AEP STUDY CIRCLE

Since 2001...

IIT-JEE | NEET | CBSE

ACCENTS EDUCATIONAL PROMOTERS

OFFLINE-ONLINE LEARNING ACADEMY

XII CBSE

pHySiCs

GAUSS'S THEOREM

UNIT:01 CH:IV

CBSE:PHYSICS

IIT MAINS & ADV

CBSE

NDA

NEET

HELPLINE:
+91-9939586130
+91-7739650505

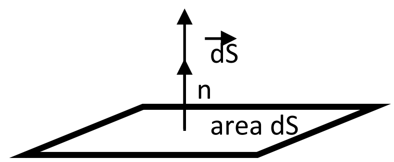
2ND FLOOR SATKOUDI COMPLEX, THANA CHOWK, RAMGARH-829122

www.aepstudycircle.com

GAUSS'S THEOREM

UNIT:01 CH:IV

AREA VECTOR: [Area is scalar quantity. But in some cases, it has been treated as a vector quantity.]

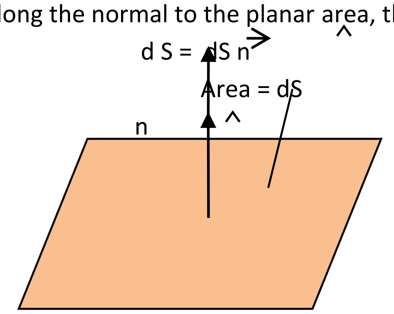


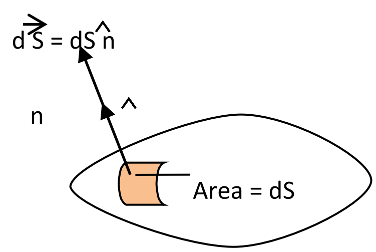
An area vector 'dS' can be represented as vector 'dS'

- ♦♦ The arrow representing the area vector and is drawn perpendicular to the area element.
- ♦♦ The length of area vector dS represents the magnitude of the area element dS. In case if, **n** is a unit vector along the normal to the area element dS, then

$\vec{dS} = n \ |dS|$

Explanation: We come across many situations where we need to know not only the magnitude of a surface area but also its direction. The direction of a planar area vector is specified by the normal to the plane. In Fig. (a), planar area element dS has been represented by a normal vector $\vec{dS}$. The length of vector d S represents the magnitude dS of the area element. If $\hat{n}$ is a unit vector along the normal to the planar area, then $d\vec{S} = dS\,\hat{n}$





[(a) A planar area element (b) An area element of a curved surface]

In case of a curved surface, we can imagine it to be divided into a large number of very small area elements. Each small area element of the curved surface can be treated as a planar area. By convention, the direction of the vector associated with every area element of a closed surface is along the outward drawn normal.

As shown in Fig. (b), the area element d S at any point on the closed surface is equal to $dS\,\hat{n}$, where $\vec{dS}$ is the magnitude of the area element and n is a unit vector in the direction of outward normal.

- **ELECTRIC FLUX: [ϕ] ---- "*The electric flux through a surface or area held inside an electric field represents the total no. of electric lines of force crossing in a direction normal to the surface".***

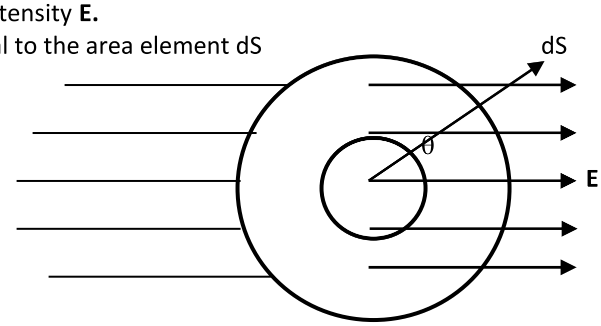
♦♦ **Electric flux is a scalar quantity**. And electric flux is given by the product of surface area and normal component of $\vec{E}$

RELATION BETWEEN ELECTRIC FIELD INTENSITY AND ELECTRIC FLUX:

Suppose we have a surface having an area 'S' is placed inside electric field intensity **E.**

Consider a small area element dS of the surface 'S' directed along the normal to the area element dS

Suppose E makes an angle 'θ' with the area vector dS.



∴. Component of electric field along normal to the area element dS (along to the area vector dS) = E Cos θ .

∴. Electric flux crossing the area element 'dS' in a direction along normal to it is $d\phi = E \cos\theta \ dS$

or, $d\phi = \vec{E}.\vec{dS}$

∴. Total electric flux through the surface 'S' is

$$\phi = \int_S \vec{E}.\vec{dS}$$

Definition: "*Electric field linked with the surface in an electric field may be defined as surface integral of the electric Field over that surface".*

UNIT OF ELECTRIC FLUX: $\phi = N/C \times m^2 = N\,m^2\,C^{-1}$

♦♦ In case, the surface S is a closed surface, then the total electric flux through the closed surface is given by ,

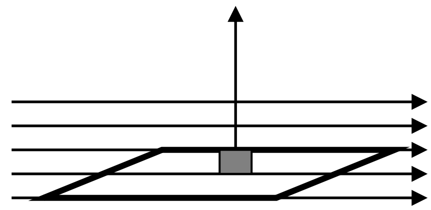
$$\Phi = \oint_S \vec{E} \cdot \vec{dS}$$

indicates that the surface of integration is a closed surface .

♦♦ Electric field parallel to a surface does not produce electric flux.

Here, $\theta = 90^0$

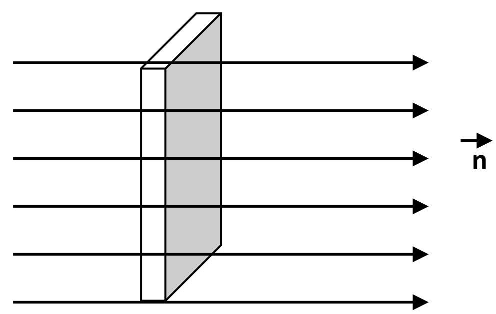
∴ Electric flux. $\phi = \int E\,dS\cos 90^0 = 0$



♦♦ Electric field normal to a surface produce maximum flux.

Here, $\theta = 0^0$

∴ Electric flux. $\phi = \int E\,dS\cos 0^0 =$ **Maximum value.**



♦♦ **ϕ Though area element dS**

[1] affected by the charge present outside the surface

[2] ϕ may change if the position of the inside charge Is changed.

[3] Denoted by ϕ

Φ Through closed surface

[1] Depends only upon the charges enclosed by the surface.

[2]Does not depends upon the location of the inside charge.

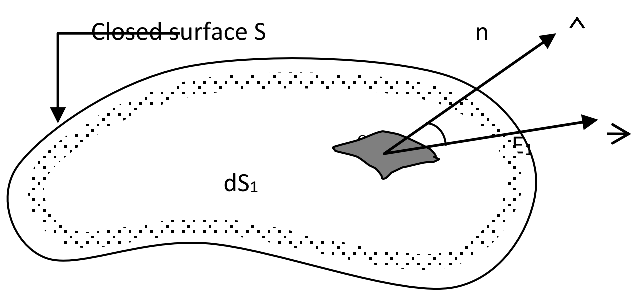
[3] Denoted by Φ

♦♦ In case the field E is non-uniform, we consider a closed surface S lying inside the field, as shown in Fig. We can divide the surface S into small area elements: $\Delta S_1, \Delta S_2, \Delta S_3, \ldots\ldots, \Delta S_N$. Let the corresponding electric fields at these elements be $E_1, E_2, \ldots. E_N$.

Then the electric flux through the surface S will be

$$\phi_E = \vec{E}_1 . \Delta\vec{S}_1 + \vec{E}_2 . \Delta\vec{S}_2 + \ldots\ldots + \vec{E}_N\ \Delta\vec{S}_N$$

$$= \sum_{i=1}^{N} E_i . \Delta S_i$$



[Electric flux through a closed surface S]

When the number of area elements becomes infinitely large ($N \to \infty$) and $\Delta S \to 0$, the above sum approaches a surface integral taken over the closed surface. Thus

$$\phi_E = \lim_{N \to \infty} \sum_{i=1}^{N} \vec{E}_i \cdot \Delta \vec{S}_i = \oint_S \vec{E} \cdot d\vec{S}$$

Thus, the electric flux through any surface S, open or closed, is equal to the surface integral of the electric field E taken over the surface S.

- ♦ **Electric flux** is a scalar quantity.
- ♦ **Unit of ϕ_E** = Unit of E × unit of S

∴ SI unit of electric flux = $NC^{-1} . m^2 = Nm^2 C^{-1}$.

Equivalently, SI unit of electric flux

$= Vm^{-1} . m^2 = Vm$

GAUSS'S THEOREM:

"It states that the total electric flux through a closed surface enclosing a charge is equal to $1/\xi_0$ times the magnitude of a charge enclosed".

Where, ξ_0 = Absolute permittivity of free space.

If a closed surface encloses an electric charge 'q', then according to G'S theorem,

Total electric flux through the closed surface is $\Phi = q/\xi_0$

But,

$$\Phi = \oint \vec{E} \cdot d\vec{S}$$

$$\therefore \quad \boxed{\oint_S \vec{E} \cdot d\vec{S} = q/\xi_0}$$

Another Definition: **"If a closed surface encloses a charge, then surface integral of the electric field (due to enclosed charge) over the closed surface is equal to *$1/\xi_0$ times* the charge enclosed".**

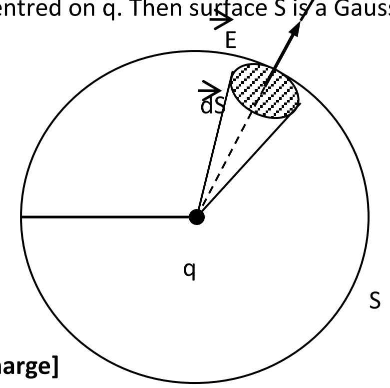
- ♦ The charge inside 'S' may be **point charge** as well as **continuous charge distribution**.
- ♦ There is ***no contribution of total electric flux from the charges outside 'S'.***
- ♦ Location of charge '***q' inside 'S' does not affect the value of the surface integral.***

[◙ Gauss's **theorem:** This theorem gives a relationship between the total flux passing through any closed surface and the net charge enclosed within the surface.]

Gauss theorem *states that the total flux through a closed surface is $1/\varepsilon_0$ times the net charge enclosed by the closed surface. Mathematically, it can be expressed as*

$$\phi_E = \oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

Proof: For the sake of simplicity, we prove Gauss's theorem for an isolated positive point charge q. As shown in Fig., suppose the surface S is a sphere of radius r centred on q. Then surface S is a Gaussian surface.



[Flux through a sphere enclosing a point charge]

Electric field at any point on S is

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

This field points radially outward at all points on S. Also, any area element points radially outwards, so it is parallel to $\vec{E}$, i.e., $\theta = 0°$

∴ Flux through area dS is

$$d\phi_E = \vec{E} \cdot d\vec{S} = E\, dS \cos 0° = EdS$$

Total flux through surface S is

$$\phi_E = \oint_S d\phi_E = \oint_S E\, dS$$

$$= E \oint dS$$

$$= E \times \text{Total area of sphere}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \cdot 4\pi r^2$$

or $$\phi_E = \frac{q}{\varepsilon_0}$$ This proves Gauss's theorem.

◘◘◘◘◘..

☑ ◘...Gauss's theorem is valid for a closed surface of any shape and for any general charge distribution.

☑ ◘...If the net charge enclosed by a closed surface is zero (q = 0), then flux through it is also zero.

$$\phi_E = \frac{q}{\varepsilon_0} = 0$$

☑ ◘...The net flux through a closed surface due to a charge lying outside the closed surface is zero.

☑ ◘...The charge q appearing in the Gauss's theorem includes the sum of all the charges located anywhere inside the inside the closed surface.

☑ ◘...The electric field E appearing in Gauss's theorem is due to all the charges, both inside and outside the closed surface. However, the charge q appearing in the theorem is only contained within the closed surface.

☑ ◘...Gauss's theorem is based on the inverse square dependence on distance contained in the coulomb's law. In fact, it is applicable to any field obeying inverse square law. It will not hold in case of any departure from inverse square law.

☑ ◘...For a medium of absolute permittivity ε or dielectric constant κ, the Gauss's theorem can be expressed as

$$\oint \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon} = \frac{q}{\kappa\varepsilon_0}$$

◘ ◘ **GAUSSIAN SURFACE**: **Gaussian surface around a charge distribution (may be a point charge, a line charge, a surface charge, or a volume charge) is a closed surface such that electric field intensity at all the points on the surface is same & the electric flux through the surface is along the normal to the surface.**

◘...While selecting the G. surface, we *shall avoid charges on the surface itself*.

◘...◘... **SPECIAL CASES**:

◘... **(I) If the closed surface does not contain any charge i.e., q = 0, then**

$$\Phi = \oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\xi_0} = \frac{0}{\xi_0} = 0$$

◘... **[II] If there are point charges $q_1, q_2, q_3 \ldots q_n$ lying inside the surface, each will contribute to the electric flux (Independent of others).**

∴ By principle of superposition,

$$\phi = \phi_1 + \phi_2 + \phi_3 + \ldots\ldots\ldots + \phi_n$$

$$= \frac{q_1}{\xi_0} + \frac{q_2}{\xi_0} + \frac{q_3}{\xi_0} + \ldots\ldots\ldots + \frac{q_n}{\xi_0}$$

$$= 1/\xi_0 \,\Sigma q_i \qquad = Q/\xi_0$$

[where, $\Sigma q_i = Q$, algebraic sum of all the charges inside the closed surface i.e., Total electric flux over a closed surface in vacuum is $1/\xi_0$ times the total charges within the surface (no matter how the charges are distributed).

♦♦ **If the medium, surrounding the charges has a dielectric constant K (= $\xi_r = \xi/\xi_0$)**

then, $\Phi = \dfrac{Q}{K\xi_0} = \dfrac{Q}{\xi_r \xi_0} = \dfrac{Q}{\xi}$

♦♦ If there is no net charge within the closed surface, then, Q = 0, then, $\boxed{\phi = 0}$

G'S theorem holds good for any closed surface, regardless of its shape and size. But we choose spherical surface for all the application of G'S theorem because--------

--- [1] The dot product of E. dS = E dS cos 0^0 = E dS, because at all the point on spherical G. Surface, θ = 0.

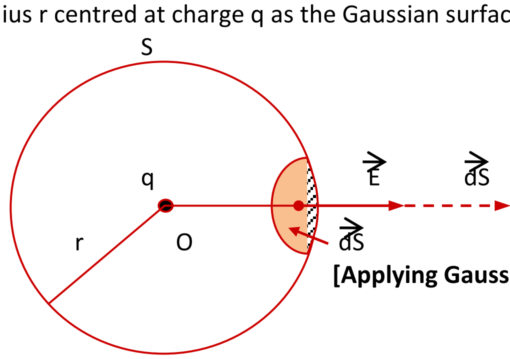
--- [2] The magnitude of E is constant at all the points on the spherical G. Surface. Therefore, E can be brought out of the integral sign as constant.

--- [3] The remaining integral i.e., ∫ dS is surface area of the surface = 4 π r^2 (Which can be obtained without actually doing the integration).

♦♦ **From G's theorem we can calculate the no. of electric lines of force that radiate outwards from 1C of positive charge as ---------- $\Phi = \dfrac{q}{\xi} = \dfrac{1}{8.85 \times 10^{-12}} = 1.13 \times 10^{11}$**

COULOMB'S LAW FROM GAUSS'S THEOREM

Deduction of Coulomb's law from Gauss's theorem: As shown in Fig., consider an isolated positive point charge q. We select a spherical surface S of radius r centred at charge q as the Gaussian surface.



[Applying Gauss's theorem to a point charge]

By symmetry, E has same magnitude at all points on S. Also, E and dS at any point on S are directed radially outward. Hence flux through area dS is

$d\phi_E = \vec{E}.\vec{dS} = EdS \cos 0° = EdS$

Net flux through closed surface S is

$\phi_E = \oint \vec{E}.\vec{dS} = \oint E\, dS = E \oint dS$

$= E \times$ total surface area of S $= E \times 4\pi r^2$

[Since E is constant, as it is **same at** all the points on the Surface.]

Using Gauss's theorem, $\phi_E = \dfrac{q}{\varepsilon_0}$

or $E \times 4\pi r^2 = \dfrac{q}{\varepsilon_0}$

or $E = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{q}{r^2}$

The force on the point charge q_0 if placed on surface S will be

$F = q_0 E = \dfrac{1}{4\pi\varepsilon_0} \dfrac{qq_0}{r^2}$

This proves the Coulomb's law.

Examples based on Electric Flux and Gauss's Theorem

❖ **FORMULAE USED**

1. Electric flux through a plane surface area S held in a uniform electric field E is

$$\phi_E = \vec{E}.\vec{S} = ES\cos\theta$$

where θ is the angle which the normal to the outward drawn normal to surface area $\vec{S}$ makes with the field $\vec{E}$.

2. According to Gauss's theorem, the total electric flux through a closed surface S enclosing charge q is

$$\phi_E = \oint_S \vec{E}.d\vec{S} = \frac{q}{\varepsilon_0}$$

3. Flux density = $\frac{\text{Total flux}}{\text{Area}} = \frac{\phi_E}{S}$

UNITS USED

Electric flux ϕ_E is in $Nm^2\,C^{-1}$ and flux density in NC^{-1}.

CONSTANT USED

Permittivity constant of free space is

$$\varepsilon_0 = \frac{1}{4\pi \times 9 \times 10^{-9}} = 8.85 \times 10^{-12}\ C^2\,N^{-1}\,m^{-2}$$

Q. 1. ***If $\vec{E} = 6\hat{i} + 3\hat{j} + 4k$, calculate the electric flux through a surface of area 20 units in Y – Z plane.***

Sol. Electric field vector, $\vec{E} = 6\hat{i} + 3\hat{j} + 4k$

As the area vector $\vec{S}$ in the Y – Z plane points along outward drawn normal i.e., along positive X-direction, so

$\vec{S} = 20\,\hat{i}$

Flux, $\phi_E = \vec{E}.\vec{S} = (6\hat{i} + 3\hat{j} + 4k).\,20\,\hat{i}$ = **20 units.**

Q. 2. ***A circular plane sheet of radius 10 cm is placed in a uniform electric field of $5 \times 10^5\ NC^{-1}$, making an angle of 60^o with the field. Calculate electric flux through the sheet.***

Sol. Here $r = 10$ cm $= 0.1$ m,

$E = 5 \times 10^5\ NC^{-1}$

As the angle between the plane sheet and the electric field is 60^o, angle made by the normal to the plane sheet and the electric field is

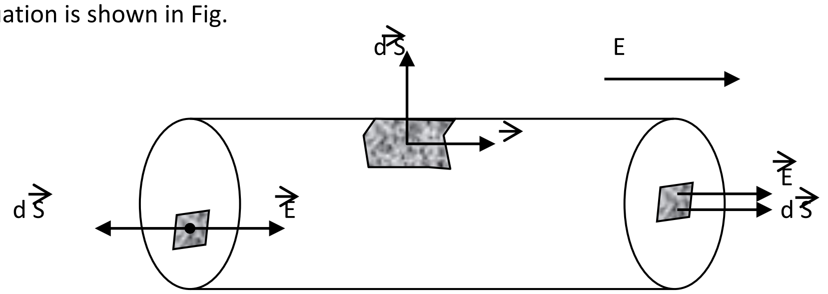
$\theta = 90^o - 60^o = 30^o$

Flux, $\phi_E = ES\cos\theta = E \times \pi r^2 \times \cos\theta$

$= 5 \times 105 \times 3.14 \times (0.1)^2 \times \cos 30^o$ = $\mathbf{1.36 \times 10^4\ Nm^2\ C^{-1}}$.

Q. 3. ***A cylinder is placed in a uniform electric field $\vec{E}$ with its axis parallel to the field. Show that the total electric flux through the cylinder is zero.***

Sol. The situation is shown in Fig.



Flux through the entire cylinder,

$$\phi = \underbrace{\int \vec{E}.d\vec{S}}_{\text{Left plane face}} + \underbrace{\int \vec{E}.d\vec{S}}_{\text{right plane face}} + \underbrace{\int \vec{E}.d\vec{S}}_{\text{curved surface}}$$

$= \int E\,dS\cos 180^o + \int E\,dS\cos 0^o + \int E\,dS\cos 90^o$

$= -E\int dS + E\int dS + 0$

$= -E \times \pi r^2 + E \times \pi r^2 = \mathbf{0}$.

Q. 4. ***Calculate the number of electric lines of force originated form a charge of 1 C.***

Sol. The number of lines of force originating from a charge of 1 C

= Electric flux through a closed surface enclosing a charge of 1 C.

$= \frac{q}{\varepsilon_0} = \frac{1}{8.85 \times 10^{-12}} = 1.129 \times 10^{11}$

Q. 5. A positive charge of 17.7 μC is placed at the centre of a hollow sphere of radius 0.5 m. Calculate the flux density through the surface of the sphere.

Sol. From Gauss's theorem,

Flux, $\phi = \dfrac{q}{\varepsilon_0} = \dfrac{17.7 \times 10^{-6}}{8.85 \times 10^{-12}}$

$= 2 \times 10^{6}\ Nm^2\ C^{-1}$

Flux density $= \dfrac{\text{Total flux}}{\text{Area}}$

$= \dfrac{2 \times 10^6}{4\pi\,(0.5)^2} = \mathbf{6.4 \times 10^5\ NC^{-1}}$**.**

Q. 6. Calculate the electric flux through each of the six faces of a closed cube of length l, if a charge q is placed (a) at its centre and (b) at one of its vertices.

Sol. (a) By symmetry, the flux through each of the six faces of the cube will be same when charge q is placed at its centre.

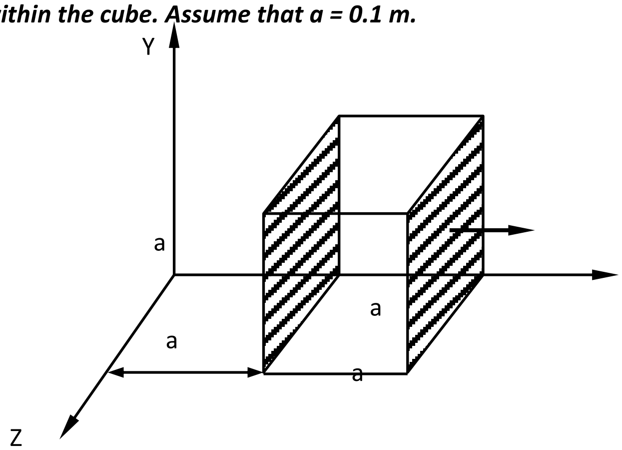
$\therefore \quad \phi_E = \dfrac{1}{6} \cdot \dfrac{q}{\varepsilon_0}$

(b) When charge q is placed at one vertex, the flux through each of the three faces meeting at this vertex will be zero, as $\vec{E}$ is parallel to these faces. As only passes through the remaining from the charge q so the flux through each such face is

$\phi_E = \dfrac{1}{3} \cdot \dfrac{1}{8} \cdot \dfrac{q}{\varepsilon_0} = \dfrac{1}{24} \cdot \dfrac{q}{\varepsilon_0}$

Q. 7. The electric field components is Fig. are $E_x = \alpha\, x^{1/2}$, $E_z = 0$, in which $\alpha = 800\ N/Cm^2$. Calculate (i) the flux ϕ_E through the cube and (ii) the charge within the cube. Assume that a = 0.1 m.

a



Sol. (i) The electric field is acting only in X-direction and its Y-and Z-components are zero. For the four non-shaded faces, the angle between $\vec{E}$ and $\Delta\vec{S}$ is $+\pi/2$. So flux $\phi = \vec{E} \cdot \Delta\vec{S}$ is zero through each of these faces.

The magnitude of the electric field at the left face is

$E_L = \alpha x^{1/2} = \alpha\, a^{1/2}$ [x = a at the left face]

Flux, $\phi_L = \vec{E}_L \cdot \Delta\vec{S} = E_L\, \Delta S \cos\theta$

$= E_L\, a^2 \cos 180^\circ = -\,E_L\, a^2$ [θ = 180° for the left face]

The magnitude of the electric field at the right face is

$E_R = \alpha\, x^{1/2} = \alpha\,(2a)^{1/2}$ [x = 2a at the right face]

Flux, $\phi_R = E_R\, \Delta S \cos 0^\circ = E_R\, a^2$ [θ = 0° for the right face]

Net flux through the cube

$\phi_E = \phi_L + \phi_R = \phi_R a^2 = E_L a^2$

$= a^2\,(E_R - E_L) = \alpha a^2\,[(2a)^{1/2} = a^{1/2}]$

$= \alpha a^{5/2}\,[\sqrt{2} - 1] = 800\,(0.1)^{5/2}\,(\sqrt{2} - 1)$

$= 1.05\ Nm^2\ C^{-1}$

(ii) By Gauss's theorem, the total charge inside the cube is

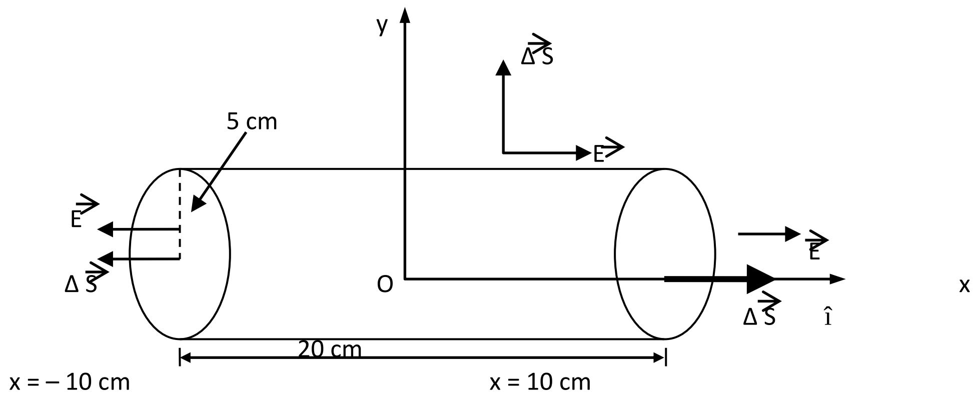
$q = \varepsilon_0\, \phi_E = \dfrac{1}{4\pi \times 9 \times 10^9} \times 1.05 = \mathbf{9.27 \times 10^{-12}\ C}$

Q. 8. An electric field is uniform, and in the positive x direction for negative x. It is given that

$E = 200\,\hat{i}\ NC^{-1}$ for x > 0

And $E = -200\,\vec{i}\ NC^{-1}$ for x < 0.

A right circular cylinder of length 20 cm and radius 5 cm has its centre at the origin and its axis along the x –axis so that one face is at x = +10 cm and the other is at x = – 10 cm. (i) What is the net outward flux through each flat face? (ii) What is the flux through the side of the cylinder? (iii) What is the net outward flux through the cylinder? (iv) What is the net charge inside the cylinder?



Sol. (i) On the left face: E = $-200\,\hat{i}$ NC^{-1}, $\Delta S = \Delta S\,\hat{i} = -\pi\,(0.05)^2\,\hat{i}\ m^2$

The outward flux through the left face is

$\phi_E = \vec{E}\,.\,\Delta\vec{S}$

$= +200 \times \pi\,(0.05)^2\,\hat{i}\,.\,\hat{i}\ NC^{-1}$

$=$ **$+ 1.57\ Nm^2\ C^{-1}$** $[\hat{i}\,.\,\hat{i} = 1]$

On the right face: $\vec{E} = 200\,\hat{i}\ NC^{-1}$

$\Delta\vec{S} = \Delta S\,\hat{i} = \pi\,(0.05)^2\,\hat{i}\ m^2$

The outward flux through the right face is

$\phi_E = \vec{E}.\,\Delta\vec{S} =$ **$+ 1.57\ Nm^2\ C^{-1}$.**

(ii) For any point on the side of the cylinder $\vec{E} \perp \Delta\vec{S}$,

∴ Flux through the side of the cylinder,

$\phi_E = \vec{E}\,.\,\Delta\vec{S} = E\,\Delta S \cos 90^o =$ **0**.

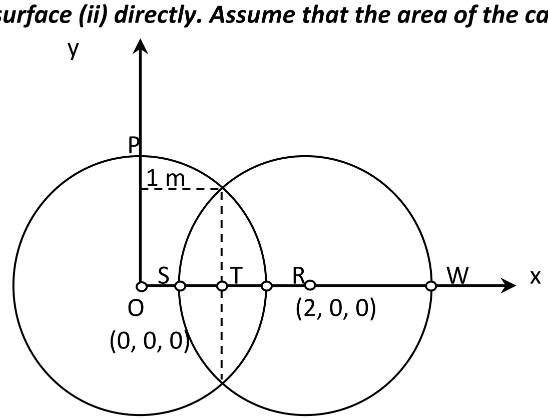
(iii) Net outward flux through the cylinder,

$\phi_E = 1.57 + 1.57 + 0 = 3.14\ Nm^2\ C^{-1}$.

(iv) By Gauss's theorem, the net charge inside the cylinder is

$q = \varepsilon_0\phi_E = 8.854 \times 10^{-12} \times 3.14 = 2.78 \times 10^{-11}$ C.

Q. 8. You are given an charge + Q at the origin O. Consider a sphere S with centre (2, 0, 0) of radius $\sqrt{2}$ m. Consider another sphere of radius $\sqrt{2}$ m centred at the origin. Consider the spherical caps (i) PSQ (ii) PRQ (iii) PWQ, with normal outward to the respective spheres, and (iv)The flat circle PTQ with normal along the x-axis.
(a) What is the sign of electric flux through each of the surfaces (i) – (iv)?
(b) What is the relation between the magnitudes of fluxes through surfaces (i) – (iv)?
(c) Calculate the flux through the surface (ii) directly. Assume that the area of the cap (ii) is A.



Sol. For the charge +Q situated at origin O, the field E points along +ve x-direction i.e., towards right.

(a) The outward drawn normal on cap PSQ points towards left while it points towards right for caps PRQ, PWQ and circle PTQ. So the flux is negative for (i) and positive for the rest.

(b) The same electric field lines crossing (i) also cross (ii), (iii). Also, by Gauss's law, the fluxes through (iii) and (iv) add up to zero. Hence, all magnitudes of fluxes are equal.

(c) Given area of the cap (ii) = $\vec{A}$

Electric field through cap (ii) is

$$E = \frac{1}{4\pi\varepsilon_0} = \frac{Q}{r^2} = 9 \times 10^9 \times \frac{Q}{(\sqrt{2})^2} = 4.5 \times 10^9 \text{ Q NC}^{-1}$$

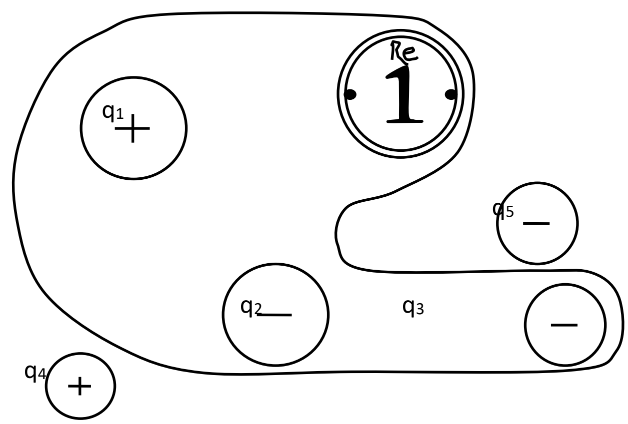
Electric flux through the cap (ii) is ϕ_E = EA= **4.5×10^9 QA NC^{-1} m^2.**

Q. 9. ***Fig. 1.92 shows five charged lumps of plastic and an electrically neutral coin. The cross-section of a Gaussian surface S is indicated. What is the net electric flux through the surface if***

$q_1 = q_4 = +3.1$ n C,

$q_2 = q_5 = -5.9$ n C

and $q_3 = -3.1$ nC ?

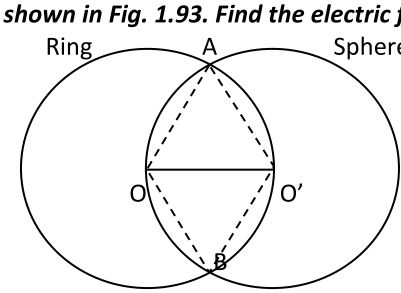


Sol. The neutral coin and the outside charges q_4 and q_5 make no contribution towards the net charge enclosed by surface S. Applying Gauss's theorem, we get

$$\phi_E = \frac{q}{\varepsilon_0} = \frac{q_1 + q_2 + q_3}{\varepsilon_0}$$

$$= \frac{+3.1 \times 10^{-9} - 5.9 \times 10^{-9} - 3.1 \times 10^{-9}}{8.85 \times 10^{-12}} = \mathbf{-667.67 \text{ Nm}^2 \text{ C}^{-2}}$$

Q. 10. ***A charge Q is distributed uniformly on a ring of radius r. A sphere of equal radius r is constructed with its centre at the periphery of the ring, as shown in Fig. 1.93. Find the electric flux through the sphere.***



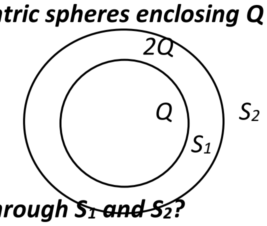
Sol. Clearly, OA = OO' = O'A = r. Thus ΔOAO' is equilateral. Hence ∠AOO' = 60° and ∠ AOB = 120°. Obviously, one-third portion AO'B of the ring lies in the sphere.

∴ Charge enclosed by the sphere, $q = \frac{Q}{3}$

From Gauss's theorem, electric flux through the sphere is

$$\phi = \frac{q}{\varepsilon_0} = \frac{Q}{3\varepsilon_0}$$

Q. 11. ***S_1 and S_2 are two hollow concentric spheres enclosing Q and 2 Q respectively as shown in the Fig.***



(i) What is the ratio of the electric flux through S_1 and S_2?

(ii) How will the electric flux through the sphere S_1 change, if the medium of dielectric constant 5 is introduced in the space inside S_1 in place of air?

Sol. (i) By Gauss's Theorem, flux through S_1 is $\phi_1 = \frac{Q}{\varepsilon_0}$

Flux through S_2, $\phi_2 = \frac{2Q + Q}{\varepsilon_0} = \frac{3Q}{\varepsilon_0}$

Ratio of electric flux through S_1 and S_2 is

$$\frac{\phi_1}{\phi_2} = \frac{Q/\varepsilon_0}{3Q/\varepsilon_0} = \frac{1}{3} = 1:3$$

(ii) If a medium of dielectric constant κ is introduced in the space inside S_1, then flux through S_1 becomes

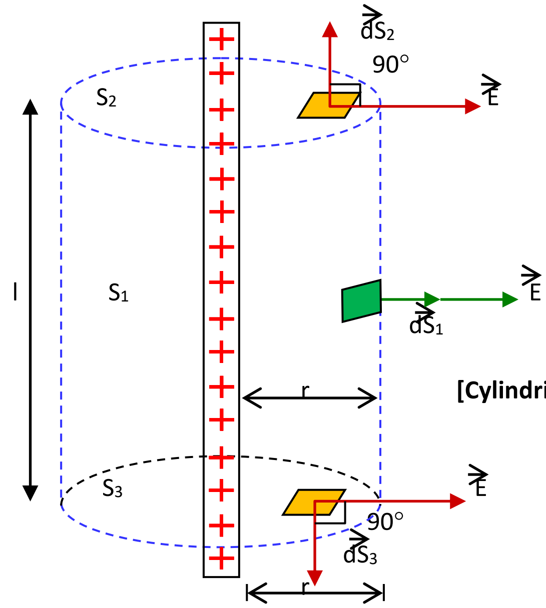
$$\phi_1' = \oint \vec{E}\,' \cdot d\vec{S} = \oint \frac{\vec{E}}{\kappa} \cdot d\vec{S}$$

$$= \frac{1}{\kappa} \oint \vec{E} \cdot d\vec{S} = \frac{1}{\kappa} \cdot \frac{Q}{\varepsilon_0}$$

But $\kappa = 5$, so $\phi_1' = \dfrac{Q}{5\varepsilon_0}$

Application of Gauss's theorem

Electric field due to an infinitely long straight charged wire: Consider a thin infinitely long straight wire having a uniform linear charge density λ Cm^{-1}. By symmetry, the field $\vec{E}$ of the line charge is directed radially outwards and its magnitude is same at all points equidistant from the line charge. To determine the field at a distance r from the line charge, we choose a cylindrical Gaussian surface of radius r, length l and with its axis along the line charge. As shown in Fig. it has curved surface S_1 and flat circular ends $\vec{S}_2$ and $\vec{S}_3$. Obviously, $dS_1 \parallel E$, $dS_2 \perp E$ and $dS_3 \perp E$. So only the curved surface contributes towards the total flux.



[Cylindrical Gaussian surface for line charge]

$$\phi_E = \oint \vec{E} \cdot d\vec{S}$$

$$= \int_{S_1} \vec{E} \cdot d\vec{S}_1 + \int_{S_2} \vec{E} \cdot d\vec{S}_2 + \int_{S_3} \vec{E} \cdot d\vec{S}_3$$

$$= \int_{S_1} E dS_1 \cos 0° + \int_{S_2} E dS_2 \cos 90° + \int_{S_3} E dS_3 \cos 90°$$

$$= E \int dS_1 + 0 + 0 = E \times \text{area of curved surface}$$

or $\quad \phi_E = E \times 2\pi r l$

Charge enclosed by the Gaussian surface, $q = \lambda l$

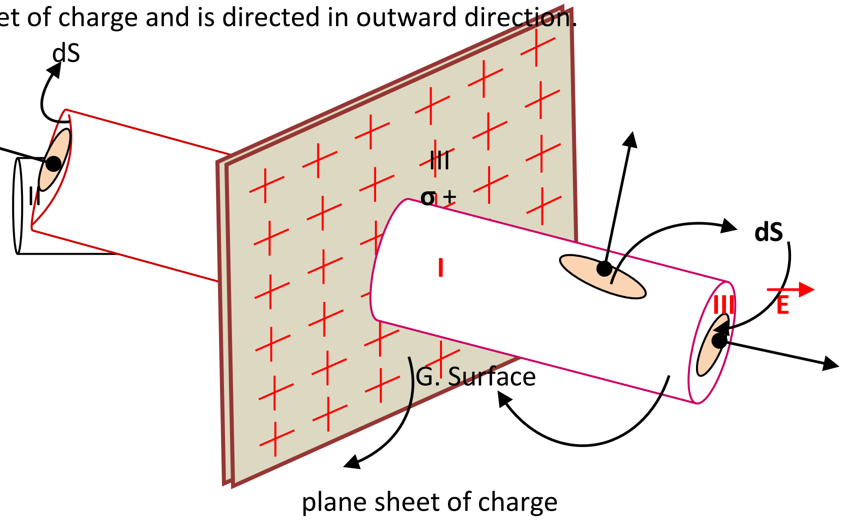
Using Gauss's theorem, $\phi_E = q/\varepsilon_0$, we get

or $\quad E \cdot 2\pi r l = \dfrac{\lambda l}{\varepsilon_0} \quad$ or $\quad E = \dfrac{\lambda}{2\pi \varepsilon_0 r}$

Thus, **the electric field of a line charge is inversely proportional to the distance from the line charge**.

...Electric field due to an INFINITE PLANE SHEET OF CHARGE:

Consider an infinite thin sheet (the same charge shows up on its two sides) of positive charge having a uniform charge Density. The electric field is perpendicular to the plane sheet of charge and is directed in outward direction.



Charge enclosed by the G . Surface,

$q = \sigma S$

According to G'S theorem ,

$$\oint_S \vec{E} \cdot \vec{dS} = \frac{q}{\xi_0}$$

$$\phi_E = \oint E . d S$$

$$= \int_{S_1} \vec{E} . \vec{dS}_1 + \int_{S_2} \vec{E} . \vec{dS}_2 + \int_{S_3} \vec{E} . \vec{dS}_3$$

$$= \int_{S_1} EdS_1 \cos 0° + \int_{S_2} EdS_2 \cos 90° + \int_{S_3} EdS_3 \cos 0°$$

Angle between E and dS is 90^0, for surface I

$$= E \int dS + 0 + E \int dS$$

$$= ES \quad + \quad ES \quad = \quad 2\ ES$$

also, $\phi = \frac{q}{\xi_0}$

$\therefore \quad 2\ ES = \frac{q}{\xi_0}$

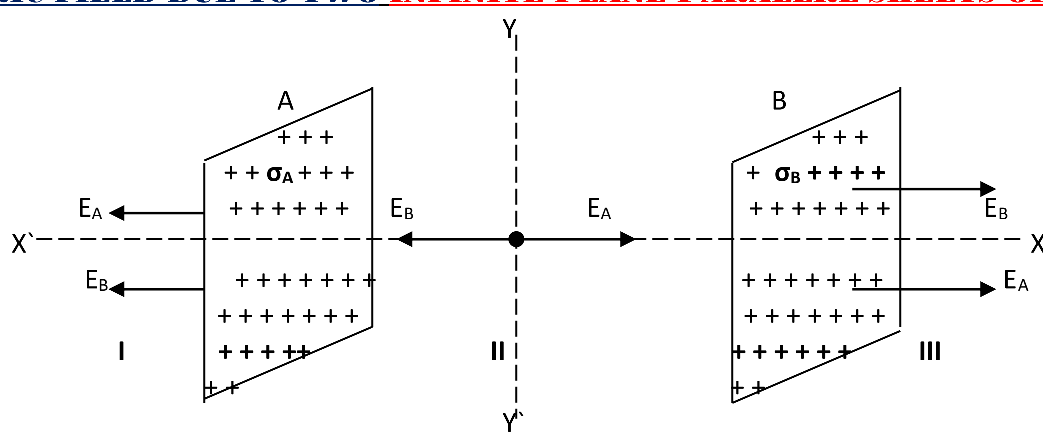
$\therefore \quad 2\ E\cancel{S} = \frac{\sigma \cancel{S}}{\xi_0}$ $\boxed{E = \frac{\sigma}{2\ \xi_0}}$

♦ ♦ For a finite thickness,

$\therefore \quad 2\ E\ S = \frac{2\ \sigma\ S}{\xi_0} \Rightarrow \boxed{E = \frac{\sigma}{\xi_0}}$

♦ ♦ $\vec{E}$ is independent of the distance of the point of observation.

...ELECTRIC FIELD DUE TO TWO INFINITE PLANE PARALLRL SHEETS OF CHARGE:



Consider two infinite plane parallel sheets of A and B, having charge densities σ_A and σ_B (such that $\sigma_A > \sigma_B$).
The two sheets divide the space into three regions, I, II & III.
Conventionally, E pointing from left to right (+ive x direction is taken as positive, while the electric field form right to left (--ive x direction is taken as negative).
We know that electric field intensity due to infinite sheet of shape is $E = \frac{\sigma}{2\xi_0}$

▶ **In region I** Therefore, Electric field (due to A) => $E_A = \frac{\sigma_A}{2\xi_0}$

And „ „ („ to B) => $E_B = \frac{\sigma_B}{2\xi_0}$

▶ **In region I** Net electric field due to A & B is
$E_I = -E_A + (-E_B) = -\frac{\sigma_A}{2\xi_0} - \frac{\sigma_B}{2\xi_0} = \frac{-1}{2\xi_0}(\sigma_A + \sigma_B)$.

▶ **In region II** Net electric field, $E_{II} = E_A + (-E_B)$
$= \frac{\sigma_A}{2\xi_0} - \frac{\sigma_B}{2\xi_0} = \frac{1}{2\xi_0}(\sigma_A - \sigma_B)$.

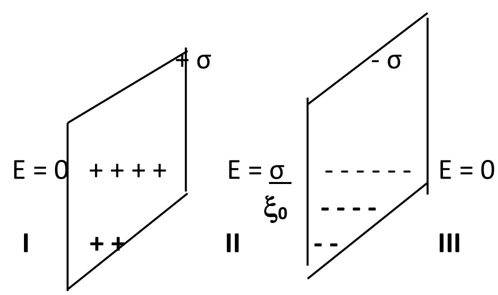
▶ **In region III** Net electric field, $E_{III} = E_A + E_B$
$= \frac{1}{2\xi_0}(\sigma_A + \sigma_B)$.

➲ Special case: If $\sigma_A = \sigma$ and $\sigma_B = -\sigma$ then,

$E_I = \frac{-1}{2\xi_0}(\sigma - \sigma) = 0.$

$E_{II} = \frac{1}{2\xi_0}[\sigma - (-\sigma)] = \frac{2\sigma}{2\xi_0} = \frac{\sigma}{\xi_0}$

$E_{III} = \frac{1}{2\xi_0}(\sigma - \sigma) = 0$



♦♦ If the plane sheet are of finite thickness then ,

$E_I = \frac{-1}{\xi_0}(\sigma_A + \sigma_B)$.

$E_{II} = \frac{1}{\xi_0}(\sigma_A - \sigma_B)$.

$E_{III} = \frac{1}{\xi_0}(\sigma_A + \sigma_B)$.

i.e., $2ES = \frac{2\sigma S}{\xi_0}$

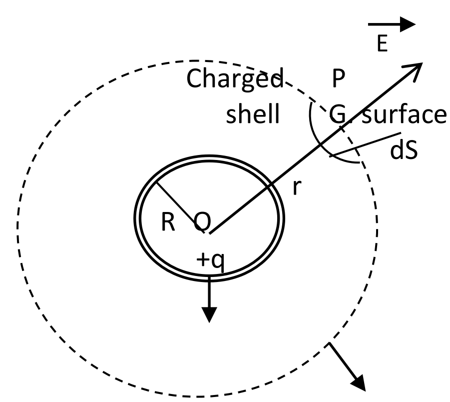
Therefore, $E = \frac{\sigma}{\xi_0}$

◘ ELECTRIC FIELD DUE TO UNIFORMLY CHARGED SPHERICAL SHELL:

Consider a thin of radius 'R' and centre O. Let 'q' be the charge of spherical shell.
Calculation of E at P distant 'r' from the centre of spherical shell ---

[a] Point 'P' lies outside the spherical shell.

Draw a G. surface of radius r such that r > R.
Let the electric field at point P = $\vec{E}$.



A / G 'theorem, $\oint \vec{E} \cdot d\vec{S} = \frac{q}{\xi_0}$.

$\oint_S E\, dS \cos\theta = \frac{q}{\xi_0}$. ---------------[i]

Since, $\vec{E}$ and $d\vec{S}$ are directed along the same direction. $\therefore \theta = 0^0$

From [i] $\oint E\, dS \cos 0^0 = q / \xi_0$

$\oint E\,dS = q/\xi_0 \Rightarrow E\oint dS = q/\xi_0$ [Since, E is same at all the points on the G. surface]

$E \times 4\pi r^2 = q/\xi_0$

Or, $\boxed{E = \frac{1}{4\pi\xi_0}\cdot\frac{q}{r^2} = K\frac{q}{r^2}}$ (Electric field due to point charge 'q' at a distance 'r') ________________ [ii]

Clearly , Electric field at any point outside te spherical shell is such that as if the entire charge were concentrated at the Center of the shell.

$E = kq/r^2$ can expressed in term of surface charge density as,

If σ = Uniform surface charge density of the spherical shell , then $\sigma = \frac{q}{4\pi R^2}$

or, $q = \sigma \times 4\pi R^2$

Putting $q = \sigma \times 4\pi r^2$ in $E = \frac{1}{4\pi\xi_0}\cdot\frac{q}{r^2}$ we get,

$$E = \frac{1}{4\pi\xi_0}\frac{\sigma \times 4\pi R^2}{r^2}$$

$$\boxed{E = \frac{\sigma}{\xi_0}\frac{R^2}{r^2}}$$

[b] Point 'P' lies on the surface of spherical shell.

then , r = R

From [ii] ,

$$E = \frac{1}{4\pi\xi_0}\cdot\frac{q}{r^2}$$

$$\boxed{E = \frac{1}{4\pi\xi_0}\cdot\frac{q}{R^2}}$$ since, r = R

Also, $E = \frac{1}{4\pi\xi_0}\frac{\sigma \times 4\pi R^2}{R^2}$

$$\boxed{E = \frac{\sigma}{\xi_0}}$$

Hence, E is maximum at the surface of the uniformly charged thin spherical shell.

[c] Point 'P' lies on the surface of spherical shell,

then, r < R, and therefore G .Surface will not enclose any charge.

According to G 'S theorem, $\oint_S \vec{E}\,d\vec{S} = \frac{q}{\xi_0}$ ---------------[i]

$E \times 4\pi R^2 = 0/\xi_0.$ Since q = 0

$$\boxed{E = 0}$$

Examples based on Application of Gauss's Theorem

❖ **FORMULAE USED**

1. Electric field of a long straight wire of uniform linear charge density λ, $E = \frac{\lambda}{2\pi\varepsilon_0 r}$

where r is the perpendicular distance of the observation point from the wire.

2. Electric field of an infinite plane sheet of uniform surface charge density σ, $E = \frac{\sigma}{2\varepsilon_0}$

3. Electric field of two positively charged parallel plates with charge densities σ_1 and σ_2 such that $\sigma_1 > \sigma_2 > 0$,

$E = \pm\frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)$ [Outside the plates]

$E = \frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)$ [Inside the plates]

4. Electric field of two equally and oppositely charged parallel plates,

$E = 0$ [For outside points]

$E = \frac{\sigma}{\varepsilon_0}$ [For inside points]

5. Electric field of a thin spherical shell of charge density σ and radius R,

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2} \quad \text{for } r > R \text{ [Outside points]}$$

$$E = 0 \quad \text{For } r < R \text{ [Inside points]}$$

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R^2} \quad \text{for } r = R \text{ [At the surface]}$$

Here $q = 4\pi r^2 \sigma$

6. Electric field of a solid sphere of uniform charge density ρ and radius R.

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2} \quad \text{For } r > R \quad \text{[Outside points]}$$

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qr}{R^3} \quad \text{For } r < R \quad \text{[Inside points]}$$

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R^2} \quad \text{For } r = R \quad \text{[At the surface]}$$

Here $q = 4/3\ \pi R^3 \rho$

❖ **UNITS USED**

Here charges are in coulomb, r and R in metre, λ in Cm^{-1}, σ in Cm^{-2}, ρ in Cm^{-3} and electric field E in NC^{-1} or Vm^{-1}.

Q. 1. ***Two long straight parallel wires carry charges λ_1 and λ_2 per unit length. The separation between their axes is d. Find the magnitude of the force exerted on unit length of one due to the charge on the other.***

Sol. Electric field at the location of wire 2 due to charge on 1 is

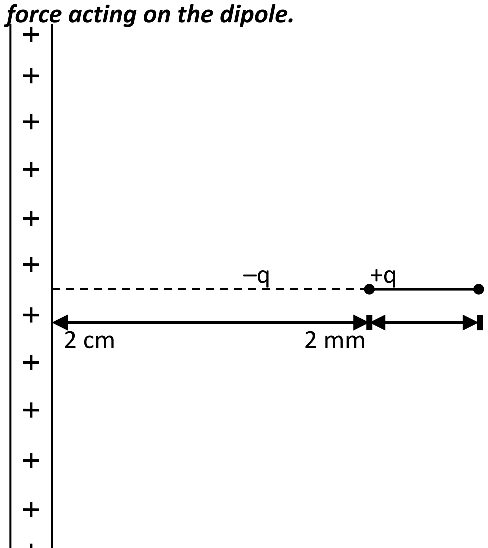
$$E = \frac{\lambda_1}{2\pi\varepsilon_0 d}$$

Force per unit length of wire 2 due to the above field

$$f = E \times \text{charge on unit length of wire 2} = E\lambda_2$$

or
$$f = \frac{\lambda_1\lambda_2}{2\pi\varepsilon_0 d}$$

Q. 2. ***An electric dipole consists of charges $\pm 2 \times 10^{-8}$ C, separated by a distance of 2 mm. It is placed near a long line charge of density 4.0×1^{-4} cm^{-1}, as shown in Fig., such that the negative charge is at a distance of 2 cm from the line charges. Calculate the force acting on the dipole.***



Sol. Electric field due to a line charge at distance r due to a line charge

$$E = \frac{1}{4\pi\varepsilon_0}\ \frac{2\lambda}{r}$$

Force exerted by this field on charge q,

$$F = qE = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2q\lambda}{r}$$

Force exerted on negative charge (r = 0.02m),

$$F_1 = \frac{9 \times 10^9 \times 2 \times 2 \times 10^{-9} \times 4 \times 10^{-4}}{0.02}\ \text{N}$$

= 7.2 N, acting towards the line charge.

Q. 3. ***An electron is revolving around a long line charge having charge density 2×10^{-8} Cm^{-1}. Find the kinetic energy of the electron, assuming that it is independent of the radius of electron's orbit.***

Sol. The electrostatic force exerted by the line charge on the electron provides the centripetal force for the revolution of electron.

$\therefore$ Force exerted by electric field = Centripetal force

$$eE = \frac{mv^2}{r}$$

Here v is the orbital velocity of the electron

But $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$

$\therefore$ $\dfrac{e\lambda}{2\pi\varepsilon_0 r} = \dfrac{mv^2}{r}$ or $v^2 = \dfrac{e\lambda}{2\pi\varepsilon_0 m}$

Kinetic energy of the electron will be $E_k = \frac{1}{2}mv^2 = \dfrac{e\lambda}{4\pi\varepsilon_0}$ $= 9 \times 10^9 \times 1.6 \times 10^{-19} \times 2.0 \times 10^{-8} = 2.88 \times 10^{-17}$ J.

Q. 4. ***A charge of 17.7×10^{-4} C is distributed uniformly over a large sheet of area 200 m^2. Calculate the electric field intensity at a distance of 20 cm from it in air.***

Sol. Surface charge density of the sheet,

$$\sigma = \frac{q}{A} = \frac{17.7 \times 10^{-4}\ \text{C}}{200\ \text{m}^2} = 8.85 \times 10^{-6}\ \text{Cm}^{-2}$$

Electric field at a distance of 20 cm from it in air,

$$E = \frac{\sigma}{\varepsilon_0} = \frac{8.85 \times 10^{-6}}{8.85 \times 10^{-12}} = \mathbf{10^6\ NC^{-1}}$$

Q. 5. ***A charged particle having a charge of – 2.0×10^{-6} C is placed close to a non-conducting plate having a surface charge density of 4.0×10^{-6} Cm^{-2}. Find the force of attraction between the particle and the plate.***

Sol. Here $q = -2.0 \times 10^{-6}$ C, $\sigma = 4.0 \times 10^{-6}$ Cm^{-2}

Field produced by charged plate,

$$E = \frac{\sigma}{2\varepsilon_0}$$

Force of attraction between the charged particle and the plate, $\therefore$ $F = qE = \dfrac{\sigma q}{2\varepsilon_0} = \dfrac{4 \times 10^{-6} \times 2.0 \times 10^{-6}}{2 \times 8.85 \times 10^{-12}} = \mathbf{0.45\ N.}$

Q. 6. ***A particle of mass 9×10^{-5} g is kept over a large horizontal sheet of charge density 5×10^{-5} Cm^{-2}. What charge should be given to the particle, so that if released, it does not fall?***

Sol. Here $m = 9 \times 10^{-5}$ g $= 9 \times 10^{-8}$ kg, $\sigma = 5 \times 10^{-5}$ Cm^{-2}

The particle must be given a positive charge q. It will not fall if

Upward force exerted on the particle by electric field = Weight of the particle

Or $qE = mg$

Or $q \cdot \dfrac{\sigma}{2\varepsilon_0} = mg$ or $q = \dfrac{2\varepsilon_0 mg}{\sigma}$

$$= \frac{2 \times 8.85 \times 10^{-12} \times 9 \times 10^{-8} \times 9.8}{5 \times 10^{-5}} = \mathbf{3.12 \times 10^{-13}\ C}$$

Q. 7. ***A large plane sheet of charge having surface charge density 5.0×10^{-16} Cm^{-2} lies in the X-Y plane. Find the electric flux through a circular area of radius 0.1 m, if the normal to the circular area makes an angle of 60° with the Z-axis. Given that $\varepsilon_0 = 8.85 \times 10^{-12}$ C^2 N^{-1} m^{-2}.***

Sol. Here $\sigma = 5.0 \times 10^{-6}$ Cm^{-2}, r = 0.1 m, $\theta = 60°$

Field due to a plane sheet of charge, $E = \dfrac{\sigma}{2\varepsilon_0}$

Flux through circular area,

$$\phi_E = E\Delta S\cos\theta = \frac{\sigma}{2\varepsilon_0} \times \pi r^2 \cos\theta = \frac{5.0 \times 10^{-16} \times 3.14 \times (0.1)^2 \cos 60°}{2 \times 8.85 \times 10^{-12}} = \mathbf{4.44 \times 10^{-7}\ Nm^2\ C^{-1}}.$$

Q. 8. ***A spherical conductor of radius 12 cm has a charge of 1.6×10^{-7} C distributed uniformly over its surface. What is the electric field (i) inside the sphere, (ii) just outside the sphere, (iii) at a point 18 cm from the centre of the sphere?***

Sol. Here $q = 1.6 \times 10^{-7}$ C,

R = 12 cm = 0.12 m

(i) Inside the sphere, E = 0. This is because the charge resides on the outer surface of the spherical conductor.

(ii) Just outside the sphere, r = R = 0.12 m. Here the charge may be assumed to be concentrated at the centre of the sphere.

∴ $E = \frac{1}{4\pi \varepsilon_0} \cdot \frac{q}{R^2} = \frac{9 \times 10^9 \times 1.6\, f\, 10^{-7}}{(0.12)^2} = \mathbf{10^5\ NC^{-1}}$

(iii) At a point 18 cm from the centre, r = 18 cm = 0.18 m.

∴ $E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} = \frac{9 \times 10^9 \times 1.6 \times 10^{-7}}{(0.18)^2} = \mathbf{4.44 \times 10^4\ NC^{-1}}$.

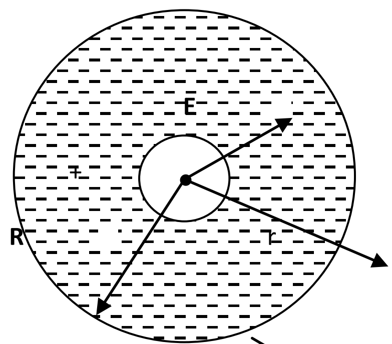
Q. 9. ***An early model of an atom considered it to have a positively charged point nucleus of charge Ze, surrounded by a uniform density of negative charge up to a radius R. The atom as a whole is neutral. For this model, what is the electric field at a distance r from the nucleus?***

Sol. Fig. 1.106 shows the charge distribution for the given model of the atom.

As the atom is neutral, the total negative charge in a sphere of sphere of radius R must be – Ze. If ρ is the negative charge density, then we must have

$$Ze + \frac{4}{3}\pi R^3 \rho = 0$$

Or $\rho = -\frac{3\,Ze}{4\pi R^3}$



[An early model of atom]

By spherical symmetry of the charge distribution, the electric field $\vec{E}$ depends only on radius distance r and not on the direction of $\vec{r}$. It should point radially inwards or outwards. So we imagine a spherical Gaussian surface of radius r centred at the nucleus.

(i) For r < R. Flux through the Gaussian surface,

q = positive nuclear charge
+ Negative charge in a sphere of radius r

$$= Ze + \frac{4\pi r^3}{3}\rho$$

$$= Ze + \frac{4\pi r^3}{3}\left(-\frac{3Ze}{4\pi R^3}\right)$$

$$= Ze\left(1 - \frac{r^3}{R^3}\right)$$

Applying Gauss's theorem, $\phi_E = q/\varepsilon_0$, we get

$$E \times 4\pi r^2 = \frac{Ze}{\varepsilon_0}\left(1 - \frac{r^3}{R^3}\right)$$

Or $E = \frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right)$ (r < R)

The field E points radially outward.

(ii) For r > R. As the atom is neutral, the total charge enclosed by the Gaussian surface is zero. By Gauss's theorem.

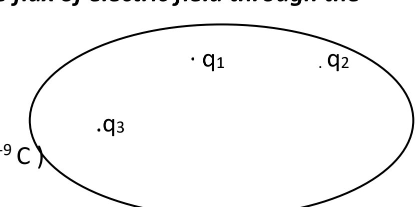
$$E \times 4\pi r^2 = 0$$

Or E = 0. (r > R)

(iii) At r = R. Both of the above cases give the same result: E = 0

Q . ***Given figure shows three charges enclosed by a Gaussian surface . What is the flux of electric field through the Gaussian Surface if q_1 = +3 nC , q_2 = -6nC and q_3 = -3 nC ?***

Sol : $q_1 = 3\text{ nC} = 3 \times 10^{-9}\text{ C}$; $q_2 = -6\text{ nC} = -6 \times 10^{-9}\text{ C}$; $q_3 = -3\text{ nC} = -3 \times 10^{-9}\text{ C}$



Total charge in the Gaussian surface, $q = q_1 + q_2 + q_3$

$= 3 \times 10^{-9}\text{ C} + (-6 \times 10^{-9}\text{ C}) + (-3 \times 10^{-9}\text{ C})$

$= -6 \times 10^{-9}\text{ C}$

∴Electric flux through G .Surface , $\phi = q/\xi_0 = \frac{-6 \times 10^{-9}\text{ C}}{8.85 \times 10^{-12}\text{ C}} = -6.78 \times 10^{2}\text{ N m}^2\text{ C}^{-1}$.