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24
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XI PROPERTIES

PROPERTIES OF TRIANGLES
MATHEMATICS

YOUR GATEWAY TO EXCELLENCE IN
IIT-JEE, NEET AND CBSE EXAMS

PROPERTIES OF TRIANGLES

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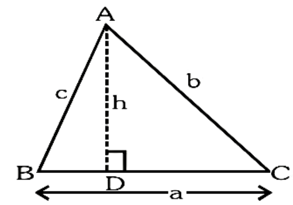
PROPERTIES OF TRIANGLE

The process of calculating the sides and angles of triangle using given information is called solution of triangle. In a ΔABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

1. SINE FORMULAE :



In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.

Illustration 1 : Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

(A) $\frac{3}{2+\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2-\sqrt{3}}$ (D) $\frac{1}{2+\sqrt{3}}$

Solution : Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120 , 30 , 30 .

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side. Now from

sine formula $\frac{a}{\sin 120°} = \frac{b}{\sin 30°} = \frac{c}{\sin 30°}$

$\Rightarrow \quad \frac{a}{\sqrt{3}/2} = \frac{b}{1/2} = \frac{c}{1/2}$

$\Rightarrow \quad \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = k$ (say)

then $a = \sqrt{3}k$, perimeter $= (2+\sqrt{3})k$

$\therefore$ required ratio $= \frac{\sqrt{3}k}{(2+\sqrt{3})k} = \frac{\sqrt{3}}{2+\sqrt{3}}$ **Ans. (B)**

Illustration 2 : In triangle ABC, if b = 3, c = 4 and $\angle B = \pi/3$, then number of such triangles is -

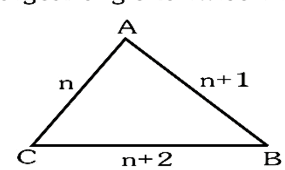
(A) 1 (B) 2 (C) 0 (D) infinite

Solution : Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

$\Rightarrow \frac{\sin \pi/3}{3} = \frac{\sin C}{4} \Rightarrow \frac{\sqrt{3}}{6} = \frac{\sin C}{4} \Rightarrow \sin C = \frac{2}{\sqrt{3}} > 1$ which is not possible.

Hence there exist no triangle with given elements. **Ans. (C)**

Illustration 3 : The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.



Solution : Let the sides be n, n + 1, n + 2 cms.

i.e. AC = n, AB = n + 1, BC = n + 2

Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180$

$\Rightarrow \quad 3\angle B + \angle C = 180 \quad \Rightarrow \angle C = 180 - 3\angle B$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \quad \Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180-3B)}{n+1}$$

$$\Rightarrow \quad \underset{(i)}{\frac{\sin 2B}{n+2}} = \underset{(ii)}{\frac{\sin B}{n}} = \underset{(iii)}{\frac{\sin 3B}{n+1}}$$

from (i) and (ii);

$$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \quad \Rightarrow \quad \cos B = \frac{n+2}{2n} \qquad \ldots\ldots\ldots \text{(iv)}$$

and from (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \quad \Rightarrow \quad \frac{\sin B}{n} = \frac{\sin B(3-4\sin^2 B)}{n+1}$$

$$\Rightarrow \quad \frac{n+1}{n} = 3 - 4(1-\cos^2 B) \qquad \ldots\ldots\ldots\ldots \text{(v)}$$

from (iv) and (v), we get

$$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2 \quad \Rightarrow \quad \frac{n+1}{n} + 1 = \left(\frac{n^2+4n+4}{n^2}\right)$$

$$\Rightarrow \quad \frac{2n+1}{n} = \frac{n^2+4n+4}{n^2} \quad \Rightarrow \quad 2n^2 + n = n^2 + 4n + 4$$

$$\Rightarrow \quad n^2 - 3n - 4 = 0 \quad \Rightarrow \quad (n-4)(n+1) = 0$$

$n = 4$ or -1

where $n \neq -1$

$\therefore$ $n = 4$. Hence the sides are 4, 5, 6 **Ans.**

Do yourself - 1 :

(i) If in a ΔABC, $\angle A = \frac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

(ii) Show that, in any ΔABC : $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.

(iii) If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, show that a^2, b^2, c^2 are in A.P.

(iv) If in a ΔABC, $\angle A = 3\angle B$, then prove that $\sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}$.

2. COSINE FORMULAE :

(a) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ (b) $\cos B = \frac{c^2+a^2-b^2}{2ca}$ (c) $\cos C = \frac{a^2+b^2-c^2}{2ab}$

or $a^2 = b^2 + c^2 - 2bc\cos A$

Illustration 4 : In a triangle ABC, if B = 30 and $c = \sqrt{3}\,b$, then A can be equal to -

(A) 45 (B) 60 (C) 90 (D) 120

Solution : We have $\cos B = \frac{c^2+a^2-b^2}{2ca} \Rightarrow \frac{\sqrt{3}}{2} = \frac{3b^2+a^2-b^2}{2\times\sqrt{3}b\times a}$

$\Rightarrow$ $a^2 - 3ab + 2b^2 = 0$ $\Rightarrow$ $(a-2b)(a-b) = 0$

$\Rightarrow$ Either $a = b$ $\Rightarrow$ $A = 30$

or $a = 2b$ $\Rightarrow$ $a^2 = 4b^2 = b^2 + c^2$ $\Rightarrow$ $A = 90$. **Ans. (C)**

Illustration 5 : In a triangle ABC, $(a^2 - b^2 - c^2)\tan A + (a^2 - b^2 + c^2)\tan B$ is equal to -

(A) $(a^2 + b^2 - c^2)\tan C$ (B) $(a^2 + b^2 + c^2)\tan C$

(C) $(b^2 + c^2 - a^2)\tan C$ (D) none of these

Solution : Using cosine law :

The given expression is equal to $-2bc\cos A\tan A + 2ac\cos B\tan B$

$= 2abc\left(-\frac{\sin A}{a} + \frac{\sin B}{b}\right) = 0$ **Ans. (D)**

Illustration 6 : If in a triangle ABC, $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ac}$, find the $\angle A$ =

(A) 90 (B) 60 (C) 30 (D) none of these

Solution : We have $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ac}$

Multiplying both sides of abc, we get

$\Rightarrow$ $2bc\cos A + ac\cos B + 2ab\cos C = a^2 + b^2$

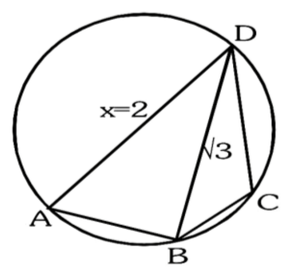
$\Rightarrow$ $(b^2 + c^2 - a^2) + \frac{(a^2 + c^2 - b^2)}{2} + (a^2 + b^2 - c^2) = a^2 + b^2$

$\Rightarrow$ $c^2 + a^2 - b^2 = 2a^2 - 2b^2$ $\Rightarrow$ $b^2 + c^2 = a^2$

$\therefore$ ΔABC is right angled at A. $\Rightarrow$ $\angle A = 90$ **Ans. (A)**

Illustration 7 : A cyclic quadrilateral ABCD of area $\frac{3\sqrt{3}}{4}$ is inscribed in unit circle. If one of its side AB = 1, and the diagonal $BD = \sqrt{3}$, find lengths of the other sides.

Solution : AB = 1, $BD = \sqrt{3}$, OA = OB = OD = 1



The given circle of radius 1 is also circumcircle of Δ ABD

$\Rightarrow$ R = 1 for ΔABD

$\Rightarrow \frac{a}{\sin A} = 2R \Rightarrow A = 60$

and hence C = 120

Also by cosine rule on ΔABD, $\left(\sqrt{3}\right)^2 = 1^2 + x^2 - 2x\cos 60°$

$\Rightarrow x = 2$

Now, area ABCD = $\Delta ABD + \Delta BCD$

$\Rightarrow \frac{3\sqrt{3}}{4} = \frac{1}{2}(1.2.\sin 60°) + \frac{1}{2}(c.d.\sin 120°)$

$\Rightarrow cd = 1$, or $c^2d^2 = 1$

Also by cosine rule on triangle BCD we have

$\left(\sqrt{3}\right)^2 = c^2 + d^2 - 2cd\cos 120° = c^2 + d^2 + cd$

$\Rightarrow c^2 + d^2 = 2$ or $cd = 1$

$\Rightarrow c^2$ and d^2 are the roots of $t^2 - 2t + 1 = 0$

$\therefore c^2 = d^2 = 1$ $\therefore$ BC = 1 = CD and AD = x = 2.

Do yourself - 2 :

(i) If a : b : c = 4 : 5 : 6, then show that $\angle C = 2\angle A$.

(ii) In any ΔABC, prove that

(a) $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

(b) $\frac{b^2}{a}\cos A+\frac{c^2}{b}\cos B+\frac{a^2}{c}\cos C=\frac{a^4+b^4+c^4}{2abc}$

3. PROJECTION FORMULAE :

(a) $b\cos C + c\cos B = a$ **(b)** $c\cos A + a\cos C = b$ **(c)** $a\cos B + b\cos A = c$

Illustration 8 : In a $\Delta ABC, c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}=\frac{3b}{2}$, then show a, b, c are in A.P.

Solution : Here, $\frac{c}{2}(1+\cos A)+\frac{a}{2}(1+\cos C)=\frac{3b}{2}$

$\Rightarrow$ $a + c + (c\cos A + a\cos C) = 3b$

$\Rightarrow$ $a + c + b = 3b$ {using projection formula}

$\Rightarrow$ $a + c = 2b$

which shows a, b, c are in A.P.

Do yourself - 3 :

(i) In a ΔABC, if $\angle A=\frac{\pi}{4}$, $\angle B=\frac{5\pi}{12}$, show that $a+c\sqrt{2}=2b$.

(ii) In a ΔABC, prove that : (a) $b(a\cos C - c\cos A) = a^2 - c^2$ (b) $2\left(b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}\right)=a+b+c$

4. NAPIER'S ANALOGY (TANGENT RULE) :

(a) $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$ **(b)** $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$ **(c)** $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

Illustration 9 : In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution : Here, $\tan\left(\frac{A-B}{2}\right)=\frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ .... (i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$ .... (ii)

from (i) & (ii) ;

$\frac{1}{3}\tan\left(\frac{A+B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right) \Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$

{as $A + B + C = \pi$ $\therefore$ $\tan\left(\frac{B+C}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}$ }

$\Rightarrow$ $\frac{a-b}{a+b}=\frac{1}{3}$ or $3a - 3b = a + b$

$2a = 4b$ or $\frac{a}{b} = \frac{2}{1} \Rightarrow \frac{b}{a} = \frac{1}{2}$

Thus the ratio of the sides opposite to the angles is b : a = 1 : 2. Ans.

Do yourself - 4 :

(i) In any ΔABC, prove that $\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$

(ii) If ΔABC is right angled at C, prove that : (a) $\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B) = \frac{a^2-b^2}{a^2+b^2}$

(iii) If in a ΔABC, two sides are a = 3, b = 5 and $\cos(A - B) = \frac{7}{25}$, find $\tan\frac{C}{2}$.

5. **HALF ANGLE FORMULAE :**

$s = \frac{a+b+c}{2}$ = semi-perimeter of triangle.

(a) (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ (ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{\Delta}{s(s-b)}$ (iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\Delta}{s(s-c)}$

(d) **Area of Triangle**

$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$, where p_1, p_2, p_3 are altitudes from vertices A,B,C respectively.

Illustration 10 : If in a triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to -

(A) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (B) $\frac{2ab}{a+b}\sin\frac{C}{2}$ (C) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (D) $\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Solution : $\Delta CAB = \Delta CAD + \Delta CDB$

$\Rightarrow \frac{1}{2}ab\sin C = \frac{1}{2}b.CD.\sin\left(\frac{C}{2}\right) + \frac{1}{2}a.CD\ \sin\left(\frac{C}{2}\right)$

$\Rightarrow CD(a+b)\sin\left(\frac{C}{2}\right) = ab\left(2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\right)$

So $CD = \frac{2ab\cos(C/2)}{(a+b)}$

and in ΔCAD, $\frac{CD}{\sin\angle DAC} = \frac{b}{\sin\angle CDA}$ (by sine rule)

$\Rightarrow CD = \frac{b\sin\angle DAC}{\sin(B+C/2)}$

Ans. (C,D)

Illustration 11 : If Δ is the area and 2s the sum of the sides of a triangle, then show $\Delta \le \frac{s^2}{3\sqrt{3}}$.

Solution : We have, $2s = a + b + c$, $\Delta^2 = s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

$$\frac{(s-a)+(s-b)+(s-c)}{3} \ge \{(s-a)(s-b)(s-c)\}^{1/3}$$

or $$\frac{3s-2s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$$

or $$\frac{s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$$

or $$\frac{\Delta^2}{s} \le \frac{s^3}{27} \Rightarrow \Delta \le \frac{s^2}{3\sqrt{3}}$$ **Ans.**

Do yourself - 5 :

(i) Given a = 6, b = 8, c = 10. Find

(a) sinA (b) tanA (c) $\sin\frac{A}{2}$ (d) $\cos\frac{A}{2}$ (e) $\tan\frac{A}{2}$ (f) Δ

(ii) Prove that in any ΔABC, $(abcs)\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2} = \Delta^2$.

(iii) Show that if $\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right) = \frac{2}{3}\cot\frac{B}{2}$, then a, b, c are in A.P.

6. m-n THEOREM :

$(m + n) \cot\theta = m \cot\alpha - n \cot\beta$

$(m + n) \cot\theta = n \cot B - m \cot C$.

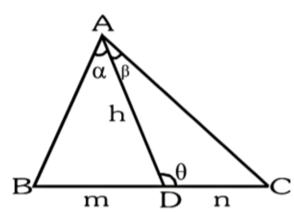


Illustration 12 : The base of a Δ is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that : $\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right) = 4\left(1+\frac{1}{t_2^2}\right)$

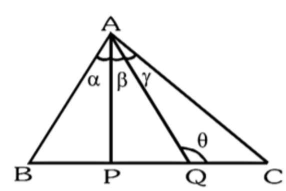
Solution : Let the points P and Q divide the side BC in three equal parts :

Such that BP = PQ = QC = x

Also let,

$\angle BAP = \alpha$, $\angle PAQ = \beta$, $\angle QAC = \gamma$

and $\angle AQC = \theta$



From question, $\tan\alpha = t_1$, $\tan\beta = t_2$, $\tan\gamma = t_3$.

Applying

m : n rule in triangle ABC we get,

$(2x + x)\cot\theta = 2x\cot(\alpha + \beta) - x\cot\gamma$ .... (i)

from ΔAPC, we get

$(x + x)\cot\theta = x\cot\beta - x\cot\gamma$ .... (ii) dividing (i) and (ii), we get

$$\frac{3}{2} = \frac{2\cot(\alpha+\beta) - \cot\gamma}{\cot\beta - \cot\gamma}$$

or $$3\cot\beta - \cot\gamma = \frac{4(\cot\alpha.\cot\beta - 1)}{\cot\beta + \cot\alpha}$$

or $$3\cot^2\beta - \cot\beta\cot\gamma + 3\cot\alpha.\cot\beta - \cot\alpha.\cot\gamma = 4\cot\alpha.\cot\beta - 4$$

or $$4 + 4\cot^2\beta = \cot^2\beta + \cot\alpha\,.\,\cot\beta + \cot\beta\,.\,\cot\gamma + \cot\gamma\,.\,\cot\alpha$$

or $$4(1 + \cot^2\beta) = (\cot\beta + \cot\alpha)(\cot\beta + \cot\gamma)$$

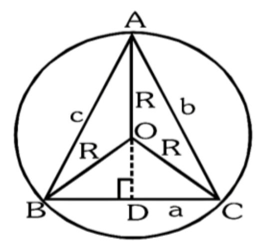
or $$4\left(1 + \frac{1}{t_2^2}\right) = \left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right)$$

Do yourself - 6 :

(i) The median AD of a ΔABC is perpendicular to AB, prove that tanA + 2tanB = 0

7. RADIUS OF THE CIRCUMCIRCLE 'R' :

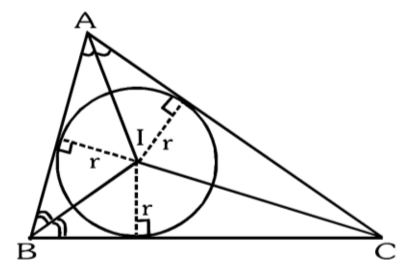
Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.



$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$

8. RADIUS OF THE INCIRCLE 'r' :

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.



$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

Illustration 13 : In a triangle ABC, if a : b : c = 4 : 5 : 6, then ratio between its circumradius and inradius is-

(A) $\frac{16}{7}$ (B) $\frac{16}{9}$ (C) $\frac{7}{16}$ (D) $\frac{11}{7}$

Solution :

$$\frac{R}{r} = \frac{abc}{4\Delta}\Big/\frac{\Delta}{s} = \frac{(abc)s}{4\Delta^2} \Rightarrow \frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)} \quad \text{....(i)}$$

$\because$ a : b : c = 4 : 5 : 6 $\Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$ (say)

$\Rightarrow$ a = 4k, b = 5k, c = 6k

$\therefore$ $s = \frac{a+b+c}{2} = \frac{15k}{2}$, $s - a = \frac{7k}{2}$, $s - b = \frac{5k}{2}$, $s - c = \frac{3k}{2}$

using (i) in these values $\frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)} = \frac{16}{7}$

Ans. (A)

Illustration 14 : If A, B, C are the angles of a triangle, prove that : $\cos A + \cos B + \cos C = 1+\frac{r}{R}$.

Solution : $\cos A + \cos B + \cos C = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)+\cos C$

$= 2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2} = 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]$

$= 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right] \qquad \left\{\because \frac{C}{2}=90^\circ-\left(\frac{A+B}{2}\right)\right\}$

$= 1+2\sin\frac{C}{2}.2\sin\frac{A}{2}.\sin\frac{B}{2} = 1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

$= 1+\frac{r}{R}$ {as, r = 4R sin A/2 . sinB/2 . sinC/2}

$\Rightarrow$ $\cos A + \cos B + \cos C = 1+\frac{r}{R}$. Hence proved.

Do yourself - 7 :

(i) If in ΔABC, a = 3, b = 4 and c = 5, find

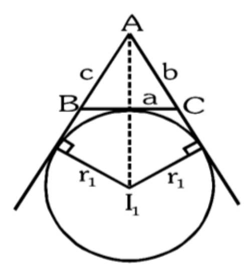
(a) Δ (b) R (c) r

(ii) In a ΔABC, show that :

(a) $\frac{a^2-b^2}{c} = 2R\sin(A-B)$ (b) $r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{\Delta}{4R}$ (c) $a+b+c = \frac{abc}{2Rr}$

(iii) Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that $\Delta : \Delta' = \left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right):\pi$

9. RADII OF THE EX-CIRCLES :



Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to $\angle A$ of ΔABC and so on, then -

(a) $r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(b) $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$

(c) $r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

I_1, I_2 and I_3 are taken as ex-centre opposite to vertex A, B, C repsectively.

Illustration 15 : Value of the expression $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 0

Solution : $\frac{(b-c)}{r_1}+\frac{(c-a)}{r_2}+\frac{(a-b)}{r_3}$

$\Rightarrow \quad (b-c)\left(\frac{s-a}{\Delta}\right)+(c-a)\left(\frac{s-b}{\Delta}\right)+(a-b).\left(\frac{s-c}{\Delta}\right)$

$\Rightarrow \quad \frac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$

$=\frac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}=\frac{0}{\Delta}=0$

Thus, $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$ **Ans. (D)**

Illustration 16 : If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Solution : We have, $r_1 - r = r_2 + r_3$

$\Rightarrow \quad \frac{\Delta}{s-a}-\frac{\Delta}{s}=\frac{\Delta}{s-b}+\frac{\Delta}{s-c} \quad \Rightarrow \quad \frac{s-s+a}{s(s-a)}=\frac{s-c+s-b}{(s-b)(s-c)}$

$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{2s-(b+c)}{(s-b)(s-c)}$ {as, $2s = a + b + c$}

$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{a}{(s-b)(s-c)} \quad \Rightarrow \quad s^2-(b+c)s+bc=s^2-as$

$\Rightarrow \quad s(-a+b+c)=bc \quad \Rightarrow \quad \frac{(b+c-a)(a+b+c)}{2}=bc$

$\Rightarrow \quad (b+c)^2-(a)^2=2bc \quad \Rightarrow \quad b^2+c^2+2bc-a^2=2bc$

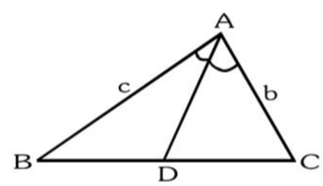
$\Rightarrow \quad b^2+c^2=a^2$

$\therefore \quad \angle A = 90$. **Ans.**

> **Do yourself - 8 :**
>
> **(i)** In an equilateral ΔABC, R = 2, find
>
> (a) r (b) r_1 (c) a
>
> **(ii)** In a ΔABC, show that
>
> (a) $r_1r_2+r_2r_3+r_3r_1=s^2$ (b) $\frac{1}{4}r^2s^2\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=\frac{r+r_1+r_2-r_3}{4\cos C}=R$
>
> (c) $\sqrt{rr_1r_2r_3}=\Delta$

10. ANGLE BISECTORS & MEDIANS :

An angle bisector divides the base in the ratio of corresponding sides.



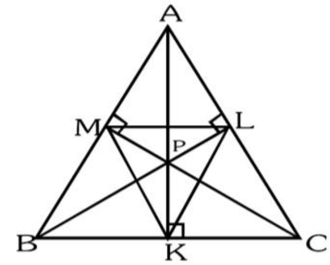
$\frac{BD}{CD}=\frac{c}{b} \quad \Rightarrow \quad BD=\frac{ac}{b+c} \quad \& \quad CD=\frac{ab}{b+c}$

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ and $\beta_a=\frac{2bc\cos\frac{A}{2}}{b+c}$

Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

11. ORTHOCENTRE :



(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.

(b) The distances of the orthocentre from the angular points of the ΔABC are 2R cosA, 2R cosB, & 2R cosC.

(c) The distance of P from sides are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB.

Do yourself - 9 :

(i) If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$

(ii) If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(a) $p_1 p_2 p_3 = \frac{a^2 b^2 c^2}{8R^3}$ (b) $\Delta = \sqrt{\frac{1}{2} R p_1 p_2 p_3}$

(iii) In a ΔABC, AD is altitude and H is the orthocentre prove that AH : DH = (tanB + tanC) : tanA

(iv) In a ΔABC, the lengths of the bisectors of the angle A, B and C are x, y, z respectively. Show that

$\frac{1}{x}\cos\frac{A}{2} + \frac{1}{y}\cos\frac{B}{2} + \frac{1}{z}\cos\frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Also show that $\frac{a}{b+c} = \sqrt{1 - \frac{x^2}{bc}}$

12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre is $= R\sqrt{1 - 8\cos A \cos B \cos C}$

(b) The distance between circumcentre and incentre is $= \sqrt{R^2 - 2Rr}$

(c) The distance between incentre and orthocentre is $= \sqrt{2r^2 - 4R^2 \cos A \cos B \cos C}$

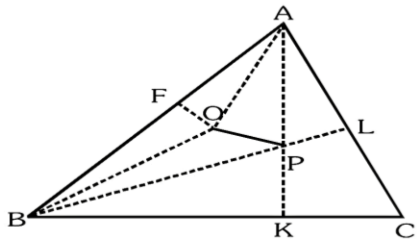
(d) The distances between circumcentre & excentres are

$OI_1 = R\sqrt{1 + 8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R^2 + 2Rr_1}$ & so on.

Illustration 17 : Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1 - 8\cos A \cos B \cos C}$.

Solution : Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpendicular to AB, we have $\angle OAF = 90 - \angle AOF = 90 - C$. Also $\angle PAL = 90 - C$.

Hence, $\angle OAP = A - \angle OAF - \angle PAL = A - 2(90 - C) = A + 2C - 180$



= A + 2C – (A + B + C) = C – B.

Also OA = R and PA = 2RcosA.

Now in ΔAOP,

$OP^2 = OA^2 + PA^2 - 2OA.\ PA\ \cos OAP$

$= R^2 + 4R^2\cos^2 A - 4R^2 \cos A\cos(C - B)$

$= R^2 + 4R^2 \cos A[\cos A - \cos(C - B)]$

$= R^2 - 4R^2 \cos A[\cos(B + C) + \cos(C - B)] = R^2 - 8R^2 \cos A \cos B \cos C.$

Hence $OP = R\sqrt{1 - 8\cos A\cos B\cos C}$. **Ans.**

Do yourself - 10 :

(i) Show that in an equilateral triangle, circumcentre, orthocentre and incentre overlap each other.

(ii) If the incentre and circumcentre of a triangle are equidistant from the side BC, show that cosB + cosC = 1.

13. SOLUTION OF TRIANGLES :

The three sides a,b,c and the three angles A,B,C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

* If the three sides a,b,c are given, angle A is obtained from $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ .B and C can be obtained in the similar way.

* If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ gives $\frac{B-C}{2}$. Also

$\frac{B+C}{2} = 90° - \frac{A}{2}$, so that B and C can be evaluated. The third side is given by a = b $\frac{\sin A}{\sin B}$

or $a^2 = b^2 + c^2 - 2bc\cos A$.

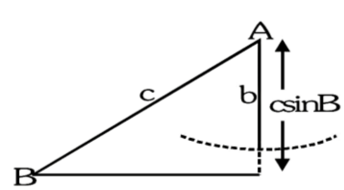
* If two sides b and c and an angle opposite the one of them (say B) are given then

$\sin C = \frac{c}{b}\sin B,\ A = 180° - (B + C)$ and $a = \frac{b\sin A}{\sin B}$ given the remaining elements.

Case I :

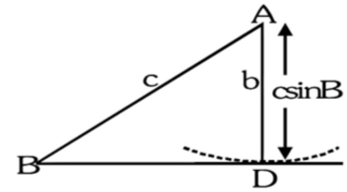
$b < c \sin B$.

We draw the side c and angle B. Now it is obvious from the figure that there is no triangle possible.



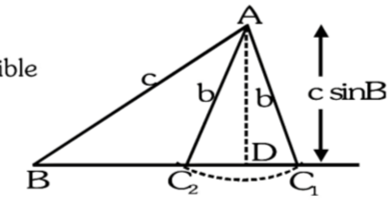
Case II :

b = c sin B and B is an acute angle, there is only one triangle possible.
and it is right-angled at C.



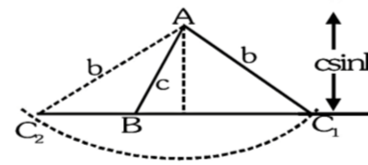
Case III :

b > c sin B, b < c and B is an acute angle, then there are two triangles possible for two values of angle C.



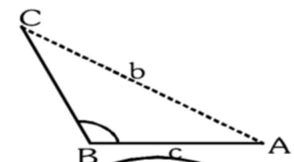
Case IV :

b > c sin B, c < b and B is an acute angle, then there is only one triangle.



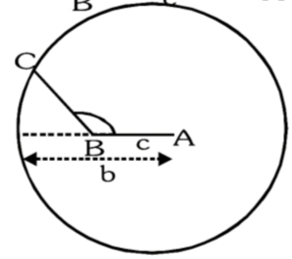
Case V :

b > c sin B, c > b and B is an obtuse angle.
For any choice of point C, b will be greater than c which is a contradication as c > b (given). So there is no triangle possible.



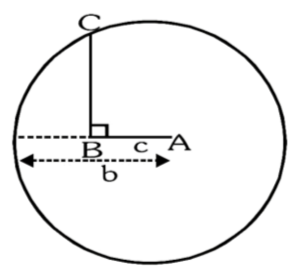
Case VI :

b > c sin B, c < b and B is an obtuse angle.
We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



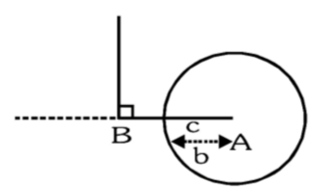
Case VII :

b > c and B = 90 .
Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



Case VIII :

b $\leq$ c and B = 90 .
The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.



This is, sometimes, called an ambiguous case.

Alternative Method :

By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 - (2c \cos B)a + (c^2 - b^2) = 0 \Rightarrow a = c \cos B \pm \sqrt{(c \cos B)^2 - (c^2 - b^2)}$

$\Rightarrow \quad a = c \cos B \pm \sqrt{b^2 - (c \sin B)^2}$

This equation leads to following cases :

Case-I : If $b < c\sin B$, no such triangle is possible.

Case-II: Let $b = c \sin B$. There are further following case :

(a) B is an obtuse angle $\Rightarrow$ cosB is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow$ cosB is positive. There exists only one such triangle.

Case-III: Let $b > c \sin B$. There are further following cases :

(a) B is an acute angle $\Rightarrow$ cosB is positive. In this case triangle will exist if and only if $c\cos B > \sqrt{b^2-(c\sin B)^2}$ or $c > b \Rightarrow$ Two such triangle is possible. If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cosB is negative. In this case triangle will exist if and only if $\sqrt{b^2-(c\sin B)^2} > |c\cos B| \Rightarrow b > c$. So in this case only one such triangle is possible. If $b < c$ there exists no such triangle.

This is called an ambiguous case.

* If one side a and angles B and C are given, then A = 180 – (B + C), and $b = \dfrac{a\sin B}{\sin A}, c = \dfrac{a\sin C}{\sin A}$.

* If the three angles A,B,C are given, we can only find the ratios of the sides a,b,c by using sine rule (since there are infinite similar triangles possible).

Illustration 18 : In the ambiguous case of the solution of triangles, prove that the circumcircles of the two triangles are of same size.

Solution : Let us say b,c and angle B are given in the ambiguous case. Both the triangles will have b and its opposite angle as B. so $\dfrac{b}{\sin B} = 2R$ will be given for both the triangles. So their circumradii and therefore their sizes will be same.

Illustration 19 : If a,b and A are given in a triangle and c_1, c_2 are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2\cos 2A = 4a^2\cos^2 A$.

Solution :
$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$
$\Rightarrow c^2 - 2bc\cos A + b^2 - a^2 = 0.$

$c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$.

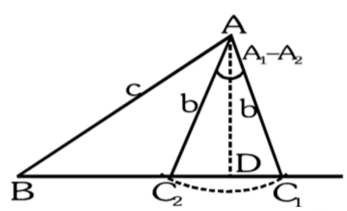
$\Rightarrow c_1^2 + c_2^2 - 2c_1c_2\cos 2A = (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$

$= 4b^2\cos^2 A - 2(b^2 - a^2)2\cos^2 A = 4a^2\cos^2 A.$

Illustration 20 : If b,c,B are given and $b < c$, prove that $\cos\left(\dfrac{A_1 - A_2}{2}\right) = \dfrac{c\sin B}{b}$.

Solution : $\angle C_2AC_1$ is bisected by AD.

$\Rightarrow$ In ΔAC_2D, $\cos\left(\dfrac{A_1-A_2}{2}\right) = \dfrac{AD}{AC_2} = \dfrac{c\sin B}{b}$



Hence proved.

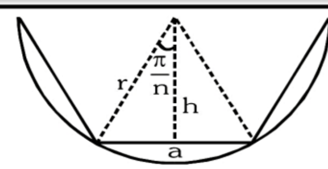
Do yourself - 11 :

(i) If b,c,B are given and b<c, prove that $\sin\left(\frac{A_1 - A_2}{2}\right) = \frac{a_1 - a_2}{2b}$

(ii) In a ΔABC, b,c,B (c > b) are gives. If the third side has two values a_1 and a_2 such that $a_1 = 3a_2$, show that $\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}$.

14. REGULAR POLYGON :

A regular polygon has all its sides equal. It may be inscribed or circumscribed.

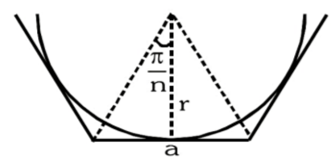


(a) Inscribed in circle of radius r :

(i) $a = 2h \tan\frac{\pi}{n} = 2r\sin\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr\sin\frac{\pi}{n}$ and $A = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$

(b) Circumscribed about a circle of radius r :



(i) $a = 2r\tan\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides circumscribed about a given circle of radius r is given by $P = 2nr\tan\frac{\pi}{n}$ and $A = nr^2\tan\frac{\pi}{n}$

Do yourself - 12 :

(i) If the perimeter of a circle and a regular polygon of n sides are equal, then prove that $\frac{\text{area of the circle}}{\text{area of polygon}} = \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$.

(ii) The ratio of the area of n-sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is 4 : 3. Find the value of n.

15. IMPORTANT POINTS :

(a) (i) If a cos B = b cos A, then the triangle is isosceles.

(ii) If a cos A = b cos B, then the triangle is isosceles or right angled.

(b) In right angle triangle

(i) $a^2 + b^2 + c^2 = 8R^2$ (ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(c) In equilateral triangle

(i) $R = 2r$ (ii) $r_1 = r_2 = r_3 = \frac{3R}{2}$

(iii) $r : R : r_1 = 1 : 2 : 3$ (iv) area $= \frac{\sqrt{3}a^2}{4}$ (v) $R = \frac{a}{\sqrt{3}}$

(d) (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.

(ii) The orthocentre of right angled triangle is the vertex at the right angle.

(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1 except in case of equilateral triangle. In equilateral triangle, all these centres coincide

(e) Area of a cyclic quadrilateral $= \sqrt{s(s-a)(s-b)(s-c)(s-d)}$

where a, b, c, d are lengths of the sides of quadrilateral and $s = \frac{a+b+c+d}{2}$

Illustration 21 : For a ΔABC, it is given that $\cos A + \cos B + \cos C = 3/2$. Prove that the triangle is equilateral.

Solution : If a, b, c are the sides of the ΔABC, then $\cos A + \cos B + \cos C = 3/2$

$$\Rightarrow \frac{b^2+c^2-a^2}{2bc}+\frac{a^2+c^2-b^2}{2ac}+\frac{a^2+b^2-c^2}{2ab}=\frac{3}{2}$$

$$\Rightarrow ab^2 + ac^2 - a^3 + bc^2 + ba^2 - b^3 + ca^2 + cb^2 - c^3 = 3abc$$

$$\Rightarrow ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc = a^3 + b^3 + c^3 - 3abc$$

$$\Rightarrow a(b-c)^2+b(c-a)^2+c(a-b)^2=\frac{(a+b+c)}{2}\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}$$

$$\Rightarrow (a+b-c)(a-b)^2 + (b+c-a)(b-c)^2 + (c+a-b)(c-a)^2 = 0 \quad \text{.......... (i)}$$

as we know $a + b > c$, $b + c > a$, $c + a > b$

$\therefore$ each term on the left side of equation (i) has positive coefficient multiplied by perfect square, each must be separately zero.

$\Rightarrow$ $a = b = c$.

Hence Δ is equilateral. **Ans.**

Illustration 22 : In a triangle ABC, if $\cos A + 2\cos B + \cos C = 2$. Prove that the sides of the triangle are in A.P.

Solution : $\cos A + 2\cos B + \cos C = 2$ or $\cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow 2\cos\left(\frac{A+C}{2}\right).\cos\left(\frac{A-C}{2}\right)=4\sin^2 B/2$$

$$\Rightarrow \cos\left(\frac{A-C}{2}\right)=2\sin\frac{B}{2} \qquad \left\{\text{as } \cos\left(\frac{A+C}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)=\sin\frac{B}{2}\right\}$$

$$\Rightarrow \cos\left(\frac{A-C}{2}\right)=2\cos\left(\frac{A+C}{2}\right)$$

$$\Rightarrow \cos\frac{A}{2}.\cos\frac{C}{2}+\sin\frac{A}{2}.\sin\frac{C}{2}=2\cos\frac{A}{2}.\cos\frac{C}{2}-2\sin\frac{A}{2}.\sin\frac{C}{2}$$

$$\Rightarrow \cot\frac{A}{2}.\cot\frac{C}{2}=3 \Rightarrow \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=3$$

$$\Rightarrow \frac{s}{(s-b)}=3 \Rightarrow s=3s-3b \Rightarrow 2s = 3b$$

$\Rightarrow$ $a + c = 2b$, $\therefore$ a, b, c are in A.P. **Ans.**

ANSWERS FOR DO YOURSELF

1 : (i) 90

4 : (iii) $\frac{1}{3}$

5 : (i) (a) $\frac{3}{5}$ (b) $\frac{3}{4}$ (c) $\frac{1}{\sqrt{10}}$ (d) $\frac{3}{\sqrt{10}}$ (e) $\frac{1}{3}$ (f) 24

7 : (i) (a) 6 (b) $\frac{5}{2}$ (c) 1

8 : (i) (a) 1 (b) 3 (c) $2\sqrt{3}$

12 : (ii) 6

PRACTICE SET

A SINGLE CORRECT CHOICE TYPE

Each of these questions has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. In a ΔABC if $r_1 < r_2 < r_3$, then

(a) $a < b < c$ (b) $a > b > c$
(c) $b < a < c$ (d) $a < c < b$

2. If the sines of the angles A and B of a triangle ABC satisfy the equation $c^2x^2 - c(a+b)x + ab = 0$, then the triangle is ($a,b,c$, are sides of Δ)

(a) acute angled
(b) obtuse angled
(c) right angled
(d) no such triangle is possible

3. In ΔABC, $a \ge b \ge c$. If $\dfrac{a^3+b^3+c^3}{\sin^3 A+\sin^3 B+\sin^3 C} = 7$, then the maximum possible value of a is

(a) 7 (b) 49
(c) $\sqrt[3]{7}$ (d) $\dfrac{1}{\sqrt{7}}$

4. Two sides of a triangle are given by the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$. The angle between the sides is $\dfrac{\pi}{3}$. The perimeter of the triangle is

(a) $6+\sqrt{3}$ (b) $2\sqrt{3}+\sqrt{6}$
(c) $2\sqrt{3}+\sqrt{10}$ (d) none of these

5. If in a triangle ABC, $\sin A$, $\sin B$, $\sin C$ are in $A.P.$, then

(a) the altitudes are in $A.P.$ (b) the altitudes are in $H.P.$
(c) the altitudes are in $G.P.$ (d) the medians are in $A.P.$

6. If a, b, c be the sides of a triangle and

$P = \dfrac{(a+b+c)^2}{ab+bc+ca}$, then

(a) $P \in [1, 2]$ (b) $P \in [3, 4)$
(c) $P \in (2, 4]$ (d) none of these

7. In an obtuse angled triangle, the obtuse angle is $\dfrac{3\pi}{4}$ and the other two angles are equal to two values of θ satisfying a $\tan\theta + b\sec\theta = c$, where $|b| \le \sqrt{a^2+c^2}$, then $a^2 - c^2$ is equal to

(a) ac (b) $2ac$
(c) $\dfrac{a}{c}$ (d) none of these

8. Let a, b and A are given and c_1 and c_2 are two values of c then the value of $c_1{}^2 + c_2{}^2 - 2c_1c_2\cos 2A =$

(a) $a^2\sin^2 A$ (b) $a^2\cos^2 A$
(c) $2(a^2+b^2)$ (d) $4a^2\cos^2 A$

9. With the notations of the previous question and suppose that B_1 is the acute angle value of the two values of B obtained then $|c_1 - c_2| =$

(a) $2a\sin B_1$ (b) $a\sin B_1$
(c) $2a\cos B_1$ (d) $a\cos B_1$

10. If the angle C be the right angle in the ΔABC, then $\tan^{-1}\dfrac{a}{b+c} + \tan^{-1}\dfrac{b}{c+a} =$

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{3\pi}{4}$ (d) $\dfrac{\pi}{3}$

MARK YOUR RESPONSE					
	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ
	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ

11. The cosine of the obtuse angle formed by medians drawn from the vertices of the acute angles of an isosceles right angled triangle is

(a) $-\frac{3}{5}$ (b) $-\frac{4}{5}$ (c) $-\frac{2}{5}$ (d) $-\frac{1}{5}$

12. ABC is a triangle with incentre I. Let P and Q be the feet of perpendiculars from A to BI and CI respectively. Then $\frac{AP}{BI}+\frac{AQ}{CI}=$

(a) $\cot A$ (b) $\cot\frac{A}{2}$

(c) $\tan\frac{A}{2}$ (d) $\tan\frac{B}{2}\tan\frac{C}{2}$

13. The area (δ) and angle θ of a triangle are given, when the side opposite to the given angle is minimum, then the length of the remaining two sides are

(a) $\sqrt{\frac{2\Delta}{\sin\theta}}, \sqrt{\frac{3\Delta}{\sin\theta}}$ (b) $\sqrt{\frac{2\Delta}{\sin\theta}}, \sqrt{\frac{2\Delta}{\sin\theta}}$

(c) $\sqrt{\frac{4\Delta}{\sin\theta}}, \sqrt{\frac{2\Delta}{\sin\theta}}$ (d) $\sqrt{\frac{6\Delta}{\sin\theta}}, \sqrt{\frac{6\Delta}{\sin\theta}}$

14. The median AD of a triangle ABC is perpendicular to AB. Which one of the following relations is correct :

(a) $\tan C+2\tan A=0$ (b) $\tan A+2\tan B=0$

(c) $\tan B+2\tan A=0$ (d) $\tan B+2\tan C=0$

15. O is the circumcentre of the trinagle ABC and R_1, R_2, R_3 are the radii of the circumcircles of the triangles OBC, OCA and OAB respectively. Then $\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}$ is equal to

(a) $\frac{abc}{R}$ (b) $\frac{abc}{R^3}$

(c) $\frac{a+b+c}{R}$ (d) $\frac{a^2+b^2+c^2}{R^2}$

16. The radius of the circle passing through the centre of the in-circle of ΔABC and through the end points of BC is given by

(a) $\frac{a}{2}\cos A$ (b) $\frac{a}{2}\sec\frac{A}{2}$

(c) $\frac{a}{2}\sin A$ (d) $a\sec\frac{A}{2}$

17. In-circle of radius 4 cm of a triangle ABC touches the side BC at D. If $BD = 6$ cm, $DC = 8$ cm, then the area of the triangle ABC is

(a) 42 cm^2 (b) 84 cm^2

(c) $\frac{84}{3}$ cm^2 (d) none of these

18. If the smallest side of a right triangle with integer sides is 23 the perimeter of the triangle is

(a) 22×23 (b) 23×24

(c) 23×13 (d) 23×32

19. Let ΔABC be equilateral. On side AB produced, we choose a point P such that A lies between P and B. Denote 'a' as the length of sides of ΔABC; r_1 as the radius of incircle of ΔPAC; and r_2 as the exradius of ΔPBC with respect to side BC. Then r_1+r_2 is equal to

(a) $2a$ (b) $\frac{\sqrt{3}}{2}a$ (c) $\frac{\sqrt{5}}{4}a$ (d) $\sqrt{2}a$

20. ABC is a right angle triangle in which $\angle B = 90°$ and $BC = a$. If n points $L_1, L_2, L_3 .... L_n$ on AB are such that AB is divided in $(n+1)$ equal parts and $L_1M_1, L_2M_2, ..., L_nM_n$ are line segments parallel to BC and $M_1, M_2, M_3, ... M_n$ are on AC. Then the sum of the lengths of the sides of L_1M_1, L_2M_2, ..., L_nM_n is

(a) $\frac{a(n+1)}{2}$ (b) $\frac{a(n-1)}{2}$

(c) $\frac{an}{2}$ (d) $\frac{an(n+1)}{2}$

21. In a triangle ABC, if $\frac{a^2+b^2}{a^2-b^2}\sin(A-B)=1$, and C is not a right angle, then $\cos(A-B)=$

(a) $\tan\left(\frac{C}{2}+\frac{\pi}{4}\right)$ (b) $\tan\left(\frac{C}{2}-\frac{\pi}{4}\right)$

(c) $\cos\left(\frac{C}{2}+\frac{\pi}{4}\right)$ (d) $\sin\left(\frac{C}{2}-\frac{\pi}{4}\right)$

22. If in triangle ABC, $r_1 = 2r_2 = 3r_3$, D is the middle point of BC. Then $\cos(\angle ADC)$ is equal to

(a) $\frac{7}{25}$ (b) $-\frac{7}{25}$ (c) $\frac{24}{25}$ (d) $-\frac{24}{25}$

MARK YOUR RESPONSE					
	11. ⓐⓑⓒⓓ	12. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ	14. ⓐⓑⓒⓓ	15. ⓐⓑⓒⓓ
	16. ⓐⓑⓒⓓ	17. ⓐⓑⓒⓓ	18. ⓐⓑⓒⓓ	19. ⓐⓑⓒⓓ	20. ⓐⓑⓒⓓ
	21. ⓐⓑⓒⓓ	22. ⓐⓑⓒⓓ			

23. If in a right angle traingle ABC, $4\sin A\cos B - 1 = 0$ and tan A is real, then

(a) angles are in $A.P.$ (b) angles are in $G.P.$

(c) angles are in $H.P.$ (d) none of these

24. In a triangle ABC, which of the following is possible

(a) $\sin 2A + \sin 2B + \sin 2C = 0$

(b) $\cos 2A + \cos 2B + \cos 2C = -1$

(c) $\sin A + \sin B + \sin C = 0$

(d) $\cos A + \cos B + \cos C = 1$

25. If a, b, c are the sides of a triangle such that $b.\ c = \lambda^2$, for some positive λ, then

(a) $a \geq 2\lambda \sin\frac{A}{2}$ (b) $b \geq 2\lambda \sin\frac{B}{2}$

(c) $c \geq 2\lambda \sin\frac{C}{2}$ (d) all are correct

26. If O is point inside the triangle ABC such that

$\angle OBC = \frac{A}{2}$, $\angle OCA = \frac{B}{2}$, $\angle OAB = \frac{C}{2}$ then

$\frac{\sin(A - C/2)\sin(B - A/2)\sin(C - B/2)}{\sin A/2 \sin B/2 \sin C/2}$ is equal to

(a) $\cos A/2 \cos B/2 \cos C/2$ (b) $\sin A \sin B \sin C$

(c) 1 (d) $\cos A \cos B \cos C$

27. Given the height h and the angle bisector l dawn from the vertex of the right angle of a triangle, then an acute angle θ of the triangle is given by

(a) $\cos\theta = \frac{h + \sqrt{l^2 - h^2}}{\sqrt{2}h}$ (b) $\cos\theta = \frac{h - \sqrt{l^2 - h^2}}{\sqrt{2}h}$

(c) $\cos\theta = \frac{h}{l}$ (d) $\cos\theta = \frac{h - \sqrt{l^2 - h^2}}{\sqrt{2}\,l}$

28. The angles of a triangle ABC are in $A.P$. The largest angle is twice the smallest and the median to the largest side divides the angle at the vertex in the ratio 2 : 3. If the length of the median is $2\sqrt{3}$ cm, then the length of the largest side is

(a) $2\cos 42°$ (b) $4\sin 32°$

(c) $8\sin 42°$ (d) $8\cos 42°$

29. A quadrilateral $ABCD$ in which $AB = a$, $BC = b$, $CD = c$ and $DA = d$ is such that one circle can be inscribed in it and another circle circumscribed about it then $\cos A =$

(a) $\frac{ad + bc}{ad - dc}$ (b) $\frac{ad - bc}{ad + bc}$

(c) $\frac{ac + bd}{ac - bd}$ (d) $\frac{ac - bd}{ac + bd}$

30. If A, B, C, D are the angles of a quadrilateral, then

$\frac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D}$ is equal to

(a) $\tan A \tan B \tan C \tan D$

(b) $\cot A \cot B \cot C \cot D$

(c) $\tan^2 A + \tan^2 B + \tan^2 C + \tan^2 D$

(d) $\Sigma \tan A \tan B \tan C$

31. A cyclic quadrilateral $ABCD$ is inscribed in a unit circle. If side $AB = 1$ and the diagonal $BD = \sqrt{3}$, then the length of the side AD is

(a) 1 (b) 2 (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{2}$

32. Two flagstaffs stand on a horizontal plane. A and B are two points on the line joining their feet and between them. The angles of elevation of the tops of the flagstaffs as seen from A are 30° and 60° and as seen from B are 60° and 45°. If AB is 30m, then the distance between the flagstaffs in metres is

(a) $30 + 15\sqrt{3}$ (b) $45 + 15\sqrt{3}$

(c) $60 - 15\sqrt{3}$ (d) $60 + 15\sqrt{3}$

33. A vertical pole of height h is placed at a point on an inclined plane of angle 15° at distance 'h' from its base. When the angle of elevation of sun is 30°, the shadow of pole falling along the line of the greatest slope just reaches a point which is at a distance 'a' away from the base of the plane.

Then $\frac{a}{h} =$

(a) $\frac{4\sqrt{6} + 3 + \sqrt{3}}{3\sqrt{2}}$ (b) $\frac{4\sqrt{6} + 3 - 2\sqrt{3}}{4\sqrt{2}}$

(c) $\frac{4\sqrt{6} - 3 + \sqrt{3}}{3\sqrt{2}}$ (d) $\frac{\sqrt{6} - \sqrt{3} + 1}{\sqrt{2}}$

MARK YOUR RESPONSE					
	23. ⓐⓑⓒⓓ	24. ⓐⓑⓒⓓ	25. ⓐⓑⓒⓓ	26. ⓐⓑⓒⓓ	27. ⓐⓑⓒⓓ
	28. ⓐⓑⓒⓓ	29. ⓐⓑⓒⓓ	30. ⓐⓑⓒⓓ	31. ⓐⓑⓒⓓ	32. ⓐⓑⓒⓓ
	33. ⓐⓑⓒⓓ				

34. Two vertical poles 20m and 80 m high stand apart on horizontal plane. The height of the point of intersection of the lines joining the top of each pole to the foot of the other is

(a) 15 m (b) 16 m (c) 18 m (d) 12 m

35. OAB is a triangle in the horizontal plane through the foot P of the tower at the middle point of the side OB of the triangle. If $OA = 2$m, $OB = 6$m. $AB = 5$m and $\angle AOB$ is equal to the angle subtended by the tower at A then the height of the tower is

(a) $\sqrt{\dfrac{11\times 39}{25\times 3}}$ (b) $\sqrt{\dfrac{11\times 39}{25\times 2}}$

(c) $\sqrt{\dfrac{11\times 25}{39\times 2}}$ (d) none of these

36. A pole 50 m high stands on a building 250 m high. To an obsever at a height of 300 m, the building and the pole subtend equal angles. The horizontal distance of the observer from the pole is

(a) 25 m (b) 50 m

(c) $25\sqrt{6}$ m (d) $25\sqrt{3}$ m

37. In a cubical hall $ABCDPQRS$ with each side 10 m, G is the cenre of the wall $BCRQ$ and T is the mid point of the side AB. The angle of elevation of G at the point T is

(a) $\sin^{-1}\dfrac{1}{\sqrt{3}}$ (b) $\cos^{-1}\dfrac{1}{\sqrt{3}}$

(c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{4}$

38. Two objects P and Q subtend an angle of 30° at A. Length of 20 m and 10 m are measured from A at right angles to AP and AQ respectively to points R and S at each of which PQ subtends angles of 30°, the length of PQ is

(a) $\sqrt{300-200\sqrt{3}}$ (b) $\sqrt{500-200\sqrt{3}}$

(c) $\sqrt{500\sqrt{3}-200}$ (d) $\sqrt{300}$

39. If each side of length a of an equilateral triangle subtends an angle of 60° at the top of a tower h metre high situated at the centre of the triangle, then

(a) $3a^2 = 2h^2$ (b) $2a^2 = 3h^2$

(c) $a^2 = 3h^2$ (d) $3a^2 = h^2$

40. In a triangular plot ABC with $BC = 7$m, $CA = 8$ m and $AB = 9$m; a lamp post is situated at the middle point E of the side AC and subtends an angle $\tan^{-1}3$ at the point B, the height of the lamp post is

(a) 21 m (b) 24 m

(c) 27 m (d) can not be determined

41. At each end of a horizontal base of length $2a$, the angular height of a certain peak is 15° and that at the mid point of the base is 45°, the height of the peak is

(a) $\dfrac{(\sqrt{3}-1)a}{2\sqrt{3}}$ (b) $\dfrac{\sqrt{3}(\sqrt{3}-1)a}{2^{1/3}}$

(c) $\dfrac{\sqrt{3}-1}{6}.3^{3/4}a$ (d) $\dfrac{\sqrt{3}-1}{6}a$

42. In a triangle ABC, if $\sin A$, $\sin B$, $\sin C$ are in $A.P.$ then the maximum value of $\tan\dfrac{B}{2}$ is

(a) $\dfrac{1}{3}$ (b) $\sqrt{3}$ (c) $\dfrac{1}{\sqrt{3}}$ (d) 1

43. For a triangle ABC, the minimum value of $\dfrac{4R^2}{\Delta}\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$ is ____.

(a) 1 (b) 2 (c) 4 (d) 16

44. If in a ΔABC, $\angle B = \dfrac{2\pi}{3}$ then the $\cos A + \cos C$ lie in

(a) $[-\sqrt{3}, \sqrt{3}]$ (b) $(-\sqrt{3}, \sqrt{3})$

(c) $\left(\dfrac{3}{2}, \sqrt{3}\right]$ (d) $\left[\dfrac{3}{2}, \sqrt{3}\right]$

45. X is the circumcentre of triangle ABC, and Y is a point on the side BC, within the circumcircle. If Y approaches to C then the circumradius of triangle CXY approaches to value

(a) $R\cos A$ (b) $R\sec A$

(c) $\dfrac{R}{2}\sec A$ (d) $2R\sec A$

46. If in a triangle ABC, $\tan\dfrac{A}{2}$, $\tan\dfrac{B}{2}$, $\tan\dfrac{C}{2}$ are in H.P., then the minimum value of $\cot\dfrac{B}{2}$ is

(a) $\dfrac{1}{\sqrt{3}}$ (b) 1 (c) $\sqrt{3}$ (d) $\sqrt{3}-1$

MARK YOUR RESPONSE					
	34. ⓐⓑⓒⓓ	35. ⓐⓑⓒⓓ	36. ⓐⓑⓒⓓ	37. ⓐⓑⓒⓓ	38. ⓐⓑⓒⓓ
	39. ⓐⓑⓒⓓ	40. ⓐⓑⓒⓓ	41. ⓐⓑⓒⓓ	42. ⓐⓑⓒⓓ	43. ⓐⓑⓒⓓ
	44. ⓐⓑⓒⓓ	45. ⓐⓑⓒⓓ	46. ⓐⓑⓒⓓ		

47. If the angles of a triangle ABC are in A.P. and $A : C = 4 : 1$, then C is equal to

(a) $\frac{\pi}{15}$ (b) $\frac{\pi}{5}$ (c) $\frac{2\pi}{15}$ (d) $\frac{2\pi}{5}$

48. In a triangle ABC, if $\sin A \sin B \sin C = p$, $\cos A \cos B \cos C = q$, then $\tan A$, $\tan B$, $\tan C$ are the root of the equation

(a) $qx^3 + px^2 - (1+q)x + p = 0$
(b) $qx^3 - px^2 + (1+q)x - p = 0$
(c) $qx^3 - (1+p)x^2 + qx - p = 0$
(d) $qx^3 - (1+p)x^2 + (1+q)x - p = 0$

49. If the angles A, B, C of a triangle ABC, are the solution of the equation $\tan^3 x - 3k\tan^2 x - 3\tan x + k = 0$, then the triangle

(a) does not exist (b) is acute angled
(c) is equilateral (d) is isosceles

50. If $\sin\theta$ and $-\cos\theta$ are the roots of the equation $ax^2 - bx - c = 0$, where a, b, c are the sides of a triangle ABC, then $\cos B$ is equal to

(a) $1 - \frac{c}{2a}$ (b) $1 - \frac{c}{a}$
(c) $1 + \frac{c}{2a}$ (d) $1 + \frac{c}{3a}$

51. In a triangle ABC, the maximum value of $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$ is

(a) $\frac{s}{2R}$ (b) $\frac{R}{2s}$ (c) $\frac{s}{2r}$ (d) $\frac{r}{2s}$

52. In a right angled triangle ABC, with $A = \frac{\pi}{2}$, a circle is drawn touching the sides AB, AC and the incircle of the triangle. Its radius is equal to

(a) $(2-\sqrt{2})r$ (b) $(3-\sqrt{2})r$
(c) $(3-2\sqrt{2})r$ (d) None of these

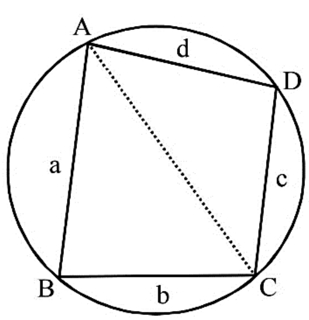
MARK YOUR RESPONSE					
	47. ⓐⓑⓒⓓ	48. ⓐⓑⓒⓓ	49. ⓐⓑⓒⓓ	50. ⓐⓑⓒⓓ	51. ⓐⓑⓒⓓ
	52. ⓐⓑⓒⓓ				

B COMPREHENSION TYPE

This section contains groups of questions. Each group is followed by some multiple choice questions based on a paragraph. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

PASSAGE-1

Suppose $ABCD$ be a cyclic quadrilateral inscribed in a circle of radius R. Angles of the quadrilateral are given by A, B, C and D. Then, $A + C = B + D = \pi$



We represent the side by a, b, c, d in a cyclic manner as shown in figure.

If sides are given then the angles of the quadrilateral can be obtained by using the formula $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$

The area S of the quadrilateral is given by

$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $s = \frac{a+b+c+d}{2}$

1. If a quadrilateral with side lengths a, b, c, d can be inscribed in one circle and circumscribed about another circle then its area is

(a) $\sqrt{abcd}$ (b) $\frac{ac + bd}{2}$
(c) $\sqrt{\frac{a^4 + b^4 + c^4 + d^4 - abcd}{4}}$
(d) Such a quadrilateral is not possible

MARK YOUR RESPONSE					
	1. ⓐⓑⓒⓓ				

2. The sides of a quadrilateral which can be inscribed in a circle are 3, 3, 4 and 4 cm, then the radius of the incircle of the quadrilateral is

(a) $\frac{5}{2}$ cm (b) $\frac{12}{7}$ cm

(c) $\frac{6}{7}$ cm (d) Incircle does not exist

3. In the previous problem, the radius of the circumcircle is

(a) $\frac{5}{2}$ cm (b) $\frac{12}{7}$ cm (c) $\frac{6}{7}$ cm (d) 5 cm

PASSAGE-2

When any two sides and the angle opposite to one of them are given then either no triangle, or one triangle or two triangles are possible.
Let the sides a, b and the angle A be given

Then, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow c^2 - (2b\cos A)c + b^2 - a^2 = 0$

This is a quadratic equation in c. So, two values of c will be obtained real, coincident or imaginary. Values of c from the above equation are given by $c_1 = b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}$, say c_1 and c_2

The discriminant of the above equation is

$D = 4b^2\cos^2 A - 4(b^2 - a^2) = 4(a^2 - b^2\sin^2 A)$

We can have following cases :

(i) If $D < 0$, i.e. $a < b\sin A$, then no triangle is possible

(ii) If $D = 0$, i..e., $a = b\sin A$, then only one triangle is possible provided A is acute. In case A is obtuse then no triangle is possible as then c_1 and c_2 will be negative.

(iii) If $D > 0$, i.e., $a > b\sin A$, then two triangles are possible provided c_1 and c_2 are both positive.

4. If A is acute then two different triangles are possible if and only if

(a) $a < b\sin A$ (b) $a > b\sin A$ and $a < b$

(c) $a > b\sin A$ and $a > b$ (d) $a > b\sin A$ and $a = b$

5. If $a > b\sin A$ and $a = b$ then

(a) No triangle is possible
(b) Only one triangle is possible
(c) Two distinct triangles are possible
(d) Any of the (a), (b), (c) may be true

6. If $a = 2\sqrt{2}$, $b = 6$ and $A = 45°$ then

(a) No triangle is possible
(b) One triangle is possible
(c) Two trianlges are possible
(d) Either no triangle or two triangles are possible

PASSAGE-3

Let ABC be a triangle and AD, BE, CF be the altitudes from A, B, C respectively to the opposite sides. Let the triangle DEF be completed. Then the sides of the triangle DEF are respectively $EF = a\cos A$, $FD = b\cos B$, $DE = c\cos C$.

7. Ratio of areas of ΔDEF and ΔABC is

(a) $2\cos A\cos B\cos C$ (b) $2\sin A\sin B\sin C$

(c) $2\tan A\tan B\tan C$ (d) none of these

8. Circumradius of ΔDEF is

(a) $r/2$ (b) $R/2$

(c) $(r+R)/4$ (d) none of these

9. In-radius of ΔDEF is

(a) $2R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (b) $2R\cos A\cos B\cos C$

(c) $2R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ (d) $2R\sin A\cos B\cos C$

10. Ratio of the perimeters of ΔDEF and ΔABC is

(a) 1/2 (b) $2r/R$ (c) r/R (d) R/r

PASSAGE-4

Let r and R represent the in radius and circum radius of a triangle ABC of which r_1, r_2, r_3 are respectively the radii of excircles opposite to vertices A, B and C. Perimeter of triangle is $2s$.

11. The cubic equation with r_1, r_2, r_3 as three roots is given by

(a) $x^3 - x^2(R + r) + sx - rs^2 = 0$
(b) $x^3 - x^2(R - 2r) + s^2x - rs^2 = 0$
(c) $x^3 - x^2(4R + r) + s^2x - rs^2 = 0$
(d) $x^3 - 4x^2(R + r) + s^2x - rs^2 = 0$

12. The expression $(s + r_1)(s + r_2)(s + r_3)$ equals to

(a) $2s^2(s + r + 2R)$ (b) $2s^2(s + 2R)$

(c) $R(s^2 + r^2)$ (d) none of these

13. Let r_1, r_2, r_3 are three consecutive terms of an A.P. then $\frac{(4R+r)^3}{rs^2}$ is greater than or equal to

(a) 8 (b) 27 (c) 1 (d) 4

Mark Your Response					
	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ	6. ⓐⓑⓒⓓ
	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ
	12. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ			

C — REASONING TYPE

In the following questions two Statements (1 and 2) are provided. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. Mark your responses from the following options:

(a) Both Statement-1 and Statement-2 are true and Statement-2 is the correct explanation of Statement-1.
(b) Both Statement-1 and Statement-2 are true and Statement-2 is not the correct explanation of Statement-1.
(c) Statement-1 is true but Statement-2 is false.
(d) Statement-1 is false but Statement-2 is true.

1. **Statement-1** : If A, B, C, D are angles of a cyclic quadrilateral then
$\sin A + \sin B + \sin C + \sin D = 0$
Statement-2 : If A, B, C, D are angles of Cyclic quadrilateral then
$\cos A + \cos B + \cos C + \cos D = 0$

2. **Statement-1** : The orthocentre of the given triangle is coincident with the incentre of the pedal triangle of the given triangle
Statement-2 : Pedal triangle is the ex-central triangle of the given triangle.

3. A circle is inscribed in an equilateral triangle of side a and a square is inscribed in this circle.
Statement-1 : Area of the square is $\frac{a^2}{6}$ square unit
Statement-2 : Length of the side of square is equal to the radius of

4. A circle is inscribed in a right angled triangle with right angle at B.
Statement 1 : The diameter of the circle equals to $AB + BC - AC$
Statement 2 : Radius of circle is equal to $\frac{\text{area of triangle}}{\text{semiperimeter}}$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	

D — MULTIPLE CORRECT CHOICE TYPE

Each of these questions has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

1. If the sides a, b, c of a triangle ABC form successive terms of $G.P.$ with common ratio $r\,(>1)$, then which of the following is / are correct

(a) $r < \frac{\sqrt{5}+1}{2}$
(b) $A < B < \frac{\pi}{3}$
(c) $B > \frac{\pi}{3}$
(d) $C > \frac{\pi}{3}$

2. Sides of a triangle ABC are in A.P. If $a <$ minimum $\{b, c\}$, then $\cos A$ may be equal to

(a) $\frac{4b-3c}{2b}$
(b) $\frac{3c-4b}{2c}$
(c) $\frac{4c-3b}{2b}$
(d) $\frac{4c-3b}{2c}$

3. Which of the following in a ΔABC is / are true?

(a) $\cos A + \cos B + \cos C > 1$
(b) $\cos A + \cos B + \cos C < 1$
(c) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > 0$
(d) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} > 0$

4. In a ΔABC, the incircle touches the sides BC, CA, and AB at P, Q and R respectively and its radius is 4 units. If the lengths BP, CQ and AR are consecutive integers then

(a) sides are also consecutive integers
(b) Sides are in $A.P.$
(c) Perimeter of the triangle is 42 unit
(d) diameter of the circumcircle is 65 unit

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	

5. In a triangle ABC, if $(a+b+c)(b+c-a)=x\,b\,c$ then x can be equal to

(a) 1 (b) 2 (c) 3 (d) 4

6. If in a triangle ABC, $a^2+b^2+c^2=ca+ab\sqrt{3}$ then

(a) A, B, C are in A.P. (b) triangle is isoceles

(c) triangle is right angled (d) $a=2c$

7. If in triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to

(a) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (b) $\frac{a+b}{ab}\cos\frac{C}{2}$

(c) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (d) $\frac{b\sin A}{\sin\left(B+\frac{C}{2}\right)}$

8. In a triangle ABC, if $\sin A.\sin(B-C)=\sin C\sin(A-B)$, then

(a) $\tan A, \tan B, \tan C$ are in A.P.

(b) $\cot A, \cot B, \cot C$ are in A.P.

(c) $\cos 2A, \cos 2B, \cos 2C$ are in A.P.

(d) $\sin 2A, \sin 2B, \sin 2C$ are in A.P.

9. There exists a triangle satisfying

(a) $b\sin A=a,\ A<\frac{\pi}{2}$ (b) $b\sin A>a,\ A>\frac{\pi}{2}$

(c) $b\sin A>a,\ A<\frac{\pi}{2}$ (d) $b\sin A<a,\ A<\frac{\pi}{2}, b>a$

10. A semi-circle is described with its diameter lying the side on AB of the triangle ABC. If this circle touches the sides AC and CB, then its radius is

(a) $\frac{\Delta}{c}$ (b) $\frac{\Delta}{s}$

(c) $\frac{2\Delta}{a+b}$ (d) $\frac{2abc}{s(a+b)}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

11. In a triangle ABC, if $\sec A, \sec B, \sec C$ are in H.P. then

(a) a, b, c are in H.P.

(b) $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in H.P.

(c) r_1, r_2, r_3 are in A.P.

(d) $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

12. If in a triangle ABC, $b\cos^2\frac{A}{2}+a\cos^2\frac{B}{2}=\frac{3c}{2}$, then

(a) $c^2\geq ab$ (b) $2c>\sqrt{ab}$

(c) $\frac{a+c}{2c-a}+\frac{b+c}{2c-b}\leq 4$ (d) $\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\geq 3$

13. In ΔABC, if $\frac{c+a}{12}=\frac{a+b}{14}=\frac{b+c}{18}$, then

(a) $r_1=\frac{11}{7}r$ (b) $r_2=11r$

(c) $r_3=\frac{11}{4}r$ (d) $R=\frac{80}{21}r$

14. If $\sin A, \sin B$ are the roots of the equation $c^2x^2-c(a+b)x+ab=0$ where A, B, C are the angles and a, b, c are the sides of a ΔABC then the triangle

(a) is obtuse angled

(b) is acute angled

(c) is right angled

(d) satisfies $c(\sin A+\cos A)=a+b$

15. If in a triangle ABC, $\tan A+\tan B+\tan C=10$, then ΔABC

(a) cannot be acute angled

(b) is acute angled

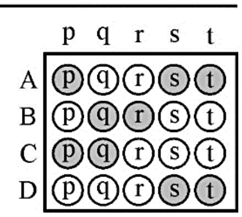
(c) cannot be obtuse angled

(d) is obtuse angled

MARK YOUR RESPONSE					
	5. ⓐⓑⓒⓓ	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ
	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ	12. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ	14. ⓐⓑⓒⓓ
	15. ⓐⓑⓒⓓ				

E MATRIX-MATCH TYPE

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labeled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example:
If the correct matches are A–p, s and t; B–q and r; C–p and q; and D–s and t; then the correct darkening of bubbles will look like the given.



1. In a triangle ABC, AD is perpendicular to BC and DE is perpendicular to AB.

	Column-I		**Column-II**
(A)	Area of $\Delta\, ADB$	p.	$\left(\frac{b^2}{4}\right)\sin 2C$
(B)	Area of $\Delta\, ADC$	q.	$\left(\frac{c^2}{4}\right)\cos^2 B \sin 2B$
(C)	Area of $\Delta\, ADE$	r.	$\left(\frac{c^2}{4}\right)\sin 2B$
(D)	Area of $\Delta\, BDE$	s.	$\left(\frac{c^2}{4}\right)\sin^2 B \sin 2B$

2. In a triangle ABC, a = 7, b = 8, c = 9, BD is the median and BE is the altitude from the vertex B, then

	Column-I		**Column-II**
(A)	Length of BD is	p.	2
(B)	Length of BE is	q.	6
(C)	Length of ED is	r.	$\sqrt{45}$
(D)	Length of CE is	s.	7

3. **Match the following columns :**

	Column-I		**Column-II**
(A)	If α, β, γ be the lengths of medians of triangle ABC then $\frac{\alpha^2+\beta^2+\gamma^2}{a^2+b^2+c^2}$ is equal to	p.	1
(B)	Let the point P lies interior of an equilateral triangle ABC of side length 2 and its distances from the sides BC, CA and AB are respectively x, y and z, then $x+y+z$ is equal to	q.	$\sqrt{3}$
(C)	In a triangle ABC. A, B, C are in A.P. and a, b, c are in G.P. then $\frac{a^2b+b^2c+c^2a}{a^3+b^3+c^3}$ is equal to	r.	$\frac{3}{4}$
(D)	In triangle ABC, the least value of $\sqrt{\frac{abc(a+b+c)}{\Delta}}$ is	s.	4

4. **Match the following column:**

	Column-I		**Column-II**
(A)	If in a triangle ABC, $\frac{r}{r_1}=\frac{1}{4}$, then the value of $\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ is equal to	p.	$\frac{3}{4}$
(B)	In a triangle the least value of $\frac{r_1 r_2 r_3}{r^3}$ is	q.	1
(C)	If the sides a, b, c of a triangle ABC are in A.P. then the ratio $\frac{b}{c}$ can be equal to	r.	3
(D)	Let P be an interior point of the triangle ABC and the lines AP, BP and CP when produced meet the opposite sides in D, E and F respectively then $\frac{PD}{AD}+\frac{PE}{BE}+\frac{PE}{CF}$ is equal to	s.	27

MARK YOUR RESPONSE

	1. p q r s	2. p q r s	3. p q r s	4. p q r s
A	(p)(q)(r)(s)	(p)(q)(r)(s)	(p)(q)(r)(s)	(p)(q)(r)(s)
B	(p)(q)(r)(s)	(p)(q)(r)(s)	(p)(q)(r)(s)	(p)(q)(r)(s)
C	(p)(q)(r)(s)	(p)(q)(r)(s)	(p)(q)(r)(s)	(p)(q)(r)(s)
D	(p)(q)(r)(s)	(p)(q)(r)(s)	(p)(q)(r)(s)	(p)(q)(r)(s)

5. **Match the Column I with Column II**

Column-I		Column-II
(A) if the medians from B and C are perpendicular to each other, the minimum value of $\cot B + \cot C$ is	p.	0
(B) if $R = \frac{13}{2}, r = 2$ and $r_1 = 3$, then its area in square units is	q.	$\frac{2}{3}$
(C) if $a(bc^2 + ca^2 + ab^2)$ $= b^2c + c^2a + a^2b + 3abc$ then $\cos A$ is equal to	r.	$\frac{1}{2}$
(D) if $\sin 4A + \sin 4B + \sin 4C$ $+ 8\cos A = 0$, then $\cos A$ is equal to	s.	30

6. In a triangle ABC, right angled at A, the radius of the in-scribed circle is 2 cm. Radius of the circle touching the side BC and also sides AB and AC produced is 15 cm, Match the entries of column 1 with corresponding entries in column II.

Column-I		Column-II
(A) Length of BC is	p.	5
(B) Perimeter of the triangle is	q.	$\frac{13}{2}$
(C) Radius of circumcircle is	r.	13
(D) $\sec B + \sec C$ is equal to	s.	30

MARK YOUR RESPONSE

5.

	p	q	r	s
A	ⓟ	ⓠ	ⓡ	ⓢ
B	ⓟ	ⓠ	ⓡ	ⓢ
C	ⓟ	ⓠ	ⓡ	ⓢ
D	ⓟ	ⓠ	ⓡ	ⓢ

6.

	p	q	r	s
A	ⓟ	ⓠ	ⓡ	ⓢ
B	ⓟ	ⓠ	ⓡ	ⓢ
C	ⓟ	ⓠ	ⓡ	ⓢ
D	ⓟ	ⓠ	ⓡ	ⓢ

F NUMERIC/INTEGER ANSWER TYPE

The answer to each of the questions is either numeric (eg. 304, 40, 3010 etc.) or a single-digit integer, ranging from 0 to 9.

The appropriate bubbles below the respective question numbers in the response grid have to be darkened.

For example, if the correct answers to a question is 6092, then the correct darkening of bubbles will look like the given.

For single digit integer answer darken the extreme right bubble only.

0	**0**	0	0
1	1	1	1
2	2	2	**2**
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	**9**	9

1. Let ABC be a right angled triangle, right angle at B. If D and E be the points on CB such that $\angle ADB = 2\angle ACB$ and $\angle AEB = 3\angle ACB$ and $K = \frac{DE}{CD}$, then $[K]$ is equal to ($[K]$ represents the greatest integer less than or equal to K)

2. The diagonals of a parallelogram are inclined to each other at an angle of 45°, while its sides a and b $(a > b)$ are inclined to each other at an angle of 30°. The value of the least integer greater than or equal to $\frac{a}{b}$ is equal to

MARK YOUR RESPONSE

1.

0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9

2.

0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9

3. Let ABC be a triangle of area Δ and $A'B'C'$ be the triangle formed by the altitudes of ΔABC as its sides with area Δ' and $A''B''C''$ be the triangle formed by the altitudes of $\Delta A'B'C'$ as its sides with area Δ''. If $\Delta' = 30$ and $\Delta'' = 20$ then the value of Δ is

4. If p_1, p_2, p_3 are the altitudes of a triangle which circumscribes a circle of diameter $\frac{4}{3}$ units, then the least value of $p_1 + p_2 + p_3$ is equal to

5. In a triangle ABC, $\frac{1}{R^2}(r^2 + r_1^2 + r_2^2 + r_3^2 + a^2 + b^2 + c^2)$ is equal to _____.

6. For a triangle ABC, with altitudes h_1, h_2, h_3 and in radius r, the minimum value of $\frac{h_1 + r}{h_1 - r} + \frac{h_2 + r}{h_2 - r} + \frac{h_3 + r}{h_3 - r}$ is _____.

MARK YOUR RESPONSE

3.	4.	5.	6.
0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
2 2 2 2	2 2 2 2	2 2 2 2	2 2 2 2
3 3 3 3	3 3 3 3	3 3 3 3	3 3 3 3
4 4 4 4	4 4 4 4	4 4 4 4	4 4 4 4
5 5 5 5	5 5 5 5	5 5 5 5	5 5 5 5
6 6 6 6	6 6 6 6	6 6 6 6	6 6 6 6
7 7 7 7	7 7 7 7	7 7 7 7	7 7 7 7
8 8 8 8	8 8 8 8	8 8 8 8	8 8 8 8
9 9 9 9	9 9 9 9	9 9 9 9	9 9 9 9

Answerkey

A. Single Correct Choice Type

1	(a)	**10**	(a)	**19**	(b)	**28**	(d)	**37**	(a)	**46**	(c)
2	(c)	**11**	(b)	**20**	(c)	**29**	(b)	**38**	(b)	**47**	(c)
3	(c)	**12**	(b)	**21**	(b)	**30**	(a)	**39**	(b)	**48**	(b)
4	(b)	**13**	(b)	**22**	(b)	**31**	(b)	**40**	(a)	**49**	(a)
5	(b)	**14**	(b)	**23**	(a)	**32**	(d)	**41**	(c)	**50**	(c)
6	(b)	**15**	(b)	**24**	(b)	**33**	(d)	**42**	(c)	**51**	(b)
7	(b)	**16**	(b)	**25**	(a)	**34**	(b)	**43**	(b)	**52**	(c)
8	(d)	**17**	(b)	**26**	(c)	**35**	(b)	**44**	(c)		
9	(c)	**18**	(b)	**27**	(d)	**36**	(c)	**45**	(c)		

B. Comprehension Type

1	(a)	**4**	(b)	**7**	(a)	**10**	(c)	**13**	(b)
2	(b)	**5**	(b)	**8**	(b)	**11**	(c)		
3	(a)	**6**	(a)	**9**	(b)	**12**	(a)		

C. Reasoning Type

1	(d)	**2**	(c)	**3**	(c)	**4**	(a)

D. Multiple Correct Choice Type

1	(a,b,d)	**4**	(a,b,c)	**7**	(c, d)	**10**	(c, d)	**13**	(a, b, d)
2	(a, d)	**5**	(a,b,c)	**8**	(b, c)	**11**	(b, c)	**14**	(c,d)
3	(a, c, d)	**6**	(a,c,d)	**9**	(a, d)	**12**	(a, b, d)	**15**	(b,c)

E. Matrix-Match Type

1. **A - r; B - p; C - s; D - q**
2. **A - s; B - r; C - p ; D - q**
3. **A - r; B - q; C - p; D - s**
4. **A - p; B - s; C - p, q; D - q**
5. **A - q; B - s; C - r, q; D - p**
6. **A - r ; B - s ; C - q ; D - p**

F. Numeric/Integer Answer Type

1	0	**2**	2	**3**	45	**4**	6	**5**	16	**6**	6

Solutions

A SINGLE CORRECT CHOICE TYPE

1. **(a)** Given $r_1 < r_2 < r_3 \Rightarrow \frac{\Delta}{s-a} < \frac{\Delta}{s-b} < \frac{\Delta}{s-c}$

$\Rightarrow s-a > s-b < s-c \Rightarrow -a > -b > -c \Rightarrow a < b < c$

2. **(c)** Sum of the roots of the equation is given by

$$\sin A + \sin B = \frac{c(a+b)}{c^2} = \frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}$$

$\sin C = 1$

$\Rightarrow$ the triangle is right angled

3. **(c)** If D is the diameter of the circumscribed circle of ΔABC, then $a = D\sin A$, $b = D\sin B$, $c = D\sin C$

$$\therefore \frac{a^3+b^3+c^3}{\Sigma\sin^3 A} = 7 \Rightarrow \frac{D^3(\Sigma\sin^3 A)}{\Sigma\sin^3 A} = 7 \ \therefore\ D = \sqrt[3]{7}$$

Since no side of triangle can exceed the diameter of the circle, the maximum possible value of a is $\sqrt[3]{7}$.

4. **(b)** Let the sides a and B be the roots of $x^2 - 2\sqrt{3}x + 2 = 0$

Then $a+b = 2\sqrt{3}$ and $ab = 2$. Also $C = \frac{\pi}{3}$

$$\therefore \cos C = \frac{a^2+b^2-c^2}{2ab} \Rightarrow \frac{1}{2} = \frac{a^2+b^2-c^2}{2ab}$$

$\Rightarrow a^2+b^2-c^2 = ab$

$\therefore (a+b)^2 - 2ab - c^2 = ab$

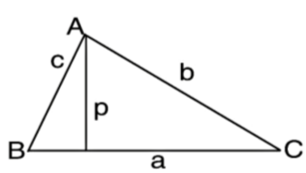
$\Rightarrow 12 - c^2 = 3\times 2 \Rightarrow c = \sqrt{6}$

$\therefore$ Perimeter $= a+b+c = 2\sqrt{3}+\sqrt{6}$

5. **(b)** Let the altitudes from A, B, C be p, q, r respectively. Then

$p = b\sin C$, $q = c\sin A$, $r = a\sin B$

$\therefore p : q : r = b\sin C : c\sin A : a\sin B$



$= \sin B\sin C : \sin C\sin A : \sin A\sin B$

$$= \frac{1}{\sin A} : \frac{1}{\sin B} : \frac{1}{\sin C}$$

$\therefore \sin A, \sin B, \sin C$ are in A.P.

$\Rightarrow p, q, r$ are in H. P.

6. **(b)** a, b, c are sides of a triangle

$\therefore a+b > c, b+c > a, c+a > b$

$\therefore a > |b-c|, b > |c-a|, c > |a-b|$ square and add

$a^2+b^2+c^2 < 2(ab+bc+ca)$

$\Rightarrow a^2+b^2+c^2+2(ab+bc+ac) < 4(ab+bc+ca)$

$$\Rightarrow \frac{(a+b+c)^2}{ab+bc+ca} < 4 \Rightarrow P < 4$$

Again $(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$

$$\Rightarrow \frac{(a+b+c)^2}{ab+bc+ca} \ge 3 \Rightarrow P \ge 3$$

$\therefore 3 \le P < 4$ or $P \in [3, 4)$

7. **(b)** $a\tan\theta + b\sec\theta = c \Rightarrow b^2\sec^2\theta = (c - a\tan\theta)^2$

$\Rightarrow b^2(1+\tan^2\theta) = c^2 - 2ca\tan\theta + a^2\tan^2\theta$

$\Rightarrow (a^2-b^2)\tan^2\theta - 2ca\tan\theta + c^2 - b^2 = 0 \quad \ldots(1)$

Roots of equation (1) are $\tan\alpha$ and $\tan\beta$, where α and β are the two angles of the triangle.

We have $\tan\alpha + \tan\beta = \frac{2ca}{a^2-b^2}$ and

$$\tan\alpha.\tan\beta = \frac{c^2-b^2}{a^2-b^2}$$

$$\therefore \tan(\alpha+\beta) = \frac{\frac{2ca}{a^2-b^2}}{1-\frac{c^2-b^2}{a^2-b^2}} = \frac{2ca}{a^2-c^2}$$

$$\therefore \tan\left(\pi - \frac{3\pi}{4}\right) = \frac{2ca}{a^2-c^2} \Rightarrow a^2-c^2 = 2ca$$

8. **(d)** $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$\Rightarrow c^2 - (2b\cos A)c + b^2 - a^2 = 0$

It is a quadratic in c, whose roots are c_1 and c_2, so

$c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$

$\therefore c_1^2 + c_2^2 - 2c_1c_2\cos 2A$

$= (c_1+c_2)^2 - 2c_1c_2(1+\cos 2A)$

$= (2b\cos A)^2 - 2(b^2-a^2)(2\cos^2 A)$

$= 4b^2\cos^2 A - 4b^2\cos^2 A + 4a^2\cos^2 A = 4a^2\cos^2 A.$

9. **(c)** $(c_1 - c_2)^2 = (c_1 + c_2)^2 - 4c_1c_2$

$$= 4b^2\cos^2 A - 4b^2 + 4a^2$$

$$= 4a^2 - 4b^2\sin^2 A = 4a^2 - 4a^2\sin^2 B$$

$$(b\sin A = a\sin B) = 4a^2\cos^2 B$$

Therefore $|c_1 - c_2| = 2a\cos B_1$, where B_1 is acute.

10. **(a)** We have $\tan^{-1}\dfrac{a}{b+c} + \tan^{-1}\dfrac{b}{c+a}$

$$= \tan^{-1}\frac{\frac{a}{b+c}+\frac{b}{c+a}}{1-\frac{a}{b+c}\cdot\frac{b}{c+a}} = \tan^{-1}\frac{ac+a^2+b^2+bc}{bc+c^2+ac}$$

$$= \tan^{-1}\frac{c(a+b)+(a^2+b^2)}{c^2+c(a+b)}$$

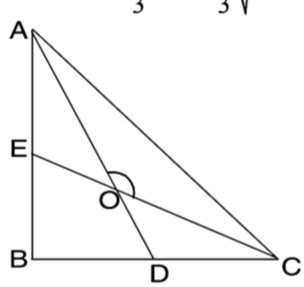
$$= \tan^{-1}\frac{c(a+b)+c^2}{c^2+c(a+b)} = \tan^{-1}(1) = \frac{\pi}{4}$$

[$\because c$ is $90° \therefore a^2+b^2=c^2$]

11. **(b)** Let $\angle ABC$ be the right angled triangle, with $AB = BC = a$ so that

$AC = \sqrt{2}a$. Let the medians AD and CE intersect at O.

Then, $CO = AO = \dfrac{2}{3}AD = \dfrac{2}{3}\sqrt{a^2+\dfrac{a^2}{4}} = \dfrac{a\sqrt{5}}{3}$



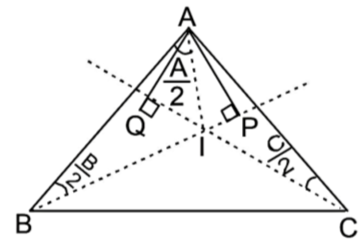
$$\therefore \cos(\angle AOC) = \frac{\frac{5a^2}{9}+\frac{5a^2}{9}-2a^2}{2\times\frac{\sqrt{5}a}{3}\times\frac{\sqrt{5}a}{3}} = -\frac{4}{5}$$

12. **(b)** In ΔAPB, $\sin\dfrac{B}{2} = \dfrac{AP}{AB}$

In ΔAQC, $\sin\dfrac{C}{2} = \dfrac{AQ}{AC}$

Now, in ΔABI, we have

$$\frac{BI}{\sin\frac{A}{2}} = \frac{AB}{\sin\left(\pi-\frac{A}{2}-\frac{B}{2}\right)} = \frac{AB}{\cos\frac{C}{2}}$$



and in ΔACI, we have $\dfrac{CI}{\sin\frac{A}{2}} = \dfrac{AC}{\cos\frac{B}{2}}$

$$\text{So, } \frac{AP}{BI}+\frac{AQ}{CI} = \frac{\sin\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}} + \frac{\sin\frac{C}{2}\cos\frac{B}{2}}{\sin\frac{A}{2}}$$

$$= \frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{\sin\frac{A}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} = \cot\frac{A}{2}$$

13. **(b)** Let θ be the angle opposite to side c of ΔABC

$$\Rightarrow c^2 = a^2+b^2-2ab\cos\theta = (a-b)^2+2ab(1-\cos\theta)$$

Also $\Delta = \dfrac{1}{2}ab\sin\theta \Rightarrow 2ab = \dfrac{4\Delta}{\sin\theta}$

$$\Rightarrow c^2 = (a-b)^2 + 4\Delta\frac{1-\cos\theta}{\sin\theta} = (a-b)^2 + 4\Delta\tan\frac{\theta}{2}$$

For minimum of c, $a = b$

$$\Rightarrow 2ab = 2a^2 = \frac{4\Delta}{\sin\theta} \Rightarrow a = \sqrt{\frac{2\Delta}{\sin\theta}} = b$$

14. **(b)** AD is median, $\therefore BD = CD$

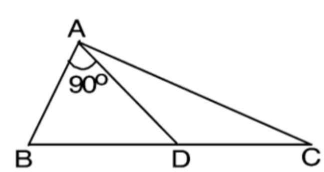
$\angle CAD = A - 90°$ and

From ΔABD, we have

$$\frac{BD}{\sin 90°} = \frac{AD}{\sin B} \Rightarrow BD = \frac{AD}{\sin B} \quad \ldots(1)$$

From the ΔACD, we have

$$\frac{CD}{\sin(A-90°)} = \frac{AD}{\sin C} \Rightarrow CD = -\frac{AD\cos A}{\sin(A+B)} \quad \ldots(2)$$

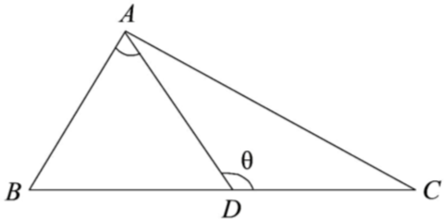


$$\because BD = CD \Rightarrow \frac{AD}{\sin B} = -\frac{AD\cos A}{\sin(A+B)}$$

$$\Rightarrow \sin(A+B) = -\cos A\sin B$$

$$\Rightarrow \sin A\cos B = -2\cos A\sin B \Rightarrow \tan A + 2\tan B = 0$$

ALTERNATE SOLUTION



Use $m-n$ cot rule in ΔABC, we get

$(1+1)\cot\theta = 1.\cot 90° - 1.\cot(A-90°)$

$= 1.\cot B - 1 - \cot C$

$\Rightarrow \tan A = \cot B - \cot C$

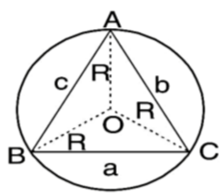
$\Rightarrow \tan A + \tan B \tan C = \tan C - \tan B$

$\Rightarrow \tan A + \tan B + \tan C = \tan C - \tan B$

$\Rightarrow \tan A + 2\tan \mathrm{B} = 0$

15. **(b)** We know that $R = \dfrac{abc}{4\Delta}$. Let Δ_1, Δ_2 and Δ_3 represent the areas of triangles OBC, OCA and OAB respectively. Then

$$R_1 = \frac{a.R.R}{4\Delta_1} \Rightarrow \frac{a}{R_1} = \frac{4\Delta_1}{R^2}$$



Similarly,

$$\frac{b}{R_2} = \frac{4\Delta_2}{R^2} \text{ and } \frac{c}{R_3} = \frac{4\Delta_3}{R^2}$$

$$\therefore \frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{4}{R^2}(\Delta_1 + \Delta_2 + \Delta_3) = \frac{4\Delta}{R^2}$$

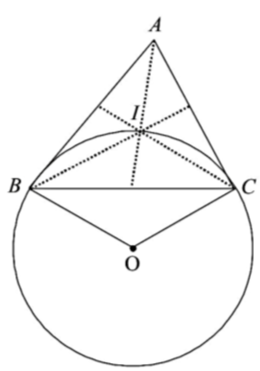
$$= \frac{4}{R^2}.\frac{abc}{4R} = \frac{abc}{R^3}$$

16. **(b)** $\angle BOC = 2\pi - (\pi + A)$

$$\left[\because \angle BIC = \pi - \frac{B+C}{2} = \frac{\pi}{2} + \frac{A}{2}\right] = \pi - A$$

$$\Rightarrow a^2 = R^2 + R^2 - 2R^2\cos(\pi - A)$$

$$\therefore a^2 = 2R^2(1+\cos A) \;\; R = \frac{a}{\sqrt{2.2\cos^2\dfrac{A}{2}}} = \frac{a}{2}\sec\frac{A}{2}$$

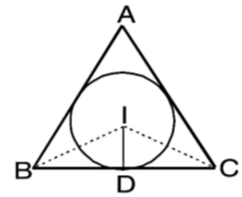


17. **(b)** We have $\angle IBD = \dfrac{B}{2}$, $\angle ICD = \dfrac{C}{2}$

$BD = 6$ cm, $CD = 8$cm

$\Rightarrow BC = a = 14$ cm.

Also $ID = r = 4$ cm



Now, $\tan\dfrac{B}{2} = \dfrac{4}{6} = \dfrac{2}{3}$

and $\tan\dfrac{C}{2} = \dfrac{4}{8} = \dfrac{1}{2}$

$$\tan\frac{B}{2}\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \times \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{s-a}{s}$$

$$\therefore \frac{2}{3}\times\frac{1}{2} = \frac{s-14}{s} \Rightarrow s = 21 \text{ cm}$$

$\therefore$ Area, $\Delta = rs = 4 \times 21 = 84 \text{ cm}^2$

18. **(b)** Let the other side be a and hypotenuse b, then

$b^2 - a^2 = 23^2 \Rightarrow (b-a)(b+a) = 23^2$

$\therefore b - a = 1$ and $b + a = 23^2$

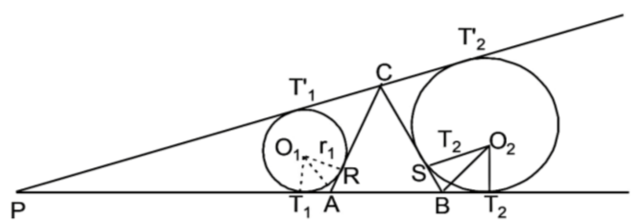
$\therefore$ Perimeter $= a + b + 23 = 24 \times 23$

19. **(b)** See the figure, we have $\angle T_1O_1R = 60°$, since it is the supplement of $\angle T_1AR = 120°$,

(exterior angle of ΔABC)

Hence, $\angle AO_1R = 30°$

Similarly, $\angle BO_2S = 30°$



Further, $T_1T_2 = T_1A + AB + BT_2 = RA + AB + SB$

$$= r_1\tan 30° + a + r_1\tan 30° = \frac{r_1 + r_2}{\sqrt{3}} + a$$

and $T_1'T_2' = T_1'C + CT_2' = CR + CS$

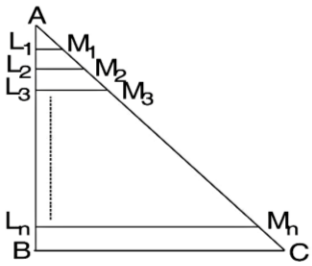
$$= (a - RA) + (a - SB) = 2a - \frac{r_1 + r_2}{\sqrt{3}}$$

Since, common external tangents to two circles are equal

$$T_1T_2 = T_1'T_2' \Rightarrow r_1 + r_2 = \frac{a\sqrt{3}}{2}.$$

20. **(c)** We have

$$\frac{AL_1}{AB} = \frac{L_1M_1}{BC} \Rightarrow \frac{1}{n+1} = \frac{L_1M_1}{a} \Rightarrow L_1M_1 = \frac{a}{n+1}$$



$$\frac{AL_2}{AB}=\frac{L_2M_2}{BC}\Rightarrow L_2M_2=\frac{2a}{n+1},\ldots etc$$

Required sum

$$=\frac{a}{n+1}+\frac{2a}{n+1}+\ldots+\frac{na}{n+1}=\frac{a}{n+1}[1+2+\ldots+n]=\frac{na}{2}$$

21. **(b)** $\frac{a^2+b^2}{a^2-b^2}\sin(A-B)=1$

$$\Rightarrow\frac{\sin^2 A+\sin^2 B}{\sin^2 A-\sin^2 B}.\sin(A-B)=1$$

$$\Rightarrow\frac{\sin^2 A+\sin^2 B}{\sin(A+B)\sin(A-B)}\times\sin(A-B)=1$$

$\Rightarrow\sin^2 A+\sin^2 B=\sin(A+B)=\sin C$

$\Rightarrow 1-\cos 2A+1-\cos 2B=2\sin C$

$\Rightarrow\cos 2A+\cos 2B=2(1-\sin C)$

$\Rightarrow 2\cos(A+B)\cos(A-B)=2(1-\sin C)$

$$\Rightarrow\cos(A-B)=\frac{1-\sin C}{-\cos C}\qquad\left(\text{if } C\neq\frac{\pi}{2}\right)$$

$$\Rightarrow\cos(A-B)=\frac{\left(\sin\frac{C}{2}-\cos\frac{C}{2}\right)^2}{\sin^2\frac{C}{2}-\cos^2\frac{C}{2}}=\frac{\sin\frac{C}{2}-\cos\frac{C}{2}}{\sin\frac{C}{2}+\cos\frac{C}{2}}$$

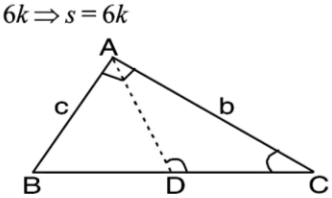
$$=\frac{\tan(C/2)-1}{\tan(C/2)+1}=\tan\left(\frac{C}{2}-\frac{\pi}{4}\right)$$

22. **(b)** $r_1=2r_2=3r_3\ \Rightarrow\frac{\Delta}{s-a}=\frac{2\Delta}{s-b}=\frac{3\Delta}{s-c}=\frac{\Delta}{k}$(say)

then $s-a=k,\ s-b=2k,\ s-c=3k$

$\Rightarrow 3s-(a+b+c)=6k\Rightarrow s=6k$

$$\Rightarrow\frac{a}{5}=\frac{b}{4}=\frac{c}{3}=k$$



so that $a^2=b^2+c^2$

$\Rightarrow ABC$ is right angle triangle with $A=90°$ since D is the middle point of BC

$AD=DC$ (radius of the circumcirle)

$\Rightarrow\angle DAC=C\quad\Rightarrow\angle ADC=180°-2C$

$\Rightarrow\cos\angle ADC=\cos(180°-2C)=-\cos 2C$

$=-[2\cos^2 C-1]=1-2\cos^2 C$

$$=1-2\times\frac{16}{25}=-\frac{7}{25}\left[\because\text{ from }\Delta ABC\ \cos C=\frac{b}{a}=\frac{4}{5}\right]$$

23. **(a)** Since $4\sin A\cos B=1$, so A and B can not be $\frac{\pi}{2}$

[As if $B=\frac{\pi}{2}$, then $\cos B=0$ and if $A=\frac{\pi}{2}$ then $\tan A$ is not defined]

$$\text{so } C=\frac{\pi}{2}\Rightarrow B=\frac{\pi}{2}-A\Rightarrow 4\sin A\cos\left(\frac{\pi}{2}-A\right)=1$$

$$\Rightarrow\sin^2 A=\frac{1}{4}\Rightarrow\sin A=\frac{1}{2}\Rightarrow A=\frac{\pi}{6}\Rightarrow B=\frac{\pi}{3}$$

so angles are $\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$ which are in A.P.

24. **(b)** For (a) $\sin 2A+\sin 2B+\sin 2C=0$

$\Rightarrow 4\sin A\sin B\sin C=0$ which is not possible as sine of any angle of the triangle can not be zero.

For (b) $\cos 2A+\cos 2B+\cos 2C=-1$

$\Rightarrow -1-4\cos A\cos B\cos C=-1$

$\Rightarrow\cos A\cos B\cos C=0$

Which is possible if the triangle is right angled.

For (c) $\sin A+\sin B+\sin C=0$

$$\Rightarrow 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=0$$

Which is again *not possible* as no angle of the triangle can be integral multiple of π.

For (d) $\cos A+\cos B+\cos C=1$

$$\Rightarrow 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1\Rightarrow\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=0$$

Which is again *not possible*, as no angle of the triangle can be an integral multiple of 2π.

25. **(a)** We have, $\frac{b+c}{2}\geq\sqrt{bc}=\sqrt{\lambda^2}\Rightarrow b+c\geq 2\lambda$

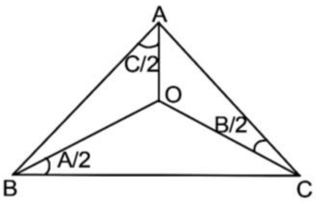
$$\text{Now, }\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{b+c}{\sin B+\sin C}$$

$$\Rightarrow\frac{a}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{b+c}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$\Rightarrow a=\frac{(b+c)\sin\frac{A}{2}}{\cos\left(\frac{B-C}{2}\right)}$$

$$\because\quad 0\le\cos\frac{B-C}{2}\le 1 \text{ and } b+c\ge 2\lambda\Rightarrow a\ge 2\lambda\sin\frac{A}{2}$$

26. **(c)** We have from $\Delta\, OAC$



$$\frac{\sin\left(A-\frac{C}{2}\right)}{\sin\left(\frac{B}{2}\right)}=\frac{OC}{OA}$$

Similarly, $\dfrac{\sin\left(B-\frac{A}{2}\right)}{\sin\left(\frac{C}{2}\right)}=\dfrac{OA}{OB}$

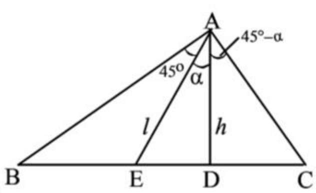
and $\dfrac{\sin\left(C-\frac{B}{2}\right)}{\sin\left(\frac{A}{2}\right)}=\dfrac{OB}{OC}$

so that the given expression is equal to 1.

27. **(d)** Let ABC be the triangle right angled at A

$AD=h, AE=l,$

Let $\angle EAD=\alpha$



Then $\cos\alpha=\dfrac{h}{l}$

$\because\quad \angle DAC=45°-\alpha$

$C=90°-(45°-\alpha)=45°+\alpha$

and $B=45°-\alpha$

So, $\cos C=\cos(45°+\alpha)$

$$=\frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)=\frac{1}{\sqrt{2}}\left(\frac{h-\sqrt{l^2-h^2}}{l}\right)$$

and $\cos B=\cos(45°-\alpha)$

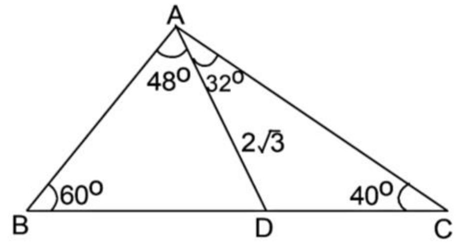
$$=\frac{1}{\sqrt{2}}(\cos\alpha+\sin\alpha)=\frac{1}{\sqrt{2}}\left(\frac{h+\sqrt{l^2-h^2}}{l}\right)$$

28. **(d)** A, B, C be in A. P. Then $2B=A+C$ and $A+B+C=180°$

$\Rightarrow B=60°$ and $A+C=120°$

Let $A=2C\Rightarrow A=80°, C=40°$

Let AD be the median of length $2\sqrt{3}$ cm



Then $\angle BAD+\angle DAC=80°$

Since $\angle BAD:\angle DAC=3:2$

$\therefore\ \angle BAD=48°,\ \angle DAC=32°$

From $\Delta\, BAD$, $\dfrac{BD}{\sin 48°}=\dfrac{AD}{\sin 60°}$

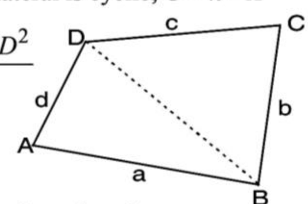
$$\Rightarrow BD=\frac{2\sqrt{3}\sin 48°}{\frac{\sqrt{3}}{2}}=4\sin 48°$$

$\therefore\ BC=8\sin 48°=8\cos 42°$

29. **(b)** Since a circle can be inscribed in the quadrilateral, we have $a+c=b+d$

and since the quadrilateral is cyclic, $C=\pi-A$

$$\cos A=\frac{a^2+d^2-BD^2}{2ad}$$



$$\Rightarrow 2ad\cos A=a^2+d^2-[b^2+c^2-2bc\cos C]$$

$$=a^2+d^2-b^2-c^2+2bc\cos A$$

$$\Rightarrow 2\,(ad+bc)\cos A=a^2+d^2-b^2-c^2$$

$$\Rightarrow \cos A=\frac{a^2+b^2-b^2-c^2}{2(ad+bc)}$$

Now, $a+c=b+d\Rightarrow a-d=b-c$

$$\Rightarrow a^2+d^2-b^2-c^2=2\,(ad-bc) \text{ and } \cos A=\frac{ad-bc}{ad+bc}$$

30. **(a)** We have,

$$\tan(A+B)=\tan(360°-C-D)=-\tan(C+D)$$

$$\Rightarrow\frac{\tan A+\tan B}{1-\tan A\tan B}=-\frac{\tan C+\tan D}{1-\tan C\tan D}$$

$$\Rightarrow(\tan A+\tan B)(1-\tan C\tan D)+$$
$$(1-\tan A\tan B)(\tan C+\tan D)=0$$

$$\Rightarrow\tan A+\tan B+\tan C+\tan D=\Sigma\tan A\tan B\tan C$$

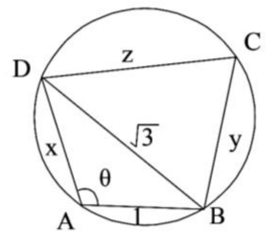
$$\frac{\tan A+\tan B+\tan C+\tan D}{\tan A\tan B\tan C\tan D}$$

$$=\frac{1}{\tan A}+\frac{1}{\tan B}+\frac{1}{\tan C}+\frac{1}{\tan D}$$

$$\Rightarrow \frac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D}$$
$$= \tan A \tan B \tan C \tan D$$

31. **(b)** Let $\angle BAD = \theta$ ad $AD = x$, then

$$\frac{BD}{\sin\theta} = 2R \Rightarrow \frac{\sqrt{3}}{\sin\theta} = 2\times 1$$



$$\therefore \sin\theta = \frac{\sqrt{3}}{2} \text{ or } \theta = 60°.$$

Applying cosine rule in ΔABD, we get

$$\cos 60° = \frac{1^2 + x^2 - (\sqrt{3})^2}{2.1.x}$$

$$\Rightarrow x^2 - x - 2 = 0 \Rightarrow x = 2$$

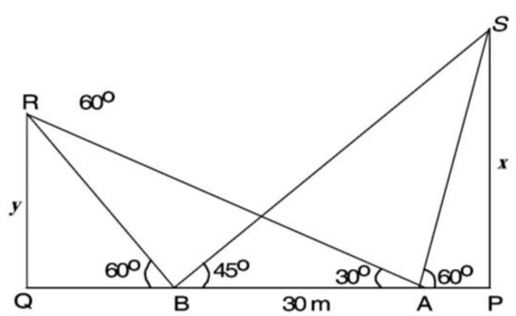
32. **(d)** Let x and y be the heights of the flagstaffs at P and Q respectively

Then, $AP = x\cot 60° = \frac{x}{\sqrt{3}}$, $AQ = y\cot 30° = y\sqrt{3}$

$$BP = x\cot 45° = x,\ BQ = y\cot 60° = \frac{y}{\sqrt{3}}$$

$$\Rightarrow AB = BP - AP = x - \frac{x}{\sqrt{3}} \qquad [\because AB = 30\ m]$$

$$\Rightarrow 30\sqrt{3} = (\sqrt{3}-1)x \Rightarrow x = 15(3+\sqrt{3})$$



Similarly $30 = y\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) \Rightarrow y = 15\sqrt{3}$

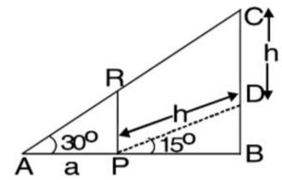
so that $PQ = BP + BQ = x + \frac{y}{\sqrt{3}}$

$$= 15(3+\sqrt{3}) + 15 = (60 + 15\sqrt{3})\text{m}.$$

33. **(d)** CD is the pole on an inclined plane PBD. Triangle APR is similar to triangle ABC.

$$\therefore \frac{AP}{AB} = \frac{RP}{BC}$$

$$\Rightarrow \frac{a}{a + h\cos 15°} = \frac{a\tan 30°}{h + h\sin 15°}$$



$$\Rightarrow \frac{1}{\sqrt{3}}(a + h\cos 15°) = h(1 + \sin 15°)$$

$$\Rightarrow \frac{a}{h} = \sqrt{3}(1 + \sin 15°) - \cos 15°$$

$$= \sqrt{3}\left[1 + \frac{\sqrt{3}-1}{2\sqrt{2}}\right] - \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{3} + 1}{\sqrt{2}}$$

$$\left(\because \sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}, \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

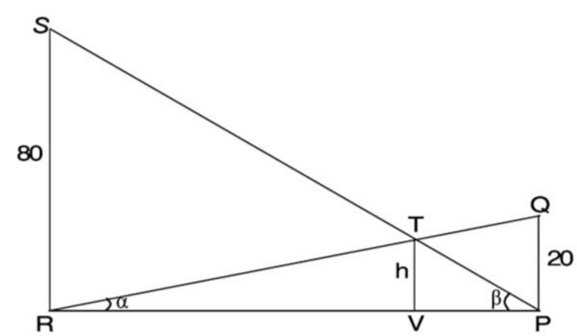
34. **(b)** Let PQ and RS be the poles of height 20m and 80m subtending angles α and β at R and P respectively. Let h be the height of the point T, the intersection of QR and PS.

Then $PR = h\cot\alpha + h\cot\beta = 20\cot\alpha = 80\cot\beta$

$$\Rightarrow \cot\alpha = 4\cot\beta \text{ or } \frac{\cot\alpha}{\cot\beta} = 4$$

Again $h\cot\alpha + h\cot\beta = 20\cot\alpha$

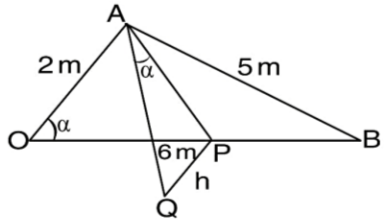
$\Rightarrow (h - 20)\cot\alpha = -h\cot\beta$



$$\Rightarrow \frac{\cot\alpha}{\cot\beta} = \frac{h}{20-h} = 4 \Rightarrow h = 80 - 4h \Rightarrow h = 16\text{m}$$

35. **(b)** Let PQ be the tower of height h at the middle point P of the side OB of the triangle OAB, where $OA = 2, OB = 6, AB = 5$ and $\angle AOB = \angle PAQ = \alpha$ Then $AP = h\cot\alpha$, $OP = 3$

From triangle OAB, $\cos\alpha = \frac{2^2 + 6^2 - 5^2}{2\times 2\times 6} = \frac{5}{8}$

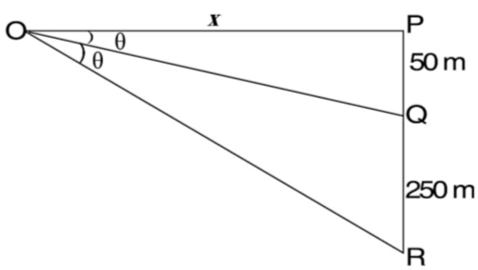


and from $\Delta\, OAP$, $\cos\alpha = \dfrac{2^2+3^2-h^2\cot^2\alpha}{2\times2\times3}$

$$\Rightarrow \frac{5}{8}=\frac{13-h^2\times\frac{25}{39}}{12} \Rightarrow \frac{25h^2}{39}=13-\frac{15}{2}=\frac{11}{2}$$

$$\Rightarrow h^2=\frac{11\times39}{25\times2}$$

36. (c) Let PQ be the pole on the building QR and O be the observer.
Then, $PQ=50$, $QR=250 \Rightarrow PR=300$
so the observer is at the same height as the top P of the pole. Let $OP=x$. Then from right angled triangles OPQ and OPR,



$$\tan\theta=\frac{50}{x} \text{ and } \tan 2\theta=\frac{300}{x}$$

so that $\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{300}{x}$

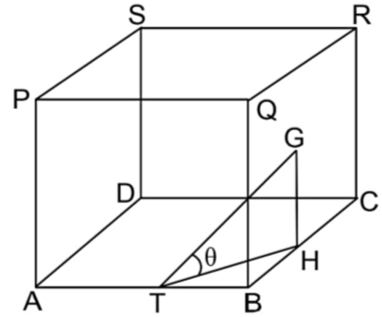
$$\Rightarrow \frac{2\times\frac{50}{x}}{1-\left(\frac{50}{x}\right)^2}=\frac{300}{x} \Rightarrow 3\left\{1-\left(\frac{50}{x}\right)^2\right\}=1$$

$$\Rightarrow \left(\frac{50}{x}\right)^2=\frac{2}{3} \Rightarrow x=\frac{50\sqrt3}{\sqrt2}=25\sqrt6$$

37. (a) Let H be the mid point of BC since
$\angle TBH=90°, TH^2=BT^2+BH^2=5^2+5^2=50$

Also since

$\angle THG=90°,\ TG^2=TH^2+GH^2=50+25=75$



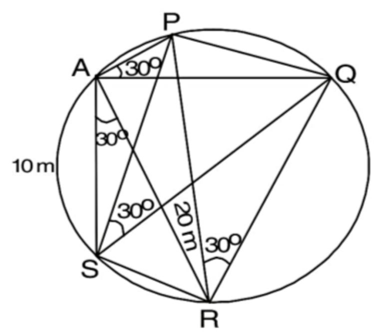
Let θ be the required angle of elevation of G at T.

Then, $\sin\theta=\dfrac{GH}{TG}=\dfrac{5}{5\sqrt3}=\dfrac{1}{\sqrt3}$

$$\Rightarrow \theta=\sin^{-1}\frac{1}{\sqrt3}$$

38. (b) Since PQ subtends the same angle of 30° at each of the points
A, R and S, the points P, Q, A, R and S lie on a circle
And $AR\perp AP$ and $AS\perp AQ$

$\angle RAS=90°-\angle RAQ=\angle PAQ=30°$



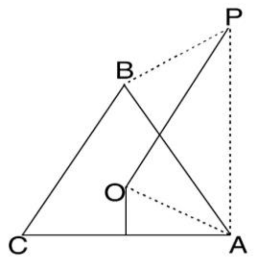
Also it is given $AR=20$m, $AS=10$ m.
Then PQ and RS are chords of the same circle making an angle of 30° at a points
on the circumference of the circle and hence are equal in length.
Now from ΔSAR,
$RS^2=20^2+10^2-2\times20\times10\cos30°$

$$\Rightarrow PQ^2=500-200\sqrt3 \quad \Rightarrow PQ=\sqrt{500-200\sqrt3}$$

39. (b) Let O be the centre of equilateral trianlge ABC and OP be the tower of height h
Then each of the triangles PAB, PBC and PCA are equilateral and thus
$PA=PB=AB=a$
In triangle ABC, OA is the bisector of angle A, so

$$\frac{OA}{a/2}=\sec30°$$

$\Rightarrow OA = \frac{a}{2} \cdot \frac{2}{\sqrt{3}} = \frac{a}{\sqrt{3}}$



Now, from right angled triangle POA.

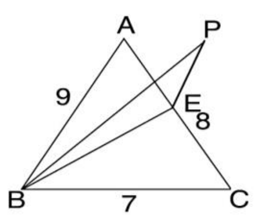
$\Rightarrow PA^2 = OP^2 + OA^2$

$\Rightarrow a^2 = h^2 + \frac{a^2}{3} \Rightarrow \frac{2a^2}{3} = h^2 \Rightarrow 2a^2 = 3h^2$

40. (a) In triangle ABC

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{2 \times 7 \times 8} = \frac{2}{7}$$

In triangle BEC, $BC = 7$, $CE = 4$ and $\cos C = \frac{2}{7}$



so that $BE^2 = 4^2 + 7^2 - 2 \times 4 \times 7 \times \frac{7}{2} = 49$

$\Rightarrow BE = 7$

if h is the height of the lamp post EP at E then

$\frac{PE}{BE} = \frac{h}{7} = \tan(\tan^{-1} 3) = 3$

$\Rightarrow h = 21$ m.

41. (c) Let OP be the peak of height h.

AB be the horizontal base of length $2a$ with C as its middle point

then $AC = CB = a$ Also $\angle PAO = \angle PBO = 15°$ and $\angle PCO = 45°$

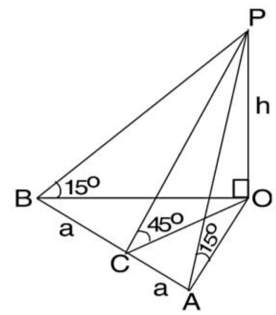
then $OA = OB = h\cot 15°$, $OC = h \cot 45° = h$

Since OAB, is isosceles. OC is perpendicular on AB and from right angled triangle

OCB, , $OB^2 = OC^2 + BC^2$

$\Rightarrow h^2 \cot^2 15° = h^2 + a^2$

$\Rightarrow h^2 [\cot^2 15° - 1] = a^2$



$$\Rightarrow h^2 = \frac{a^2}{\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 - 1} = \frac{a^2(\sqrt{3}-1)^2}{2(2\sqrt{3})}$$

$$\Rightarrow h = (\sqrt{3}-1)\frac{a}{2 \times 3^{1/4}} = \frac{\sqrt{3}-1}{6} \times 3^{3/4} a$$

42. (c) Here $2 \sin B = \sin A + \sin C = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$

or $4\sin\frac{B}{2}\cos\frac{B}{2} = 2\cos\frac{B}{2}\cos\frac{A-B}{2}$

$\Rightarrow \sin\frac{B}{2} = \frac{1}{2}\cos\frac{A-C}{2}$

$$\Rightarrow \tan\frac{B}{2} = \frac{\cos\frac{A-C}{2}}{\sqrt{4 - \cos^2\frac{A-C}{2}}}$$

or $4\tan^2\frac{B}{2} - \cos^2\frac{A-C}{2}\tan^2\frac{B}{2} = \cos^2\frac{A-C}{2}$

$$\Rightarrow \frac{4\tan^2\frac{B}{2}}{1+\tan^2\frac{B}{2}} = \cos^2\frac{A-C}{2} \le 1$$

or $4\tan^2\frac{B}{2} \le 1 + \tan^2\frac{B}{2} \Rightarrow \tan^2\frac{B}{2} \le \frac{1}{3}$

43. (b) We know that

$$\frac{4R^2}{\Delta}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$= \frac{R^2}{\Delta}(\sin A + \sin B + \sin C)$$

$$= \frac{R.2R}{2\Delta}(\sin A + \sin B + \sin C)$$

$$= \frac{R(a+b+c)}{2\Delta} = \frac{R2s}{2\Delta} = \frac{R}{r} \ge 2$$

44. **(c)** Given $\angle B = \frac{2\pi}{3} \Rightarrow A + C = \frac{\pi}{3}$

Now $\cos A + \cos C = 2\cos\frac{A+C}{2}\cos\frac{A-C}{2}$

$= 2\left(\frac{\sqrt{3}}{2}\right)\cos\frac{A-C}{2} = \sqrt{3}\cos\frac{A-C}{2}$

Now $0 \le A - C < \frac{\pi}{3} \Rightarrow \frac{\sqrt{3}}{2} < \cos\frac{A-C}{2} \le 1$

$\therefore \cos A + \cos C \in \left(\frac{3}{2}, \sqrt{3}\right]$

45. **(c)** The circumradius XC makes angle $\frac{\pi}{2} - A$ with BC.

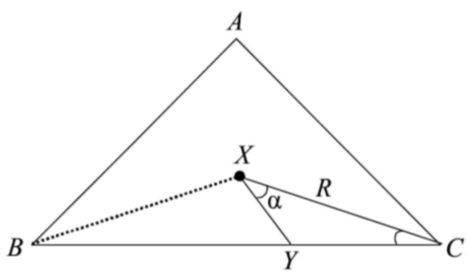
Let $\angle YXC = \alpha$. Apply sine rule to ΔXYC then

$$\frac{R}{\sin\left(\alpha + \frac{\pi}{2} - a\right)} = \frac{XY}{\sin\left(\frac{\pi}{2} - A\right)} = \frac{YC}{\sin\alpha} = 2R_1$$

where R_1 is the circumradius of ΔXYC.
When $Y \to C$, $\alpha \to 0$ and hence

$$2R_1 = \frac{R}{\sin\left(\frac{\pi}{2} - A\right)} = \frac{R}{\cos A}$$

$$\Rightarrow R_1 = \frac{R}{2}\sec A$$



46. **(c)** Since $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

$2\cot\frac{B}{2} = \cot\frac{A}{2} + \cot\frac{C}{2} \ge 2\sqrt{\cot\frac{A}{2}\cot\frac{C}{2}}$

$\Rightarrow 2\cot\frac{B}{2} \ge 2\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}\sqrt{\tan\frac{B}{2}}$

or $\cot\frac{B}{2} \ge \sqrt{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}\sqrt{\tan\frac{B}{2}}$

or $\cot\frac{B}{2} \ge \sqrt{3\cot\frac{B}{2}}\sqrt{\tan\frac{B}{2}}$ or $\cot\frac{B}{2} \ge \sqrt{3}$.

47. **(c)** Here $2B = A + C$ and $\frac{A}{C} = \frac{4}{1}$

or $\frac{A}{4} = \frac{C}{1} = \frac{A+C}{5} = \frac{2B}{5} = k$ (say)

So that $4k + k + \frac{5k}{2} = \pi$ or $k = \frac{2\pi}{15} = C$

48. **(b)** We have

$\tan A + \tan B + \tan C = \tan A \tan B \tan C = \frac{p}{q}$

and $\tan A \tan B + \tan B \tan C + \tan C \tan A$

$$= \frac{\sin A \sin B \cos C + \sin B \sin C \cos A + \sin A \sin C \cos B}{\cos A \cos B \cos C}$$

$$= \frac{\sin B \sin(A+C) + \sin A \sin C \cos B}{q}$$

$$= \frac{\sin^2 B + \sin A \sin C \cos B}{q}$$

$$= \frac{1 - \cos^2 B + \sin A \sin C \cos B}{q}$$

$$= \frac{1 + \cos B \cos(A+C) + \sin A \sin C \cos B}{q}$$

$$= \frac{1 + \cos A \cos B \cos C}{q} = \frac{1+q}{q}.$$

Hence, $\tan A$, $\tan B$, $\tan C$ are roots of the equation
$qx^3 - px^2 + (1+q)x - p = 0$

49. **(a)** We have
$\tan A + \tan B + \tan C = 3k$
and $\tan A \tan B \tan C = -k$
$\Rightarrow \tan A + \tan B + \tan C = -3\tan A \tan B \tan C$
which is not possible.

50. **(c)** We have $\sin\theta - \cos\theta = \frac{b}{a}$ and $\sin\theta\cos\theta = \frac{c}{a}$

$\Rightarrow 1 - 2\sin\theta\cos\theta = \frac{b^2}{a^2}$

or $1 - \frac{2c}{a} = \frac{b^2}{a^2}$

$\Rightarrow a^2 - b^2 = 2ac.$

Hence $\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{2ac + c^2}{2ac} = 1 + \frac{c}{2a}$.

51. **(b)** For a triangle ABC

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$$

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s^2}=\frac{\Delta}{s^2}=\frac{r}{s}\le\frac{R}{2s}.$$

52. **(c)** Let O be the incentre of the given triangle ABC. Let O' be the centre of the circle touching the incircle and the sides AB and AC. Let r_1 be its radius. We have

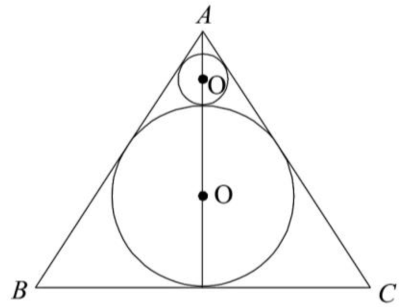
$$AO'=r_1\operatorname{cosec}\frac{A}{2}=\sqrt{2}\,r_1$$

and $AO=r\operatorname{cosec}\dfrac{A}{2}=\sqrt{2}\,r$

Hence $OO'=r_1+r=AO-AO'=\sqrt{2}(r-r_1)$

$$\Rightarrow r_1(1+\sqrt{2})=r\left(\sqrt{2}-1\right)$$

or $\quad r_1=\dfrac{r(\sqrt{2}-1)}{1+\sqrt{2}}=r(2+1-2\sqrt{2})=r(3-2\sqrt{2})$



B COMPREHENSION TYPE

1. **(a)** In ΔAOE,

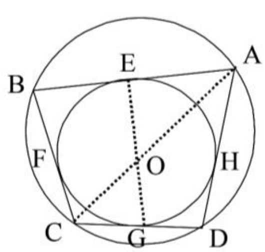
$$\tan\frac{A}{2}=\frac{OE}{AE}\Rightarrow AE=r\cot\frac{A}{2}\qquad (OE=r)$$

Similarly $BE=r\cot\dfrac{B}{2}$

$a=r\left(\cot\dfrac{A}{2}+\cot\dfrac{B}{2}\right)$, similarly

$b=r\left(\cot\dfrac{B}{2}+\cot\dfrac{C}{2}\right)$,

$c=r\left(\cot\dfrac{C}{2}+\cot\dfrac{D}{2}\right)$ and $d=r\left(\cot\dfrac{D}{2}+\cot\dfrac{A}{2}\right)$



However in the cyclic quadrilateral $C=\pi-A$ and $D=\pi-B$, so

$b=r\left(\cot\dfrac{B}{2}+\tan\dfrac{A}{2}\right)$, $c=r\left(\tan\dfrac{A}{2}+\tan\dfrac{B}{2}\right)$

and $d=r\left(\tan\dfrac{B}{2}+\cot\dfrac{A}{2}\right)$

So,

$$s=\frac{a+b+c+d}{2}=r\left(\tan\frac{A}{2}+\cot\frac{A}{2}+\tan\frac{B}{2}+\cot\frac{B}{2}\right)$$

$$\therefore s-a=r\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=c,$$

similarly $s-b=d$, $s-c=a$ and $s-d=b$.
Therefore Area of the quadrilateral

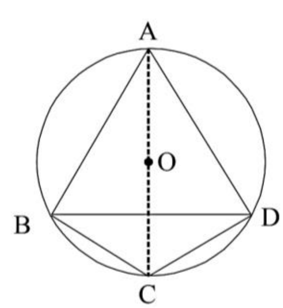
$$=\sqrt{(s-a)(s-b)(s-c)(s-d)}=\sqrt{abcd}$$

2. **(b)** The area of the quadrilateral can be given by $\dfrac{1}{2}(ra+rb+rc+rd)$, where r is the radius of desired circle

$$\therefore \frac{1}{2}r(a+b+c+d)=\sqrt{abcd}\Rightarrow r=\frac{2\sqrt{abcd}}{a+b+c+d}$$

Putting the values of a, b, c, d in the formula obtained in (2), we get $r=\dfrac{2\sqrt{3\times3\times4\times4}}{3+3+4+4}=\dfrac{2\times3\times4}{14}=\dfrac{12}{7}$ cm

3. **(a)** ΔABD and ΔBCD are isosceles, so, the diagonal AC is the diameter of the circumcircle.



Now, $AC = \sqrt{3^2 + 4^2} = 5$.

Therefore radius of the circumcircle $= \frac{5}{2}$

4. **(b)** For real roots c_1 and c_2, $D > 0$, i.e., $a > b \sin A$
Consider the smaller root, say

$c_1 = b\cos A - \sqrt{a^2 - b^2 \sin^2 A}$

$c_1 > 0$ if $b \cos A > \sqrt{a^2 - b^2 \sin^2 A}$, i.e., if

$b^2 \cos^2 A > a^2 - b^2 \sin^2 A$ and $\cos A > 0$
or if $b^2 > a^2$ and $\cos A > 0$.
Hence, two different triangles are possible if $a > b \sin A$, $b > a$ and A is acute.

ALTERNATIVELY
The equation $c^2 - (2b\cos A)c + b^2 - a^2 = 0$ has two distinct positive roots c_1 and c_2 if and only if discriminant > 0, $c_1 + c_2 > 0$ and $c_1 c_2 > 0$
$\Rightarrow a > b\sin A, 2b\cos A > 0$ and $b^2 - a^2 > 0$
$\Rightarrow a > b \sin A$ and $b > a$

5. **(b)** If $a = b$, then $a^2 = b^2(\cos^2 A + \sin^2 A)$

$\Rightarrow a^2 - b^2 \sin^2 A = b^2 \cos^2 A$

$\therefore \sqrt{a^2 - b^2 \sin^2 A} = b \cos A \Rightarrow G = 0$
but $c_2 > 0$. So, only one triangle is possible.

6. **(a)** We have $b\sin A = 6 \sin 45° = 3\sqrt{2}$

So, $a = 2\sqrt{2} < b \sin A$, so no triangle will form.

7. **(a)** Area of $\Delta DEF = \frac{1}{2}\, a \cos A.\, b \cos B.\, \sin(\pi - 2C)$

$= \frac{1}{2}\, ab \cos A \cos B \sin 2C$

$= 2 \cos A \cos B \cos C.\, \frac{ab}{2}\, \sin C$

$= 2 \cos A \cos B \cos C \times$ area of ΔABC.

8. **(b)** Circum radius of $DEF = \frac{a\cos A. b\cos B. c\cos C}{4\text{area of}\Delta\ DEF}$

$= \frac{abc \cos A \cos B \cos C}{2ab \cos A \cos B \sin 2C} = \frac{c}{4 \sin C} = \frac{R}{2}$

9. **(b)** In radius of $\Delta DEF = \frac{\text{Area of}\Delta DEF}{\text{semi-perimeter of }\Delta DEF}$

$= \frac{2ab \cos A \cos B \cos C \sin C}{(a\cos A + b\cos B + c\cos C)}$

$= \frac{2ab \cos A \cos B \cos C \sin C}{2R(\sin A \cos A + \cos B \sin B + \sin C \cos C)}$

$= \frac{8R^2 \cos A \cos B \cos C \sin A \sin B \sin C}{R(\sin 2A + \sin 2B + \sin 2C)}$

$= \frac{8R \cos A \cos B \cos C \sin A \sin B \sin C}{4 \sin A \sin B \sin C}$

$= 2R \cos A \cos B \cos C$

10. **(c)** Perimeter of $\Delta DEF = a \cos A + b \cos B + c \cos C$
$= 2R (\sin A \cos A + \sin B \cos B + \sin C \cos C)$
$= R (\sin 2A + \sin 2B + \sin 2C)$
$= 4 R \sin A \sin B \sin C$

$= 32 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

$= 8r \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

$= 2 r (\sin A + \sin B + \sin C)$

$= \frac{r}{R}(a + b + c)$

11. **(c)** $r_1 + r_2 + r_3 - r = \Delta\left[\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}\right]$

$= \Delta\left[\frac{2s - a - b}{(s-a)(s-b)} + \frac{s - s + c}{s(s-c)}\right]$

$= \Delta c\left[\frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$

$= \frac{\Delta c}{\Delta^2}\left[2s^2 - s(a+b+c) + ab\right] = \frac{abc}{\Delta} = 4R$

$\therefore r_1 + r_2 + r_3 = r + 4R$
$r_1 r_2 + r_2 r_3 + r_3 r_1$

$= \Delta^2\left[\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\right]$

$= \frac{\Delta^2(s - c + s - a + s - b)}{(s-a)(s-b)(s-c)} = s^2$

$r_1 r_2 r_3 = \frac{\Delta^2}{(s-a)(s-b)(s-c)} = \Delta s = rs^2$

$\therefore r_1, r_2, r_3$ are roots of the equations
$x^3 - x^2(r_1 + r_2 + r_3) + x(r_1 r_2 + r_2 r_3 + r_3 r_1) - r_1 r_2 r_3 = 0$
$\Rightarrow x^3 - x^2(4R + r) + x(s^2) - rs^2 = 0$

12. **(a)** We have
$x^3 - (4R + r)x^2 + s^2 x - rs^2 = (x - r_1)(x - r_2)(x - r_3)$
$\Rightarrow (-s)^3 - (4R + r)(-s)^2 + s^2(-s) - rs^2$
$= (-s - r_1)(-s - r_2)(-s - r_3)$
$\therefore (s + r_1)(s + r_2)(s + r_3) = 2s^2(s + r + 2R)$

13. **(b)** r_1, r_2, r_3 are in A.P. $\Rightarrow r_2 = \frac{4R + r}{3}$

Also, $r_2 \geq \sqrt{r_1 r_3} \Rightarrow r_2^3 \geq r_1 r_2 r_3 = rs^2$
$\therefore (4R + r)^3 \geq 27\, rs^2$

C REASONING TYPE

1. **(d)** $A+C=180;\ B+D=180$

$\cos A = -\cos C \Rightarrow \cos B + \cos D = 0$

$\cos A + \cos C = 0$

$= \cos A + \cos B + \cos C + \cos D = 0$

but $A + C = 180° \Rightarrow \sin A = \sin C$

$B + D = 180° \Rightarrow \sin B = \sin D$

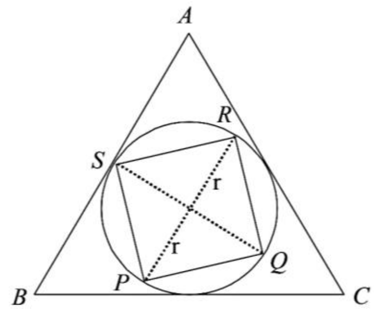
only II is correct

2. **(c)** The incentre of the pedal triangle of the given triangle is coincident with the orthocentre of the given triangle. Since pedal triangle is formed with the feet of the altitudes of the triangle so the given triangle is the excentral triangle of the pedal triangle.

3. **(c)** $r = \dfrac{\Delta}{s} = \dfrac{a}{2\sqrt{3}}$

Now $PQ^2 + QR^2 = 4r^2 \Rightarrow PQ^2 = \dfrac{a^2}{6}$

$\therefore$ Area $= \dfrac{a^2}{6}$



4. **(a)** $r = \dfrac{\Delta}{s} = \dfrac{ac}{2s} = \dfrac{ac}{a+b+c} \qquad \left(\because \Delta = \dfrac{1}{2}ac\right)$

$= \dfrac{ac(a+c-b)}{(a+c)^2 - b^2} = \dfrac{ac(a+c-b)}{a^2+c^2+2ac-b^2}$

$= a+c-b \qquad (\because a^2+c^2=b^2)$

D MULTIPLE CORRECT CHOICE TYPE

1. **(a,b,d)** Let the sides of a triangle be a, ar, ar^2

$\because$ ar^2 is the greater side ($r > 1$).

$\therefore a + ar > ar^2$

$\because r^2 - r - 1 < 0 \Rightarrow \dfrac{1-\sqrt{5}}{2} < r < \dfrac{1+\sqrt{5}}{2}$

$\Rightarrow 1 < r < \dfrac{1+\sqrt{5}}{2}$

Therefore (a) is correct.

Also $r^2 < \dfrac{1}{4}(6+2\sqrt{5}) = \dfrac{1}{2}(3+\sqrt{5})$

and $r^4 < \dfrac{1}{4}(14+6\sqrt{5}) = \dfrac{1}{2}(7+3\sqrt{5})$

$\therefore 1 + r^2 - r^4 < 1 + \dfrac{1}{2}(3+\sqrt{5}) - \dfrac{1}{2}(7+3\sqrt{5})$

$= -1-\sqrt{5} < r$

$\cos C = \dfrac{a^2 + a^2r^2 - a^2r^4}{2a^2r} = \dfrac{1+r^2-r^4}{2r} < \dfrac{1}{2}$

$\therefore \cos C < \cos\dfrac{\pi}{3} \Rightarrow C > \dfrac{\pi}{3}$

Therefore (d) is correct.

Also, $\cos B = \dfrac{a^2 + a^2r^4 - a^2r^2}{2a^2r^2} = \dfrac{1+r^4-r^2}{2r^2}$

$= \dfrac{1}{2}\left[r^2 + \dfrac{1}{r^2} - 1\right] = \dfrac{1}{2}\left[\left(r - \dfrac{1}{2}\right)^2 + 1\right] > \dfrac{1}{2}$

$\therefore \cos B > \cos\dfrac{\pi}{3} \Rightarrow B < \dfrac{\pi}{3}$

Also,

$a < ar < ar^2 \Rightarrow A < B < C \Rightarrow A < B < \dfrac{\pi}{3} < C$

Hence (b) is also correct and (c) is incorrect.

2. **(a, d)** Sides are in A.P. and $a <$ minimum $\{b, c\}$

$\Rightarrow$ Order of A.P. can be b, c, a, or c, b, a

Case I : If $2c = a + b$

$\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{b^2+c^2-(2c-b)^2}{2bc} = \dfrac{4b-3c}{2b}$

Case II : If $2b = a + c$

$\cos A = \dfrac{(b^2+c^2)-(2b-c)^2}{2bc} = \dfrac{4c-3b}{2c}$

3. **(a, c, d)** In a triangle, $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}$ are acute angles.

$\therefore \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} > 0$ and

$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > 0$. Also, $\cos A + \cos B + \cos C$

$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > 1$

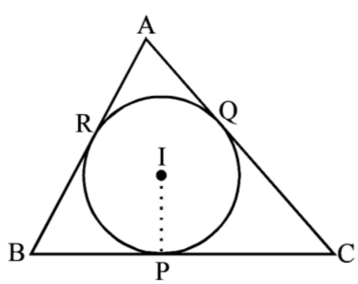
4. **(a,b,c)** Let $BP = n, CQ = n+1, AR = n+2$
Then $BP = BR = n$
$CQ = CP = n+1$ and $AR = AQ = n+2$

$\therefore BC = 2n+1, CA = 2n+3, AB = 2n+2$ and

$S = \frac{1}{2}[2n+1+2n+3+2n+2] = 3n+3$

$\Delta = \sqrt{(3n+3)(n+2)(n)(n+1)}$ and inradius

$= \frac{\Delta}{s} = 4$



$\therefore \sqrt{\frac{n(n+2)}{3}} = 4 \Rightarrow n^2 + 2n - 48 = 0 \Rightarrow n = 6$

So, the sides are 13, 14, 15. and perimeter
$= 2s = 42$ unit

$\Delta = \sqrt{31 \times 8 \times 6 \times 7} = 7 \times 3 \times 4 = 84$ unit

$\therefore$ radius of circumcircle

$R = \frac{13 \times 14 \times 15}{4 \times 84} = \frac{65}{8}$ cm

5. **(a,b,c)** $(a+b+c)(b+c-a) = xbc \Rightarrow 2s(2s-2a) = xbc$

$\Rightarrow \frac{4s(s-a)}{bc} = x$ or $4\cos^2\frac{A}{2} = x \Rightarrow 0 < x < 4$

6. **(a,c,d)** $a^2 + b^2 + c^2 - ca - ab\sqrt{3} = 0$

$\Rightarrow a^2 - 2.a\frac{\sqrt{3}}{2}.b + b^2 + \frac{3a^2}{4} - \frac{3a^2}{4} + c^2 - ca = 0$

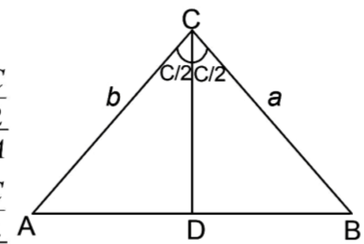
$\Rightarrow \left(\frac{\sqrt{3}a}{2} - b\right)^2 + \left(\frac{a}{2}\right)^2 + c^2 - 2.\frac{ca}{2} = 0$

$\left(\frac{\sqrt{3}a}{2} - b\right)^2 + \left(\frac{a}{2} - c\right)^2 = 0$

$\Rightarrow \frac{b}{a} = \frac{\sqrt{3}}{2}$ and $\frac{c}{a} = \frac{1}{2}$

$\therefore \angle A = 90°, \angle B = 60°, \angle C = 30°$

7. **(c, d)** $\frac{CD}{\sin A} = \frac{AD}{\sin\frac{C}{2}}$



$AD = CD\frac{\sin\frac{C}{2}}{\sin A}$

$BD = CD\frac{\sin\frac{C}{2}}{\sin B}$

$AD + BD = c \Rightarrow CD.\sin\frac{C}{2}\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right) = c$

$CD = c\frac{\sin A.\sin B}{(\sin A + \sin B)\sin\frac{C}{2}}$

$= 2c\frac{\sin A.\sin B.\cos\frac{C}{2}}{(\sin A + \sin B)\sin C}$

$\frac{2c.a.b\cos\frac{C}{2}}{(a+b)c} = \frac{2ab}{a+b}\cos\frac{C}{2}$

Also, $\frac{CD}{\sin B} = \frac{BC}{\sin\left\{\pi - \left(B + \frac{C}{2}\right)\right\}}$

$\therefore CD = \frac{a\sin B}{\sin\left(B + \frac{C}{2}\right)} = \frac{b\sin A}{\sin\left(B + \frac{C}{2}\right)}$

8. **(b, c)** The given equation can be rewritten as
$\sin(B+C)\sin(B-C) = \sin(A+B)\sin(A-B)$
or $\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$
$\Rightarrow 2\sin^2 B = \sin^2 A + \sin^2 C$

or $(1 - \cos 2B) = \frac{1 - \cos 2A}{2} + \frac{1 - \cos 2C}{2}$

$\Rightarrow 2\cos 2B = \cos 2A + \cos 2C$
Hence $\cos 2A, \cos 2B, \cos 2C$ in A.P.
Now the given equation can also be written as
$\sin A\sin B\cos C - \sin A\sin C\cos B = \sin C\sin A\cos B - \sin C\sin B\cos A$
Dividing by $\sin A\sin B\sin C$, we get
$\cot C - \cot B = \cot B - \cot A$
$\Rightarrow \cot A + \cot C = 2\cot B$
$\Rightarrow \cot A, \cot B, \cot C$ are in A.P.

9. (a, d) $b \sin A$ = the altidue from the vertex C to the opposite side and hence $b \sin A$ cannot be greater than a whether $A > \frac{\pi}{2}$ or $A < \frac{\pi}{2}$.

$b \sin A = a$ only if $B = \frac{\pi}{2}$.

Also $\frac{b}{\sin B} = \frac{a}{\sin A}$

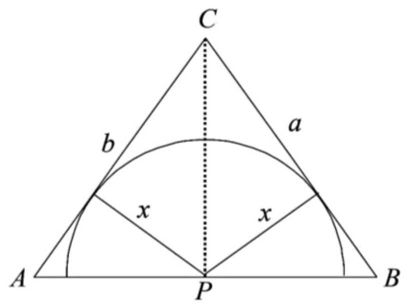
$\Rightarrow b\sin A = a \sin B < a$ for $B \neq \frac{\pi}{2}$

$= a$ for $B = \frac{\pi}{2}$

Moreover, for $B > A$, $A < \frac{\pi}{2}$, $B < \frac{\pi}{2}$ $\sin B > \sin A$

and hence $b \sin A = a \sin B > a \sin A \Rightarrow b > a$.

10. (c, d) Let P be the centre and x the radius of the semi-circle.



Area of $\Delta ABC = \Delta$ = area of ΔPCB + area of ΔAPC

or $\Delta = \frac{1}{2}xa + \frac{1}{2}xb$

$$\Rightarrow x = \frac{2\Delta}{a+b} = \frac{2abc}{4R(a+b)}$$

$$= \frac{abc}{2R(a+b)} \frac{(a+b+c)}{2s}$$

$$= \frac{abc}{2s(a+b)}(\sin A + \sin B + \sin C)$$

$$= \frac{4abc \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2s(a+b)} = \frac{2abc \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{s(a+b)}.$$

11. (b, c) Here $\cos A$, $\cos B$, $\cos C$ are in A.P.

$\Rightarrow 2\cos B = \cos A + \cos C$

$$= 2\cos\frac{A+C}{2}\cos\frac{A-C}{2}$$

$$\Rightarrow \left(1 - 2\sin^2\frac{B}{2}\right) = \sin\frac{B}{2}\cos\frac{A-C}{2}$$

or $\cos^2\frac{B}{2} = \sin\frac{B}{2}\left[\sin\frac{B}{2} + \cos\frac{A-C}{2}\right]$

$$= \sin\frac{B}{2}\left[\cos\frac{A+C}{2} + \cos\frac{A-C}{2}\right]$$

$$= 2\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2}.$$

$$\Rightarrow \cot\frac{B}{2} = \frac{2\cos\frac{A}{2}\cos\frac{C}{2}}{\sin\left(\frac{A+C}{2}\right)}$$

$$= \frac{2\cos\frac{A}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{C}{2} + \cos\frac{A}{2}\sin\frac{C}{2}}$$

$$\Rightarrow \cot\frac{B}{2} = \frac{2}{\tan\frac{A}{2} + \tan\frac{C}{2}}$$

$$\Rightarrow \tan\frac{A}{2} + \tan\frac{C}{2} = 2\tan\frac{B}{2}$$

$\Rightarrow \tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in A.P.

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= 2\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

or $\frac{\Delta}{s(s-a)} + \frac{\Delta}{s(s-c)} = 2\frac{\Delta}{s(s-b)}$

or $r_1 + r_2 = 2r_2 \Rightarrow r_1, r_2, r_3$ are in A.P.

12. (a, b, d) $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3c}{2}$

$$\Rightarrow \frac{b}{2}(1+\cos A) + \frac{a}{2}(1+\cos B) = \frac{3c}{2}$$

$\Rightarrow b + a + (b\cos A + a\cos B) = 3c$

$\Rightarrow b + a + c = 3c \Rightarrow a + b = 2c$

Thus $a + b \geq 2\sqrt{ab}$

$\Rightarrow 2c \geq 2\sqrt{ab} > \sqrt{ab}$

Also $2c \geq 2\sqrt{ab} \Rightarrow c^2 \geq ab$

Moreover, $\dfrac{a+c}{2c-a}+\dfrac{b+c}{2c-b}=\dfrac{a+c}{b}+\dfrac{b+c}{a}$

$$=\frac{a}{b}+\frac{c}{b}+\frac{b}{a}+\frac{c}{a}\geq 4\left(\frac{c^2ab}{a^2b^2}\right)^{1/4}\geq 4 \text{ and}$$

$$\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\geq 3\left(\frac{acb}{cba}\right)^{1/3}=3$$

13. (a,b,d) $\dfrac{c+a}{12}=\dfrac{a+b}{14}=\dfrac{b+c}{18}=\dfrac{s}{11}$

$$\Rightarrow\ b+c=\frac{18}{11}s$$

$$\Rightarrow 2s-a=\frac{18}{11}s\ \Rightarrow s-a=\frac{7}{11}s$$

$$\therefore\ r_1=\frac{\Delta}{s-a}=\frac{11\Delta}{7s}=\frac{11}{7}r$$

similarly, $r_2=11r$ and $r_3=\dfrac{11r}{3}$

Further

$$r_1+r_2+r_3-r=4R$$

$$\Rightarrow\left(\frac{11}{7}+11+\frac{11}{3}-1\right)r=4R$$

$$\Rightarrow R=\frac{80}{21}r$$

14. (c,d) Given that $\sin A+\sin B$

$$=\frac{c(a+b)}{c^2}=\frac{a+b}{c}=\frac{\sin A+\sin B}{\sin C}$$

$\Rightarrow \sin C=1\Rightarrow C=\pi/2$

$\Rightarrow A+B=\pi/2\Rightarrow\ B=\dfrac{\pi}{2}-A$

$\Rightarrow \sin B=\cos A$

Thus we have $\sin A+\cos A=\dfrac{a+b}{c}$.

15. (b,c) In ΔABC, $\tan A.\tan B.\tan C=\tan A+\tan B+\tan C=10$ (given)

$\Rightarrow \tan A$, $\tan B$, $\tan C$ are positive $\Rightarrow \Delta ABC$ is acute angled.

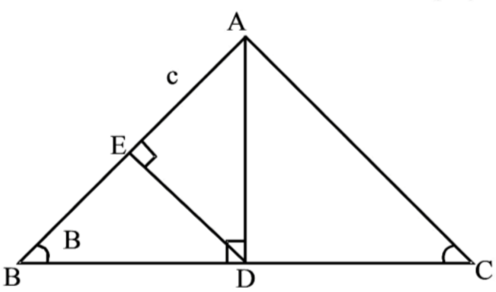
E MATRIX-MATCH TYPE

1. A - r; B - p; C - s; D - q

$\angle ADE=B$, $AD=c\sin B=b\sin C$, $BD=c\cos B$, $DC=b\cos C$

(A) Area of $\Delta ABD=\dfrac{1}{2}.c\sin B.c\cos B=\left(\dfrac{c^2}{4}\right)\sin 2B$

(B) Area of $\Delta ADC=\dfrac{1}{2}.b\sin C.b\cos C=\left(\dfrac{b^2}{4}\right)\sin 2C$



(C) Area of $\Delta ADE=\dfrac{c^2}{4}\sin^2 B.\sin 2B$

[since $AE=AD\sin B=c\sin^2 B$, $ED=c\sin B\cos B$]

(D) Area of $\Delta BDE=\dfrac{1}{2}.c^2\cos^2 B.\cos B.\sin B$.

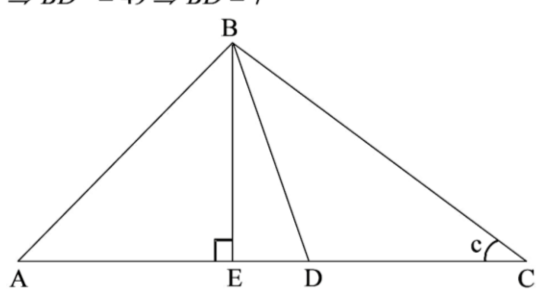
$$=\left(\frac{c^2}{4}\right)\cos^2 B\sin 2B$$

2. A - s; B - r; C - p ; D - q

$a=7, b=8, c=9\quad AB^2+BC^2=2(AD^2+BD^2)$

$\Rightarrow 81+49=2(16+BD^2)=130\Rightarrow 16+BD^2=65$

$\Rightarrow BD^2=49\Rightarrow BD=7$



ABD is an isosceles triangle,

$\Rightarrow ED=\dfrac{AD}{2}=2,\ BE=\sqrt{BD^2-ED^2}=\sqrt{49-4}=\sqrt{45}$

$\Rightarrow CE=CD+DE=4+2=6$

3. A - r; B - q; C - p; D - s

(A) $\alpha=\dfrac{1}{2}\sqrt{2b^2+2c^2-a^2},\ \beta=\dfrac{1}{2}\sqrt{2c^2+2a^2-b^2},$

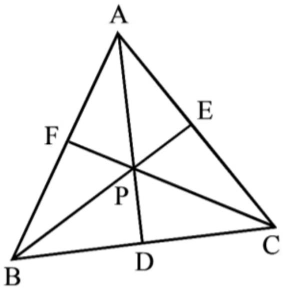
$$=\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

$$\therefore\ \alpha^2+\beta^2+\gamma^2=\frac{1}{4}(3a^2+3b^2+3c^2)$$

$$\Rightarrow\frac{\alpha^2+\beta^2+\gamma^2}{a^2+b^2+c^2}=\frac{3}{4}$$

(B) $ar(\Delta ABC) = ar(\Delta PBC) + ar(\Delta CPA) + ar(\Delta APB)$

$$\frac{\sqrt{3}}{4} \times 4 = \frac{1}{2}[x \times 2 + y \times 2 + z \times 2]$$



$\therefore\ x + y + z = \sqrt{3}$

(C) $2B = A + C \Rightarrow B = \frac{\pi}{3}$ and $A + C = \frac{2\pi}{3}$

$$b^2 = ac \Rightarrow \sin^2 B = \sin A \sin C \Rightarrow \sin A \sin C = \frac{3}{4}$$

$$\therefore\ \cos(A - C) - \cos(A + C) = \frac{3}{2} \Rightarrow \cos(A - C) = 1$$

$$\therefore\ A = C \Rightarrow A = B = C = \frac{\pi}{3}$$

so the triangle is equilateral.

(D) $\frac{\sqrt{abc(a+b+c)}}{\Delta} = \frac{1}{\Delta}\sqrt{4R\Delta\,.\,2s}$

$$= \sqrt{\frac{8Rs}{\Delta}} = \sqrt{\frac{8R}{r}} \geq \sqrt{8.2} = 4$$

4. A - p; B - s; C - p, q; D - q

(A) We have,

$$\frac{r}{r_1} = \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{4}$$

$$\therefore\ \tan\frac{A}{2}\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)$$

$$= 1 - \tan\frac{B}{2}\tan\frac{C}{2} = 1 - \frac{1}{4} = \frac{3}{4}$$

(B) We have, $(r_1 r_2 r_3)^{1/3} \geq \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} = 3r$

$$\therefore\ \frac{r_1 r_2 r_3}{r^3} \geq 27$$

(C) $a + b > c \Rightarrow 2b - c + b > c \Rightarrow \frac{b}{c} > \frac{2}{3}$

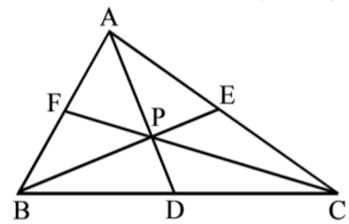
Also, $b + c > a \Rightarrow b + c > 2b - c \Rightarrow \frac{b}{c} < 2$

Again, $c + a > b \Rightarrow 2b > b \ \therefore\ \frac{b}{c} \in \left(\frac{2}{3}, 2\right)$

(D) We have, $\frac{PD}{AD} = \frac{ar(\Delta BPC)}{ar(\Delta BAC)}$, ...etc.

$$\therefore\ \frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF}$$

$$= \frac{ar(\Delta BPC) + ar(\Delta CPA) + ar(\Delta APB)}{ar(\Delta ABC)} = 1$$



5. A - q; B - s; C - r, q; D - p

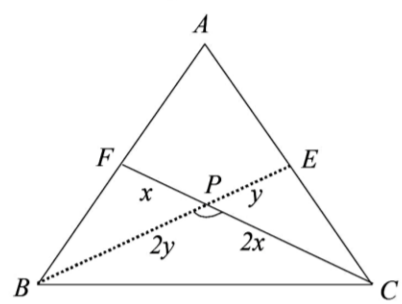
(A) Let the medians BE and CF meet at P. Let $EP = y$
$\Rightarrow PB = 2y$
Let $FP = x \Rightarrow PC = 2x$

Now, $\tan \angle FBP = \frac{x}{2y}$ and $\tan \angle CPB = \frac{2x}{2y}$

Also $B = \angle FBP + \angle CPB$

$$\text{Hence } \tan B = \frac{\frac{x}{2y} + \frac{x}{y}}{1 - \frac{x^2}{2y^2}} = \frac{3xy}{2y^2 - x^2}$$

$$\Rightarrow \cot B = \frac{2y^2 - x^2}{3xy}$$



Similarly, $\cot C = \frac{2x^2 - y^2}{3xy}$ so that

$$\cot B + \cot C = \frac{x^2 + y^2}{3xy} \geq \frac{2xy}{3xy} = \frac{2}{3}$$

(B) Since $r_1 + r_2 + r_3 = r + 4R$,

$$r_2 + r_3 = 4 \times \frac{13}{2} + 2 - 3 = 25$$

Also, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$\Rightarrow \frac{1}{2} = \frac{2}{3} + \frac{r_2 + r_3}{r_2 r_3} \Rightarrow r_2 r_3 = 150 \Rightarrow r_2 = 10, r_3 = 15$

Moreover, $s^2 = \frac{r_1 r_2 r_3}{r} = \frac{3 \times 15 \times 10}{2} = 225 \Rightarrow s = 15.$

Hence, $\Delta = rs = 30$ sq. units

(C) The given equation can be written as

$(a+b-c)(b-c)^2 + (b+c-a)(c-a)^2 + (c+a-b)(a-b)^2 = 0$

Since all the terms are non negative,

$(a+b+c)(b-c)^2 = 0, (b+c-a)(c-a)^2 = 0, (c+a-b)(a-b)^2 = 0$

$\Rightarrow b = c = a \Rightarrow$ the triangle is equilateral.

(D) We have

$0 = \sin 4A + \sin 4B + \sin 4C + 8 \cos A$

$= -4 \sin 2A \sin 2B \sin 2C + 8 \cos A$

$= 8[-\sin A \sin 2B \sin 2C + 1] \cos A$

$\Rightarrow$ either $\cos A = 0$ i.e., $A = \frac{\pi}{2}$

or $\sin A \sin 2B \sin 2C = 1 \Rightarrow A = \frac{\pi}{2}, B = \frac{\pi}{4}, C = \frac{\pi}{4}$

6. **A - r ; B - s ; C - q ; D - p**

$r = \frac{\Delta}{s} = 2$ and $r_1 = \frac{\Delta}{s-a} = 15$

Thus $\frac{s-a}{s} = \frac{2}{15}$ or $s = \frac{15}{13}a$

So, $\frac{\Delta}{s} = 2 \Rightarrow \Delta = \frac{30}{13}a$

or $\frac{1}{2}bc = \frac{30}{13}a$ or $bc = \frac{60}{13}a$ ....(1)

Also, $b + c = 2s - a = \frac{17}{13}a$ ....(2)

Solving (1) and (2) we get $b = \frac{12}{13}a$ and $c = \frac{5}{13}a$

Also, $\frac{\Delta}{s} = \frac{\frac{bc}{2}}{\frac{a+b+c}{2}} = \frac{bc}{a+b+c} \Rightarrow 2 = \frac{2}{13}a \Rightarrow a = 13$

F NUMERIC/INTEGER ANSWER TYPE

1. **Ans : 0**

Let $\angle ACB = \theta$, then $\angle ADB = 2\theta$ and $\angle AEB = 3\theta$

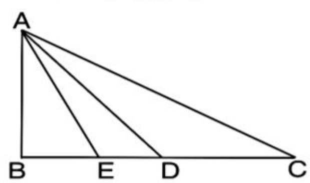
$\therefore \angle EAD = \angle DAC = \theta$

Given $\frac{DE}{DC} = k$

Now, on applying $m - n$ cot theorem, we get

$(k+1)\cot(180° - 2\theta) = k\cot\theta - \cot\theta$

$\Rightarrow \frac{k+1}{1-k} = \frac{\tan 2\theta}{\tan\theta} = \frac{2}{1-\tan^2\theta}$



$\Rightarrow 1 - \tan^2\theta = \frac{2(1-k)}{1+k} \Rightarrow \tan^2\theta = \frac{3k-1}{1+k} > 0 \Rightarrow k > \frac{1}{3}$

and $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan\theta \frac{3 - \tan^2\theta}{1 - 3\tan^2\theta}$

$= \sqrt{\frac{3k-1}{k+1}} \frac{3 - \frac{3k-1}{k+1}}{1 - 3.\frac{3k-1}{k+1}}$

$= \sqrt{\frac{3k-1}{k+1}} \frac{4}{4(1-2k)} = \frac{1}{(1-2k)}\sqrt{\frac{3k-1}{k+1}}$

$\therefore 0 < 3\theta < \frac{\pi}{2}$, therefore, $\tan 3\theta > 0$

$\Rightarrow 1 - 2k > 0 \Rightarrow k < \frac{1}{2}$

Thus, $k \in \left(\frac{1}{3}, \frac{1}{2}\right) \Rightarrow [k] = 0$

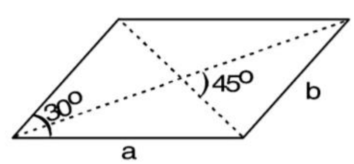
2. **Ans : 2**

Let the sides be a and b, $a > b$ and diagonals d_1 and d_2, $d_1 > d_2$

Then,

$d_1^2 = a^2 + b^2 - 2ab\cos 150° = a^2 + b^2 + \sqrt{3}ab$ ...(1)

$d_2^2 = a^2 + b^2 - 2ab\cos 30° = a^2 + b^2 - \sqrt{3}ab$ ...(2)



Equating area of parallelogram

$\frac{1}{2}d_1 d_2 \sin 45° = 2\left(\frac{1}{2}ab\sin 30°\right)$

$\Rightarrow d_1 d_2 = \sqrt{2}ab$ ...(3)

From (3) $d_1^2 d_2^2 = 2a^2b^2$

$\Rightarrow (a^2 + b^2 + \sqrt{3}ab)(a^2 + b^2 - \sqrt{3}ab) = 2a^2b^2$

[from (1) and (2)]

$$\Rightarrow (a^2+b^2)^2-3a^2b^2=2a^2b^2 \Rightarrow a^4+b^4-3a^2b^2=0$$

$$\Rightarrow \left(\frac{a^2}{b^2}\right)^2-3\left(\frac{a^2}{b^2}\right)+1=0$$

$\therefore \quad \dfrac{a^2}{b^2}=\dfrac{3\pm\sqrt{5}}{2}$ Taking positive sign (as $a>b$)

$$\therefore \quad \frac{a^2}{b^2}=\frac{6+2\sqrt{5}}{4}=\frac{(1+\sqrt{5})^2}{4} \Rightarrow \frac{a}{b}=\frac{\sqrt{5}+1}{2}$$

3. Ans : 45

Let h_a, h_b, h_c be sides of $\Delta A'B'C'$ and h_a', h_b', h_c' be sides of $A''B''C''$

Then $\dfrac{1}{2}ah_a=\dfrac{1}{2}bh_b=\dfrac{1}{2}ch_c=\Delta$ ....(1)

Also, $\dfrac{1}{2}h_ah'_a=\dfrac{1}{2}h_bh'_b=\dfrac{1}{2}h_ch'_c=\Delta'$ ...(2)

$\therefore \quad h'_a=\dfrac{2\Delta'}{h_a}=\dfrac{2\Delta'}{\frac{2\Delta}{a}}=\dfrac{a\Delta'}{\Delta}$ from (1)

$$\text{Now } \Delta''^2=\left(\frac{h'_a+h'_b+h'_c}{2}\right)\left(\frac{h'_a+h'_b-h'_c}{2}\right)\left(\frac{h'_a-h'_b+h'_c}{2}\right)\left(\frac{-h'_a+h'_b+h'_c}{2}\right)$$

$$=\frac{1}{2^4}\left[\frac{a\Delta'}{\Delta}+\frac{b\Delta'}{\Delta}+\frac{c\Delta'}{\Delta}\right]\left[\frac{a\Delta'}{\Delta}+\frac{b\Delta'}{\Delta}-\frac{c\Delta'}{\Delta}\right]\left[\frac{a\Delta'}{\Delta}-\frac{b\Delta'}{\Delta}+\frac{c\Delta'}{\Delta}\right]\left[-\frac{a\Delta'}{\Delta}+\frac{b\Delta'}{\Delta}+\frac{c\Delta'}{\Delta}\right]$$

$$=\frac{(\Delta')^4}{2^4\Delta^4}(a+b+c)(a+b-c)(a-b+c)(-a+b+c)=\frac{(\Delta')^4\Delta^2}{\Delta^4}$$

$$\therefore \quad \Delta^2=\frac{(\Delta')^4}{(\Delta'')^2}=\frac{(30)^4}{(20)^2}=\frac{3^4\times 10^2}{2^2}$$

$$\therefore \quad \Delta=\frac{3^2\times 10}{2}=45$$

4. Ans : 6

$$p_1=\frac{2\Delta}{a},\ p_2=\frac{2\Delta}{b},\ p_3=\frac{2\Delta}{c}$$

$$\therefore \quad p_1+p_2+p_3\ge 3(p_1p_2p_3)^{1/3}$$

$$=3\left\{\frac{(2\Delta)^3}{abc}\right\}^{1/3}=6\Delta\left(\frac{1}{abc}\right)^{1/3}$$

$$=6rs\left(\frac{1}{abc}\right)^{1/3}=6r\frac{a+b+c}{2}\left(\frac{1}{abc}\right)^{1/3}$$

$$=9r\left(\frac{a+b+c}{3}\right)\left(\frac{1}{abc}\right)^{1/3}\ge 9r \quad (AM/GM\ge 1)$$

(Equality occurs when $p_1=p_2=p_3$ and $a=b=c$, i.e. when ΔABC is equilateral)

$$\therefore \quad p_1+p_2+p_3\ge 9\times\frac{2}{3}=6$$

5. Ans 16

Here $r^2+r_1^2+r_2^2+r_3^2+a^2+b^2+c^2$

$$=r^2+(r_1+r_2+r_3)^2-2(r_1r_2+r_2r_3+r_3r_1)+(a+b+c)^2-2(ab+bc+ca)$$

$$=r^2+(4R+r)^2-2s^2+4s^2-2(ab+bc+ca)$$

$$=2r^2+16R^2+8rR+2s^2-2(ab+bc+ca)$$

$$=\frac{2\Delta^2}{s^2}+16R^2+\frac{8\Delta R}{s}+2s^2-2(ab+bc+ca)$$

$$=16R^2+\frac{2(s-a)(s-b)(s-c)+2abc}{s}+2s^2-2(ab+bc+ca)$$

$$=16R^2+\frac{2[s^3-s^2(a+b+c)+s(ab+bc+ca)]}{s}+2s^2-2(ab+bc+ca)$$

$$=16R^2+\frac{2[2s^3-2s^3]}{s}=16R^2.$$

6. Ans : 6

We have $\dfrac{1}{2}ah_1=\Delta=rs$

$$\Rightarrow \frac{h_1}{r}=\frac{2s}{a}=\frac{a+b+c}{a}$$

$$\Rightarrow \frac{h_1+r}{h_1-r}=\frac{2a+b+c}{b+c}=\frac{2(a+b+c)}{b+c}-1$$

$$\text{Hence } \sum\frac{h_1+r}{h_1-r}=2(a+b+c)\left[\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right]-3$$

$$\ge 2(a+b+c).3\frac{3}{(b+c)+(c+a)+(a+b)}-3$$

(A.M. ≥ H.M.)

i.e., $\displaystyle\sum\frac{h_1+r}{h_1-r}\ge 6$.