{e^2}-\frac{12}{b^2}=1$$

$$\Rightarrow \frac{12}{b^2}=\frac{5}{e^2}-1 \Rightarrow \frac{12}{b^2}=\frac{5-e^2}{e^2}$$

$$\Rightarrow b^2=\frac{12e^2}{5-e^2} \quad \text{...(vi)}$$

From Eqs. (v) and (vi), we get

$$16e^4-16e^2=5\left(\frac{12e^2}{5-e^2}\right)$$

$$\Rightarrow 16(e^2-1)(5-e^2)=60$$

$$\Rightarrow 4(5e^2-e^4-5+e^2)=15$$

$$\Rightarrow 4e^4-24e^2+35=0$$

3. We know that in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2=a^2(e^2-1)$, the length of conjugate axis is $2b$ and distance between the foci is $2ae$.

$\therefore$ According the problem, $2b=5$ and $2ae=13$

Now, $\quad b^2=a^2(e^2-1)$

$$\Rightarrow \left(\frac{5}{2}\right)^2=a^2e^2-a^2$$

$$\Rightarrow \frac{25}{4}=\frac{(2ae)^2}{4}-a^2$$

$$\frac{25}{4}=\frac{169}{4}-a^2 \quad [\because 2ae=13]$$

$$\Rightarrow a^2=\frac{169-25}{4}=\frac{144}{4}=36 \Rightarrow a=6$$

Now, $2ae=13 \Rightarrow 2\times6\times e=13 \Rightarrow e=\frac{13}{12}$

4. Now, taking option (c).

Let $\quad x=a\frac{e^t+e^{-t}}{2} \Rightarrow \frac{2x}{a}=e^t+e^{-t} \quad \text{...(i)}$

and $\quad \frac{2y}{a}=e^t-e^{-t} \quad \text{...(ii)}$

On squaring and subtracting Eq. (ii) from Eq. (i), we get

$$\frac{4x^2}{a^2}-\frac{4y^2}{b^2}=4 \Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

5. Put the value of $(x,y)\equiv(\tan\theta+\sin\theta,\tan\theta-\sin\theta)$ in the given option, we get the required result.

On putting the value of x and y in option (c), we get

$$[(\tan\theta+\sin\theta)^2-(\tan\theta-\sin\theta)^2]^2$$
$$=16(\tan\theta+\sin\theta)\times(\tan\theta-\sin\theta)$$

$$\Rightarrow [\tan^2\theta+\sin^2\theta-\tan^2\theta-\sin^2\theta+4\tan\theta\sin\theta]^2$$
$$=16(\tan^2\theta-\sin^2\theta)$$

$$\Rightarrow (4\tan\theta\cdot\sin\theta)^2=16(\tan^2\theta-\sin^2\theta)$$

$$\Rightarrow 16\tan^2\theta\sin^2\theta=16\tan^2\theta(1-\cos^2\theta)$$

$$\Rightarrow 16\tan^2\theta\sin^2\theta=16\tan^2\theta\sin^2\theta$$

Hence, the option (c) satisfies.

6. We have, $\quad 16(x^2-2x)-3(y^2-4y)=44$

$$\Rightarrow 16(x-1)^2-3(y-2)^2=48$$

$$\Rightarrow \frac{(x-1)^2}{3}-\frac{(y-2)^2}{16}=1$$

This equation represents a hyperbola with eccentricity

$$e=\sqrt{1+\frac{16}{3}}=\sqrt{\frac{19}{3}}$$

7. The given equation may be written as

$$\frac{x^2}{\frac{32}{3}} - \frac{y^2}{8} = 1 \Rightarrow \frac{x^2}{\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^2} - \frac{y^2}{(2\sqrt{2})^2} = 1$$

On comparing the given equation with the standard equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$$a^2 = \left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^2 \text{ and } b^2 = (2\sqrt{2})^2$$

∴ Length of transverse axis of a hyperbola

$$= 2a = 2 \times \frac{4\sqrt{2}}{\sqrt{3}} = \frac{8\sqrt{2}}{\sqrt{3}}$$

8. Let equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Length of transverse axis is $2a$ and

Length of latusrectum is $\frac{2b^2}{a}$.

Now, difference $E = \left|2a - \frac{2b^2}{a}\right| = \frac{2}{a}\left|2a^2 - a^2e^2\right|$

∴ Difference $= 2a\,|2 - e^2|$

9. Given equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and equation of conjugate hyperbola is $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$

Since, e and e' are the eccentricities of the respective hyperbola, then

$$e^2 = 1 + \frac{b^2}{a^2}, (e')^2 = 1 + \frac{a^2}{b^2}$$

$$\therefore \quad \frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

10. $(x - 3)^2 + (y + 1)^2 = (4x + 3y)^2$

$\Rightarrow (x - 3)^2 + (y + 1)^2 = 25\left(\frac{4x + 3y}{5}\right)^2$

$\Rightarrow \quad PS = 5PM$

⇒ Directrix is $4x + 3y = 0$ and focus $(3, -1)$.

So, equation of transverse axis is $y + 1 = \frac{3}{4}(x - 3)$

$\Rightarrow \quad 3x - 4y = 13$

11. Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The coordinates of the foci are $(ae, 0)$ and $(-ae, 0)$.

∴ $2ae = 16 \Rightarrow 2a\sqrt{2} = 16 \Rightarrow a = 4\sqrt{2}$

Also, $b^2 = a^2(e^2 - 1) = 32(2 - 1) = 32$

Thus, $a^2 = 32$ and $b^2 = 32$

Hence, the required equation is

$$\frac{x^2}{32} - \frac{y^2}{32} = 1 \Rightarrow x^2 - y^2 = 32$$

12. Given, S (6,4) and S' (–4,4) and eccentricity, $e = 2$

∴ $SS' = \sqrt{(6 + 4)^2 + (4 - 4)^2} = 10$

But $SS' = 2ae$

∴ $2a \times 2 = 10 \Rightarrow a = \frac{5}{2}$

and we know that,

$$b^2 = a^2(e^2 - 1) \Rightarrow b^2 = \frac{25}{4}(4 - 1) = \frac{75}{4}$$

Centre of hyperbola is $\left[\frac{6 + (-4)}{2}, \frac{4 + 4}{2}\right] = (1, 4)$

∴ Equation of hyperbola is $\frac{(x - 1)^2}{\frac{25}{4}} - \frac{(y - 4)^2}{\frac{75}{4}} = 1$

$$\Rightarrow \quad \frac{4(x - 1)^2}{25} - \frac{4(y - 4)^2}{75} = 1$$

13. Let equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(i)

According to the given condition,

$2ae = 16$ and $e = \sqrt{2}$

$\Rightarrow \quad 2a(\sqrt{2}) = 16 \Rightarrow a = \frac{8}{\sqrt{2}} \Rightarrow a^2 = 32$

∵ $b^2 = a^2(e^2 - 1) = 32(2 - 1) = 32$

On putting the values of a^2 and b^2 in Eq. (i), we get

$$\frac{x^2}{32} - \frac{y^2}{32} = 1$$

$\Rightarrow \quad x^2 - y^2 = 32$

14. Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Given, $e = \frac{3}{2}$ and foci $= (\pm ae, 0) = (\pm 2, 0)$

∴ $e = \frac{3}{2}$ and $ae = 2$

$\Rightarrow a \times \frac{3}{2} = 2 \Rightarrow a^2 = \frac{16}{9}$

∵ $b^2 = a^2(e^2 - 1)$

∴ $b^2 = \frac{16}{9}\left(\frac{9}{4} - 1\right) = \frac{16}{9} \times \frac{5}{4} = \frac{20}{9}$

On putting the values of a^2 and b^2 in Eq. (i), we get

$$\frac{x^2}{16/9} - \frac{y^2}{20/9} = 1 \Rightarrow \frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$$

15. The centre of hyperbola is the mid-point of vertices.

So, coordinates of centre are $\left(\frac{0 + 10}{2}, \frac{0 + 0}{2}\right)$, i.e. (5, 0).

One of the foci is (18, 0).

∴ Other foci of hyperbola is (–8, 0).

[∵ centre of hyperbola is also mid-point of foci of hyperbola]

Let $2a$ and $2b$ be the lengths of transverse and conjugate axes and e be the eccentricity.

Then, the equation of hyperbola is

$$\frac{(x-5)^2}{a^2}-\frac{y^2}{b^2}=1$$

Distance between two vertices $=2a$

i.e. $$2a=\sqrt{(10-0)^2+(0-0)^2}=10$$

$\Rightarrow$ $$a=5$$

Now, distance between two foci $=2ae$

i.e. $$2ae=\sqrt{(18+8)^2+(0-0)^2}=26$$

$\Rightarrow$ $$ae=\frac{26}{2}=13$$

In hyperbola, $(ae)^2=a^2+b^2$

$\Rightarrow$ $$b^2=(ae)^2-a^2$$

$\Rightarrow$ $$b^2=(13)^2-(5)^2=144$$

$\therefore$ Equation of hyperbola is

$$\frac{(x-5)^2}{25}-\frac{y^2}{144}=1$$

16. We have, $x=t^2+1$ and $y=2t$

On eliminating t, we get

$$y^2=4x-4$$

and $$x=2s \text{ and } y=\frac{2}{s}$$

On substituting this $y^2=4x-4$, we get

$$2s^3-s^2-1=0$$

$\Rightarrow$ $$(s-1)(2s^2+s+1)=0$$

$\Rightarrow$ $$s=1$$

Putting $s=1$ in $x=2s$, $y=\frac{2}{s}$, we get

$$x=2,\ y=2$$

So, point is (2, 2).

17. We have, $9x^2-16y^2-18x+32y-151=0$

$\Rightarrow$ $$9(x^2-2x+1)-16(y^2-2y+1)=151+9-16$$

$\Rightarrow$ $$9(x-1)^2-16(y-1)^2=144$$

$\Rightarrow$ $$\frac{(x-1)^2}{16}-\frac{(y-1)^2}{9}=1$$

(a) Length of transverse axes $=2a=2\times 4=8$

(b) Length of latusrectum $=\frac{2b^2}{a}=\frac{2\times 9}{4}=\frac{9}{2}$

(c) Equation of directrix is $x=\pm\frac{a}{e}+1$

$\Rightarrow$ $$x=\pm\frac{4}{5}\times 4+1 \quad \left[\because e=\sqrt{\frac{a^2+b^2}{a^2}}=\frac{5}{4}\right]$$

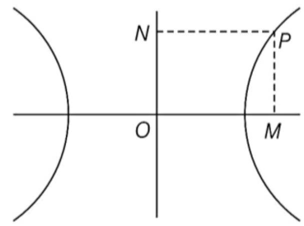
$\therefore$ $$x=\frac{21}{5}$$

and $$x=-\frac{11}{5}$$

18. We have, transverse axis $=x+2y-3=0$ and conjugate axis $=2x-y+4=0$ both are perpendicular,

and $$2a=\sqrt{2} \text{ and } 2b=\frac{2}{\sqrt{3}}$$

$\Rightarrow$ $$a=\frac{1}{\sqrt{2}} \text{ and } b=\frac{1}{\sqrt{3}}$$



We know that,

Equation of the hyperbola referred to two perpendicular lines,

i.e. $$\frac{PN^2}{a^2}-\frac{PM^2}{b^2}=1$$

$\Rightarrow$ $$\frac{\left(\frac{2x-y+4}{\sqrt{5}}\right)^2}{\frac{1}{2}}-\frac{\left(\frac{x+2y-3}{\sqrt{5}}\right)^2}{\frac{1}{3}}=1$$

$\therefore$ $$\frac{2}{5}(2x-y+4)^2-\frac{3}{5}(x+2y-3)^2=1$$

19. Given equation of hyperbola is $\frac{x^2}{9}-\frac{y^2}{4}=1$.

Here, $a^2=9$, $b^2=4$

and equation of line is $y=-x+\sqrt{2}\,p$ ...(i)

If the line $y=mx+c$ touches the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad \text{then } c^2=a^2m^2-b^2 \qquad \text{...(ii)}$$

From Eq. (i), we get

$$m=-1, c=\sqrt{2}p$$

On putting these values in Eq. (ii), we get

$$(\sqrt{2}p)^2=9(1)-4 \Rightarrow 2p^2=5$$

20. Let equation of tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y=mx+\sqrt{a^2m^2-b^2}$$

i.e. $$mx-y+\sqrt{a^2m^2-b^2}=0$$

$\therefore$ Required product

$$=\left|\frac{mae+\sqrt{a^2m^2-b^2}}{\sqrt{m^2+1}}\right|\left|\frac{-mae+\sqrt{a^2m^2-b^2}}{\sqrt{m^2+1}}\right|$$

$$=\left|\frac{a^2m^2-b^2-m^2a^2e^2}{m^2+1}\right|$$

$$=\left|\frac{m^2a^2(1-e^2)-b^2}{m^2+1}\right|=\left|\frac{-m^2b^2-b^2}{m^2+1}\right| \quad [\because b^2=a^2(e^2-1)]$$

$=b^2$

21. Equation of given hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

Since, $P(3,3)$, be a point on Eq. (i).

So, $$\frac{9}{a^2}-\frac{9}{b^2}=1 \quad \text{...(ii)}$$

Now, equation of normal at point P to the hyperbola is

$$\frac{a^2x}{3}+\frac{b^2y}{3}=a^2+b^2 \quad \text{...(iii)}$$

The normal Eq. (iii) intersect the X-axis at (9, 0), so

$$3a^2=a^2+b^2 \Rightarrow b^2=2a^2 \quad \text{...(iv)}$$

$\because$ The eccentricity (e) of hyperbola (i) is

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+2}=\sqrt{3} \Rightarrow e^2=3$$

From Eqs. (ii) and (iv), we get

$$\frac{9}{a^2}-\frac{9}{2a^2}=1 \Rightarrow 18-9=2a^2 \Rightarrow a^2=\frac{9}{2}$$

$\therefore$ The ordered pair $(a^2, e^2)=\left(\frac{9}{2},3\right)$.

22. Let the equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

$\therefore \quad ae=2 \Rightarrow a^2e^2=4$

$\Rightarrow a^2+b^2=4 \Rightarrow b^2=4-a^2$

$\therefore \quad \frac{x^2}{a^2}-\frac{y^2}{4-a^2}=1$

Since, $(\sqrt{2},\sqrt{3})$ lie on hyperbola.

$$\therefore \quad \frac{2}{a^2}-\frac{3}{4-a^2}=1$$

$\Rightarrow \quad 8-2a^2-3a^2=a^2(4-a^2)$

$\Rightarrow \quad 8-5a^2=4a^2-a^4$

$\Rightarrow \quad a^4-9a^2+8=0$

$\Rightarrow \quad (a^4-8)(a^4-1)=0$

$\Rightarrow \quad a^4=8, a^4=1$

$\therefore \quad a=1$

Now, equation of hyperbola is $\frac{x^2}{1}-\frac{y^2}{3}=1$.

$\therefore$ Equation of tangent at $(\sqrt{2},\sqrt{3})$ is given by

$$\sqrt{2}x-\frac{\sqrt{3}y}{3}=1 \Rightarrow \sqrt{2}x-\frac{y}{\sqrt{3}}=1$$

which passes through the point $(2\sqrt{2},3\sqrt{3})$.

23. Let the slopes of the two tangents to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ be } cm \text{ and } \frac{c}{m}$$

The equations of tangents are

$$y=cmx+\sqrt{a^2c^2m^2-b^2} \quad \text{...(i)}$$

and $$my-cx=\sqrt{a^2c^2-b^2m^2} \quad \text{...(ii)}$$

On squaring and subtracting Eq. (ii) from Eq. (i), we get

$$(y-cmx)^2-(my-cx)^2=a^2c^2m^2-b^2-a^2c^2+b^2m^2$$

$$\Rightarrow \quad (1-m^2)(y^2-c^2x^2)=-(1-m^2)(a^2c^2+b^2)$$

$$\Rightarrow \quad y^2+b^2=c^2(x^2-a^2)$$

24. Equation of normal to the hyperbola at the point $(5\sec\theta, 4\tan\theta)$ is

$$5x\cos\theta+4y\cot\theta=25+16 \quad \text{...(i)}$$

This line is perpendicular to the line $2x+y=1$

$\therefore \quad m_1m_2=-1$

$$\Rightarrow \quad \left(\frac{-5\cos\theta}{4\cot\theta}\right)(-2)=-1 \Rightarrow \sin\theta=-\frac{2}{5}$$

$$\therefore \quad \cos\theta=\sqrt{1-\frac{4}{25}}=\mp\frac{\sqrt{21}}{5} \text{ and } \cot\theta=\mp\frac{\sqrt{21}}{2}$$

From Eq. (i),

$$5x\frac{\sqrt{21}}{5}-\frac{4y\sqrt{21}}{2}=41 \Rightarrow \sqrt{21}(x-2y)=41$$

25. Let $P(h,k)$ be the point of intersection of tangents

$\Rightarrow y=mx\pm\sqrt{a^2m^2-b^2}$ passes through (h,k)

$\Rightarrow \quad k-mh=\pm\sqrt{a^2m^2-b^2}$

$\Rightarrow \quad (k-mh)^2=a^2m^2-b^2$

$\Rightarrow \quad k^2+m^2h^2-2khm=a^2m^2-b^2$

$\Rightarrow \quad m^2(h^2-a^2)-2khm+(k^2+b^2)=0$

Given, $m_1m_2=c^2$

$$\Rightarrow \quad \frac{k^2+b^2}{h^2-a^2}=c^2$$

$\Rightarrow \quad k^2+b^2=c^2(h^2-a^2)$

$\therefore$ Locus of (h,k) is

$$y^2+b^2=c^2(x^2-a^2)$$

26. Tangent at $(a\sec\phi, b\tan\phi)$ on the I st hyperbola is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1 \quad \text{...(i)}$$

Similarly, tangent at any point $(b\tan\theta, a\sec\theta)$ on IInd hyperbola is

$$\frac{y}{a}\sec\theta-\frac{x}{b}\tan\theta=1 \quad \text{...(ii)}$$

Since, Eqs. (i) and (ii) are common tangents, then they should be identical.

$$\Rightarrow \quad \frac{\sec\theta}{a}=-\frac{\tan\phi}{b}$$

$$\Rightarrow \quad \sec\theta=-\frac{a}{b}\tan\phi \quad \text{...(iii)}$$

$$\Rightarrow \quad -\frac{\tan\theta}{b}=\frac{\sec\phi}{a}$$

$$\Rightarrow \quad \tan\theta=-\frac{b}{a}\sec\phi \quad \text{...(iv)}$$

$\because \quad \sec^2\theta-\tan^2\theta=1$

$$\Rightarrow \quad \frac{a^2}{b^2}\tan^2\phi-\frac{b^2}{a^2}\sec^2\phi=1 \quad \text{[from Eqs. (iii) and (iv)]}$$

$$\Rightarrow \quad \frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}(1+\tan^2\phi) = 1$$

$$\Rightarrow \quad \left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\tan^2\phi = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow \quad \tan^2\phi = \frac{b^2}{a^2 - b^2}$$

$$\therefore \quad \sec^2\phi = 1 + \tan^2\phi = \frac{a^2}{a^2 - b^2}$$

Now, substituting $\sec\phi$ and $\tan\phi$ in Eq. (i), we get

$$\pm\frac{x}{\sqrt{a^2 - b^2}} \mp \frac{y}{\sqrt{a^2 - b^2}} = 1$$

$$\therefore \quad x \mp y = \pm\sqrt{a^2 - b^2} \quad \text{or } y = \pm x \pm \sqrt{a^2 - b^2}$$

27. Equation of the tangents at $P(a\sec\theta, b\tan\theta)$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

$\therefore$ Equation of the normal at P is

$$ax + b\,\text{cosec}\,\theta\, y = (a^2 + b^2)\sec\theta \quad \text{...(i)}$$

Similarly, the equation of normal at $Q(a\sec\phi, b\tan\phi)$ is

$$ax + b\,\text{cosec}\,\phi\, y = (a^2 + b^2)\sec\phi \quad \text{...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$y = \frac{a^2 + b^2}{b}\cdot\frac{\sec\theta - \sec\phi}{\text{cosec}\,\theta - \text{cosec}\,\phi}$$

So, $$k = y = \frac{a^2 + b^2}{b}\cdot\frac{\sec\theta - \sec\left(\frac{\pi}{2} - \theta\right)}{\text{cosec}\,\theta - \text{cosec}\left(\frac{\pi}{2} - \theta\right)}$$

$$= \frac{a^2 + b^2}{b}\cdot\frac{\sec\theta - \text{cosec}\,\theta}{\text{cosec}\,\theta - \sec\theta}$$

$$= -\left[\frac{a^2 + b^2}{b}\right]$$

28. Given that, $y = mx + \frac{25\sqrt{3}}{3}$ ...(i)

and $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ...(ii)

Here, Eq. (i) is normal to Eq. (ii), then

$$\frac{(a^2 + b^2)^2}{c^2} = \frac{a^2}{m^2} - \frac{b^2}{1}$$

$$\Rightarrow \quad \frac{(16+9)^2 \times 9}{625 \times 3} = \frac{16}{m^2} - \frac{9}{1}$$

$$\Rightarrow \quad \frac{16}{m^2} = 12 \Rightarrow m = \pm\frac{2}{\sqrt{3}}$$

29. The equation of hyperbola is

$$4x^2 - 9y^2 = 36$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{4} = 1 \quad \text{...(i)}$$

The equation of the chords of contact of tangents from (x_1, y_1) and (x_2, y_2) to the given hyperbola are

$$\frac{xx_1}{9} - \frac{yy_1}{4} = 1 \quad \text{...(ii)}$$

and $$\frac{xx_2}{9} - \frac{yy_2}{4} = 1 \quad \text{...(iii)}$$

Lines (ii) and (iii) are at right angles.

$$\frac{9}{4}\cdot\frac{x_1}{y_1} \times \frac{4}{9}\cdot\frac{x_2}{y_2} = -1$$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2} = -\left(\frac{9}{4}\right)^2 = -\frac{81}{16}$$

30. Let (h, k) be the point whose chord of contact w.r.t. hyperbola $x^2 - y^2 = 9$ is $x = 9$. We know that chord of (h, k) w.r.t. hyperbola $x^2 - y^2 = 9$ is $T = 0$

$$\Rightarrow \quad hx - ky - 9 = 0$$

But it is the equation of line $x = 9$. This is possible only when $h = 1, k = 0$.

Again, equation of pair of tangents is

$$T^2 = SS_1$$

$$\Rightarrow \quad (x-9)^2 = (x^2 - y^2 - 9)(1-9)$$

$$\Rightarrow \quad x^2 - 18x + 81 = (x^2 - y^2 - 9)(-8)$$

$$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$$

31. Equation of chord of hyperbola $x^2 - y^2 = a^2$ with mid-point as (h, k) is given by

$$xh - yk = h^2 - k^2$$

$$\Rightarrow \quad y = \frac{h}{k}x - \frac{(h^2 - k^2)}{k}$$

This will touch the parabola $y^2 = 4ax$, if

$$-\left(\frac{h^2 - k^2}{k}\right) = \frac{a}{h/k}$$

$$\Rightarrow \quad ak^2 = -h^3 + k^2h$$

$\therefore$ Locus of the mid-point is $x^3 = y^2(x - a)$

32. Let (h, k) is mid-point of chord.

Then, its equation is

$$3hx - 2ky + 2(x+h) - 3(y+k) = 3h^2 - 2k^2 + 4h - 6k$$

$$\Rightarrow \quad x(3h+2) + y(-2k-3) = 3h^2 - 2k^2 + 2h - 3k$$

Since, this line is parallel to $y = 2x$.

$$\therefore \quad \frac{3h+2}{2k+3} = 2$$

$$\Rightarrow \quad 3h + 2 = 4k + 6$$

$$\Rightarrow \quad 3h - 4k = 4$$

Thus, locus of mid-point is $3x - 4y = 4$

33. We know that asymptotes of rectangular hyperbola are mutually perpendicular, thus other asymptote should be $4x + 3y + \lambda = 0$

Also, intersection point of asymptotes is also the centre of the hyperbola.

Hence, intersection point of $4x+3y+\lambda=0$ and $3x-4y-6=0$ is $\left(\frac{18-4\lambda}{25}, \frac{-12\lambda-96}{100}\right)$ and it should lie on the line $x-y-1=0$

$\therefore \quad \frac{18-4\lambda}{25} - \frac{-12\lambda-96}{100} - 1 = 0 \Rightarrow \lambda = 17$

Hence, equation of other asymptote is $4x+3y+17=0$

34. Given that, $xy = hx + ky \Rightarrow (x-k)(y-h) = hk$

On shifting origin to (k, h) the above equation reduces to

$$XY = hk = c^2 \quad \text{(say)}$$

where, $\quad x = X + k$ and $y = Y + h$

Then, the equation of the asymptotes are $X=0$ and $Y=0$

i.e. $\quad x = k, \quad y = h$

35. Given equation is $x^2 - 2y^2 - 2 = 0$, it can be rewritten as

$$\frac{x^2}{2} - \frac{y^2}{1} = 1$$

Here, $a^2 = 2$, $b^2 = 1$

We know that equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the product of length of perpendicular drawn from any point on the hyperbola to the asymptotes is

$$\frac{a^2b^2}{a^2+b^2} = \frac{2(1)}{2+1} = \frac{2}{3}$$

36. Equation of asymptotes are

$$x + 2y = 3 \quad \text{...(i)}$$

and $$x - y = 0 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$x = 1, y = 1$$

$\therefore$ Centre of hyperbola is (1, 1) because asymptotes passes through the centre of the hyperbola.

37. Equation of asymptotes of the hyperbola are

$$x^2 + 2xy - 3y^2 = 0$$

The angle between asymptotes is

$$\theta = \tan^{-1}\left[\frac{1-1(-3)}{1-3}\right]$$

$$= \tan^{-1}\left(\frac{1+3}{-2}\right) = \tan^{-1}(\pm 2)$$

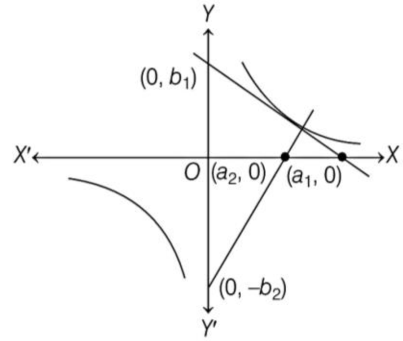
38. We know that angle between two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.

Equation of given hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Here, $a = 4$ and $b = 3$

$\therefore \quad$ Required angle $= 2\tan^{-1}\left(\frac{3}{4}\right)$

39. $\because$ Tangent and normal are at 90°.



$\therefore$ Product of slopes is -1.

$\Rightarrow \quad \left(-\frac{b_1}{a_1}\right)\left(-\frac{b_2}{a_2}\right) = -1 \Rightarrow a_1a_2 + b_1b_2 = 0$

40. The given equation of rectangular hyperbola is

$$xy = 18 \quad \text{...(i)}$$

On comparing Eq. (i), with general equation of rectangular hyperbola $xy = \frac{a^2}{2}$, we get

$$\frac{a^2}{2} = 18 \Rightarrow a^2 = 36 \Rightarrow a = 6$$

$\therefore$ Length of the transverse axis of rectangular hyperbola is $2a = 2\times 6 = 12$

41. Any point on parabola $y^2 = 8x$ is $(2t^2, 4t)$. The equation of tangent at that point is

$$yt = x + 2t^2 \quad \text{...(i)}$$

Given that, $$xy = -1 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$y(yt - 2t^2) = -1 \Rightarrow ty^2 - 2t^2y + 1 = 0$$

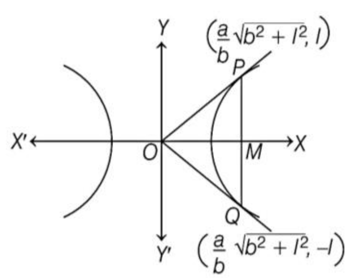
$\because$ It is common tangent. It means they are intersect only at one point and the value of discriminant is equal to zero.

i.e. $\quad 4t^4 - 4t = 0 \Rightarrow t = 0, 1$

$\therefore$ The common tangent is $y = x + 2$, (when $t = 0$, it is $x = 0$ which can touch $xy = -1$ at infinity only)

Round II

1. $\because$ PQ is the double ordinate. Let $MP = MQ = l$.



Given that ΔOPQ is an equilateral, then $OP = OQ = PQ$

$\Rightarrow \quad (OP)^2 = (OQ)^2 = (PQ)^2$

$\Rightarrow \quad \frac{a^2}{b^2}(b^2+l^2)+l^2 = \frac{a^2}{b^2}(b^2+l^2)+l^2 = 4l^2$

$\Rightarrow \quad \frac{a^2}{b^2}(b^2+l^2) = 3l^2$

$\Rightarrow \quad a^2 = l^2\left(3-\frac{a^2}{b^2}\right) \Rightarrow l^2 = \frac{a^2b^2}{(3b^2-a^2)} > 0$

$\therefore \quad 3b^2 - a^2 > 0 \Rightarrow 3b^2 > a^2$

$\Rightarrow 3a^2(e^2-1) > a^2 \Rightarrow e^2 > 4/3$

$\therefore \quad e > \frac{2}{\sqrt{3}}$

2. Equation of normal at $P(a\sec\phi, b\tan\phi)$ is

$a x\cos\phi + by\cot\phi = a^2+b^2.$

Then, coordinates of L and M are

$\left(\frac{a^2+b^2}{a}\cdot\sec\phi, 0\right)$ and $\left(0, \frac{a^2+b^2}{b}\tan\phi\right)$ respectively.

Let mid-point of ML be $Q(h,k)$, then

$$h = \frac{(a^2+b^2)}{2a}\sec\phi$$

$$\therefore \quad \sec\phi = \frac{2ah}{(a^2+b^2)} \quad \text{...(i)}$$

and

$$k = \frac{(a^2+b^2)}{2b}\tan\phi$$

$$\therefore \quad \tan\phi = \frac{2bk}{(a^2+b^2)} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\sec^2\phi - \tan^2\phi = \frac{4a^2h^2}{(a^2+b^2)^2} - \frac{4b^2k^2}{(a^2+b^2)^2}$$

Hence, required locus is

$$\frac{x^2}{\left(\frac{a^2+b^2}{2a}\right)^2} - \frac{y^2}{\left(\frac{a^2+b^2}{2b}\right)^2} = 1$$

Let eccentricity of this curve be e_1.

$\Rightarrow \quad \left(\frac{a^2+b^2}{2b}\right)^2 = \left(\frac{a^2+b^2}{2a}\right)^2 (e_1^2-1)$

$\Rightarrow \quad a^2 = b^2(e_1^2-1)$

$\Rightarrow \quad a^2 = a^2(e^2-1)(e_1^2-1) \quad [\because b^2 = a^2(e^2-1)]$

$\Rightarrow e^2e_1^2 - e^2 - e_1^2 + 1 = 1$

$\Rightarrow \quad e_1^2(e^2-1) = e^2 \Rightarrow e_1 = \frac{e}{\sqrt{(e^2-1)}}$

3. We have, $\frac{2b^2}{a} = 8$ and $2b = ae$

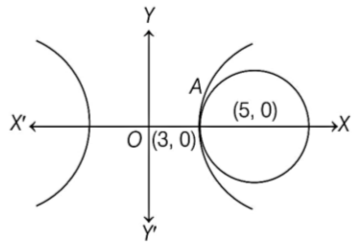
$\Rightarrow \quad b^2 = 4a$ and $2b = ae$

Consider, $2b = ae \Rightarrow 4b^2 = a^2e^2 \Rightarrow 4a^2(e^2-1) = a^2e^2$

$\Rightarrow \quad 4e^2 - 4 = e^2 \quad [\because a \neq 0]$

$\Rightarrow \quad 3e^2 = 4 \Rightarrow e = \frac{2}{\sqrt{3}} \quad [\because e > 0]$

4. Given hyperbola is



$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \text{...(i)}$$

$\therefore \quad e^2 = 1 + \frac{16}{9} = \frac{25}{9} \Rightarrow e = \frac{5}{3}$

Hence, its foci are $(\pm 5, 0)$.

The equation of the circle with (5, 0) as centre is

$$(x-5)^2 + y^2 = r^2 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we have

$$16x^2 - 9[r^2 - (x-5)^2] = 144$$

$\Rightarrow \quad 25x^2 - 90x - 9r^2 + 81 = 0$

Since, the circle touches the hyperbola, above equation must have equal roots. Hence,

$$90^2 - 4(25)(81 - 9r^2) = 0$$

$\Rightarrow \quad 9 - (9 - r^2) = 0$

$\Rightarrow r = 0$, which is not possible.

Hence, the circle cannot touch at two points.

It can only be tangent at the vertex. Hence,

$$r = 5 - 3 = 2$$

5. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $e = \sqrt{1+\frac{b^2}{a^2}}$

$\therefore$ For the given hyperbola, $e = \sqrt{1+\frac{\sin^2\theta}{\cos^2\theta}} > 2$

$[\because a^2 = \cos^2\theta \text{ and } b^2 = \sin^2\theta]$

$\Rightarrow \quad 1 + \tan^2\theta > 4$

$\Rightarrow \quad \tan^2\theta > 3$

$\Rightarrow \quad \tan\theta \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

$[x^2 > 3 \Rightarrow |x| > \sqrt{3} \Rightarrow x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)]$

But $\theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \tan\theta \in (\sqrt{3}, \infty) \Rightarrow \theta \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

Now, length of latusrectum $= \frac{2b^2}{a} = 2\frac{\sin^2\theta}{\cos\theta}$

$= 2\sin\theta\tan\theta$

Since, both $\sin\theta$ and $\tan\theta$ are increasing functions in $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

$\therefore$ Least value of latus rectum is

$> 2\sin\frac{\pi}{3}\cdot\tan\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2}\cdot\sqrt{3} = 3 \quad \left(\text{at } \theta = \frac{\pi}{3}\right)$

and greatest value of latusrectum is $< \infty$

Hence, latusrectum length $\in (3, \infty)$.

6. We know that, the equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in slope form is

$$y=mx\pm\sqrt{a^2m^2-b^2}$$

$\because$ Equation of tangent to the hyperbola $x^2-y^2=3$ is $y=mx\pm\sqrt{3m^2-3}$, since this tangent is parallel to the line $2x+y+8=0$.

$\because$ $\quad m=-2\Rightarrow y=-2x\pm 3$

or $\quad 2x+y\pm 3=0 \quad$ ...(i)

Let (x_1, y_1) be the point of contact

$$xx_1-yy_1=3 \quad \text{...(ii)}$$

Since, Eqs. (i) and (ii) are identical, then

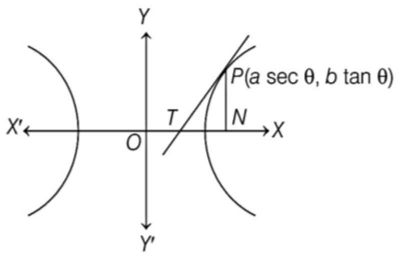
$$\frac{x_1}{2}=-\frac{y_1}{1}=\pm 1$$

$\Rightarrow \quad x_1=\pm 2$ and $y_1=\pm 1$

$\therefore$ Point of contact $=(2,-1)$

7. Tangent at point P is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$



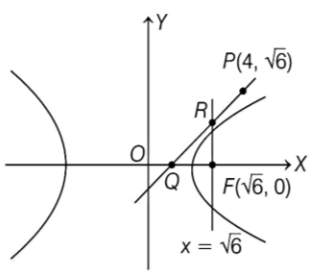
It meets the X-axis at point $T(a\cos\theta, 0)$ and foot of perpendicular from P to X-axis is $N(a\sec\theta, 0)$.

From the diagram, we have

$$OT=a\cos\theta$$

and $\quad ON=a\sec\theta\Rightarrow OT\times ON=a^2$

8.



$$\frac{x^2}{4}-\frac{y^2}{2}=1$$

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{\frac{3}{2}}$$

Focus $F(ae, 0)\Rightarrow F(\sqrt{6}, 0)$

Equation of tangent at P to the hyperbola is

$$2x-y\sqrt{6}=2$$

Tangent meet X-axis at $Q(1, 0)$

and latus rectum $x=\sqrt{6}$ at $R\left(\sqrt{6}, \frac{2}{\sqrt{6}}(\sqrt{6}-1)\right)$

$$\therefore \quad \text{Area of } \Delta QFR=\frac{1}{2}(\sqrt{6}-1)\cdot\frac{2}{\sqrt{6}}(\sqrt{6}-1)$$

$$=\frac{7}{\sqrt{6}}-2$$

9. Director circle of circle

$$x^2+y^2=a^2 \text{ is } x^2+y^2=2a^2$$

The semi-transverse axis is $\sqrt{3}a$.

Radius of the circle is $\sqrt{2}a$.

Hence, director circle and hyperbola do not intersect.

10. Let point on the hyperbola $x^2-y^2=4$ be $(2\sec\theta, 2\tan\theta)$.

Equation of tangent at $(2\sec\theta, 2\tan\theta)$

$$\frac{x}{2}\sec\theta-\frac{y}{2}\tan\theta=1$$

X-intercept, i.e. $\quad a_1=2\cos\theta$

Y-intercept, i.e. $\quad b_1=-2\cot\theta$

Now, the equation of normal at $(2\sec\theta, 2\tan\theta)$ is

$$2x\cos\theta+2y\cot\theta=a^2+b^2$$

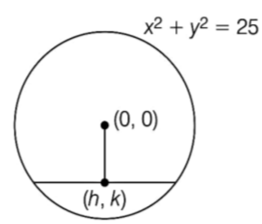
X-intercept, i.e. $\quad a_2=\frac{a^2+b^2}{2\cos\theta}$

Y-intercept, i.e. $\quad b_2=\frac{a^2+b^2}{2\cot\theta}$

$$\therefore \quad a_1a_2+b_1b_2=2\cos\theta\times\frac{a^2+b^2}{2\cos\theta}+(-2\cot\theta)\left(\frac{a^2+b^2}{2\cot\theta}\right)$$

$$=a^2+b^2-a^2-b^2=0$$

11.



Equation of chord

$$y-k=-\frac{h}{k}(x-h)$$

$\Rightarrow \quad ky-k^2=-hx+h^2$

$\Rightarrow \quad hx+ky=h^2+k^2$

$\Rightarrow \quad y=-\frac{hx}{k}+\frac{h^2+k^2}{k}$

Tangent to $\frac{x^2}{9}-\frac{y^2}{16}=1$

$$c^2=a^2m^2-b^2$$

$\Rightarrow \quad \left(\frac{h^2+k^2}{k}\right)=9\left(-\frac{h}{k}\right)^2-16$

$\Rightarrow \quad (x^2+y^2)^2=9x^2-16y^2$

12. Equation of given parabola $y^2 = 12x$ ... (i)

and hyperbola $8x^2 - y^2 = 8$ ... (ii)

Now, equation of tangent to parabola $y^2 = 12x$ having slope 'm' is $y = mx + \frac{3}{m}$ ... (iii)

and equation of tangent to hyperbola

$$\frac{x^2}{1} - \frac{y^2}{8} = 1 \text{ having slope '}m\text{' is}$$

$$y = mx \pm \sqrt{1^2 m^2 - 8} \quad \text{...(iv)}$$

Since, tangents (iii) and (iv) represent the same line

$\therefore$ $$m^2 - 8 = \left(\frac{3}{m}\right)^2$$

$\Rightarrow$ $$m^4 - 8m^2 - 9 = 0$$

$\Rightarrow$ $$(m^2 - 9)(m^2 + 1) = 0$$

$\Rightarrow$ $$m = \pm 3$$

Now, equation of common tangents to the parabola (i) and hyperbola (ii) are

$$y = 3x + 1 \text{ and } y = -3x - 1$$

$\because$ Point 'P' is point of intersection of above common tangents,

$\therefore$ $$P(-1/3, 0)$$

and focus of hyperbola $S(3, 0)$ and $S'(-3, 0)$.

Thus, the required ratio $= \frac{PS}{PS'} = \frac{3 + 1/3}{3 - 1/3} = \frac{10}{8} = \frac{5}{4}$

13. The equation of tangent in terms of slope of $y^2 = 32x$ is

$$y = mx + \frac{8}{m} \quad \text{...(i)}$$

which is also tangent of the hyperbola $9x^2 - 9y^2 = 8$

i.e., $$x^2 - y^2 = \frac{8}{9}$$

Then, $$\left(\frac{8}{m}\right)^2 = \frac{8}{9}m^2 - \frac{8}{9}$$

$\Rightarrow$ $$\frac{8}{m^2} = \frac{m^2}{9} - \frac{1}{9}$$

$\Rightarrow$ $$72 = m^4 - m^2$$

$\Rightarrow$ $$m^4 - m^2 - 72 = 0$$

$\Rightarrow$ $$(m^2 - 9)(m^2 + 8) = 0$$

$\Rightarrow$ $$m^2 = 9, \quad [\because m^2 + 8 \neq 0]$$

$\Rightarrow$ $$m = \pm 3$$

From Eq. (i), we get

$$y = \pm 3x \pm \frac{8}{3}$$

$\Rightarrow$ $$3y = \pm 9x \pm 8$$

$\Rightarrow$ $$\pm 9x - 3y \pm 8 = 0$$

$\Rightarrow$ $$9x - 3y + 8 = 0, \quad 9x - 3y - 8 = 0$$

$$-9x - 3y + 8 = 0, \quad -9x - 3y - 8 = 0$$

or $$9x - 3y + 8 = 0, \quad 9x - 3y - 8 = 0$$

$\Rightarrow$ $$9x + 3y - 8 = 0$$

and $$9x + 3y + 8 = 0$$

14. Given equation ot hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Its vertices are $A(a, 0)$ and $A' = (-a, 0)$

Normal at any point θ i.e. $(a \sec\theta, b\tan\theta)$ is

$ax\cos\theta + by\cot\theta = a^2 + b^2$

Now, G is it intersection on transverse axis i.e. X-axis.

$\therefore$ $$G = \left(\frac{(a^2 + b^2)\sec\theta}{a}, 0\right)$$

$\Rightarrow$ $$AG = a - \frac{(a^2 + b^2)\sec\theta}{a}$$

$\Rightarrow$ $$A'G = -a - \frac{(a^2 + b^2)\sec\theta}{a}$$

$\therefore$ $$AG \times A'G = a^2 - \frac{(a^2 + b^2)^2 \cdot \sec^2\theta}{a^2}$$

$$= a^2 - \frac{a^4 e^4 \cdot \sec^2\theta}{a^2} \quad [\because a^2 + b^2 = a^2e^2]$$

$$= a^2 - a^2 e^4 \cdot \sec^2\theta$$

$$= a^2(1 - e^4 \sec^2\theta)$$

$$= a^2(e^4 \sec^2\theta - 1)$$

15. We know that, $y = mx + \frac{a}{m}$ is the equation of tangent to the parabola $y^2 = 4ax$.

$\therefore$ $y = mx + \frac{1}{m}$ is a tangent to the parabola

$$y^2 = 4x \quad [\because a = 1]$$

Let, this tangent is also a tangent to the hyperbola $xy = 2$

Now, on substituting $y = mx + \frac{1}{m}$ in $xy = 2$, we get

$$x\left(mx + \frac{1}{m}\right) = 2 \Rightarrow m^2x^2 + x - 2m = 0$$

Note that tangent touch the curve exactly at one point, therefore both roots of above equations are equal.

$\Rightarrow$ $$D = 0 \Rightarrow 1 = 4(m^2)(-2m)$$

$\Rightarrow$ $$m^3 = \left(-\frac{1}{2}\right)^3 \Rightarrow m = -\frac{1}{2}$$

$\therefore$ Required equation of tangent is

$$y = -\frac{x}{2} - 2 \Rightarrow 2y = -x - 4 \Rightarrow x + 2y + 4 = 0$$

16. The equation of normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is

$$xt^3 - yt - ct^4 + c = 0 \quad \text{...(i)}$$

Since, Eq. (i) intersect at hyperbola $\left(ct_1, \frac{c}{t_1}\right)$.

$$ct_1t^3 - \frac{c}{t_1}t - ct^4 + c = 0$$

$$\Rightarrow \quad t^3 t_1 = \frac{t}{t_1} + t^4 - 1$$

$$\Rightarrow \quad t^3 t_1^2 - t - t^4 t_1 + t_1 = 0$$

$$\Rightarrow \quad (t^3 t_1^2 - t^4 t_1) + (t_1 - t) = 0$$

$$\Rightarrow \quad t^3 t_1 (t_1 - t) + 1(t_1 - t) = 0$$

$$\Rightarrow \quad (t_1 - t)(t^3 t_1 + 1) = 0$$

$$\Rightarrow \quad t_1 \neq t$$

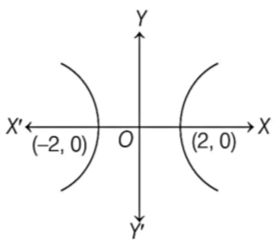
or $\quad t^3 t_1 + 1 = 0$

$$\therefore \quad t^3 t_1 = -1$$

17. Let equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where, $2ae = 4$ and $e = 2$



$$\Rightarrow \quad a = 1$$

$$a^2 e^2 = a^2 + b^2$$

$$\Rightarrow \quad 4 = 1 + b^2$$

$$\therefore \quad b^2 = 3$$

Thus, equation of hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1$$

$$\Rightarrow \quad 3x^2 - y^2 = 3$$

18. Equation of given

Hyperbola : $x^2 - y^2 \sec^2 \theta = 10 \Rightarrow \frac{x^2}{10} - \frac{y^2}{10\cos^2\theta} = 1$ ...(i)

$\therefore$ Eccentricity of hyperbola (i) is

$$e_1 = \sqrt{1 + \cos^2\theta}$$

and Ellipse : $x^2 \sec^2\theta + y^2 = 5 = \frac{x^2}{5\cos^2\theta} + \frac{y^2}{5} = 1$ ...(ii)

$\therefore$ Eccentricity of ellipse (ii) is $e_2 = \sqrt{1 - \cos^2\theta}$

It is given that

$$e_1 = \sqrt{5}\, e_2$$

$$\Rightarrow \quad 1 + \cos^2\theta = 5(1 - \cos^2\theta)$$

$$\Rightarrow \quad 6\cos^2\theta = 4 \Rightarrow \cos^2\theta = \frac{2}{3}$$

$\therefore$ Length of latus rectum of ellipse (i) $= \frac{2(5\cos^2\theta)}{\sqrt{5}}$

$$= 2\sqrt{5}\left(\frac{2}{3}\right) = \frac{4\sqrt{5}}{3}$$

19. It is given that e_1 is the eccentricity of the ellipse,

$\frac{x^2}{18} + \frac{y^2}{4} = 1$, so $e_1 = \sqrt{1 - \frac{4}{18}} = \sqrt{\frac{14}{18}} = \sqrt{\frac{7}{9}}$

and e_2 is the eccentricity of the hyperbola,

$\frac{x^2}{9} - \frac{y^2}{4} = 1$, so $e_2 = \sqrt{1 + \frac{4}{9}} = \sqrt{\frac{13}{9}}$

Now, as $(e_1, e_2) = \left(\sqrt{\frac{7}{9}}, \sqrt{\frac{13}{9}}\right)$ is a point on the ellipse,

$15x^2 + 3y^2 = k$, then

$$15\left(\frac{7}{9}\right) + 3\left(\frac{13}{9}\right) = k \Rightarrow \frac{105 + 39}{9} = k$$

$$\Rightarrow \quad \frac{144}{9} = k \Rightarrow k = 16$$

20. The equation of tangent is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

$\therefore$ Coordinates of A and B are $(a\cos\theta, 0)$ and $(0, -b\cot\theta)$ respectively.

Let coordinates of P be (h, k).

$$\therefore \quad h = a\cos\theta, \quad k = -b\cot\theta$$

$$\Rightarrow \quad \frac{k}{h} = -\frac{b}{a\sin\theta} \Rightarrow \sin\theta = -\frac{bh}{ak}$$

$$\Rightarrow \quad \frac{b^2h^2}{a^2k^2} = \sin^2\theta \Rightarrow \frac{b^2h^2}{a^2k^2} + \frac{h^2}{a^2} = 1$$

$$\Rightarrow \quad \frac{b^2}{k^2} + 1 = \frac{a^2}{h^2} \Rightarrow \frac{a^2}{h^2} - \frac{b^2}{k^2} = 1$$

Hence, the locus of P is $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

21. We have,

$$(10x - 5)^2 + (10y - 2)^2 = 9(3x + 4y - 7)^2$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{5}\right)^2 = \frac{9}{100}\ \frac{(3x + 4y - 7)^2 \times 25}{25}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{5}\right)^2 = \left(\frac{3}{2}\right)^2\left(\frac{3x + 4y - 7}{5}\right)^2$$

From this equation, foci of hyperbola is $\left(\frac{1}{2}, \frac{1}{5}\right)$

eccentricity of hyperbola $= \frac{3}{2}$

equation of directrix is $3x + 4y - 7 = 0$

equation of latusrectum is $3x + 4y = \lambda$.

[$\because$ latusrectum is parallel to directrix]

Since, latusrectum is passes through the foci

i.e. $\quad \frac{3}{2} + \frac{4}{5} = \lambda$

$$\Rightarrow \quad \lambda = \frac{23}{10}$$

$\therefore$ Equation of latusrectum is

$$30x + 40y - 23 = 0$$

Now, $\quad |b + 2c| = |40 - 46| = 6$

22. Foci of the hyperbola are (5, 12) and (24, 7).

Let $P(x, y)$ be any point on the hyperbola and S and S' be the foci of parabola.

By definition,

$|PS - PS'| =$ Length of transverse axis

$\therefore \left|\sqrt{(x-5)^2+(y-12)^2} - \sqrt{(x-24)^2+(y-7)^2}\right| = 2a$

Since, hyperbola is passing through the origin.

$\therefore \quad 2a = |\sqrt{25+144} - \sqrt{5+6+49}|$

$= |13 - 25| = 12$

Now, distance between two foci $= 2ae$

i.e. $\sqrt{(5-24)^2 + (12-7)^2} = 2ae$

$\sqrt{19^2 + 5^2} = 2ae$

$\Rightarrow \quad \sqrt{368} = 12e$

$\therefore \quad \dfrac{12e}{\sqrt{368}} = 1$

23. Let the length of each side of the equilateral ΔOPQ be l units. Then, the coordinates of P are $\left(\dfrac{\sqrt{3}\,l}{2}, \dfrac{l}{2}\right)$.

Point $P\left(\dfrac{\sqrt{3}\,l}{2}, \dfrac{l}{2}\right)$ lies on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\therefore \quad \dfrac{3l^2}{4a^2} - \dfrac{l^2}{4b^2} = 1$

$\Rightarrow \quad (3b^2 - a^2)l^2 = 4a^2b^2$

$\Rightarrow \quad (3e^2 - 4)l^2 = 4a^2(e^2 - 1)$

$\Rightarrow \quad l = 2a\sqrt{\dfrac{e^2-1}{3e^2-4}}$

Since, l is real and $e > 1$.

$\therefore \quad e^2 > \dfrac{4}{3}$

$\Rightarrow \quad e > \dfrac{2}{\sqrt{3}} = \sqrt{3}\ e > 2$

$\Rightarrow \quad k = 2$

24. Equation of tangent at point (h, k) on the hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$\frac{xh}{a^2} - \frac{yk}{b^2} = 1$$

$\Rightarrow \quad x = \dfrac{(a^2b^2 + yka^2)}{b^2h}$

This tangent cuts the circle $x^2 + y^2 = a^2$

Putting the value of x in equation of circle, we get

$$\left(\frac{a^2b^2 + yka^2}{b^2h}\right)^2 + y^2 = a^2$$

$$a^4b^4 + 2yka^4b^4 + y^2k^2a^4 + y^2b^4h^2 = a^2b^4h^2$$

$\Rightarrow (k^2a^4 + b^4h^2)y^2 + 2y\,ka^4b^2 + a^4b^4 - a^2b^4h^2 = 0$

Now, sum of roots, i.e.

$$y_1 + y_2 = \frac{-2ka^4b^2}{k^2a^4 + b^4h^2}$$

and product of roots i.e.

$$y_1y_2 = \frac{a^4b^4 - a^2b^4h^2}{k^2a^4 + b^4h^2}$$

Now, $\quad \dfrac{1}{y_1} + \dfrac{1}{y_2} = \dfrac{y_1 + y_2}{y_1y_2}$

$= \dfrac{-2ka^4b^2}{a^4b^4 - a^2b^4h^2} = \dfrac{-2ka^4b^2}{a^2b^4(a^2 - h^2)}$

$= -\dfrac{2ka^2}{b^2(a^2 - h^2)} = \dfrac{-2k}{b^2\left(1 - \dfrac{h^2}{a^2}\right)}$

$= -\dfrac{2k}{b^2\left[-\dfrac{k^2}{b^2}\right]} = \dfrac{2}{k} \quad \left[\because \dfrac{h^2}{a^2} - \dfrac{k^2}{a^2} = 1\right]$

$\therefore \quad \lambda = 2$

25. Let the tangent of point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ be $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$.

This line meets the coordinate axes at A and B

$\therefore$ Coordinates of point $A(a\cos\theta, 0)$ and $B(0, -b\cot\theta)$.

Since, $OACB$ is formed a rectangle.

$\therefore \quad C$ is $(a\cos\theta, -b\cot\theta)$

Now, C lies on $\quad \dfrac{a^2}{x^2} - \dfrac{b^2}{y^2} = \lambda$

Then, $\quad \dfrac{a^2}{a^2\cos^2\theta} - \dfrac{b^2}{b^2\cot^2\theta} = \lambda$

$\Rightarrow \quad \sec^2\theta - \tan^2\theta = \lambda$

$\therefore \quad \lambda = 1$