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HYPERBOLA

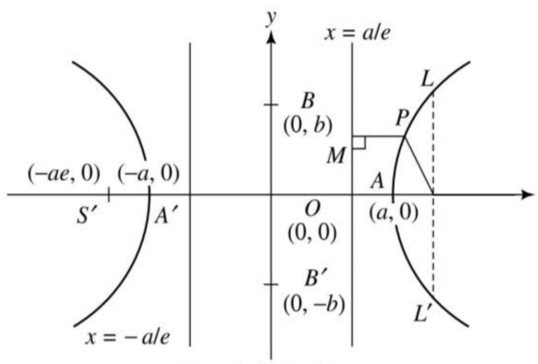
Definition A hyperbola is the locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point in the plane to its distance from a fixed line in the plane (not passing through the fixed point) is a constant and equal to e, where $e > 1$.

(i) Fixed point is called a *focus* of the hyperbola.

(ii) Fixed line is called a *directrix* of the hyperbola.

(iii) Constant ratio e, is called the eccentricity of the hyperbola.

Standard equation of a hyperbola referred to its principal axes along the coordinate axies is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$.



Hyperbola ($e > 1$)

Fig. 20.1

Let $P(x, y)$ be any point on the hyperbola.

$S(ae, 0)$ be a focus and $x = \frac{a}{e}$ be a directrix then according to the definition

$$(x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

Since $e > 1$, $e^2 - 1 > 0$ so $a^2(e^2 - 1) = b^2$ and the required equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

In the figure

(i) $A'A$ is the *transverse axis* of the hyperbola along the axis of x and is of length $2a$. $A(a, 0)$, $A^1(-a, 0)$ are the vertices of the hyperbola.

(ii) $b'b$ is the *conjugate axis* of the hyperbola along the axis of y and is of length $2b$.

(iii) $O(0, 0)$ is the centre of the hyperbola.

(iv) $e = \sqrt{\frac{a^2 + b^2}{a^2}}$ is the eccentricity of the hyperbola.

From symmetry we observe that if $S'(-ae, 0)$ be taken as the focus and $x = -\frac{a}{e}$ is taken as the directrix, same hyperbola is described. So

(v) $S'(-ae, 0)$ and $S(ae, 0)$ are the two foci of the hyperbola.

(vi) $x = -\frac{a}{e}$ and $x = \frac{a}{e}$ are the two directrices of the hyperbola.

(vii) $x = ae$ and $x = -ae$ are the two latera recta of the hyperbola.

(viii) Latus rectum $x = ae$ meets the hyperbola at the points $\left(ae, \pm\frac{b^2}{a}\right)$ so length of each latus rectum is $2b^2/a$.

Illustration 1

Find the eccentricity, length of a latus rectum, equations of the latera recta of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Solution Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a^2 = 16$, $b^2 = 9$.

$$\Rightarrow e^2 = \frac{16 + 9}{16} \Rightarrow e = \frac{5}{4}.$$

Solution: Equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Equation of the chord of contact of (5, 1) is $\frac{5x}{16} - \frac{y(1)}{4} =$

$\Rightarrow 5x - 4y = 16.$ (1)

If (α, β) is the mid-point of this chord then its equation is

$$\frac{\alpha x}{16} - \frac{\beta y}{4} - 1 = \frac{\alpha^2}{16} - \frac{\beta^2}{4} - 1 \quad (2)$$

From (1) and (2) we get

$\alpha = \frac{80}{21}, \beta = \frac{16}{21}.$

So that coordinate of the mid-point of the chord are $\left(\frac{80}{21}, \frac{16}{21}\right)$.

8. *Director circle* of a hyperbola is the locus of the point of intersection of the tangents to the hyperbola which intersect at right angles and its equation is $x^2 + y^2 = a^2 - b^2$.
9. *A diameter* of a hyperbola is the locus of mid-points of a system of parallel chords of the hyperbola and its equation is $y = \frac{b^2}{a^2 m}x$ where m is the slope of the parallel chords of the hyperbola which are bisected by it.

   Two diameters of a hyperbola are said to be *conjugate* when each bisects the chords parallel to the other. Thus two diameters $y = mx$ and $y = m'x$ of the hyperbola are conjugate if $mm' = \frac{b^2}{a^2}$.

Illustration 7

Find the points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$ from which two perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$.

Solution: Director circle of $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$ or

$$\frac{x^2}{2a^2} + \frac{y^2}{2a^2} = 1 \quad (1)$$

Equation of the hyperbola can be written as

$$\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1 \quad (2)$$

Since this represents the director circle of the given circle. Subtracting (2) from (1) we get

$$\left(\frac{1}{2a^2} + \frac{1}{2b^2}\right)y^2 = 0 \Rightarrow y = 0$$

So the required points are $\left(\pm\sqrt{2}a, 0\right)$.

Illustration 8

If $y = mx + 2\sqrt{5}$ is a tangent to the hyperbola $4x^2 - y^2 = 16$. Find the equation of a diameter of the hyperbola bisecting the chords parallel of the tangent.

Solution: Equation of the hyperbola can be written as $\frac{x^2}{4} - \frac{y^2}{16} = 1$

$y = mx + 2\sqrt{5}$ touches the hyperbola.

$\Rightarrow (2\sqrt{5})^2 = 4m^2 - 16 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3$

So the required equation of a diameter is

$y = \frac{16}{4(\pm 3)}x \Rightarrow 4x \pm 3y = 0.$

10. *Asymptotes* If the length of the perpendicular let fall from a point on a hyperbola. to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola then the straight line is called an *asymptote* of the hyperbola.

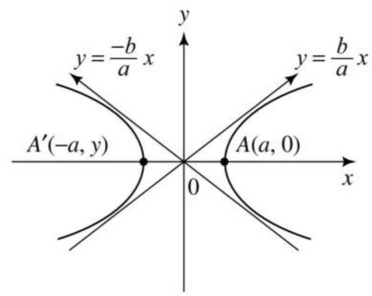


Fig. 20.2

(i) Equation of the asymptotes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

i.e. $\frac{x}{a} \pm \frac{y}{b} = 0$

(ii) Angle between two asymptotes is $2\tan^{-1}\left(\frac{b}{a}\right)$.

Asymptotes are at right angle if and only if $a = b$.

(iii) If angle between two asymptotes is 2θ, then eccentricity of the hyperbola is $\sec\theta$.

11. *Rectangular hyperbola* A hyperbola in which the lengths of the transverse and conjugate axes are equal is called a rectangular hyperbola.

(i) Equation of a rectangular hyperbola is $x^2 - y^2 = a^2$.

(ii) Eccentricity of the rectangular hyperbola is $\sqrt{2}$.

(iii) Asymptotes of a rectangular hyperbola are at right angles.

(iv) Rotation of the system of coordinates through an angle $\pi/4$ in the clockwise direction (i.e. the axes are taken along the asymptotes of the hyperbola) gives another form of the equation of the rectangular hyperbola i.e. $xy = c^2$.

(v) Some facts about rectangular hyperbola $xy = c^2$

(a) Vertices (c, c) and $(-c, -c)$

(b) Foci $(\sqrt{2}c, \sqrt{2}c)$ and $(-\sqrt{2}c, -\sqrt{2}c)$

(c) Directrices $x + y = \pm \sqrt{2}\, c$

(d) length of the latus rectum $= 2\sqrt{2}\, c$

(e) parametric equation: $x = ct$, $y = c/t$.

Illustration 9

If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = 25$ in four points, then find the product of the ordinates of these points.

Solution: The ordinates of the points of intersection of $x^2 + y^2 = a^2$ and the hyperbola $xy = 25$ are given by

$$\left(\frac{25}{y}\right)^2 + y^2 = a^2 \text{ or } y^4 - a^2y^2 + (25)^2 = 0$$

If y_1, y_2, y_3, y_4 are the roots of this equation then $y_1 y_2 y_3 y_4 = (25)^2 = 625$.

12. *Conjugate hyperbola* Two hyperbolas such that the transverse and conjugate axes of one hyperbola are, respectively, the conjugate and transverse axis of the other are called conjugate hyperbolas of each other.

(a) Equation of the hyperbola conjugate to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$$

(b) If e_1 and e_2 are the eccentricities of the hyperbola and its conjugate, then $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$.

(c) The foci of a hyperbola and its conjugate are concentric and form the vertices of a square.

13. *Similar hyperbolas* Two hyperbolas are said to be similar if they have the same eccentricity.

14. *Equal hyperbola* two hyperbolas are equal if they are similar and have the same latus rectum.

15. *Some more properties of the hyperbola $x^2/a^2 - y^2/b^2 = 1$*

(i) If PN be the ordinate of a point P on the hyperbola and the tangent at P meets the tranverse axis in T, then $ON. OT = a^2$, O being the origin.

(ii) If PM be drawn perpendiculars to the conjugate axis from a point P on the hyperbola and the tangent at P meets the conjugate axis in T, then $OM. OT = -b^2$; O, being the origin.

(iii) If the normal at P on the hyperbola meets the transverse axis in G, then $SG = eSP$; S being a foci and e the eccentricity of the hyperbola.

(iv) The tangent and normal at any point of a hyperbola bisect the angle between the focal radii to that point.

(v) The locus of the feet of the perpendiculars from the foci on a tangent to a hyperbola is the auxillary circle.

(vi) The product of the perpendiculars from the foci on any tangent to a hyperbola is constant and equal to b^2.

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: If e_1 and e_2 are respectively the eccentricities of the conics $\frac{x^2}{25} - \frac{y^2}{11} = 1$ and $\frac{x^2}{16} + \frac{y^2}{7} = 1$ then e_1e_2 is equal to

(a) $\frac{10}{9}$ (b) $\frac{4}{3}$

(c) $\frac{9}{10}$ (d) $\frac{8}{5}$

Ans. (c)

Solution: $e_1 = \sqrt{\frac{25+11}{25}} = \frac{6}{5}$, $e_2 = \sqrt{\frac{16-7}{16}} = \frac{3}{4}$.

$$\Rightarrow e_1e_2 = \frac{6}{5} \times \frac{3}{4} = \frac{9}{10}$$

Example 2: If $x - \sqrt{5}\, y + c = 0$ is a tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$, then the value of c is

(a) $\pm 3\sqrt{5}$ (b) $\pm\sqrt{5}$

(c) $\pm 2\sqrt{5}$ (d) none of these

Ans. (b)

Solution: Equation of the tangent is $y = \frac{1}{\sqrt{5}}x + \frac{c}{\sqrt{5}}$

so $\left(\frac{c}{\sqrt{5}}\right)^2 = 25\left(\frac{1}{\sqrt{5}}\right)^2 - 4 \Rightarrow c = \pm\sqrt{5}$

Example 3: If the tangent at the point $P(a \sec\theta, b \tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point where a directrix of the hyperbola meets the positive side of the transverse axis, then θ is equal to

(a) $\cos^{-1}(1/e)$ (b) $\tan^{-1}(1/e)$
(c) $\cot^{-1}(1/e)$ (d) $\sec^{-1}(1/e)$

Where e is the eccentricity of the hyperbola.

Ans. (a)

Solution: Equation of the tangent at $P(a\sec\theta, b\tan\theta)$ to the hyperbola is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \qquad (1)$$

which passes through the point of intersections of the directrix $x = \frac{a}{e}$ and $y = 0$, the transverse axis of the hyperbola.

i.e. $\left(\frac{a}{e}, 0\right)$. So from (1) we have

$$\frac{a}{a \times e}\sec\theta = 1 \Rightarrow \sec\theta = e \Rightarrow \cos\theta = \frac{1}{e}$$

$\Rightarrow \theta = \cos^{-1}(1/e)$.

Example 4: The circle described on the line joining the foci of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ as a diameter passes through an end of the latus rectum of the parabola $y^2 = 4ax$, the length of the latus rectum of the parabola is

(a) $2\sqrt{5}$ units (b) 5 units
(c) $4\sqrt{5}$ units (d) $5\sqrt{5}$ units

Ans. (c)

Solution: Eccentricity of the hyperbola is given by $e = \sqrt{\frac{16+9}{16}} = \frac{5}{4}$.

Coordinates of the foci are $\left(\pm 4 \times \frac{5}{4}, 0\right) = (\pm 5, 0)$.

So the circle described on the line joining the foci is $x^2 + y^2 = 25$.

If it passes through an end $(a, \pm 2a)$ of the latus rectum of the parabola $y^2 = 4ax$ then $a^2 + 4a^2 = 25 \Rightarrow a^2 = 5 \Rightarrow a = \sqrt{5}$.

Length of the latus rectum $= 4a = 4\sqrt{5}$.

Example 5: The curve represented by $x = 5\left(t + \frac{1}{t}\right)$, $y = \left(t - \frac{1}{t}\right)$, $t \neq 0$ is

(a) a point of straight lines
(b) an ellipse
(c) a hyperbola
(d) a rectangular hyperbola.

Ans. (c)

Solution: $\left(\frac{x}{5}\right)^2 - y^2 = 4$

$$\Rightarrow \frac{x^2}{100} - \frac{y^2}{4} = 1$$

which is a hyperbola.

Example 6: If the distance between two directrices of a rectangular hyperbola is 15, then the distance between its foci in units is:

(a) $15\sqrt{2}$ (b) 30
(c) 60 (d) 45

Ans. We know that the eccentricity of a rectangular hyperbola is $\sqrt{2} = e$. Distance between the directricess $= \frac{2a}{e}$, where $2a$ is the length of the transverse axis.

So $\frac{2a}{e} = 15 \Rightarrow 2a = 15\sqrt{2}$

Distance between the foci $= 2ae = 15\sqrt{2} \times \sqrt{2} = 30$.

Example 7: If two perpendicular tangents can be drawn from the point (α, β) to the hyperbola $x^2 - y^2 = a^2$, then (α, β) lies on

(a) $y = \pm x$ (b) $x^2 + y^2 = a^2$
(c) $x^2 + y^2 = 2a^2$ (d) $y^2 = 4ax$

Ans. (a)

Solution: An equation of a tangent to the hyperbola $x^2 - y^2 = a^2$ is $y = mx + a\sqrt{m^2 - 1}$ (1)

Equation of the tangent perpendicular to (1) is

$$y = -\frac{1}{m}x + a\sqrt{\frac{1}{m^2} - 1}$$

$$\Rightarrow my + x = a\sqrt{1 - m^2} \qquad (2)$$

We must have $m^2 - 1 \geq 0$ and $1 - m^2 \geq 0$

$\Rightarrow m^2 = 1 \Rightarrow m = \pm 1$ and thus (α, β) lies on $y = \pm x$.

Example 8: The tangent at an extremity (in the first quadrant) of latus rectum of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ meets x-axis and y-axis at A and B respectively. Then $(OA)^2 - (OB)^2$, where O is the origin, is equal to

(a) $-\frac{20}{9}$ (b) $\frac{16}{9}$
(c) 4 (d) $-\frac{4}{3}$

Ans. (a)

◎ **Solution:** Coordinates of the extremity of latus rectum in the first quadrant is $P\left(ae, \frac{b^2}{a}\right) = \left(3, \frac{5}{2}\right)$. Equation of the tangent at P is $\frac{3x}{4} - \frac{5}{2}\left(\frac{y}{5}\right) = 1$ or $3x - 2y = 4$.

which meets x-axis at $A\left(\frac{4}{3}, 0\right)$ and y-axis at $B(0, -2)$.

So $(OA)^2 - (OB)^2 = \frac{16}{9} - 4 = -\frac{20}{9}$.

Example 9: If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then eccentricity e of the hyperbola satisfies.

(a) $e = \frac{2}{\sqrt{3}}$ (b) $e = \frac{\sqrt{3}}{2}$

(c) $e > \frac{2}{\sqrt{3}}$ (d) $1 < e < \frac{2}{\sqrt{3}}$

Ans. (c)

◎ **Solution:** As $\angle POQ = 60°$. OP makes an angle of 30° with the positive side of x-axis.

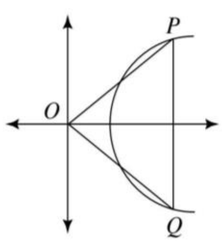


Fig. 20.3

Equation of OP is $y = \frac{1}{\sqrt{3}}x$, which meets the hyperbola at points for which

$$\frac{3y^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow (3b^2 - a^2)y^2 = a^2b^2$$

For real values of y, $3b^2 > a^2$

$$\Rightarrow 3a^2(e^2 - 1) > a^2 \Rightarrow 3e^2 > 4 \Rightarrow e > \frac{2}{\sqrt{3}}.$$

Example 10: If the tangent and normal to the hyperbola $x^2 - y^2 = 4$ at a point cut off intercepts a_1 and a_2 respectively on the x-axis, and b_1 and b_2 respectively on the y-axis, then the value of $a_1a_2 + b_1b_2$ is

(a) –1 (b) 0

(c) 4 (d) 1

Ans. (b)

◎ **Solution:** Let the coordinates of a point P on the hyperbola be (x_1, y_1). Equation of the tangent at P to the hyperbola is

$$xx_1 - yy_1 = 4 \Rightarrow a_1 = \frac{4}{x_1}, b_1 = -\frac{4}{y_1}.$$

Equation of the normal is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

$$y_1x + x_1y = 2x_1y_1$$

$\Rightarrow$ $a_2 = 2x_1, b_2 = 2y_1$

So that $a_1a_2 + b_1b_2 = 0$.

Example 11: The rectangular hyperbola $xy = 16$ and the circle $x^2 + y^2 = 32$ meet at a point P in the first quadrant. Equation of the common tangent to two curves at P is

(a) $x + y - 4 = 0$ (b) $x + y + 4 = 0$

(c) $x + y - 8 = 0$ (d) $x + y + 8 = 0$

Ans. (c)

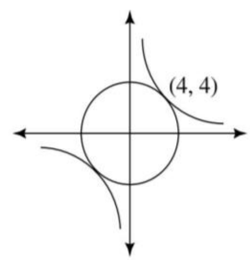


Fig. 20.4

◎ **Solution:** Any point on the hyperbola $xy = 16$ is $\left(4t, \frac{4}{t}\right)$ which meets the circle $x^2 + y^2 = 32$.

If $16t^2 + \frac{16}{t^2} = 32$

$\Rightarrow t^4 - 2t^2 + 1 = 0 \Rightarrow t^2 = 1, t = \pm 1$

Curves meet at the point (4, 4) in the first quadrant,

Equation of the tangent at (4, 4) to both the curves is $4x + 4y = 32$ or $x + y = 8$.

Example 12: If the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes through the focus of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then eccentricity of the hyperbola is

(a) $\sqrt{2}$ (b) $\sqrt{3}$

(c) 3/2 (d) $\sqrt{3/2}$

Ans. (b)

◎ **Solution:** Let e_1 and e_2 be the eccentricities of the ellipse and hyperbola respectively. Since the hyperbola passes through the focus

$$(\pm ae_1, 0) \Rightarrow b = \pm ae_1$$

For the ellipse $b^2 = a^2(1 - e_1^2) \Rightarrow e_1 = 1/\sqrt{2}$

For the hyperbola $a^2 = b^2(e_2^2 - 1) \Rightarrow e_2 = \sqrt{3}$.

Example 13: Which one of the following is independent of α $(0 < \alpha < \pi/2)$ for the hyperbola $\dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = 1$

(a) eccentricity (b) equation of a directrix
(c) abscissa of foci (d) abscissa of vertices

Ans. (c)

◎ **Solution:** Eccentricity

$$= \sqrt{\frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}} = \sqrt{\sec^2\alpha}$$

Equation of the directrix is $x = \pm\dfrac{\cos\alpha}{\sec\alpha} = \pm\cos^3\alpha$

Abscissa of foci = $\cos\alpha\,(\sec\alpha) = 1$
which is independent of α
abscissa of vertices = $\pm\cos\alpha$.

Example 14: If e_1 and e_2 are the eccentricities of the hyperbolas $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$ and $\dfrac{y^2}{25} - \dfrac{x^2}{16} = 1$ respectively, then $\dfrac{1}{e_1^2} + \dfrac{1}{e_2^2}$ is equal to

(a) $\dfrac{5}{4}$ (b) $\dfrac{4}{5}$
(c) 1 (d) $\dfrac{1}{2}$

Ans. (c)

◎ **Solution:** $e_1^2 = \dfrac{25+16}{25}$, $e_2^2 = \dfrac{25+16}{16}$

$$\Rightarrow \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

Example 15: The normal at $P(x_1, y_1)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meets the coordinate axes at A and B. If O, is the origin and e, the eccentricity of the hyperbola, then

(a) $OA = e^2x_1$ (b) $OB = e^2y_1$
(c) $OA = e^2y_1$ (d) $OB = e^2x_1$

Ans. (a)

◎ **Solution:** Equation of the normal at $P(x_1, y_1)$ to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$\frac{x - x_1}{\frac{x_1}{a^2}} + \frac{y - y_1}{\frac{y_1}{b^2}} = 0.$$

$$\Rightarrow OA = b^2\left(\frac{x_1}{a^2}\right) + x_1 = \frac{a^2 + b^2}{a^2}x_1 = e^2x_1$$

$$OB = a^2\left(\frac{y_1}{b^2}\right) + y_1 = \frac{a^2 + b^2}{b^2}y_1 = \frac{a^2e^2}{b^2}y_1$$

LEVEL 1

Straight Objective Type Questions

Example 16: If the latus rectum of a hyperbola subtend an angle of 60° at the other focus, then eccentricity of the hyperbola is

(a) 2 (b) $\dfrac{\sqrt{3}+1}{2}$
(c) $2\sqrt{3}$ (d) $\sqrt{3}$

Ans. (d)

◎ **Solution:** Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$F_1(-ae, 0)$, $F_2(ae, 0)$ be the foci, e being the eccentricity. Let LF_2L', the latus rectum subtend an angle of 60° at F_1, then slope of LF_1 is tan 30°.

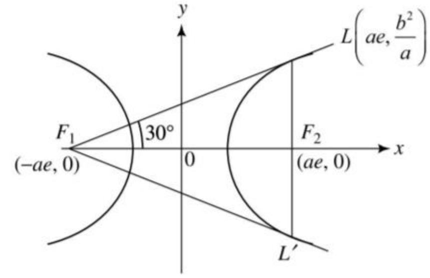


Fig. 20.5

So $\tan 30° = \dfrac{b^2/a}{2ae}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{a^2(e^2-1)}{2a^2e} = \frac{e^2-1}{2e}$$

$$\Rightarrow \quad e^2 - 1 = \frac{2}{\sqrt{3}}e \Rightarrow \left(e - \frac{1}{\sqrt{3}}\right)^2 = 1 + \frac{1}{3}$$

$$\Rightarrow \quad e - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \Rightarrow e = \sqrt{3}$$

Example 17: The locus of the foot of the perpendicular drawn from the origin to any tangent to the hyperbola $\dfrac{x^2}{36} - \dfrac{y^2}{16} = 1$ is

(a) $(x^2 + y^2)^2 = 36x^2 - 16y^2$
(b) $(x^2 - y^2)^2 = 36x^2 - 16y^2$
(c) $(x^2 + y^2)^2 = 36x^2 + 16y^2$
(d) $(x^2 - y^2)^2 = 36x^2 + 16y^2$

Ans. (a)

Solution: Any tangent to the hyperbola is

$$y = mx + \sqrt{36m^2 - 16} \quad (1)$$

Equation of the line through the origin perpendicular to the tangent is

$$y = -\frac{1}{m}x. \quad (2)$$

Eliminating m from (1) and (2) we get the required locus as

$$y = -\frac{x^2}{y} + \sqrt{36\frac{x^2}{y^2} - 16}$$

$\Rightarrow \quad (x^2 + y^2)^2 = 36x^2 - 16y^2.$

Example 18: The locus of the middle points of the portions of the tangents of the hyperbola.

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ included between the axes is

(a) $a^2x^2 - b^2y^2 = x^2y^2$
(b) $b^2x^2 - a^2y^2 = x^2y^2$
(c) $b^2x^2 - a^2y^2 + 4x^2y^2 = 0$
(d) $a^2x^2 - b^2y^2 + 4x^2y^2 = 0$

Ans. (c)

Solution: Equation of a tangent to the hyperbola is $\dfrac{x}{a}\sec\theta - \dfrac{y}{b}\tan\theta = 1$ which meets the axes at A $(a\cos\theta, 0)$ and B $(0, -b\cot\theta)$.

If (h, k) is the midpoint of AB then $h = \dfrac{a\cos\theta}{2}, k = \dfrac{-b\cot\theta}{2}$.

Eliminating θ, we get the required locus

$$\left(\frac{a}{2h}\right)^2 - \left(\frac{-b}{2k}\right)^2 = 1$$

$\Rightarrow b^2x^2 - a^2y^2 + 4x^2y^2 = 0$

Example 19: If the slope of a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $2\sqrt{2}$, then the eccentricity e of the hyperbola lies in the interval.

(a) $(1, \sqrt{2}]$ (b) $(1, 2\sqrt{2}]$
(c) $(1, 3]$ (d) $(1, 4]$

Ans. (c)

Solution: Any tangent to the hyperbola is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

we must have $a^2m^2 - b^2 \geq 0$.

$$\Rightarrow \quad m^2 \geq \frac{b^2}{a^2}$$

$$\Rightarrow \quad m^2 \geq e^2 - 1$$

$$\Rightarrow \quad 1 < e \leq \sqrt{1 + m^2}$$

$\Rightarrow \quad 1 < e \leq 3$ when $m = 2\sqrt{2}$.

Example 20: If $\dfrac{x^2}{\lambda + 3} + \dfrac{y^2}{2 - \lambda} = 1$ represents a hyperbola, then

(a) $\lambda \notin (2, 3)$ (b) $\lambda \notin (-2, 3)$
(c) $\lambda \notin (-3, 2)$ (d) $\lambda \notin (2, \infty)$

Ans. (c)

Solution: $\lambda + 3$ and $2 - \lambda$ must be of opposite signs

$\Rightarrow \lambda + 3 > 0$ and $2 - \lambda < 0$ or $\lambda + 3 < 0$ and $2 - \lambda > 0$

$\Rightarrow \lambda > -3$ and $\lambda > 2$ or $\lambda < -3$ and $\lambda < 2$

$\Rightarrow \lambda > 2$ and $\lambda < -3 \Rightarrow \lambda \notin (-3, 2)$

Example 21: The normal to the curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is

(a) ellipse (b) parabola
(c) circle (d) hyperbola or ellipse

Ans. (d)

Solution: Equation of the normal at (x, y) is $Y - y = -\dfrac{dx}{dy}(X - x)$ which meets the x-axis at $G\left(0, x + y\dfrac{dy}{dx}\right)$, then

$$x + y\frac{dy}{dx} = \pm 2x$$

$$\Rightarrow x + y\frac{dy}{dx} = 2x \Rightarrow y\,dy - x\,dx = 0 \Rightarrow x^2 - y^2 = c$$

or $y\,dy = -3x\,dx \Rightarrow 3x^2 + y^2 = c$

Thus the curve is either hyperbola or ellipse.

Example 22: If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$ $R(x_3, y_3)$, $S(x_4, y_4)$, then which of the following need not hold.

(a) $x_1 + x_2 + x_3 + x_4 = 0$
(b) $x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = c^4$
(c) $y_1 + y_2 + y_3 + y_4 = 0$
(d) $x_1 + y_2 + x_3 + y_4 = 0$

Ans. (d)

Solution: The abcissaes of the points of intersection of the given curves are connected by

$$x^2 + \frac{c^4}{x^2} = a^2 \quad \Rightarrow \quad x^4 - a^2x^2 + c^4 = 0$$

As x_1, x_2, x_3, x_4 are the roots of this equation

$x_1 + x_2 + x_3 + x_4 = 0, x_1 x_2 x_3 x_4 = c^4$

Similarly $y_1 + y_2 + y_3 + y_4 = 0, y_1 y_2 y_3 y_4 = c^4$

Example 23: If the normal to the rectangular hyperbola $xy = c^2$ at the point $(ct, c/t)$ meets the curve again at $(ct', c/t')$, then

(a) $t^3t' = 1$ (b) $t^3 t' = -1$
(c) $t t' = 1$ (d) $t t' = -1$

Ans. (b)

Solution: Equation of the tangent at $(ct, c/t)$ to the hyperbola $xy = c^2$ is

$$\frac{cx}{t} + cty = 2c^2$$

Slope of the tangent $= -1/t^2$ and slope of the normal $= t^2$

Equation of the normal at $(ct, c/t)$ is $y - c/t = t^2 (x - ct)$

If it passes through $(ct', c/t')$, then

$$\left(\frac{c}{t'} - \frac{c}{t}\right) = t^2 (ct' - ct)$$

$$\Rightarrow \quad (t - t') = t^3 t'(t' - t) \Rightarrow t^3t' = -1$$

Example 24: If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes of x and y in G and g respectively and C is the centre of the hyperbola, then $2PC =$

(a) PG (b) Pg
(c) Gg (d) none of these

Ans. (c)

Solution: Let $P(x_1, y_1)$ be any point on the hyperbola $x^2 - y^2 = 4$ then equation of the normal at P is

$$y - y_1 = \frac{-y_1}{x_1}(x - x_1)$$

$$\Rightarrow \quad x_1y + y_1x = 2x_1y_1.$$

Then coordinates of G are $(2x_1, 0)$ and of g are $(0, 2y_1)$

so that $PG = \sqrt{(2x_1 - x_1)^2 + y_1^2} = \sqrt{x_1^2 + y_1^2} = PC$

$Pg = \sqrt{x_1^2 + (2y_1 - y_1)^2} = \sqrt{x_1^2 + y_1^2} = PC$

and $Gg = \sqrt{(2x_1)^2 + (2y_1)^2} = 2\sqrt{x_1^2 + y_1^2} = 2\,PC$

Example 25: If e_1, e_2 are the eccentricities of the hyperbola $2x^2 - 2y^2 = 1$ and the ellipse $x^2 + 2y^2 = 2$ respectively. Then

(a) $e_1 + e_2 = 1$ (b) $e_1e_2 = 1$
(c) $e_1^2 + e_2^2 = 1$ (d) none of these

Ans. (b)

Solution: Equation of the hyperbola is

$$\frac{x^2}{1/2} - \frac{y^2}{1/2} = 1$$

So $\quad e_1^2 = \dfrac{a^2 + b^2}{a^2}$ where $a^2 = b^2 = \dfrac{1}{2}$

$\Rightarrow \quad e_1 = \sqrt{2}$ (Note: It is a rectangular hyperbola)

Equation of the ellipse is

$$\frac{x^2}{2} + \frac{y^2}{1} = 1$$

So $\quad e_2^2 = \dfrac{a^2 - b^2}{a^2} = \dfrac{2-1}{2} = \dfrac{1}{2}$

$\Rightarrow \quad e_2 = 1/\sqrt{2}$ hence $e_1e_2 = 1$.

Example 26: The line $2x + y = 1$ touches a hyperbola and passes through the point of intersection of a directrix and the x-axis. The equation of the hyperbola is

(a) $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$ (b) $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 2$

(c) $\dfrac{x^2}{3} - \dfrac{y^2}{1} = 1$ (d) $\dfrac{x^2}{3} - \dfrac{y^2}{1} = 2$

Ans. (a)

Solution: Let the equation of the hyperbola be $x^2/a^2 - y^2/b^2 = 1$. As the line $2x + y = 1$ touches the hyperbola $(2)^2 a^2 - b^2 = 1$

$\Rightarrow \quad 4a^2 - b^2 = 1$

$\Rightarrow \quad 4a^2 + a^2 (1 - e^2) = 1$, e being the eccentricity.

$\Rightarrow \quad 5a^2 - a^2e^2 = 1$

As the line passes through $(a/e, 0)$.

$2\,a/e + 0 = 1 \Rightarrow a = e/2.$

So that $5\dfrac{e^2}{4} - \dfrac{e^4}{4} = 1$

$\Rightarrow \quad e^4 - 5e^2 + 4 = 0$

$\Rightarrow \quad e^2 = 1$ or $e^2 = 4. \Rightarrow e = 2$ as $e^2 \neq 1$

Hence $a = 1$ and $b^2 = 3$ and the required equation is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1.$$

Example 27: The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by

(a) $x^2 - 3y^2 = 3$ (b) $3x^2 - y^2 = 3$
(c) $-x^2 + 3y^2 = 3$ (d) $-3x^2 + y^2 = 3$

Ans. (b)

Solution: Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$ae = 2 \text{ and } e = 2 \Rightarrow a = 1$$

$$b^2 = a^2(e^2 - 1) = 1(4 - 1) = 3$$

so the equation of the hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1$$

or $3x^2 - y^2 = 3$

Example 28: The eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

(a) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(b) a focus of the hyperbola is $(\sqrt{3}, 0)$

(c) the eccentricity of the hyperbola is $\sqrt{5/3}$

(d) the equation of the hyperbola is $x^2 - 3y^2 = 3$

Ans. (d)

Solution: Eccentricity of the ellipse $= \sqrt{1 - \frac{1}{4}} = \sqrt{3}/2$

Foci of the ellipse $= (\pm\sqrt{3}, 0)$

Eccentricity of the hyperbola $= \sqrt{1 + \frac{b^2}{a^2}} = \frac{2}{\sqrt{3}}$.

$\Rightarrow$ $\frac{b}{a} = \frac{1}{\sqrt{3}}$

Since the hyperbola passes through the focus of the ellipse.

$$\frac{3}{a^2} - 0 = 1 \Rightarrow a^2 = 3 \text{ and } b^2 = 1$$

and the equation of the hyperbola is

$$\frac{x^2}{3} - \frac{y^2}{1} = 1 \text{ or } x^2 - 3y^2 = 3$$

Focus of the hyperbola is $(\pm 2, 0)$.

Example 29: Let P (6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0) then the eccentricity of the hyperbola is

(a) $\sqrt{5/2}$ (b) $\sqrt{3/2}$

(c) $\sqrt{2}$ (d) $\sqrt{3}$

Ans. (b)

Solution: The equation of the normal at $P(6, 3)$

$$\frac{a^2 x}{6} + \frac{b^2 y}{3} = a^2 + b^2$$

As it passes through (9, 0)

$$\frac{9a^2}{6} = a^2 + b^2 \Rightarrow a^2 = 2b^2$$

Eccentricity of the hyperbola $= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{3}{2}}$

Example 30: If both the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is

(a) 9 (b) 7
(c) 41 (d) 12

Ans. (b)

Solution: Foci of the ellipse are $\left(\pm\sqrt{16 - b^2}, 0\right)$

and Foci of the hyperbola are $\left(\pm\sqrt{\frac{144}{25} + \frac{81}{25}}, 0\right) = (\pm 3, 0)$

Since they coincide

$$16 - b^2 = 9 \Rightarrow b^2 = 7.$$

Example 31: If the tangent at the point $(a \sec\alpha, b \tan\alpha)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at T, then the distance of T from a focus of the hyperbola is

(a) $a(e - \cos\alpha)$ (b) $b(e + \cos\alpha)$

(c) $a(e + \cos\alpha)$ (d) $\sqrt{a^2e^2 + b^2\cot^2\alpha}$

Ans. (a)

Solution: Equation of the tangent is $\frac{x}{a}\sec\alpha - \frac{y}{b}\tan\alpha = 1$

Coordinates of T are $(a\cos\alpha, 0)$ and of a focus are $(ae, 0)$.
Distance of T from the focus $(ae - a\cos\alpha)$
$= a(e - \cos\alpha)$

(Note $e > \cos\alpha$).

Example 32: The distance between the tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$, parallel to the line $y = x + 2$ is

(a) 2 (b) $2\sqrt{2}$

(c) $\sqrt{2}$ (d) 1

Ans. (c)

◎ **Solution:** Equations of the tangents are

$y = mx \pm \sqrt{4m^2 - 3}$ where $m = 1$

$$\Rightarrow \quad y = x \pm 1$$

Distance between them $= \dfrac{2}{\sqrt{1+1}} = \sqrt{2}$

⦿ **Example 33:** The locus of the middle points of the normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is

(a) $(y^2 - x^2)^3 = 4a^2x^2y^2$
(b) $(y^2 - x^2)^2 = 4a^2x^2y^2$
(c) $(y^2 + x^2)^3 = 4a^2x^2y^2$
(d) $(y^2 + x^2)^2 = 4a^2x^2y^2$

Ans. (a)

◎ **Solution:** Equation of the normal at $(a\sec\theta, a\tan\theta)$ to the hyperbola $x^2 - y^2 = a^2$ is

$$\frac{ax}{\sec\theta} + \frac{ay}{\tan\theta} = a^2 + a^2 = 2a^2 \qquad (1)$$

Let (h, k) be the middle point of this normal then its equation is

$$hx - ky - a^2 = h^2 - k^2 - a^2 \qquad (T = S_1)$$

$$\Rightarrow \quad hx - ky = h^2 - k^2 \qquad (2)$$

Comparing (1) and (2) we get

$$h\sec\theta = -k\tan\theta = \frac{h^2 - k^2}{2a}$$

$$\Rightarrow \quad \sec\theta = \frac{h^2 - k^2}{2ah},\ \tan\theta = \frac{h^2 - k^2}{-2ak}$$

So $$1 = \sec^2\theta - \tan^2\theta = \frac{(h^2 - k^2)^2}{4a^2}\left(\frac{1}{h^2} - \frac{1}{k^2}\right)$$

$$\Rightarrow \quad (h^2 - k^2)^3 + 4a^2h^2k^2 = 0$$

Locus of (h, k) is $(y^2 - x^2)^3 = 4a^2x^2y^2$

⦿ **Example 34:** If $y = mx + 6$ is a tangent to the hyperbola $\dfrac{x^2}{100} - \dfrac{y^2}{49} = 1$ and the parabola $y^2 = 4ax$, then the length of the latus rectum of the parabola is

(a) $6\sqrt{\dfrac{17}{20}}$ (b) $4\sqrt{\dfrac{17}{20}}$

(c) $24\sqrt{\dfrac{17}{20}}$ (d) $\sqrt{\dfrac{17}{20}}$

Ans. (c)

◎ **Solution:** Since $y = mx + 6$ is a tangent to the hyperbola, $(6)^2 = 100m^2 - 49$.

$$\Rightarrow \quad m^2 = \frac{17}{20} \Rightarrow m = \pm\sqrt{\frac{17}{20}}$$

Since $y = mx + 6$ is a tangent to the parabola

$$y^2 = 4ax;\ 6 = \frac{a}{m} \Rightarrow a = 6m = \pm 6\sqrt{\frac{17}{20}}$$

Length of the latus rectum of the parabola

$$= 4|a| = 24\sqrt{\frac{17}{20}}.$$

⦿ **Example 35:** P is a point on the hyperbola $\dfrac{x^2}{81} - \dfrac{y^2}{49} = 1.$ The tangent at P meets the transverse axis at T, N is the foot of the perpendicular from P to the transverse axis. If O is the origin, then $ON.OT$ is equal to.

(a) 81 (b) 49

(c) –81 (d) – 49.

Ans. (a)

◎ **Solution:** Let the coordinate of P be $(9\sec\theta, 7\tan\theta)$ then $ON = 9\sec\theta$.

Equation of the tangent at P is $\dfrac{x}{9}\sec\theta - \dfrac{y}{7}\tan\theta = 1$ which meets the transverse axis at $(9\cos\theta, 0) \Rightarrow OT = 9\cos\theta$.

Then $ON.OT = 9\sec\theta \times 9\cos\theta = 81$

⦿ **Example 36:** The product of the perpendiculars from the foci on any tangent to the hyperbola $\dfrac{x^2}{64} - \dfrac{y^2}{9} = 1$ is

(a) 8 (b) 9

(c) 16 (d) 18.

Ans. (b)

◎ **Solution:** Equation of a tangent at $(8\sec\theta, 3\tan\theta)$ to the hyperbola is

$$\frac{x}{8}\sec\theta - \frac{y}{3}\tan\theta = 1$$

If e is the eccentricity of the hyperbola; product of the perpendiculars

$$= \frac{(e\sec\theta - 1)(-e\sec\theta - 1)}{\dfrac{\sec^2\theta}{64} + \dfrac{\tan^2\theta}{9}}$$

$$= \frac{(1 - e^2\sec^2\theta) \times (64 \times 9)}{9\sec^2\theta + 64\tan^2\theta}$$

$$= \frac{\left[1 - \dfrac{64 - 9}{64}\sec^2\theta\right](64 \times 9)}{9\sec^2\theta + 64\tan^2\theta}$$

$$= \frac{64\tan^2\theta + 9\sec^2\theta}{9\sec^2\theta + 64\tan^2\theta} \times 9 = 9.$$

Example 37: If the normal at P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the transverse axis at G, S is a foci and e the eccentricity of the hyperbola then $SG : SP$ is equal to

(a) a (b) b
(c) e (d) $1/e$.

Ans. (c)

Solution: Equation of the normal at $P(a\sec\theta, b\tan\theta)$ is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

Which meets the transverse axis at

$G\left(\frac{(a^2+b^2)\sec\theta}{a},0\right)$ whose distance from S $(ae, 0)$ is

$$SG=\frac{(a^2+b^2)\sec\theta}{a}-ae$$

$$SP=e\left[a\sec\theta-\frac{a}{e}\right]=a[e\sec\theta-1]$$

$$\frac{SG}{SP}=\frac{[a^2+a^2(e^2-1)]\sec\theta-a^2e}{a\times a[e\sec\theta-1]}$$

$$=\frac{e^2\sec\theta-e}{e\sec\theta-1}=e.$$

Example 38: If the chords of contacts of the tangents from the points (x_1, y_1) and (x_2, y_2) to the hyperbola $2x^2-3y^2=6$ are at right angle, then $4x_1x_2+9y_1y_2$ is equal to

(a) -1 (b) 0
(c) 6 (d) -12

Ans. (b)

Solution: Equation of the chords of contacts are

$$2xx_1-3yy_1=6,\ 2xx_2-3yy_2=6$$

Since they are at right angles $\frac{2x_1}{3y_1}\times\frac{2x_2}{3y_2}=-1$

$$\Rightarrow\quad 4x_1x_2+9y_1y_2=0.$$

Example 39: Consider a branch of the hyperbola $x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$ with vertex at the point A. let B be one of the end points of its latus rectum. If C is the focus of the hyperbola near A, then area of the ΔABC is:

(a) $1-\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}}-1$
(c) $1+\sqrt{\frac{2}{3}}$ (d) $\sqrt{\frac{3}{2}}+1$.

Ans. (b)

Solution: Equation of the hyperbola is

$$\frac{(x-\sqrt{2})^2}{4}-\frac{(y+\sqrt{2})^2}{2}=1$$

Which can be written as

$\frac{X^2}{4}-\frac{Y^2}{2}=1$ where $x=X+\sqrt{2}$, $y=Y-\sqrt{2}$.

Eccentricity of the hyperbola is $\sqrt{\frac{4+2}{4}}=\sqrt{\frac{3}{2}}$

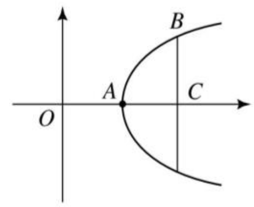


Fig. 20.6

Area of the $\Delta ABC=\frac{1}{2}(AC)(BC)$

$$=\frac{1}{2}(ae-a)\times\frac{b^2}{a}\quad\text{where } a^2=4, b^2=2$$

$$=\frac{1}{2}\left[2\sqrt{\frac{3}{2}}-2\right]\times\frac{2}{2}=\sqrt{\frac{3}{2}}-1$$

Example 40: Normal at point (5, 3) to the rectangular hyperbola $xy-y-2x-2=0$ meets the curve at the point whose coordinates are

(a) $(0, -2)$ (b) $(-1, 0)$
(c) $\left(\frac{1}{4},\frac{-10}{3}\right)$ (d) $\left(\frac{3}{4},-14\right)$.

Ans. (d)

Solution: Equation of the hyperbola can be written as $(x-1)(y-2)=4$

or $$y=2+\frac{4}{x-1}$$

$$\frac{dy}{dx}=-\frac{4}{(x-1)^2}$$

Any point on the hyperbola is

$$\left(1+2t,2+\frac{2}{t}\right)=(5,3)\Rightarrow t=2$$

So slope of the normal at (5, 3) is

$$-\left.\frac{dx}{dy}\right|_{(5,3)}=4$$

Equation of the normal at (5, 3) is $y-3=4(x-5)$ which will pass through $\left(1+2t,2+\frac{2}{t}\right)$ if $2+\frac{2}{t}=4(1+2t)-17$.

$$\Rightarrow 8t^2-15t-2=0\Rightarrow t=2,\ -\frac{1}{8}.$$

For $t=-\frac{1}{8}$, we get the required point as

$$\left(1-\frac{2}{8},2-\frac{2}{1/8}\right)=\left(\frac{3}{4},-14\right)$$

Assertion-Reason Type Questions

Example 41: Statement-1 An ellipse passes through the foci of the hyperbola $9x^2 - 4y^2 = 36$ and its major and minor axes lie along the transverse and conjugate axes respectively of the hyperbola. If the product of the eccentricities of the two conics is $\frac{1}{2}$, then the ellipse does not pass through the point $\left(\frac{1}{2}\sqrt{i3}, \frac{\sqrt{3}}{2}\right)$.

Statement-2 Product of the eccentricities of the hyperbola $9x^2 - 4y^2 = 36$ and the ellipse $\frac{x^2}{65} - \frac{y^2}{60} = 1$ is $\frac{1}{2}$.

Ans. (b)

Solution: Eccentricity e_1 of the hyperbola $9x^2 - 4y^2 = 36$ or $\frac{x^2}{4} - \frac{y^2}{9} = 36$ is $\sqrt{\frac{4+9}{4}} = \frac{\sqrt{13}}{2}$.

Foci of the hyperbola $\left(\pm\sqrt{13}, 0\right)$

If e_2 is the eccentricity of the ellipse then

$$e_1 e_2 = \frac{1}{2} \Rightarrow e_2 = \frac{1}{\sqrt{13}}.$$

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through $\left(\pm\sqrt{13}, 0\right)$

$$\frac{13}{a^2} = 1 \Rightarrow a^2 = 13 \text{ and } b^2 = a^2(1 - e_2^2) = 12$$

Equation of the ellipse is $\frac{x^2}{13} + \frac{y^2}{12} = 1$

Which clearly does not pass through $\left(\frac{1}{2}\sqrt{13}, \frac{\sqrt{3}}{2}\right)$

So statement-1 is true.

In statement-2 if e_3 is the eccentricity of the ellipse, then

$$e_3 = \sqrt{\frac{65-60}{65}} = \frac{1}{\sqrt{13}}.$$

$\Rightarrow e_1 e_3 = \frac{1}{2}$, so statement-2 is also true but does not justify statement-1.

Example 42: Statement-1 If a tangent to the hyperbola $2x^2 - 4y^2 = 8$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle where O is the origin then locus of R is a hyperbola.

Statement-2 The curve described parametrically by $x = \frac{2}{3} + 4t$ and $y = -\frac{4}{7} + \frac{4}{t}$ represents a hyperbola.

Ans. (d)

Solution: Tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ at a point (x_1, y_1) is $\frac{xx_1}{4} - \frac{yy_1}{2} = 1$

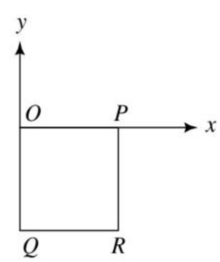


Fig. 20.7

Coordinates of P are $\left(\frac{4}{x_1}, 0\right)$ of Q are $\left(0, \frac{-2}{y_1}\right)$ and of R are $\left(\frac{4}{x_1}, \frac{-2}{y_1}\right) = (h, k)$

Also $\frac{x_1^2}{4} - \frac{y_1^2}{2} = 1$ as (x_1, y_1) lies on the hyperbola

$$\Rightarrow \frac{4}{h^2} - \frac{2}{k^2} = 1$$

Locus of (h, k) is $\frac{4}{x^2} - \frac{2}{y^2} = 1$

Which is not a hyperbola and thus statmnet-1 is false.

In statement-2 we have $\left(x - \frac{2}{3}\right)\left(y + \frac{4}{7}\right) = 16$ which represents a rectangular hyperbola and thus is true.

Example 43: $P(2, 3)$ and $Q(2 \tan\theta, 3 \sec\theta)$; $0 < \theta < \frac{\pi}{2}$ are two given points. R divides the line segment PQ externally in the ratio 2:3.

Statement-1: Locus of R is a hyperbola length of whose transverse axis is 12.
Statement-2: Locus of Q is a hyperbola length of whose transverse axis is 6.
Ans. (b)
◎ **Solution:** If coordinates of R are (x, y) then

$$x = \frac{3(2) - 2(2\tan\theta)}{3-2},\ y = \frac{3(3) - 2(3\sec\theta)}{3-2}$$

$$\Rightarrow \tan\theta = \frac{1}{4}(6 - x),\ \sec\theta = \frac{1}{6}(9 - y)$$

Eliminating θ, we get the locus of R as

$$\frac{1}{36}(y-9)^2 - \frac{1}{16}(x-6)^2 = 1$$

Which is a hyperbola, length of whose transverse axis is $2(6) = 12$
So statement-1 is true.

Locus of $Q(2\tan\theta, 3\sec\theta)$ is $\left(\frac{y}{3}\right)^2 - \left(\frac{x}{2}\right)^2 = 1$

$$\Rightarrow \frac{y^2}{9} - \frac{x^2}{4} = 1$$

Which is a hyperbola, length of whose transverse axis is $2(3) = 6$.
So statement-2 is also true but does not justify statement-1.

◉ **Example 44: Statement-1:** The line $bx - ay = 0$ does not meet the hyperbola $x^2/a^2 - y^2/b^2 = 1$.
Statement-2: The line $y = mx + c$ does not meet the hyperbola $x^2/a^2 - y^2/b^2 = 1$ if $c^2 = a^2m^2 - b^2$.
Ans. (c)
◎ **Solution:** Statement-1 is true, as solving $y = (b/a)x$ and $x^2/a^2 - y^2/b^2 = 1 \Rightarrow 0 = 1$ which is not possible.
Statement-2 is false as the line is a tangent to the hyperbola and hence meets the hyperbola at one point only. In statement-1, the line satisfies the condition of being a tangent, so it must touch the hyperbola at a point infinity.

◉ **Example 45: Statement-1:** The rectangular hyperbola $xy = 4^2$ does not intersect the parabola $y^2 = 4x$ in real point.

Statement-2: If hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes through the focus of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then eccentricity of the hyperbola is $\sqrt{3}$.
Ans. (d)
◎ **Solution:** Statement-1 is false as the point (4, 4) lies on both the curves.

In statmetn-2 if e is the eccentricity of the ellipse, then

$$\frac{a^2e^2}{b^2} = 1 \Rightarrow e^2 = \frac{b^2}{a^2} \Rightarrow \frac{a^2 - b^2}{a^2} = \frac{b^2}{a^2} \Rightarrow a^2 = 2b^2$$

eccentricity of the hyperbola is $\sqrt{\frac{b^2 + a^2}{b^2}} = \sqrt{3}$ and thus the statement-2 is true.

◉ **Example 46 Statement-1:** If a hyperbola does not have mutually perpendicular tangents, then its eccentricity is greater than $\sqrt{2}$.
Statement-2: Locus of a point from which two perpendicular tangents can be drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$.
Ans. (a)
◎ **Solution:** Statement-2 is true as $x^2 + y^2 = a^2 - b^2$ is the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ using it in statement-1, if the hyperbola does not have mutually perpendicular tangents then the points on $x^2 + y^2 = a^2 - b^2$ are not real.

$$\Rightarrow b^2 > a^2 \Rightarrow e^2 = \frac{a^2 - b^2}{a^2} > 2 \Rightarrow e > \sqrt{2}$$ and thus statement-1 is also true.

◉ **Example 47: Statement-1:** The angle between a pair of tangents drawn to the curve $7x^2 - 12y^2 = 84$ from $M(1, 2)$ is $\frac{\pi}{2}$.
Statement-2: Equation of the chord of contact of the point $M(1, 2)$ with respect to the hyperbola $7x^2 - 12y^2 = 84$ is $7x - 24y = 84$.
Ans. (b)
◎ **Solution:** Equation of the hyperbola is $\frac{x^2}{12} - \frac{y^2}{7} = 1$.

Equation of its director circle is $x^2 + y^2 = 12 - 7 = 5$ $M(1, 2)$ lies on it. So the tangents drawn from M to the hyperbola are at right angles. Thus statement-1 is true.
Equation of the chord of contact of $M(1, 2)$ is

$$\frac{x \times 1}{12} - \frac{y(2)}{7} = 1.$$

$\Rightarrow 7x - 24y = 84$. So statement-2 is also true but does not justify statement-1.

◉ **Example 48:** Let H be the rectangular hyperbola $xy = c^2$.
Statement-1: If the sum of the slopes of the normal from a point P to the hyperbola H is $\frac{1}{c^2}$, then P lies on a pair of lines parallel to y-axis.

Statement-2: Four normals can be drawn to the hyperbola H from any point P.

Ans. (c)

◎ **Solution:** Equation of the normal at any point $\left(ct, \frac{c}{t}\right)$ on the hyperbola H is

$$y - \frac{c}{t} = t^2(x - ct) \quad (1)$$

If it passes through $P(c, c)$, then $1 - \frac{1}{t} = t^2(1 - t)$

$\Rightarrow (t - 1)(t^3 + 1) = 0.$

Which gives only two real values of t, 1 and –1.

So from the point P only two normals can be drawn and the statement-2 is false.

Note: Point $P(c, c)$ lies on the hyperbola. If the point P does not lie on the hyperbola, then there will be four normals from P.

For statement-1, let $P(\alpha, \beta)$ be a point such that sum of the slopes of the normals from P to H is $\frac{1}{c^2}$, then from (1)

$$\beta - \frac{c}{t} = t^2(\alpha - ct)$$

$$\Rightarrow \quad ct^4 - \alpha t^3 + \beta t - c = 0 \quad (2)$$

If t_1, t_2, t_3, t_4 are the roots of (2). Then

$$\sum_{i=1}^{4} t_i = \frac{\alpha}{c}, \Sigma t_1 t_2 = 0$$

Sum of the slopes of the normals

$$= \sum_{i=1}^{4} t_i^2 = \left(\sum_{i=1}^{4} ti\right)^2 - 2\Sigma t_1 t_2 = \frac{\alpha^2}{c^2} = \frac{1}{c^2}$$

$\Rightarrow \alpha = \pm 1 \Rightarrow P$ lies on the lines $x = \pm 1$ and thus statement-1 is true.

◉ **Example 49:** H: $\frac{x^2}{4} - \frac{y^2}{16} = 1$ is a hyperbola.

Statement-1 Every line which cuts the hyperbola H at two distinct points has slope lying in (–2, 2).

Statement-2 The slope of the tangents of the hyperbola lies in $(-\infty, -2) \cup (2, \infty)$.

Ans. (a)

◎ **Solution:** Equation of the tangent with slope m of the hyperbola H is $y = mx \pm \sqrt{4m^2 - 16}$.

This will represent a real tangent if $4m^2 - 16 > 0$

$\Rightarrow m^2 > 4 \Rightarrow m \in (-\infty, -2) \cup (2, \infty)$

Showing that statement-2 is true.

Using it in statement-1 if any line cuts the hyperbola H at two distinct points, it is not a tangent to H and hence its slope lies in (–2, 2) thus statement-1 is also true.

◉ **Example 50: Statement-1:** If l is the length of the chord of the hyperbola $x^2 - y^2 = 8$ whose mid-point is (4, 2) then $3l^2 = 80$.

Statement-2: Length of the chord of contact of the point $\left(\frac{8}{3}, \frac{4}{3}\right)$ with respect to the hyperbola $x^2 - y^2 = 8$ is $4\sqrt{\frac{5}{3}}$.

Ans. (b)

◎ **Solution:** Equation of the chord whose mid-point is (4, 2) of the hyperbola $x^2 - y^2 = 8$ is

$$4x - 2y = 4^2 - 2^2 = 12$$

$$\Rightarrow \quad 2x - y = 6$$

Which meets the hyperbola $2x - y = 6$ at points for which.

$$x^2 - (2x - 6)^2 = 8$$

$$\Rightarrow \quad 3x^2 - 24x + 44 = 0$$

Let (x_1, y_1) and (x_2, y_2) be the end points of the chord, then $x_1 + x_2 = 8, x_1x_2 = \frac{44}{3}$.

$$y_1 = 2x_1 - 6, y_2 = 2x_2 - 6$$

$$l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$= (x_2 - x_1)^2 + 4(x_2 - x_1)^2$$

$$= 5[(x_1 + x_2)^2 - 4x_1x_2]$$

$$= 5\left[8^2 - 4\left(\frac{44}{3}\right)\right] = \frac{80}{3}.$$

Thus statement-1 is true.

In statement-2, equation of the chord of contact of $\left(\frac{8}{3}, \frac{4}{3}\right)$ to the hyperbola $x^2 - y^2 = 8$ is $x \times \frac{8}{3} - y \times \frac{4}{3} = 8$

$$\Rightarrow \quad 2x - y = 6.$$

Which is same as the chord in statement-1 and hence the required length is $\sqrt{\frac{80}{3}} = 4\sqrt{\frac{5}{3}}$.

Thus statement-2 is also true but does not justify statement-1.

LEVEL 2

Straight Objective Type Questions

Example 51: If ($a\sec\alpha$, $b\tan\alpha$) and ($a\sec\beta$, $b\tan\beta$) be the ends of a chord of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, passing through the focus (ae, 0) then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ is equal to:

(a) $\frac{1+e}{1-e}$ (b) $\frac{e+1}{e-1}$

(c) $\frac{1-e}{1+e}$ (d) $\frac{e-1}{e+1}$

Ans. (c)

Solution: Equation of the chord joining the given points is

$$\begin{vmatrix} x & y & 1 \\ a\sec\alpha & b\tan\alpha & 1 \\ a\sec\beta & b\tan\beta & 1 \end{vmatrix} = 0$$

$bx(\tan\alpha - \tan\beta) - ay(\sec\alpha - \sec\beta) + ab(\sec\alpha\tan\beta - \sec\beta\tan\alpha) = 0$

$\Rightarrow bx\sin(\alpha-\beta) - ay(\cos\beta - \cos\alpha) + ab(\sin\beta - \sin\alpha) = 0$

$$\Rightarrow bx\cos\frac{\alpha-\beta}{2} - ay\sin\frac{\alpha+\beta}{2} - ab\cos\frac{\alpha+\beta}{2} = 0$$

$$\Rightarrow \frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

If it passes through ($0e$, 0) $e\cos\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2}$

$$\Rightarrow \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = e$$

$$\Rightarrow \frac{\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha-\beta}{2}} = \frac{e-1}{e+1}$$

$$\Rightarrow -\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{e-1}{e+1}$$

$$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}$$

Example 52: Equations of the asymptotes of the hyperbola $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$ is

(a) $3x^2 + 10xy + 8y^2 + 14x + 22y + 11 = 0$
(b) $3x^2 + 10xy + 8y^2 - 14x - 22y + 13 = 0$
(c) $3x^2 - 10xy + 8y^2 + 14x + 22y + 15 = 0$
(d) $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$

Ans. (d)

Solution: Since the equation of a hyperbola differs from the combined equation of the asymptotes by a constant. Let the equation of the asymptotes be.

$$3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0$$

This will represent a pair of straight lines if

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

where $a = 3, b = 8, c = \lambda, f = 11, g = 7, h = 5$

$\Rightarrow$ $3\times8\times\lambda + 2\times11\times7\times5 - 3\times121 - 8\times49 - \lambda\times25 = 0$

$\Rightarrow$ $\lambda = 15$.

Example 53: The product of the lengths of the perpendiculars drawn from any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to its assumption is

(a) $\frac{a^2b^2}{a^2+b^2}$ (b) $\frac{1}{a^2}+\frac{1}{b^2}$

(c) $\frac{1}{a^2}-\frac{1}{b^2}$ (d) a^2b^2

Ans. (a)

Solution: Let $P(a\sec\alpha, b\tan\alpha)$ be a point on the hyperbola.

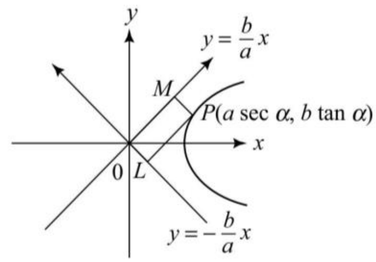


Fig. 20.8

Equation of the asymptotes is $y = \pm\frac{b}{a}x$.

Required product = $PL.PM$

$$= \frac{a^2b^2(\sec\alpha+\tan\alpha)(\sec\alpha-\tan\alpha)}{a^2+b^2}$$

$$= \frac{a^2b^2}{a^2+b^2}$$

Example 54: If (5, 12) and (24, 7) are the foci of a hyperbola passing though the origin then the eccentricity of the hyperbola is

(a) $\sqrt{386}/12$ (b) $\sqrt{386}/13$

(c) $\sqrt{386}/25$ (d) $\sqrt{386}/38$

Ans. (a)

Solution: Let $S(5, 12)$ and $S'(24, 7)$ be the two foci and $P(0, 0)$ be a point on the conic

then $SP = \sqrt{25+144} = \sqrt{169} = 13;$

$$S'P = \sqrt{(24)^2+7^2} = \sqrt{625} = 25$$

and $SS' = \sqrt{(24-5)^2+(7-12)^2} = \sqrt{19^2+5^2}$

$$= \sqrt{386}$$

since the conic is a hyperbola, $S'P - SP = 2a$, the length of transverse axis and $SS' = 2ae$, e being the eccentricity.

$$\Rightarrow \quad e = \frac{SS'}{S'P-SP} = \frac{\sqrt{386}}{12}.$$

Example 55: An equation of a tangent to the hyperbola. $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\pi/4$ with the transverse axis is

(a) $y = x + 2$ (b) $y = x + 4$

(c) $x = y + 3$ (d) $x + y + 2 = 0$

Ans. (a)

Solution: Equation of the hyperbola can be written as

$$X^2/5^2 - Y^2/4^2 = 1 \quad (1)$$

where $X = x - 3$ and $Y = y - 2$.

Equation of a tangent which makes an angle $\pi/4$. With the transverse axis $X = 0$ of (1) is

$$Y = \tan\frac{\pi}{4}X \pm \sqrt{25\tan^2\frac{\pi}{4} - 16}$$

$$\Rightarrow \quad y - 2 = x - 3 \pm \sqrt{25-16}$$

$$\Rightarrow \quad y = x + 2 \text{ or } y = x - 4$$

Example 56: Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ where $\theta + \phi = \pi/2$, be two points on the hyperbola $x^2/a^2 - y^2/b^2 = 1$. If (h, k) is the point of intersection of normals at P and Q, then k is equal to

(a) $\dfrac{a^2+b^2}{a}$ (b) $-\left[\dfrac{a^2+b^2}{a}\right]$

(c) $\dfrac{a^2+b^2}{b}$ (d) $-\left[\dfrac{a^2+b^2}{b}\right]$

Ans. (d)

Solution: Equation of the tangent at $P(a\sec\theta, b\tan\theta)$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1.$$

Therefore equation of the normal at P is

$$y - b\tan\theta = -\frac{a}{b}\sin\theta\,(x - a\sec\theta)$$

$$\Rightarrow \quad ax + b\,\text{cosec}\,\theta\, y = (a^2+b^2)\sec\theta \quad \text{(i)}$$

Similarly the equation of the normal at $Q(a\sec\phi, b\sec\phi)$ is $ax + b\,\text{cosec}\,\phi\, y = (a^2+b^2)\sec\phi$ (ii)

Subtracting (ii) from (i) we get

$$y = \frac{a^2+b^2}{b}\cdot\frac{\sec\theta - \sec\phi}{\text{cosec}\theta - \text{cosec}\phi}$$

So that $$k = y = \frac{a^2+b^2}{b}\frac{\sec\theta - \sec(\pi/2-\theta)}{\text{cosec}\theta - \text{cosec}(\pi/2-\theta)}$$

$$[\because \theta + \phi = \pi/2]$$

$$= \frac{a^2+b^2}{b}\frac{\sec\theta - \text{cosec}\,\theta}{\text{cosec}\,\theta - \sec\theta} = -\left[\frac{a^2+b^2}{b}\right]$$

Example 57: If P is a point on the rectangular hyperbola $x^2 - y^2 = a^2$, C is its centre and S, S' are the two foci, then $SP.S'P =$

(a) 2 (b) $(CP)^2$

(c) $(CS)^2$ (d) $(SS')^2$

Ans. (b)

Solution: Let the coordinate of P be (x, y).

The coordinates of the centre are (0, 0).

The eccentricity of the hyperbola is $\sqrt{1+\dfrac{a^2}{b^2}} = \sqrt{2}$

So the coordinates of the foci are $S(a\sqrt{2}, 0)$ and $S(-a\sqrt{2}, 0)$. (Fig 19.9)

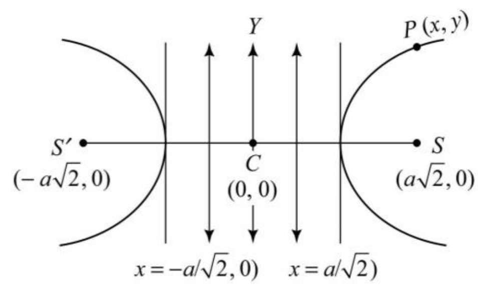


Fig. 20.9

Equation of the corresponding directrices are $x = a/\sqrt{2}$ and $S = --a/\sqrt{2}$.

By definition of the hyperbola

$$SP = e(\text{distance of } P \text{ from } x = a/\sqrt{2})$$

$$= \sqrt{2}\,|x - a/\sqrt{2}|$$

Similarly $S'P = \sqrt{2}\,|x + a/\sqrt{2}|$

So that $SP.S'.P = 2|x^2 - a^2/2|$

$$= 2x^2 - a^2 = x^2 + y^2 = (CP)^2$$

$(\because P$ lies on the hyperbola $x^2 - y^2 = a^2)$

Example 58: If P is the length of the perpendicular from a focus upon the tangent at any point P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and r is the distance of P from that focus, then $\frac{b^2}{p^2}-\frac{2a}{r}=$

(a) –1 (b) 0
(c) 1 (d) 2

Ans. (c)

Solution: Equation of the tangent at $P(a\sec\theta, b\tan\theta)$ to the hyperbola is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

Length of the perpendicular from the focus $S(ae, 0)$ to this tangent is

$$P=\left|\frac{e\sec\theta-1}{\sqrt{\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}}}\right|=\frac{ab(e\sec\theta-1)}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}$$

$$=\frac{ab(e\sec\theta-1)}{\sqrt{a^2(e^2-1)\sec^2\theta+a^2\tan^2\theta}}$$

$$=\frac{ab(e\sec\theta-1)}{a\sqrt{e^2\sec^2\theta-1}}=b\sqrt{\frac{e\sec\theta-1}{e\sec\theta+1}}$$

$$\Rightarrow \quad \frac{b^2}{p^2}=\frac{e\sec\theta+1}{e\sec\theta-1}$$

Now $r^2=(ae-a\sec\theta)^2+b^2\tan^2\theta$
$=a^2[e^2-2e\sec\theta+\sec^2\theta]+a^2(e^2-1)\tan^2\theta.$
$=a^2[e^2\sec^2\theta-2e\sec\theta+1]$
$=a^2(e\sec\theta-1)^2$

$$\Rightarrow \quad \frac{r^2}{a^2}=(e\sec\theta-1)^2\Rightarrow\frac{a}{r}=\frac{1}{e\sec\theta-1}$$

So $$\frac{b^2}{p^2}-\frac{2a}{r}=\frac{e\sec\theta+1}{e\sec\theta-1}-\frac{2}{e\sec\theta-1}=1.$$

Example 59: A hyperbola having transverse axis of length $2\sin\theta$ is confocal with the ellipse.
$3x^2+4y^2=12$, then its equation is

(a) $x^2\operatorname{cosec}^2\theta-y^2\sec^2\theta=1$
(b) $x^2\sec^2\theta-y^2\operatorname{cosec}^2\theta=1$
(c) $x^2\sin^2\theta-y^2\cos^2\theta=1$
(d) $x^2\cos^2\theta-y^2\sin^2\theta=1$

Ans. (a)

Solution: Foci of the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ is $(\pm 1, 0)$

Let the equation of the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Such that $2a=2\sin\theta\Rightarrow a=\sin\theta$.
and $a^2+b^2=a^2e^2=1\Rightarrow b=\cos\theta$.
So the required equation is

$$\frac{x^2}{\sin^2\theta}-\frac{y^2}{\cos^2\theta}=1$$

or $$x^2\operatorname{cosec}^2\theta-y^2\sec^2\theta=1$$

Example 60: If a hyperbola passing through the origin has $3x+4y-1=0$ and $4x-3y-6=0$ as its assumptions, then the equation of the transverse axis of the hyperbola is

(a) $x+y-5=0$ (b) $x-7y-5=0$
(c) $x-y-1=0$ (d) $x+y-1=0$

Ans. (b)

Solution: Assumptions are perpendicular so the hyperbola is a rectangle hyperbola.

Axes are the bisectors of the angles between the asymptotes.

Equation of the transverse axis is the bisector of the of the angle which contains the origin and is given by

$$\frac{3x+4y-1}{5}=\frac{4x-3y-6}{5}$$

$$\Rightarrow \quad x-7y-5=0$$

EXERCISE

Concept-based Straight Objective Type Questions

1. If e is the eccentricity of the hyperbola $\frac{x^2}{18}-\frac{y^2}{6}=1$ and r is the radius of the circle $x^2+y^2-6x-18y+87=0$, then the value of er is equal to:

(a) $2\sqrt{3}$ (b) $2/\sqrt{3}$
(c) 2 (d) 3

2. If m_1 and m_2 are two values of m for which the line $y = mx + 2\sqrt{5}$ is a tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{16} = 1$, then the value of $\left|m_1 + \frac{1}{m_2}\right|$ is equal to
   (a) $\frac{8}{3}$ (b) $\frac{10}{3}$
   (c) 0 (d) 9.

3. Distance between the directrices of the hyperbola $\frac{x^2}{49} - \frac{y^2}{16} = 1$ is
   (a) $\frac{\sqrt{65}}{7}$ (b) $\frac{49}{\sqrt{65}}$
   (c) $\frac{\sqrt{33}}{4}$ (d) $\frac{98}{\sqrt{65}}$

4. If A line with slope m touches the hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$ and the parabola $y^2 = 20x$. Then the value of $25m^4 - 4m^2$ is equal to
   (a) 29 (b) 21
   (c) 25 (d) 4

5. If length of the transverse axis of a hyperbola is 8 and its eccentricity is $\sqrt{5}/2$ then the length of a latus rectum of the hyperbola is
   (a) 1 (b) 2
   (c) $2\sqrt{5}$ (d) $8/\sqrt{5}$

6. A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$, meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where O is the origin), then R lies on
   (a) $\frac{4}{x^2} + \frac{2}{y^2} = 1$ (b) $\frac{2}{x^2} - \frac{4}{y^2} = 1$
   (c) $\frac{2}{x^2} + \frac{4}{y^2} = 1$ (d) $\frac{4}{x^2} - \frac{2}{y^2} = 1$

7. If $P(3\sec\theta, 2\tan\theta)$ and $Q(3\sec\phi, 2\tan\phi)$ where $\theta + \phi = \frac{\pi}{2}$, be two distinct points on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, then the ordinate of the point of intersection of the normals at P and Q is
   (a) 11/3 (b) –11/3
   (c) 13/2 (d) –13/2

8. A common tangent to $x^2 - 2y^2 = 18$ and $x^2 + y^2 = 9$ is
   (a) $y = 2x + 3\sqrt{5}$ (b) $y = \sqrt{2}x + 3\sqrt{3}$
   (c) $y = 2x + 3\sqrt{7}$ (d) $y = \sqrt{2}x + 3\sqrt{5}$

9. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then b^2 equals
   (a) 5 (b) 7
   (c) 9 (d) 1

10. Tangent drawn from the point (c, d) to the hyperbola $9x^2 - 64y^2 = 576$ make angle α and β with the x-axis. If $\tan\alpha \tan\beta = 1$, then the value of $c^2 - d^2$ is equal to
   (a) 73 (b) 55
   (c) 64 (d) 9

11. If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then the ratio of the square of its conjugate axis to the square of half the distance between the foci is
   (a) 4/3 (b) $4/\sqrt{3}$
   (c) $2/\sqrt{3}$ (d) 3/4

12. If the chords of contact of the tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $4x^2 - 5y^2 = a^2$ are at right angles. Then $16x_1x_2 + 25y_1y_2$ is equal to
   (a) –1 (b) 0
   (c) a^2 (d) 1

13. If two perpendicular tangents are drawn from a point (α, β) to the hyperbola $x^2 - y^2 = 16$, then the locus of (α, β) is
   (a) a pair of straight line (b) a circle
   (c) a parabola (d) an ellipse

14. If $\frac{x^2}{a+7} + \frac{y^2}{5-a} = 1$ represents a hyperbola, then
   (a) $a > 5$ (b) $a < -7$
   (c) $a < 5$ (d) $a < -7$ or $a > 5$

15. If e_1 is the eccentricity of the hyperbola $\frac{x^2}{49} - \frac{y^2}{36} = 1$ and e_2 is the eccentricity of the hyperbola $\frac{x^2}{36} - \frac{y^2}{49} = 1$, then
   (a) $e_1e_2 = 1$ (b) $\frac{e_1}{e_2} = 1$
   (c) $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ (d) $e_1^2 + e_2^2 = 1$

LEVEL 1

Straight Objective Type Questions

16. The line $4\sqrt{2}x - 5y = 40$ touches the hyperbola $\frac{x^2}{100} - \frac{y^2}{64} = 1$ at the point

(a) $(10, 8\sqrt{2})$ (b) $(10\sqrt{2}, 8)$
(c) $(20, 8\sqrt{3})$ (d) $\left(\frac{20}{\sqrt{3}}, \frac{8}{\sqrt{3}}\right)$

17. The line $9\sqrt{3}x + 12y = 234\sqrt{3}$ is a normal to the hyperbola $\frac{x^2}{81} - \frac{y^2}{36} = 1$ at the points.

(a) $(18, 6\sqrt{3})$ (b) $(9\sqrt{2}, 6)$
(c) $(9\sqrt{3}, 6)$ (d) $\left(\frac{18}{\sqrt{3}}, \frac{6}{\sqrt{3}}\right)$

18. Length of the latus rectum of the hyperbola $x^2 \tan\alpha - y^2 \cot\alpha = 1$, $(0 < \alpha < \pi/4)$ is

(a) $2\sqrt{\cot\alpha}$ (b) $2\sqrt{\tan\alpha}$
(c) $2(\tan\alpha)^{3/2}$ (d) $2(\cot\alpha)^{3/2}$

19. If the angle between the asymptotes of the hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$ is 2θ and e is its eccentricity then $\sin\theta$ is equal to

(a) $\frac{b}{a}$ (b) $\frac{1}{e}$
(c) $\frac{a}{be}$ (d) $\sqrt{1 - \frac{1}{e^2}}$

20. Axymptotes of the hyperbola $2xy = 4x + 3y$ pass through the point

(a) (2, 3/2) (b) (2, 3)
(c) (3/2, 2) (d) (0, 0)

21. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents

(a) a pair of straight lines
(b) an ellipse
(c) a parabola
(d) a hyperbola

22. The point $(at^2, 2bt)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for

(a) all real values of t (b) $t^2 = 2 + \sqrt{5}$
(c) $t^2 = 2 - \sqrt{5}$ (d) no real value of t

23. If the coordinates of four concyclic point on the rectangular hyperbola $xy = c^2$ are $(ct_i, c/t_i)$ $i = 1, 2, 3, 4$ then

(a) $t_1 t_2 t_3 t_4 = -1$ (b) $t_1 t_2 t_3 t_4 = 1$
(c) $t_1 t_3 = t_2 t_4$ (d) $t_1 + t_2 + t_3 + t_4 = c^2$

24. The eccentricity of a rectangular hyperbola is

(a) 2 (b) $\sqrt{2}$
(c) $2 + \sqrt{2}$ (d) none of these

25. If e and e_1 are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2} + \frac{1}{e_1^2}$ is equal to

(a) –1 (b) 0
(c) 1 (d) none of these

26. Foci of the rectangular hyperbola are (±7, 0), the equation of the hyperbola is

(a) $x^2 - y^2 = 49$ (b) $x^2 - y^2 = 98$
(c) $2x^2 - 2y^2 = 49$ (d) none of these

27. P is a point on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, Q is the point where the right directrix meets the axis of x and S is the right focus of the hyperbola. If the area of the triangle PQS is 27/10 sq. units, coordinates of P are

(a) $(4\sqrt{2}, 3)$ (b) $(4\sqrt{3}, 3\sqrt{2})$
(c) $(4\sqrt{5}, 6)$ (d) none of these

28. The normal at a point P to the parabola $y^2 = 4x$ is parallel to the tangent at Q $(\sqrt{2}, 2)$ to the hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ and meets the axis of the parabola at R. If S is the focus of the parabola, area of the triangle PSR in sq. units is

(a) $9\sqrt{2}$ (b) $10\sqrt{2}$
(c) $18\sqrt{2}$ (d) $20\sqrt{2}$

29. The difference between the length $2a$ of the transverse axis of a hyperbola of eccentricity e and the length of its latus rectum is

(a) $2a|3 - e^2|$ (b) $2a|2 - e^2|$
(c) $2a|e^2 - 1|$ (d) $a|2e^2 - 1|$

30. The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{36} = 1$ which are at right angles is

(a) $x^2 + y^2 = 20$ (b) $x^2 - y^2 = 20$
(c) $x^2 + y^2 = 52$ (d) none of these

31. If the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{36} = 1$ are perpendicular to the asymptotes of the hyperbola $\frac{x^2}{49} - \frac{y^2}{b^2} = 1$ then
(a) $7a \pm 6b = 0$ (b) $6a \pm 7b = 0$
(c) $a^2 - b^2 = 1$ (d) $a - b = 1$

32. P and Q are two points on the rectangular hyperbola $xy = c^2$ such that the abscissa of P and Q are the roots of the equations $x^2 - 6x - 16 = 0$. Equation of the chord joining P and Q is
(a) $16x - c^2y = 6c^2$ (b) $c^2x - 16y = c^2$
(c) $c^2x - 16y = 6c^2$ (d) $c^2x - 6y = 16c^2$

33. Normal at (3, 4) to the rectangular hyperbola $xy - y - 2x - 2 = 0$ meets the curve again at the points
(a) (1, 2) (b) (2, 3)
(c) (–1, 0) (d) none of these

34. Locus of the mid-points of the chords of the circle $x^2 + y^2 = 16$ which touch the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is
(a) $(x^2 - y^2)^2 = 16x^2 - 9y^2$
(b) $(x^2 + y^2)^2 = 9x^2 - 16y^2$
(c) $(x^2 + y^2)^2 = 16x^2 + 9y^2$
(d) $(x^2 - y^2)^2 = 16x^2 + 9y^2$

35. If the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\sqrt{5}$ and the distance between the foci is 12, then $b^2 - a^2$ is equal to (3/5) k^2 where k is equal to
(a) 5 (b) 3
(c) 2 (d) 6

36. If the extremities of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with positive coordinates lie on the parabola $x^2 = 3(y + 3)$, then length of the latus rectum of the hyperbola when its eccentricity is $\sqrt{3}$ is
(a) 3 (b) 6
(c) 12 (d) none of there

37. If the locus of the point of intersection of two straight lines $\sqrt{3}\,\alpha x + \alpha y - 4\sqrt{3} = 0$ and $\sqrt{3}\,x - y - 4\sqrt{3}\,\alpha = 0$ is a hyperbola with eccentricity e; for different values of α, then $e^2 + 6e - 9$ is equal to
(a) 0 (b) 7
(c) 3 (d) 4

38. Locus of the mid-points of the chords of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ that are parallel to $3x - 4y = 1$ is
(a) $4x - 3y = 0$ (b) $3x - 4y = 0$
(c) $3x + 4y = 0$ (d) $4x + 3y = 0$

39. The parametric equation:
$x = a(\sec\theta + \tan\theta)$, $y = b(\sec\theta - \tan\theta)$ represents
(a) a parabola (b) an ellipse
(c) a hyperbola (d) a rectangular hyperbola

40. If a normal to the hyperbola $x^2 - 4y^2 = 4$ having equal positive intercepts on the axes is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the distance between the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
(a) $\frac{10}{\sqrt{3}}$ (b) $\frac{5}{\sqrt{3}}$
(c) $10\sqrt{3}$ (d) $5\sqrt{3}$

Assertion-Reason Type Questions

41. A line $y = x + 4$ meets the hyperbola $xy = 16$ at A and B.
Statement 1: A circle on AB as diameter passes through the points (4, 4) and (– 4, – 4).
Statement 2: A circle on AB as diameter passes through the intersection of the circle $x^2 + y^2 = 32$ and the line $y = x$.

42. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ S and S' are its foci.
Statement-1: Product of the lengths of the perpendiculars from S and S' on the tangent at P is equal to b^2.
Statement-2: $|PS - PS'| = 2a$.

43. **Statement-1:** Two tangents drawn from any point on the hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ make complementary angles with the axis of the ellipse.

Statement-2: If two lines make complementary angles with the axis of x then the product of their slopes is 1.

44. **Statement-1:** If the foci of a hyperbola are at the points (4, 1) and (–6, 1), eccentricity is 5/4 then the length of the transverse axis is 8.

Statement-2: Distance between the foci of a hyperbola is equal to the product of its eccentricity and the length of the transverse axis.

45. **Statement-1:** If the angle between two asymptotes of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{\pi}{3}$, its eccentricity is $2\sqrt{3}$.

Statement 2: Angles between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $2\tan^{-1}\left(\frac{b}{a}\right)$ or $\pi-2\tan^{-1}\left(\frac{b}{a}\right)$.

46. **Statement-1:** Equation of a circle on the ends of a latus rectum of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ as a diameter is $16x^2+16y^2\pm 160x+319=0$

Statement-2: Focus of the parabola $y^2=20x$ coincides with a focus of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$

47. Let $P(x, y)$ be a variable point such that $\left|\sqrt{(x-3)^2+(y-2)^2}-\sqrt{(x-6)^2+(y+2)^2}\right|=3$.

Statement-1: P traces a hyperbola whose eccentricity is $\frac{5}{3}$.

Statement-2: P traces a hyperbola such that the equation of its conjugate axis is $6x-8y=27$.

48. **Statement 1:** A normal to the hyperbola with eccentricity 3, meets the transverse axis and conjugate axis at P and Q respectively. The locus of the mid-point of PQ is a hyperbola with eccentricity $\frac{3}{2\sqrt{2}}$.

Statement 2: Eccentricity of the hyperbola $8x^2-y^2=8a^2$ is $\frac{3}{2\sqrt{2}}$.

49. **Statement-1:** The locus of the point of intersection of the tangents that are at right angles to the hyperbola $\frac{x^2}{36}-\frac{y^2}{16}=1$ is the circle $x^2+y^2=52$.

Statement-2: Perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ interest on the director circle $x^2+y^2=a^2-b^2$ $(a^2>b^2)$ of the hyperbola.

50. A hyperbola H: $\frac{x^2}{9}-\frac{y^2}{4}=1$ intersects the circle, C: $x^2+y^2-8x=0$ at the points A and B.

Statement-1: $2x-\sqrt{5}\,y+4=0$ is a common tangent to both C and H.

Statement-2: Circle on AB as a diameter passes through the centre of the hyperbola H.

LEVEL 2

Straight Objective Type Questions

51. Locus of the mid-point of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which is a tangent to the circle $x^2+y^2=c^2$ is

(a) $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$

(b) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=c^2\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)$

(c) $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$

(d) $\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)=c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$

52. $H_1 : xy=c^2$ and $H_2 : xy=k^2$ are two different hyperbolas. From a point on H_1, tangents are drawn to H_2. Area of the triangle formed by the chord of contact and the asymptote to H_2 is

(a) $\frac{k^2}{c^2}$ (b) $\frac{k^4}{c^2}$

(c) $\frac{2k^4}{c^2}$ (d) none of these

53. e_1, e_2 are respectively the eccentricities of the hyperbola $x^2 - y^2 \operatorname{cosec}^2\theta = 5$ and the ellipse $x^2\operatorname{cosec}^2\theta + y^2 = 5$. If $0 < \theta < \pi/2$ and $e_1 = \sqrt{7}\, e_2$, then θ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{3}$ (d) none of these

54. If θ is an angle between the two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\cos\left(\frac{\theta}{2}\right)$ is equal to

(a) $\frac{b}{\sqrt{a^2+b^2}}$ (b) $\frac{ab}{\sqrt{a^2+b^2}}$

(c) $\frac{a}{\sqrt{a^2+b^2}}$ (d) $\sqrt{\frac{a-b}{a+b}}$.

55. A and B are two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, O is the centre. If OA is perpendicular to OB then $\frac{1}{(OA)^2} + \frac{1}{(OB)^2}$ is equal to

(a) $\frac{1}{a^2} + \frac{1}{b^2}$ (b) $\frac{1}{a^2} - \frac{1}{b^2}$

(c) $\frac{1}{b^2} - \frac{1}{a^2}$ (d) $a^2 + b^2$

56. The coordinates of a point common to a directrix and an asymptote of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ are

(a) $\left(\frac{25}{\sqrt{41}}, \frac{20}{3}\right)$ (b) $\left(\frac{20}{\sqrt{41}}, \frac{-25}{\sqrt{41}}\right)$

(c) $\left(\frac{25}{3}, \frac{20}{3}\right)$ (d) $\left(\frac{-25}{\sqrt{41}}, \frac{20}{\sqrt{41}}\right)$

57. If the normals at P, Q, R on the rectangular hyperbola $xy = c^2$ intersect at a point S on the hyperbola, then centroid of the triangle PQR is at

(a) an extremity of the latus rectum
(b) the centre
(c) a focus
(d) the point S on the hyperbola.

58. If a diameter of a hyperbola meets the hyperbola in real points then

(a) it meets the conjugate hyperbola in imaginary points.
(b) the conjugate diameter meets the given hyperbola in real points.
(c) the conjugate diameter meets the conjugate hyperbola in imaginary points.
(d) none of these.

59. An ellipse has eccentricity 1/2 and a focus at the point $P(1/2, 1)$. One of its directrix is the common tangent near to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$, the equation of the ellipse is

(a) $3x^2 + 4y^2 - 6x - 8y + 4 = 0$
(b) $3x^2 + 4y^2 - 2x - 8y + 4 = 0$
(c) $4x^2 + 3y^2 - 8x - 6y + 4 = 0$
(d) $4x^2 + 3y^2 - 8x - 2y + 4 = 0$

60. The asymptotes of $xy = hx + ky$ are

(a) $x = h, y = k$ (b) $x = -h, y = -k$
(c) $x = k, y = h$ (d) $x = -k, y = -h$

Previous Years' AIEEE/JEE Main Questions

1. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide then, the value of b^2 is:

(a) 5 (b) 7
(c) 9 (d) 1 **[2003]**

2. The locus of the point P (α, β) moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is:

(a) a parabola (b) a hyperbola
(c) an ellipse (d) a circle **[2005]**

3. The normal to curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is

(a) ellipse (b) parabola
(c) circle (d) hyperbola **[2007]**

4. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α varies?

(a) eccentricity
(b) directrix
(c) abscissae of vertices
(d) abscissae of foci. **[2007]**

5. The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by
(a) $x^2 - 3y^2 = 3$ (b) $3x^2 - y^2 = 3$
(c) $-x^2 + 3y^2 = 3$ (d) $-3x^2 + y^2 = 3$ **[2011]**

6. A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where O is the origin) then R lies on
(a) $\frac{4}{x^2} + \frac{2}{y^2} = 1$ (b) $\frac{2}{x^2} - \frac{4}{y^2} = 1$
(c) $\frac{2}{x^2} + \frac{4}{y^2} = 1$ (d) $\frac{4}{x^2} - \frac{2}{y^2} = 1$
[2013, online]

7. A common tangent to the conic $x^2 = 6y$ and $2x^2 - 4y^2 = 9$ is
(a) $x - y = \frac{3}{2}$ (b) $x + y = 1$
(c) $x + y = \frac{9}{2}$ (d) $x - y = 1$ **[2013, online]**

8. If $P(3\sec\theta, 2\tan\theta)$ and $Q(3\sec\phi, 2\tan\phi)$ where $\theta + \phi = \frac{\pi}{2}$, be two distinct points on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Then the ordinate of the points of intersection of the normals at P and Q is
(a) $\frac{11}{3}$ (b) $-\frac{11}{3}$
(c) $\frac{13}{2}$ (d) $-\frac{13}{2}$ **[2014, online]**

9. The tangent at an extremity (in the first quadrant) of latus rectum of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ meets x-axis and y-axis at A and B respectively. Then $(OA)^2 - (OB)^2$, where O is the origin, equals
(a) $-\frac{20}{9}$ (b) $\frac{16}{9}$
(c) 4 (d) $-\frac{4}{3}$ **[2014, online]**

10. An ellipse passes through the foci of the hyperbola, $9x^2 - 4y^2 = 36$ and it major and minor axes lie along the transvers and conjugate axes of the hyperbola respectively. If the product of eccentricities of the two conics is $\frac{1}{2}$, then which of the following points does not lie on the ellipse?
(a) $(\sqrt{13}, 0)$ (b) $\left(\frac{\sqrt{39}}{2}, \sqrt{3}\right)$
(c) $\left(\frac{1}{2}\sqrt{13}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(\frac{\sqrt{13}}{2}, \sqrt{6}\right)$
[2015, online]

11. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is
(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$
(c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{3}$ **[2016]**

12. A hyperbola whose transverse axis is along the major axis of the conic, $\frac{x^2}{3} + \frac{y^2}{4} = 4$ and has vertices at the foci of this conic. If the eccentricity of the hyperbola is $\frac{3}{2}$, then which of the following points does NOT lie on it?
(a) $(\sqrt{5}, 2\sqrt{2})$ (b) (0, 2)
(c) $(5, 2\sqrt{3})$ (d) $(\sqrt{10}, 2\sqrt{3})$
[2016, online]

13. Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation $9e^2 - 18e + 5 = 0$. If $S(5, 0)$ is a focus and $5x = 9$ is corresponding directrix of this hyperbola, then $a^2 - b^2$ is equal to:
(a) – 7 (b) – 5
(c) 5 (d) 7 **[2016, on line]**

Previous Years' B-Architecture Entrance Examination Questions

1. If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then the eccentricity e of hyperbola satisfies
   (a) $e = \frac{2}{\sqrt{3}}$ (b) $e = \frac{\sqrt{3}}{2}$
   (c) $e > \frac{2}{\sqrt{3}}$ (d) $1 < e < \frac{2}{\sqrt{3}}$ **[2006]**

2. If the tangent and the normal to the hyperbola $x^2 - y^2 = 4$ at a point cut off intercepts a_1 and a_2 respectively on the x-axis, and b_1 and b_2 respectively on the y-axis, then the value of $a_1 a_2 + b_1 b_2$ is
   (a) – 1 (b) 0
   (c) 4 (d) 1 **[2010]**

3. A common tangent to $x^2 - 2y^2 = 18$ and $x^2 + y^2 = 9$ is
   (a) $y = 2x + 3\sqrt{5}$ (b) $y = \sqrt{2}x + 3\sqrt{3}$
   (c) $y = 2x + 3\sqrt{7}$ (d) $y = \sqrt{2}x + 3\sqrt{5}$ **[2013]**

4. If the point R divides the line segment joining the point (2, 3) and ($2 \tan\theta$, $3 \sec\theta$); $0 < \theta < \frac{\pi}{2}$, externally in the ratio 2 : 3, then the locus of R is
   (a) an ellipse length of whose major axis is 12.
   (b) an ellipse length of whose major axis is 8.
   (c) a hyperbola length of whose transverse axis is 12.
   (d) a hyperbola length of whose transverse axis is 8. **[2015]**

5. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If the eccentricity of the hyperbola is 2, then the equation of the tangent to this hyperbola passing through the point (4, 6) is
   (a) $3x - 2y = 0$ (b) $2x - 3y + 10 = 0$
   (c) $x - 2y + 8 = 0$ (d) $2x - y - 2 = 0$ **[2016]**

Answers

Concept-based

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)
5. (b)	**6.** (d)	**7.** (d)	**8.** (b)
9. (b)	**10.** (a)	**11.** (d)	**12.** (b)
13. (a)	**14.** (d)	**15.** (c)	

Level 1

16. (b)	**17.** (a)	**18.** (c)	**19.** (d)
20. (c)	**21.** (c)	**22.** (b)	**23.** (b)
24. (b)	**25.** (c)	**26.** (c)	**27.** (a)
28. (c)	**29.** (c)	**30.** (d)	**31.** (a)
32. (c)	**33.** (c)	**34.** (c)	**35.** (d)
36. (c)	**37.** (b)	**38.** (b)	**39.** (d)
40. (a)	**41.** (a)	**42.** (b)	**43.** (a)
44. (a)	**45.** (c)	**46.** (b)	**47.** (b)
48. (c)	**49.** (d)	**50.** (a)	

Level 2

51. (a)	**52.** (c)	**53.** (c)	**54.** (c)
55. (b)	**56.** (d)	**57.** (b)	**58.** (a)
59. (b)	**60.** (c)		

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (b)	**3.** (a)	**4.** (d)
5. (b)	**6.** (d)	**7.** (a)	**8.** (d)
9. (a)	**10.** (c)	**11.** (c)	**12.** (c)
13. (a)			

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (b)	**3.** (b)	**4.** (c)
5. (d)			

Hints and Solutions

Concept-based

1. $e^2 = \frac{6+18}{18} = \frac{4}{3}$, $r^2 = 9 + 81 - 87 = 3$
   $\Rightarrow er = 2$

2. $(2\sqrt{5})^2 = 4m^2 - 16 \Rightarrow m^2 = 9, m_1 = 3, m_2 = -3$
   $\Rightarrow m_1 m_2 = -9 \Rightarrow \left|m_1 + \frac{1}{m_2}\right| = \left|\frac{-9+1}{\pm 3}\right| = \frac{8}{3}$

3. The required distance is $\frac{2a}{e} = \frac{2a^2}{\sqrt{a^2+b^2}} = \frac{98}{\sqrt{65}}$.

4. Let the equation of the line be $y = mx + c$, then $c^2 = 25m^2 - 4$ and $c = \dfrac{5}{m}$.

$\Rightarrow 25 = (25m^2 - 4)m^2$

$\Rightarrow 25m^4 - 4m^2 = 25.$

5. Length of the L.R. = $\dfrac{2b^2}{a} = 2a(e^2 - 1) = 8\left(\dfrac{5}{4} - 1\right) = 2$

6. Let the equation of the tangent be $\dfrac{xx_1}{4} - \dfrac{yy_1}{2} = 1$

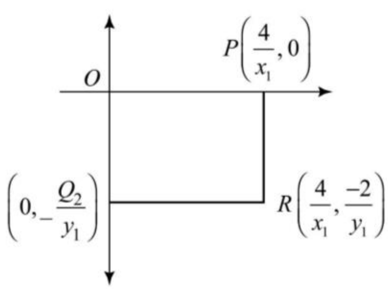
Coordinates of P are $\left(\dfrac{4}{x_1}, 0\right)$

and of Q are $\left(0, \dfrac{-2}{y_1}\right)$

$\Rightarrow$ Coordinates of R are $\left(\dfrac{4}{x_1}, \dfrac{-2}{y_1}\right) = (h, k)$ (x_1, y_1) lies on the hyperbola.

$\Rightarrow \dfrac{x_1^2}{4} - \dfrac{y_1^2}{2} = 1$

$\Rightarrow \dfrac{4}{h^2} - \dfrac{2}{k^2} = 1 \Rightarrow$ Locus of R is $\dfrac{4}{x^2} - \dfrac{2}{y^2} = 1$



7. An equation of the normal at P is

$3x \cos\theta + 2y \cot\theta = 9 + 4$

$\Rightarrow 3x + 2y \operatorname{cosec}\theta = 13 \sec\theta.$

An equation of the normal at ϕ is

$3x + 2y \operatorname{cosec}\phi = 13 \sec\phi$

Subtracting we get

$2y (\operatorname{cosec}\theta - \operatorname{cosec}\phi) = 13 (\sec\theta - \sec\phi)$

$\Rightarrow 2y (\operatorname{cosec}\theta - \sec\theta) = 13 (\sec\theta - \operatorname{cosec}\theta)$

$\Rightarrow y = -\dfrac{13}{2}.$

8. Let $y = mx + c$ be a common tangent

then $c^2 = 18m^2 - 9$ as it touches the hyperbola.

$= 9(1 + m^2)$ as it touches the circle.

$\Rightarrow m^2 = 2$, $c = 3\sqrt{3}$ and the required equation is

$y = \sqrt{2}x + 3\sqrt{3}$.

9. Foci of the ellipse are $\left(\pm\sqrt{16 - b^2}, 0\right)$

and of the hyperbola are $\left(\pm\sqrt{\left(\dfrac{12}{5}\right)^2 + \left(\dfrac{9}{5}\right)^2}, 0\right)$

If they coincide $16 - b^2 = \dfrac{144 + 81}{25} \Rightarrow b^2 = 7$

10. Any tangent to the hyperbola is

$y = mx + \sqrt{64m^2 - 9}$. If passes through (c, d)

$\Rightarrow (d - mc)^2 = 64m^2 - 9$

$\Rightarrow (c^2 - 64)m^2 - 2cd\,m + d^2 + 9 = 0.$

Product of the roots = $\dfrac{d^2 + 9}{c^2 - 64} = \tan\alpha \tan\beta = 1$

$\Rightarrow c^2 - d^2 = 64 + 9 = 73$

11. Let the equation of the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Centre is (0, 0), vertex $(a, 0)$, focus $(ae, 0)$

So $ae = 2a \Rightarrow e = 2$.

Conjugate axis = b, distance between the foci = $2ae$

So $\dfrac{b^2}{(ae)^2} = \dfrac{e^2 - 1}{e^2} = \dfrac{3}{4}$

12. Equations of the chords of contact are

$4xx_1 - 5yy_1 = a^2$ and $4xx_2 - 5yy_2 = a^2$

If they are at right angles

$\dfrac{4x_1}{5y_1} \times \dfrac{4x_2}{5y_2} = -1 \Rightarrow 16x_1x_2 + 25y_1y_2 = 0.$

13. An equation of a tangent to the hyperbola is

$y = mx + 4\sqrt{m^2 - 1}$

and equation of the tangent perpendicular to it is

$y = -\dfrac{1}{m}x + 4\sqrt{\dfrac{1}{m^2} - 1}$

$\Rightarrow my + x = 4\sqrt{1 - m^2}$.

We must have $m^2 - 1 \geq 0$, $1 - m^2 \geq 0 \Rightarrow m^2 = 1$

As (α, β) lies on both, locus of (α, β) is $y = \pm x$, which is a pair of straight lines.

14. We must have $\left.\begin{array}{l} a + 7 > 0 \text{ and } 5 - a < 0 \\ a + 7 < 0 \text{ and } 5 - a > 0 \end{array}\right\} \Rightarrow \begin{array}{c} a > 5 \\ \text{or} \\ a < -7 \end{array}$

15. $e_1^2 = \dfrac{49 + 36}{49}$ and $e_2^2 = \dfrac{49 + 36}{36}$.

$\Rightarrow \dfrac{1}{e_1^2} + \dfrac{1}{e_2^2} = 1.$

Level 1

16. Equation of the tangent at $(10 \sec \theta, 8 \tan \theta)$ on the hyperbola is $\frac{x}{10} \sec \theta - \frac{y}{8} \tan \theta = 1$

Comparing with the given equation

$\frac{x\sqrt{2}}{10} - \frac{y}{8} = 1$, we get

$\sec \theta = \sqrt{2}$, $\tan \theta = 1$

so the required point is $(10\sqrt{2}, 8)$

17. Equation of the normal at $(9 \sec \theta, 6 \tan \theta)$ on the hyperbola is $\frac{9x}{\sec\theta} + \frac{6y}{\tan\theta} = 117$.

Comparing with the given equation

$\frac{9x}{2} + \frac{6y}{\sqrt{3}} = 117$

we get $\sec \theta = 2$, $\tan \theta = \sqrt{3}$

and the required point is $(18, 6\sqrt{3})$.

18. Length of the latus rectum is $\frac{2(\tan\alpha)}{\sqrt{\cot\alpha}} = 2(\tan \alpha)^{3/2}$.

19. Equation of the asymptotes is $\frac{x}{a} \pm \frac{y}{b} = 0$.

So $\tan 2\theta = \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)}$

$\Rightarrow 2\theta = \tan^{-1} \frac{2(b/a)}{1-(b/a)^2} = 2 \tan^{-1}\left(\frac{b}{a}\right)$

$\Rightarrow \theta = \tan^{-1} \frac{b}{a} \Rightarrow \tan \theta = \frac{b}{a}$.

$\Rightarrow \sin \theta = \frac{b}{a} \times \frac{1}{\sqrt{1+\frac{b^2}{a^2}}} = \frac{b}{ae} = \sqrt{\frac{a^2(e^2-1)}{a^2e^2}}$

$= \sqrt{1 - \frac{1}{e^2}}$

20. Given hyperbola is $xy = 2x + \frac{3}{2} y$

or $\left(x - \frac{3}{2}\right)(y-2) = 3$.

Equations of the asymptotes are $x - \frac{3}{2} = 0$, $y - 2 = 0$

Both of them pass through the point $\left(\frac{3}{2}, 2\right)$

21. Eliminating t, we get $x = \left(\frac{x-y}{2}\right)^2 + \frac{x-y}{2} + 1$

$\Rightarrow (x-y)^2 = 2(x+y-2)$ which is a parabola.

22. $\frac{a^2t^4}{a^2} - \frac{4b^2t^2}{b^2} = 1 \Rightarrow t^4 - 4t^2 - 1 = 0$

$\Rightarrow t^2 = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$ But $t^2 \neq 2 - \sqrt{5}$

so $t^2 = 2 + \sqrt{5}$.

23. Equation of a circle be $x^2 + y^2 + 2gx + 2fy + k = 0$.

If the point $(ct, c/t)$ lies on it, then

$c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$

which gives four values of t say t_1, t_2, t_3, t_4 and the four concyclic points such that $t_1 t_2 t_3 t_4 = 1$.

24. For a rectangular hyperbola $b^2 = a^2$ so

$b^2 = a^2(e^2 - 1) \Rightarrow e^2 - 1 = 1$

$\Rightarrow e^2 = 2 \Rightarrow e = \sqrt{2}$

25. $e^2 = \frac{a^2+b^2}{a^2}$ and $e_1^2 = \frac{a^2+b^2}{b^2} = \frac{1}{e^2} + \frac{1}{e_1^2} = 1$

26. Equation of the hyperbola is $x^2 - y^2 = a^2$

Foci $= (\pm ae, 0) = (\pm 7, 0)$ where $e = \sqrt{2}$.

27. Eccentricity $e = 5/4$, coordinates of $Q(16/5, 0)$, $S(5, 0)$, $QS = 9/5$

If $P(x, y)$, then area of the triangle PQS

$= (1/2)(9/5)|y| = 27/10 \Rightarrow y = 3, x = 4\sqrt{2}$

28. Equation of the tangent at Q $(\sqrt{2}, 2)$ is $y = 2\sqrt{2}x - 2$

so equation of the normal at P is

$y = 2\sqrt{2}x - 2 \times 2\sqrt{2} - (2\sqrt{2})^3$

or $y = 2\sqrt{2}x - 20\sqrt{2}$

$S(1, 0)$, $R(10, 0)$, $P(8, -4\sqrt{2})$

Area of the triangle $\frac{1}{2} \times |10-1||-4\sqrt{2}|$

$= 18\sqrt{2}$

29. Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$b^2 = a^2(e^2 - 1)$

length of the latus rectum $= 2b\sqrt{e^2-1}$

$= 2a(e^2 - 1)$

30. The locus is the circle $x^2 + y^2 = 16 - 36$ which is not possible. So there is no pair of tangents that are at right angles.

31. Asymptotes to the two hyperbola are

$$\frac{x}{a} \pm \frac{y}{6} = 0 \text{ and } \frac{x}{7} \pm \frac{y}{b} = 0$$

If they are at right angles then $\left(\pm\frac{6}{a}\right)\left(\pm\frac{b}{7}\right) = -1$

$\Rightarrow 7a \pm 6b = 0$

32. Let $P(x_1, c^2/x_1)$ and $Q(x_2, c^2/x_2)$ such that $x_1 + x_2 = 6$, $x_1x_2 = -16$.

Equation of the chord PQ is

$$y - \frac{c^2}{x_1} = \frac{\frac{c^2}{x_2} - \frac{c^2}{x_1}}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow x - x_1 = -\frac{x_1x_2}{c^2}\left(y - \frac{c^2}{x_1}\right)$$

$\Rightarrow c^2x + x_1x_2\, y = c^2(x_1 + x_2)$

$\Rightarrow c^2x - 16y = 6c^2$

33. Equation of the hyperbola is $(x - 1)(y - 2) = 4$

or $y = 2 + \dfrac{4}{x-1}$

Any point on this hyperbola is $\left(1 + 2t, 2 + \dfrac{2}{t}\right)$

$$\frac{dy}{dx} = -\frac{4}{(x-1)^2}$$

Slope of the normal at $\left(1+2t, 2+\dfrac{2}{t}\right)$ is t^2.

Now $(3, 4) = \left(1+2t, 2+\dfrac{2}{t}\right) \Rightarrow t = 1$

Slope of the normal at (3, 4) is 1.

So equation of the normal at (3, 4) is $x - y + 1 = 0$

Point $\left(1+2t, 2+\dfrac{2}{t}\right)$ lies on this normal.

If $1 + 2t - 2 - \dfrac{2}{t} + 1 = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

So the required point is for $t = -1$, i.e. $(-1, 0)$.

34. Let (h, k) be the mid-point, equation of the chord is $hx + ky = h^2 + k^2$ which touches the hyperbola at $(4 \sec\theta, 3\tan\theta)$.

So its equation is $\dfrac{x}{4}\sec\theta - \dfrac{y}{3}\tan\theta = 1$.

Comparing the two equations we get

$$\frac{\sec\theta}{4} = \frac{h}{h^2+k^2} \text{ and } \frac{-\tan\theta}{3} = \frac{k}{h^2+k^2}$$

Eliminating θ, we get

$(h^2 + k^2)^2(\sec^2\theta - \tan^2\theta) = (4h)^2 - (3k)^2$

$\Rightarrow (h^2 + k^2)^2 = 16h^2 - 9k^2$.

and the required locus is $(x^2 + y^2)^2 = 16x^2 - 9y^2$

35. We have $e = \sqrt{5}$, $2ae = 12$

$$b^2 - a^2 = a^2e^2 - 2a^2 = 36 - 2\left(\frac{36}{5}\right) = \frac{3}{5}(6)^2$$

36. As $\left(ae, \dfrac{b^2}{a}\right), \left(-ae, \dfrac{b^2}{a}\right)$ lies on the parabola

$x^2 = 3(y + 3)$, we have $a^2e^2 = 3\left(\dfrac{b^2}{a} + 3\right)$

$= 3(a(e^2 - 1) + 3)$

$\Rightarrow (a - 3)(ae^2 + 3) = 0 \Rightarrow a = 3$.

So length of the latus rectum of the hyperbola

$$= \frac{2b^2}{a} = \frac{2a^2(e^2-1)}{a} = 2 \times 3 \times 2 = 12.$$

37. Point of intersection is (x, y)

$$= \left(2\left(\alpha + \frac{1}{\alpha}\right), 2\sqrt{3}\left(\frac{1}{\alpha} - \alpha\right)\right)$$

$$\Rightarrow \left(\frac{x}{2}\right)^2 - \left(\frac{y}{2\sqrt{3}}\right)^2 = 4$$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{48} = 1$$

So $e^2 = 1 + \dfrac{48}{16} = 4 \Rightarrow e = 2$

and hence $e^2 + 6e - 9 = 7$

38. Equation of the chord with mid-point (h, k) is

$\dfrac{hx}{16} - \dfrac{ky}{9} = \dfrac{h^2}{16} - \dfrac{k^2}{9}$ whose slope is $\dfrac{9h}{16k}$

so $\dfrac{9h}{16k} = \dfrac{3}{4} \Rightarrow 3h - 4k = 0$

Locus of (h, k) is $3x - 4y = 0$.

39. We have $bx + ay = 2ab\sec\theta$

$bx - ay = 2ab\tan\theta$

Eliminating θ we get

$$\left(\frac{bx+ay}{2ab}\right)^2 - \left(\frac{bx-ay}{2ab}\right)^2 = 1$$

$\Rightarrow 4abxy = 4a^2b^2$

$\Rightarrow xy = ab$

which represents a rectangular hyperbola.

40. Equation of the normal to the hyperbola at $(2\sec\theta, \tan\theta)$ is $\frac{2x}{\sec\theta}+\frac{y}{\tan\theta} = 5$.

Intercepts on axes are $\frac{5}{2}\sec\theta$ and $5\tan\theta$

$\frac{5}{2}\sec\theta = 5\tan\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

and the equation of the normal is

$\sqrt{3}x+\sqrt{3}y = 5$

$\Rightarrow y = -x + \frac{5}{\sqrt{3}}$

which will touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

if $\left(\frac{5}{\sqrt{3}}\right)^2 = a^2(-1)^2 + b^2$

$\Rightarrow a^2 + b^2 = \left(\frac{5}{\sqrt{3}}\right)^2$

Distance between the foci of the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is $2ae = 2\sqrt{a^2+b^2}$

$= 2\times\frac{5}{\sqrt{3}} = \frac{10}{\sqrt{3}}$

41. Any point on the hyperbola $xy = 16$ is $\left(4t, \frac{4}{t}\right)$

Let the coordinate of A and B be $\left(4t_1, \frac{4}{t_1}\right)$ and $\left(4t_2, \frac{4}{t_2}\right)$ respectively. So slope of $AB = -\frac{1}{t_1t_2} = 1$

$\Rightarrow t_1t_2 = -1$.

Equation of the circle on AB as diameter is

$(x-4t_1)(x-4t_2) + \left(y-\frac{4}{t_1}\right)\left(y-\frac{4}{t_2}\right) = 0$

$\Rightarrow (x-4t_1)\left(x+\frac{4}{t_1}\right)+\left(y-\frac{4}{t_1}\right)(y+4t_1) = 0$

$\Rightarrow x^2 + y^2 - 32 + 4\left(\frac{1}{t_1}-t_1\right)(x-y) = 0$

which shows that the circle passes through the points of intersection of the circle $x^2 + y^2 = 32$ and the line $x - y = 0$ i.e. $y = x$ and thus statement-2 is true.

Using it the required points of intersection are (4, 4) and (– 4, – 4).

Hence statement-1 is also true.

42. Let $P(a\sec\theta, b\tan\theta)$, equation of the tangent at P is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.

Product of the lengths of the perpendiculars from $S(ae, 0)$ and $S'(-ae, 0)$ is

$$\left|\frac{(e\sec\theta-1)(e\sec\theta+1)}{\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}}\right|$$

$$= \frac{a^2b^2(e^2\sec^2\theta-1)}{a^2[(e^2-1)\sec^2\theta+\tan^2\theta]} = b^2$$

So statement-1 is true.

For statement-2, by definition of hyperbola distance of P from a focus is e times its distance from the corresponding directrix.

So $PS = e\left(x-\frac{a}{e}\right)$ and $PS' = e\left(x+\frac{a}{e}\right)$.

$\Rightarrow |PS - PS'| = 2a$.

Thus statement-2 is also true but does not justify statement-1.

43. If the angle in statement-2 are α and β such that $\alpha+\beta = \pi/2$ then the product of the slopes is $\tan\alpha\tan\beta = 1$, and the statement-2 is true.

In statement-1, any tangent to the ellipse is $y = mx + \sqrt{a^2m^2+b^2}$ which passes through $\left(\sqrt{a^2-b^2}\sec\theta, \sqrt{a^2-b^2}\tan\theta\right)$

$\Rightarrow (a^2 - b^2)(\tan\theta - m\sec\theta)^2 = a^2m^2 + b^2$

Product of the slopes

$$= \frac{(a^2-b^2)\tan^2\theta-b^2}{(a^2-b^2)\sec^2\theta-a^2} = \frac{a^2\sin^2\theta-b^2}{a^2\sin^2\theta-b^2} = 1$$

So by statement-2, statement-1 is also true.

44. Statement-2 is true, as the distance between the foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is $2ae$ where $2a$ is the length of the transverse axis and e is the eccentricity.

Using it in statement-1, length of the transverse axis is $\frac{\sqrt{(4+6)^2}}{5/4} = \frac{10\times4}{5} = 8$

and thus statement-1 is also true.

45. Equation of the asymptotes is $\frac{x}{a} \pm \frac{y}{b} = 0$

And if θ is an angle between the asymptotes $\tan \theta = \frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{2\tan\phi}{1+\tan^2\phi}$ where $\phi = \tan^{-1}\frac{b}{a}$

$= \tan 2\phi$

$\Rightarrow \theta = 2\phi = 2\tan^{-1}\frac{b}{a}$ and the statement-1 is true.

Using in statement-1.

If $\frac{\pi}{3} = 2\tan^{-1}\frac{b}{a} \Rightarrow \tan\frac{\pi}{6} = \frac{b}{a}$

$\Rightarrow 3b^2 = a^2 \Rightarrow 3(e^2 - 1) = 1 \Rightarrow e = \frac{2}{\sqrt{3}}$

If $\frac{\pi}{3} = \pi - 2\tan^{-1}\frac{b}{a} \Rightarrow \frac{b}{a} = \tan\frac{\pi}{3} = \sqrt{3}$

$\Rightarrow b^2 = 3a^2 \Rightarrow e^2 - 1 = 3 \Rightarrow e = 2$

Showing that statement-1 is false.

46. Foci $(\pm 5, 0)$, length of the latus rectum $= \frac{2\times 9}{4}$

So the equation of a required circle is $(x \pm 5)^2 + y^2 = \left(\frac{9}{4}\right)^2$

$\Rightarrow 16(x^2 + y^2 \pm 10x + 25) = 81$

or $16x^2 + 16y^2 \pm 160x + 319 = 0$

and statement-1 is true, statement-2 is also true as the focus of $y^2 = 20x$ is $(5, 0)$ but does not justify statement-1.

47. Distance of P from the fixed points $F_1(3, 2)$ and $F_2(6, -2)$ is constant 3.

So by definition, locus of P is a hyperbola with foci F_1 and F_2, length of the transverse axes is 3. If e is the eccentricity then $F_1F_2 = 2ae$ where $2a = 3$.

$\Rightarrow e = \frac{1}{3}\sqrt{9+16} = \frac{5}{3}$ and the statement-1 is true.

In statement-2 slope of the conjugate axis is $-\frac{6-3}{-2-2}$

$= \frac{3}{4}$ and it passes through the mid-point of F_1F_2.

So its equation is

$y = \frac{3}{4}\left(x - \frac{9}{2}\right)$

$\Rightarrow 6x - 8y = 27$

Thus statement-2 is also true but does not lead to statement-1.

48. Equation of the normal at $(a \sec \theta, b \tan \theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

It meets the axes at $L\left(\frac{a^2+b^2}{a\cos\theta}, 0\right)$, $M\left(0, \frac{a^2+b^2}{b\cot\theta}\right)$

If (h, k) is the mid-point of LM

then $h = \frac{a^2+b^2}{2a\cos\theta}$, $k = \frac{a^2+b^2}{2b\cot\theta}$

$\Rightarrow \sec\theta = \frac{2h}{9a}$, $\tan\theta = \frac{2bk}{9a^2}$ $[\because a^2 + b^2 = a^2e^2 = 9a^2]$

Eliminating θ, we get

$\frac{4h^2}{81a^2} - \frac{4b^2k^2}{81a^4} = 1$

Locus of (h, k) is $\frac{x^2}{\frac{81a^2}{4}} - \frac{y^2}{\frac{81a^4}{4b^2}} = 1$

which is a hyperbola and its eccentricity

$\sqrt{1+\frac{81a^4}{4b^2}\times\frac{4}{81a^2}} = \sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{1}{e^2-1}} = \sqrt{1+\frac{1}{8}}$

$= \frac{3}{2\sqrt{2}}$

So statement-1 is true. In statement-2, the eccentricity is

$\sqrt{1+\frac{8a^2}{a^2}} = 3$

and the statement-2 is false.

49. Equation of a tangent to the hyperbola is $y = mx + \sqrt{a^2m^2 - b^2}$ and the equation of the tangent perpendicular to it is $y = -\frac{1}{m}x + \sqrt{\frac{a^2}{m^2} - b^2}$.

Eliminating m we get the required locus as $(y - mx)^2 + (x + my)^2 = a^2m^2 - b^2 + a^2 - b^2m^2$

$\Rightarrow x^2 + y^2 = a^2 - b^2$ and the statement-2 is true using which statement-1 is false.

50. Equation of a tangent with slope $\frac{2}{\sqrt{5}}$ to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is

$y = \frac{2}{\sqrt{5}}x + \sqrt{9 \times \frac{4}{5} - 4} \Rightarrow 2x - \sqrt{5}\,y + 4 = 0$

Next $2x - \sqrt{5}\,y + 4 = 0$ touches the circle $(x-4)^2 + y^2 = 16$ if the length of the perpendicular from (4, 0) on the line is 4 which is true. Hence statement-1 is true.

In statement-2, let A be $(3\sec\theta, 2\tan\theta)$

A lies on the circle $x^2 + y^2 - 8x = 0$

$\Rightarrow 13\sec^2\theta - 24\sec\theta - 4 = 0$

$\Rightarrow \sec\theta = 2 \Rightarrow \tan\theta = \pm\sqrt{3}$

So the coordinate of A are $(6, 2\sqrt{3})$ and of B are $(6, -2\sqrt{3})$ and equation of the circle on AB as diameter is $(x-6)(x-6) + (y - 2\sqrt{3})(y + 2\sqrt{3}) = 0$.

$\Rightarrow x^2 + y^2 - 12x + 24 = 0$ which does not pass through the centre (0, 0) of the hyperbola. Thus statement-2 is false.

Level - 2

51. Let (h, k) be the mid-point of the chord, then its equation is $\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

Since it touches the circle $x^2 + y^2 = c^2$

$$\frac{\frac{h^2}{a^2} - \frac{k^2}{b^2}}{\sqrt{\left(\frac{h}{a^2}\right)^2 + \left(\frac{k}{b^2}\right)^2}} = \pm c$$

$$\Rightarrow \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 = c^2\left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right)$$

Required locus is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$$

52. Equation of the chord of contact from $\left(ct, \frac{c}{t}\right)$ on H_1 to H_2 is $(ct)x + \left(\frac{c}{t}\right)y = 2k^2$.

Asymptote of H_2 are $x = 0$ and $y = 0$ and the chord meets these asymptotes at $A\left(\frac{2k^2}{ct}, 0\right)$ and $B\left(0, \frac{2k^2 t}{c}\right)$

Area of the required triangle is $\frac{1}{2}\left|\frac{2k^2}{ct}\right|\left|\frac{2k^2 t}{c}\right| = \frac{2k^4}{c^2}$.

53. $e_1^2 = \frac{5 + 5\sin^2\theta}{5} = 1 + \sin^2\theta$.

$e_2^2 = \frac{5 - 5\sin^2\theta}{5} = \cos^2\theta$.

$e_1 = \sqrt{7}\; e_2 \Rightarrow 1 + \sin^2\theta = 7 - 7\sin^2\theta$

$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$.

54. Slopes of the asymptotes are $\pm\frac{b}{a}$

$$\Rightarrow \tan\theta = \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)}$$

$$\Rightarrow \frac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)} = \frac{2(b/a)}{1 - \left(\frac{b}{a}\right)^2}$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right) = \frac{b}{a} \text{ as } \tan\left(\frac{\theta}{2}\right) > 0$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1 + \frac{b^2}{a^2}}} = \frac{a}{\sqrt{a^2 + b^2}}$$

55. Let $OA = r_1$ and the coordinates of A be $(r_1\cos\alpha, r_1\sin\alpha)$.

If $OB = r_2$ then coordinates of B are

$$\left(r_2\cos\left(\alpha + \frac{\pi}{2}\right), r_2\sin\left(\alpha + \frac{\pi}{2}\right)\right)$$

As A, B lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$r_1^2\left(\frac{\cos^2\alpha}{a^2} - \frac{\sin^2\alpha}{b^2}\right) = 1$$

$$r_2^2\left(\frac{\sin^2\alpha}{a^2} - \frac{\cos^2\alpha}{b^2}\right) = 1 \Rightarrow \frac{1}{(OA)^2} + \frac{1}{(OB)^2}$$

$$= \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

56. $e^2 = \frac{25 + 16}{25} \Rightarrow e = \frac{\sqrt{41}}{5}$

Directrix is $x = \pm\frac{25}{\sqrt{41}}$ $\qquad \left(x = \pm\frac{a}{e}\right)$

Asymptotes are $\frac{x^2}{25} - \frac{y^2}{16} = 0$

So the required coordinates are $\left(-\frac{25}{\sqrt{41}}, \frac{20}{\sqrt{41}}\right)$

57. Normal at any point $(ct, c/t)$ on the hyperbola $xy = c^2$ is $y - (c/t) = t^2 (x - ct)$ or $t^3 x - ty + c - ct^4 = 0$

If it passes through another points $(c\alpha, c/\alpha)$ on the hyperbola, then

$$t^3 \times c\alpha - t \times \frac{c}{\alpha} + c - ct^4 = 0$$

$\Rightarrow t^3\alpha^2 - t + \alpha - \alpha t^4 = 0$

$\Rightarrow (t^3\alpha + 1)(\alpha - t) = 0$

$\Rightarrow t^3\alpha + 1 = 0$ as $\alpha \neq t$

which gives three values of t say t_1, t_2, t_3 and hence three points P, Q, R on the hyperbola the normals at which pass through S.

We have $t_1 + t_2 + t_3 = 0 = \Sigma t_1 t_2$ and $t_1 t_2 t_3 = -1/\alpha$.

Centroid of the triangle PQR is

$$\left(\frac{c(t_1 + t_2 + t_3)}{3}, \frac{c(1/t_1 + 1/t_2 + 1/t_3)}{3}\right) = (0, 0)$$

which is the centre of hyperbola.

58. Equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, so equation of the conjugate hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

Equation of a diameter of the hyperbola is $y = (b^2/a^2 m)\, x$, which meets the hyperbola at points given by

$$\frac{x^2}{a^2} - \frac{b^4 x^2}{b^2 a^4 m^2} = 1 \Rightarrow x^2 = \frac{a^2 m^2}{m^2 - b^2}$$

These points are real if $m^2 > b^2$ and for this value of m, the diameter will meet the conjugate hyperbola at point for which $x^2 = \frac{-a^2 m^2}{m^2 - b^2}$ and these points are imaginary.

Note: Conjugate diameter will meet the conjugate hyperbola in real points and the hyperbola in imaginary points.

59. Equation of a common tangent is $x = \pm 1$, nearer to $P\,(1/2, 1)$ is $x = 1$.

So the directrix of the ellipse is $x = 1$ and the focus is $P\,(1/2, 1)$, $e = 1/2$.

Thus the required equation is

$(x - 1/2)^2 + (y - 1)^2 = 1/4(1 - x)^2$

$\Rightarrow 3x^2 + 4y^2 - 2x - 8y + 4 = 0$

60. Equation of the asymptotes differ from the equation of the hyperbola by a constant so the equation of the asymptotes is $xy - hx - ky + \lambda = 0$ which represents a pair of lines if $\lambda = hk$ and the required asymptotes are $x = k$ and $y = h$.

Previous Years' AIEEE/JEE Main Questions

1. Foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ are $(\pm\sqrt{16 - b^2}, 0)$

and of the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ are $\pm\sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{9}{5}\right)^2}$

If the foci of the two conics coincide then $16 - b^2 = \frac{144 + 81}{25} = 9 \Rightarrow b^2 = 7$

2. Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is $y = mx + \sqrt{a^2 m^2 - b^2}$

Comparing it with $y = \alpha x + \beta$, we get

$m = \alpha$, $\beta = \sqrt{a^2 m^2 - b^2}$

$\Rightarrow \beta^2 = a^2\alpha^2 - b^2$

$\Rightarrow a^2\alpha^2 - \beta^2 = b^2$

$\Rightarrow$ Locus of (α, β) is $a^2x^2 - y^2 = b^2$

which is a hyperbola.

3. Equation of normal at point (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x)$$

It meets the x-axis at $G\left(x + y\frac{dx}{dy}, 0\right)$

We are given

$$\left(x + y\frac{dx}{dy}\right) = 2|x|$$

$$\Rightarrow x + y\frac{dy}{dx} = \pm 2x$$

$$\Rightarrow x + y\frac{dy}{dx} = 2x \text{ or } x + y\frac{dy}{dx} = -2x$$

$\Rightarrow y\,dy = x\,dx$

or $y\,dy = -3x\,dx$

$\Rightarrow x^2 - y^2 = c$

or $3x^2 + y^2 = c$

Thus curve is either a hyperbola or an ellipse.

4. $b^2 = a^2(e^2 - 1)$
$\Rightarrow \sin^2\alpha = \cos^2\alpha\,(e^2 - 1)$
$\Rightarrow e^2 - 1 = \tan^2\alpha \Rightarrow e = \sec\alpha$
Coordinates of foci are $(\pm ae, 0)$
$= (\pm 1, 0)$
$\therefore$ abscissae of foci remains constant.

5. Let the equation of the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,
$\pm ae = \pm 2$ and $e = 2 \Rightarrow a = 1$ and $b^2 = 1(4 - 1) \Rightarrow b^2 = 3$
and the required equation is $x^2 - \dfrac{y^2}{3} = 1$
or $3x^2 - y^2 = 3$.

6. Equation of the tangent at a point (x_1, y_1) on the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$ is $\dfrac{xx_1}{4} - \dfrac{yy_1}{2} = 1$

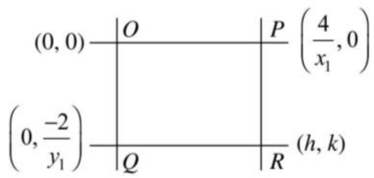


Fig. 20.10

So coordinates of P are $\left(\dfrac{4}{x_1}, 0\right)$ and of Q are $\left(0, \dfrac{-2}{y_1}\right)$

Let the coordinates of R be (h, k)

$h = \dfrac{4}{x_1},\ k = \dfrac{-2}{y_1}$

(x_1, y_1) lies on the hyperbola.

So $\dfrac{x_1^2}{4} - \dfrac{y_1^2}{2} = 1$

$\Rightarrow \dfrac{4}{h^2} - \dfrac{2}{k^2} = 1$

Locus of (h, k) is $\dfrac{4}{x^2} - \dfrac{2}{y^2} = 1$

7. Equation of a tangent to $x^2 = 6y$ is $x = my + \dfrac{3}{2m}$

$\Rightarrow y = \dfrac{1}{m}x - \dfrac{3}{2m^2}$

which touches the hyperbola $\dfrac{x^2}{\frac{9}{2}} - \dfrac{y^2}{\frac{9}{4}} = 1$

if $\left(\dfrac{-3}{2m^2}\right)^2 = \dfrac{9}{2} \times \dfrac{1}{m^2} - \dfrac{9}{4} \Rightarrow m^2 = 1$

So equation of a common tangent is $x = \pm y \pm \dfrac{3}{2}$ and the required equation is $x - y = \dfrac{3}{2}$.

8. An equation of the normal at P is $3x\cos\theta + 2y\cot\theta = 9 + 4$
$\Rightarrow 3x + 2y\,\text{cosec}\,\theta = 13\sec\theta$ (1)
An equation of normal at Q is
$3x + 2y\,\text{cosec}\,\phi = 13\sec\phi$ (2)
Subtracting (2) from (1), we get
$2y[\text{cosec}\,\theta - \text{cosec}\,\phi] = 13(\sec\theta - \sec\phi)$
$\Rightarrow 2y[\text{cosec}\,\theta - \sec\theta] = 13(\sec\theta - \text{cosec}\,\theta)$
$[\because\ \phi = \pi/2 - \theta]$

$\Rightarrow y = -\dfrac{13}{2}$

9. Coordinates of P, an extremity of latus rectum in the first quadrant are $\left(3, \dfrac{5}{2}\right)$. An equation of tangent at P is $\dfrac{3}{4}x - \dfrac{5}{2}\left(\dfrac{y}{5}\right) = 1$ or $3x - 2y = 4$

Coordinates of A are $\left(\dfrac{4}{3}, 0\right)$ and that of B are $(0, -2)$.

Now, $OA^2 - OB^2 = \dfrac{16}{9} - 4 = -\dfrac{20}{9}$.

10. Equation of hyperbola is
$\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$
If e_1 is the eccentricity of the hyperbola, then
$9 = 4(e_1^2 - 1) \Rightarrow e_1 = \sqrt{13}/2$
Foci of the hyperbola are $(\pm 2e_1, 0) = (\pm\sqrt{13}, 0)$
If e_2 is the eccentricity of the ellipse, then
$e_1e_2 = 1/2 \Rightarrow e_2 = 1/\sqrt{13}$.

Let the equation of ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

As it passes through $(\pm\sqrt{3}, 0)$,
$13/a^2 = 1 \Rightarrow a^2 = 13$.
Also, $b^2 = a^2(1 - e_2^2) = 13\left(1 - \dfrac{1}{13}\right) = 12$.

$\therefore$ equation of ellipse is $\dfrac{x^2}{13} + \dfrac{y^2}{12} = 1$.

$\left(\dfrac{1}{2}\sqrt{13}, \dfrac{\sqrt{3}}{2}\right)$ does not lie on it.

11. $\frac{2b^2}{a} = 8 \Rightarrow b^2 = 4a$

Also, $2b = \frac{1}{2}(2ae) \Rightarrow 2b = ae$

$\therefore \left(\frac{1}{2}ae\right)^2 = 4a \Rightarrow ae^2 = 16$

$\Rightarrow a\left(1+\frac{b^2}{a^2}\right) = 16$

$\Rightarrow a + \frac{b^2}{a} = 16 \Rightarrow a + 4 = 16$

$\Rightarrow a = 12$

$\therefore b^2 = 4a \Rightarrow a^2(e^2 - 1) = 4a$

$\Rightarrow a(e^2 - 1) = 4$

$\Rightarrow 12(e^2 - 1) = 4 \Rightarrow e^2 = 1 + \frac{1}{3}$

$\Rightarrow e = \frac{2}{\sqrt{3}}$

12. Equation of ellipse is

$\frac{x^2}{12}+\frac{y^2}{16} = 1$

Its major axis is along y-axis, and its foci are (0, 2) and (0, –2).

Let equation of hyperbola be $\frac{y^2}{a^2}-\frac{x^2}{b^2} = 1$, then $a = 2$ and $b^2 = a^2(e^2 - 1)$

$\Rightarrow b^2 = 4\left(\frac{9}{4}-1\right) = 5$

Thus, equation of hyperbola is

$$\frac{y^2}{4}-\frac{x^2}{5} = 1 \quad (1)$$

Note that $(\sqrt{5},2\sqrt{2})$, (0, 2) and $(\sqrt{10},2\sqrt{3})$ lie on (1)

13. As $9e^2 - 18e + 5 = 0$

$\Rightarrow 9e^2 - 15e - 3e + 5 = 0$

$\Rightarrow 3e(3e - 5) - (3e - 5) = 0$

$\Rightarrow (3e - 1)(3e - 5) = 0 \Rightarrow e = 1/3, 5/3$

As $e > 1$, $e = 5/3$

We have $ae = 5 \Rightarrow a = 3$

Also, $a^2 - b^2 = a^2 - a^2(e^2 - 1)$

$= 9(2 - 25/9) = -7$

Previous Years' B-Architecture Entrance Examination Questions

1. Equation of OP is $y = \frac{1}{\sqrt{3}}x$ which meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ at points for which $\frac{3y^2}{a^2}-\frac{y^2}{b^2} = 1$

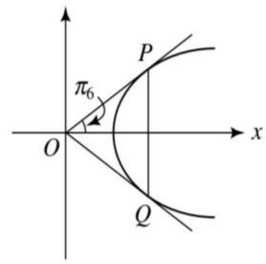


Fig. 20.11

$\Rightarrow (3b^2 - a^2)y^2 = a^2b^2$

For real values of y, $3b^2 > a^2$

$\Rightarrow 3a^2(e^2 - 1) > a^2 \Rightarrow 3e^2 > 4 \Rightarrow e > \frac{2}{\sqrt{3}}$

2. Equation of the tangent at (x_1, y_1) to the hyperbola $x^2 - y^2 = 4$ is

$xx_1 - yy_1 = 4$

then $a_1 = \frac{4}{x_1}$, $b_1 = \frac{-4}{y_1}$.

Equation of the normal at (x_1, y_1) is $y_1x + x_1y = 2x_1y_1$ then $a_2 = 2x_1$, $b_2 = 2y_1$.

Hence $a_1a_2 + b_1b_2 = 0$.

3. Let $y = mx + c$ be a common tangent to $x^2 - 2y^2 = 18$ and $x^2 + y^2 = 9$.

As it touches the circle, $c^2 = 9(1 + m^2)$ and as it touches the hyperbola, $c^2 = 18m^2 - 9$

So $18m^2 - 9 = 9 + 9m^2 \Rightarrow m^2 = 2$, $c = 3\sqrt{3}$ and the required equation is $y = \sqrt{2}x + 3\sqrt{3}$.

4. If coordinates of R are (x, y), then

$x = \frac{(3)(2)-2(2\tan\theta)}{3-2}$, $y = \frac{3(3)-2(3\sec\theta)}{3-2}$

$\Rightarrow \tan\theta = \frac{1}{4}(6 - x)$, $\sec\theta = \frac{1}{6}(9 - y)$

As $\sec^2\theta - \tan^2\theta = 1$, we get

$\frac{1}{36}(y - 9)^2 - \frac{1}{16}(x - 6)^2 = 1$

It is a hyperbola, length of whose transverse axis is $2(6) = 12$.

5. Foci of $\frac{x^2}{25}+\frac{y^2}{9} = 1$ are $(\sqrt{25-9},0)$ and $(-\sqrt{25-9},0)$ i.e. (4, 0) and (–4, 0).
   If a is the length of transverse axis of the hyperbola, then
   $ae = 4 \Rightarrow a = 2$

Also, $b^2 = a^2(e^2 - 1) = 4(4 - 1) = 12$

$\therefore$ Equation of hyperbola is $\frac{x^2}{4}-\frac{y^2}{12} = 1$

An equation of tangent at (4, 6) is $\frac{4x}{4}-\frac{6y}{12} = 1$

or $2x - y = 2$.