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PROJECTILE MOTION

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UNIT - II : KINEMATICS

CHAP:02: PROJECTILE MOTION

The motion of an object is called two dimensional, if two of the three co-ordinates are required to specify the position of the object in space changes *w.r.t* time.

In such a motion, the object moves in a plane. For example, a billiard ball moving over the billiard table, an insect crawling over the floor of a room, earth revolving around the sun *etc*.

Two special cases of motion in two dimension are 1. Projectile motion 2. Circular motion

ILLUSTRATION

In one dimensional motion, the object moves in a straight line (*e.g.,* along *X*-axis or *Y*-axis). Therefore, only one number (*x* or *y*) is required to specify the position of the object. But in two dimensional motion, the object moves in a plane —say *XY* plane. Consequently, we require two numbers (*x*, *y*) to specify the position of the object in the plane. However, in three dimensional motion, the object moves in space and we need three numbers (*x*, *y*, *z*) to locate the position of the object in space. It is because the space we live in is three dimensional.

. Two dimensional motion is very important due to two principal reasons. First, many interesting phenomena are associated with two dimensional motion. Secondly, this type of motion can be easily illustrated on a paper or blackboard. For this reason, most of our discussion in this chapter will be confined to two dimensional motion.

In one dimensional motion, we can handle vector quantities (*e.g.,* displacement, velocity, acceleration *etc.*) quite easily because their directional aspect can be taken care of by +ve and –ve signs. This is due to the fact that in one dimension, only two directions are possible. However, the situation is not that simple in two dimensional motion (*i.e.,* motion in a plane) or three dimensional motion (*i.e.,* motion in space). Here we have to use vectors to deal with vector quantities.

MOTION IN A PLANE

When an object moves in a plane, say *XY* plane, we require two rectangular coordinates (*x*, *y*) to specify its position in the plane. Consider the simple case of an object moving along a straight line *AB* in the *XY* plane as shown in Fig. 9.1. When the object is at point *A*, its coordinates are (x_1, y_1) and when it is at point *B*, its coordinates are (x_2, y_2). Thus we can locate the position of an object in a plane by rectangular coordinates (*x*, *y*). However, we can also locate the position of an object in a plane by position vector of the object. Thus in Fig. 9.1, the position vector of the object at point *A* is $\overrightarrow{OA}\ (=\vec{r}_1)$ and at point *B*, the position vector of the object is $\overrightarrow{OB}\ (=\vec{r}_2)$. The two methods are equivalent and are frequently used to describe the motion of an object in a plane.

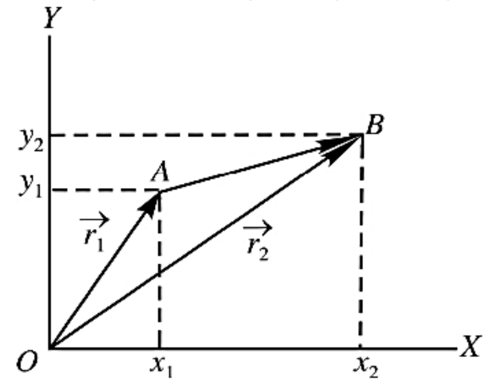


Fig. 9.1

Uniform velocity. *An object is said to be moving with a uniform velocity in a plane if it undergoes equal displacements in equal intervals of time, however small these time intervals may be.*

> Note that uniform velocity means that the magnitude as well as direction of the velocity remains the same. This is possible only if the object moves in a straight line with constant speed in the plane. *Therefore, we arrive at an important conclusion that the path of an object moving with uniform velocity in a plane is a straight line.*

MOTION WITH UNIFORM VELOCITY IN A PLANE

Consider an object moving with a uniform velocity $\vec{v}$ in the *XY* plane as shown in Fig. 9.2. Suppose at $t = 0$, the object is at point $A(x_0, y_0)$ and the position vector of the object is $\overrightarrow{OA}\ (=\vec{r}_0)$. Let at time *t*, the object be at point $B(x, y)$ and its position vector be $\overrightarrow{OB}\ (=\vec{r})$.

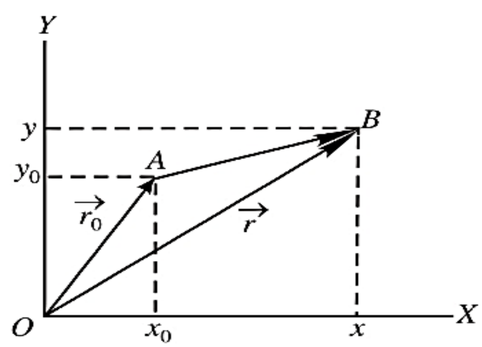


Fig. 9.2

(*i*) Displacement. The displacement of the object in the time interval $(t - 0)$ is $\overrightarrow{AB}$. According to triangle law of vectors,

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

or
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

∴ Displacement, $\overrightarrow{AB} = \vec{r} - \vec{r}_0$ ...(*i*)

Eq. (*i*) gives the displacement $\overrightarrow{AB}$ of the object in terms of its position vectors.

We can also express displacement $\overrightarrow{AB}$ in terms of rectangular components (or coordinates).

Now $\vec{r}_0 = x_0\hat{i} + y_0\hat{j}\ ;\ \vec{r} = x\hat{i} + y\hat{j}$

∴ Displacement, $\overrightarrow{AB} = \vec{r} - \vec{r}_0 = (x\hat{i} + y\hat{j}) - (x_0\hat{i} + y_0\hat{j})$

or $$\overrightarrow{AB} = (x - x_0)\hat{i} + (y - y_0)\hat{j} \qquad ...(ii)$$

Eq. (*ii*) gives the displacement $\overrightarrow{AB}$ of the object in terms of rectangular components.

(*ii*) Velocity. The velocity of the object is given by ;

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time interval}}$$

or $$\vec{v} = \frac{\vec{r} - \vec{r_0}}{t - 0} = \frac{\vec{r} - \vec{r_0}}{t}$$

$\therefore$ $$\vec{v} = \frac{\vec{r} - \vec{r_0}}{t} \qquad ...(iii)$$

Eq. (*iii*) gives the velocity $\vec{v}$ of the object in terms of its position vectors. Since the object is moving with uniform velocity ($\vec{v}$), the magnitude as well as direction of the velocity is the same at every instant.

(*iii*) Equation of motion of object. From eq. (*iii*), we have,

$$\vec{r} - \vec{r_0} = \vec{v}\, t$$

or $$\vec{r} = \vec{r_0} + \vec{v}\, t \qquad ...(iv)$$

Eq. (*iv*) gives the position $(\vec{r})$ of the object moving with uniform velocity $(\vec{v})$ at any time t. Note that $\vec{r_0}$ is the position vector of the object at $t = 0$.

In terms of rectangular components. Let us express eq. (*iv*) in terms of rectangular components (or coordinates). Suppose v_x and v_y are the magnitudes of the rectangular components of $\vec{v}$ along *X*-axis and *Y*-axis respectively. Then,

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \qquad \text{where } v = \sqrt{v_x^2 + v_y^2}$$

Since magnitude of $\vec{v}$ is constant, v_x = constant and v_y = constant.

Now $$\vec{r} = \vec{r_0} + \vec{v}\, t$$

Here $$\vec{r} = x\hat{i} + y\hat{j}\ ;\ \vec{r_0} = x_0\hat{i} + y_0\hat{j}\ ;\ \vec{v} = v_x\hat{i} + v_y\hat{j}$$

$\therefore$ $$x\hat{i} + y\hat{j} = (x_0\hat{i} + y_0\hat{j}) + (v_x\hat{i} + v_y\hat{j})t$$

or $$x\hat{i} + y\hat{j} = (x_0 + v_x t)\hat{i} + (y_0 + v_y t)\hat{j} \qquad ...(v)$$

The coefficients $\hat{i}$ and $\hat{j}$ on both sides of the eq. (*v*) must be equal.

$\therefore$ $$x = x_0 + v_x t \qquad ...(vi)$$

$$y = y_0 + v_y t \qquad ...(vii)$$

Thus eq. (*iv*) can be replaced by two scalar equations (*vi*) and (*vii*). Now eqs. (*vi*) and (*vii*) represent motion along two perpendicular straight lines—motion along *X*-axis with uniform velocity v_x and motion along *Y*-axis with uniform velocity v_y.

Therefore, two dimensional motion (i.e., motion in a plane) with uniform velocity is a combination of two one-dimensional uniform motions along perpendicular directions.

(*iv*) Path of the object. An object moving with a uniform velocity in a plane follows a straight line path. We have already given the physical explanation for it. Let us now turn to mathematical explanation.

Now $$x = x_0 + v_x t \;\; ; \;\; y = y_0 + v_y t$$

It is clear that $t = \dfrac{x - x_0}{v_x}$. Putting this value of t in the equation for y, we have,

$$y = y_0 + \frac{v_y}{v_x}(x - x_0)$$

It is clear that this is the equation of a straight line having slope = v_y / v_x. *Therefore, the path of the object moving with uniform velocity in a plane is a straight line.*

MOTION WITH UNIFORM ACCLEARATION IN A PLANE

We now discuss the motion of an object moving with uniform acceleration $\vec{a}$ in a plane. *An object is said to be moving with uniform acceleration in a plane if its velocity vector changes by equal amounts in equal intervals of time, however small these time intervals may be.* Note that uniform acceleration means that magnitude as well as direction of acceleration remains the same. As we shall see, the path of the object moving with uniform acceleration in a plane is parabolic.

(*i*) Acceleration. Suppose at $t = 0$, the velocity of the object is $\vec{v}_0$ and at $t = t$, its velocity is $\vec{v}$. Then uniform acceleration $\vec{a}$ of the object is

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Time interval}}$$

or $$\vec{a} = \frac{\vec{v} - \vec{v}_0}{t - 0} = \frac{\vec{v} - \vec{v}_0}{t}$$

$$\therefore \quad \vec{a} = \frac{\vec{v} - \vec{v}_0}{t} \quad \text{...}(i)$$

Eq. (*i*) gives the expression for the uniform acceleration $\vec{a}$ of the object. Since the object is moving with uniform (constant) acceleration, the magnitude as well as direction of acceleration remains constant at every instant of motion.

(*ii*) Equation of motion. From eq. (*i*), we have,

$$\vec{v} - \vec{v}_0 = \vec{a}\, t$$

or $$\vec{v} = \vec{v}_0 + \vec{a}\, t \quad \text{...}(ii)$$

Eq. (*ii*) gives the velocity $(\vec{v})$ of the object moving with uniform acceleration $(\vec{a})$ at any time t. Note that $\vec{v}_0$ is the velocity of the object at $t = 0$.

In terms of rectangular components. Let us express eq. (*ii*) in terms of rectangular components (or coordinates). Suppose a_x and a_y are the magnitudes of the rectangular components of $\vec{a}$ along *X*-axis and *Y*-axis respectively. Then,

$$\vec{a} = a_x\hat{i} + a_y\hat{j} \quad \text{where} \quad a = \sqrt{a_x^2 + a_y^2}$$

Since magnitude of $\vec{a}$ is constant, a_x = constant and a_y = constant.

$$\text{Now} \quad \vec{v} = \vec{v}_0 + \vec{a}\,t$$

Here $\vec{v} = v_x\hat{i} + v_y\hat{j}$; $\vec{v}_0 = v_{x0}\hat{i} + v_{y0}\hat{j}$; $\vec{a} = a_x\hat{i} + a_y\hat{j}$

$$\therefore \quad v_x\hat{i} + v_y\hat{j} = (v_{x0}\hat{i} + v_{y0}\hat{j}) + (a_x\hat{i} + a_y\hat{j})t$$

$$\text{or} \quad v_x\hat{i} + v_y\hat{j} = (v_{x0} + a_x t)\hat{i} + (v_{y0} + a_y t)\hat{j} \quad \text{...(iii)}$$

The coefficients of $\hat{i}$ and $\hat{j}$ on both sides of eq. (*iii*) must be the same.

$$\therefore \quad v_x = v_{x0} + a_x t \quad \text{...(iv)}$$

$$v_y = v_{y0} + a_y t \quad \text{...(v)}$$

Thus eq. (*ii*) can be replaced by two equations (*iv*) and (*v*). Now eqs. (*iv*) and (*v*) represent motion along two perpendicular straight lines—motion along *X*-axis with uniform acceleration a_x and motion along *Y*-axis with uniform acceleration a_y.

Therefore, two dimensional motion (i.e, motion in a plane) with uniform acceleration is a combination of two one dimensional uniformly accelerated motions along perpendicular directions.

(*iii*) Expression for displacement for uniform acceleration. Suppose at $t = 0$, the velocity of the object is $\vec{v}_0$ and at $t = t$, it is $\vec{v}$. The average velocity $\vec{v}_{av}$ during the time interval $0 - t$ is

$$\vec{v}_{av} = \left(\frac{\vec{v}_0 + \vec{v}}{2}\right) = \left(\frac{\vec{v}_0 + \vec{v}_0 + \vec{a}\,t}{2}\right) \qquad \left(\because \vec{v} = \vec{v}_0 + \vec{a}\,t\right)$$

$$\therefore \quad \vec{v}_{av} = \vec{v}_0 + \frac{1}{2}\vec{a}\,t$$

If $\vec{r}_0$ and $\vec{r}$ are the position vectors of the object at $t = 0$ and $t = t$ respectively, then $\vec{v}_{av}$ is given by ;

$$\vec{v}_{av} = \frac{\vec{r} - \vec{r}_0}{t - 0} = \frac{\vec{r} - \vec{r}_0}{t}$$

$$\text{or} \quad \vec{r} - \vec{r}_0 = \vec{v}_{av}\,t$$

$$\text{or} \quad \vec{r} = \vec{r}_0 + \vec{v}_{av}\,t = \vec{r}_0 + (\vec{v}_0 + \frac{1}{2}\vec{a}\,t)t \qquad \left(\because \vec{v}_{av} = \vec{v}_0 + \frac{1}{2}\vec{a}\,t\right)$$

$$\therefore \quad \vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}\,t^2 \quad \text{... (vi)}$$

Eq. (*vi*) can be written in terms of rectangular components as :

$$x\hat{i} + y\hat{j} = \left(x_0\hat{i} + y_0\hat{j}\right) + \left(v_{x0}\hat{i} + v_{y0}\hat{j}\right)t + \frac{1}{2}\left(a_x\hat{i} + a_y\hat{j}\right)t^2$$

or $$x\hat{i} + y\hat{j} = \left(x_0 + v_{x0}t + \frac{1}{2}a_x t^2\right)\hat{i} + \left(y_0 + v_{y0}t + \frac{1}{2}a_y t^2\right)\hat{j}$$

$\therefore$ $$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \quad \text{...(vii)}$$

and $$y = y_0 + v_{y0}t + \frac{1}{2}a_y t^2 \quad \text{...(viii)}$$

Note that eqs. (*vii*) and (*viii*) are similar to that of the position-time relation for the motion of the object with uniform acceleration in one dimension.

(*iv*) Path of the object. If we eliminate t from eqs. (*vii*) and (*viii*), we get second degree equation in x and y. Therefore, path of the object will be parabolic. *Therefore, path of an object moving with uniform acceleration in a plane is parabolic.*

Projectile Motion

A body which is in flight through the atmosphere but is not being propelled by any fuel is called projectile.

Example: (i) A bomb released from an aeroplane in level flight

(ii) A bullet fired from a gun

(iii) An arrow released from bow

(iv) A Javelin thrown by an athlete

■ The path followed by a projectile during its flight is called **"trajectory"**.

Assumptions of Projectile Motion

(1) There is no resistance due to air.

(2) The effect due to curvature of earth is negligible.

(3) The effect due to rotation of earth is negligible.

(4) For all points of the trajectory, the acceleration due to gravity '*g*' is constant in magnitude and direction.

PRINCIPLES OF PHYSICAL INDEPENDENCE OF MOTIONS

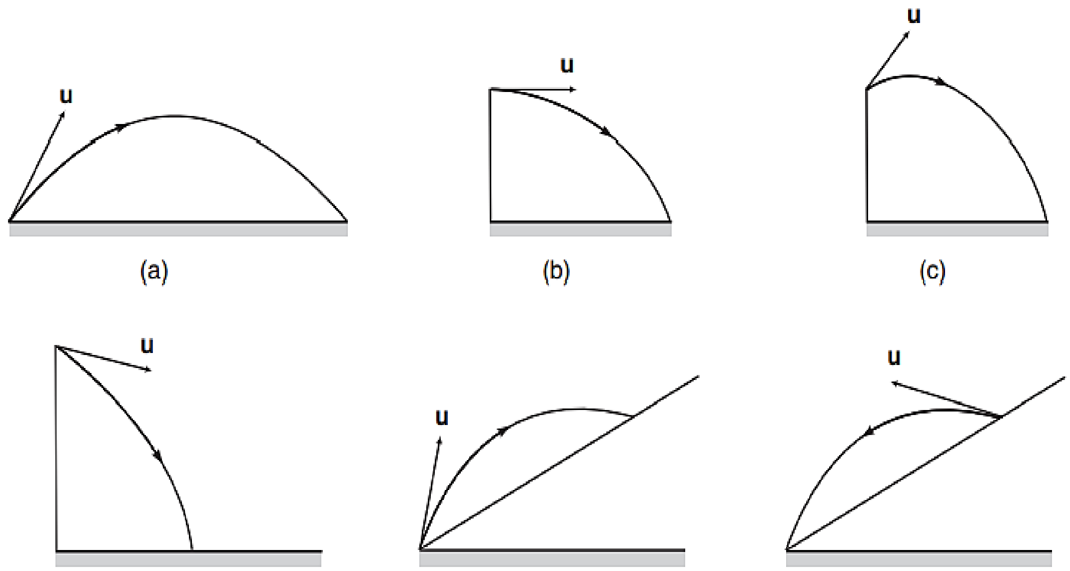
(1) The motion of a projectile is a two-dimensional motion. So, it can be discussed in two parts. Horizontal motion and vertical motion. These two motions take place independent of each other. This is called the principle of physical independence of motions.

(2) The velocity of the particle can be resolved into two mutually perpendicular components. Horizontal component and vertical component.

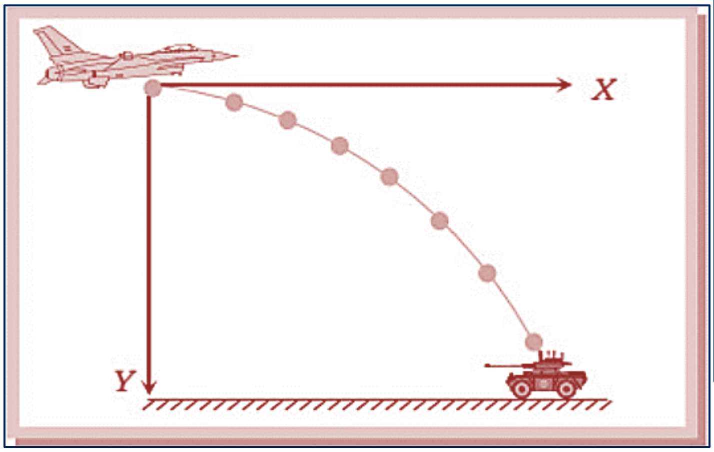
(3) The horizontal component remains unchanged throughout the flight. The force of gravity continuously affects the vertical component.

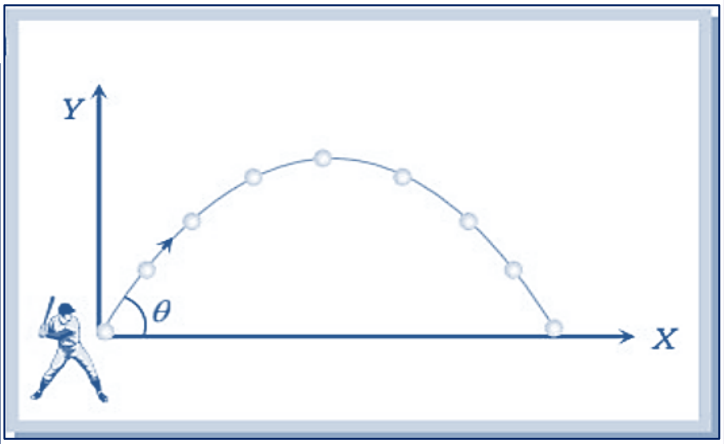
(4) The horizontal motion is a uniform motion and the vertical motion is a uniformly accelerated retarded motion.

The different types of projectile motion are as shown below.



(a) (b) (c) (d) (e) (f)





HORIZONTAL PROJECTILE

A body be projected horizontally from a certain height 'y' vertically above the ground with initial velocity u. If friction is considered to be absent, then there is no other horizontal force which can affect the horizontal motion. The horizontal velocity therefore remains constant and so the object covers equal distance in horizontal direction in equal intervals of time.

Consider a projectile (say a ball) is thrown horizontally with the **initial velocity u** from the top of a tower of ***height 'h'***. As it moves, it covers horizontal distance due to **uniform horizontal velocity** $\vec{u}$ and vertically downward distance because of **constant acceleration due to gravity 'g'**. Under the combined effect of these two motion the projectile moves along the path OP.................

let it takes **time 't'** to reach the ground at point P.

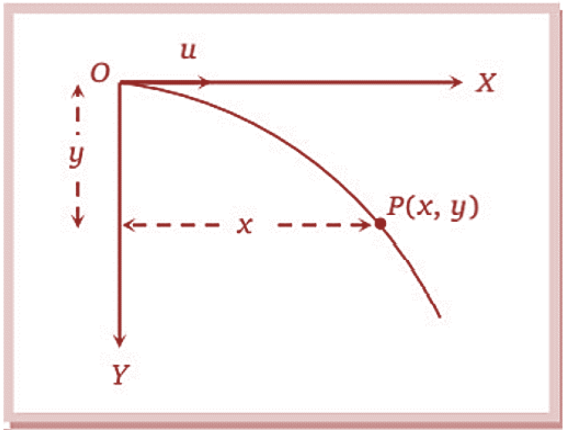
∴ **Horizontal distance** travelled by it = x.

also, **Vertical distance** travelled by it = y.

The motion of projectile is in plane and the Eq^n describing the motion are

$x = v_{0x}t + \frac{1}{2} a_x t^2$ .... (i)

$y = v_{0y}t + \frac{1}{2} a_y t^2$ .... (ii)



Motion along horizontal direction OX

$v_{ox} = u$, $a_x = 0$ (There is no change in velocity in horizontal direction).

From (i), $x = ut + \frac{1}{2} \times 0 \times t^2$

∴ **$x = ut$** or **$t = x/u$**

Motion along Vertical direction (OY)

Here, $v_{oy} = 0$ & $a_y = g$, (acclʳ due to gravity)

From eq. (ii)

$$y = v_{oy}t + \frac{1}{2} a_y t^2$$

$$\therefore \quad y = 0 + \frac{1}{2} g t^2$$

$$y = \frac{1}{2} g t^2$$

But, $t = x/u$

$$\therefore \quad y = \frac{1}{2} g \; x^2/u^2$$

But, $\frac{1}{2} g/u^2$ = Constant = k

$y = kx^2$, Equation of parabola.

Hence, path of projectile projected horizontally from a certain in height from the ground is a parabolic path.

Time of flight: "The time taken by projectile to complete its trajectory is called time of flight".

If a body is projected horizontally from a height h with velocity u and

$$h = 0 + \frac{1}{2} g T^2 \text{ (for vertical motion)}$$

$$T = \sqrt{\frac{2h}{g}}$$

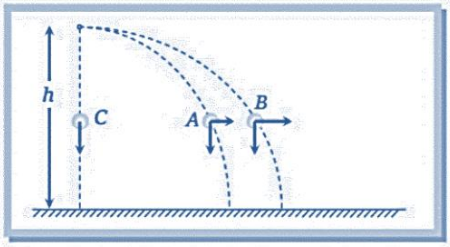
When the body is projected horizontally from the top of a tower, the time taken by it to reach the ground is independent of the velocity of projection but depends upon the Height of the tower.

(6) If projectiles A and B are projected horizontally with different initial velocity from same height and third particle C is dropped from same point then

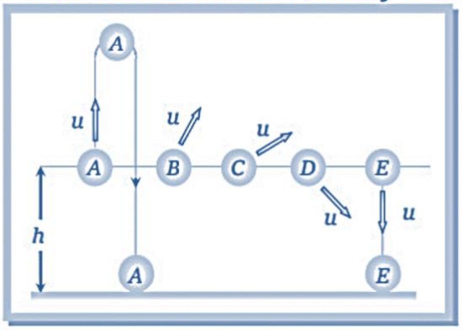
(i) All three particles will take equal time to reach the ground.

(ii) Their net velocity would be different but all three particle possess same vertical component of velocity.

(iii) The trajectory of projectiles A and B will be straight line *w.r.t.* particle C.



(7) If various particles thrown with same initial velocity but indifferent direction then



(i) They strike the ground with same speed at different times irrespective of their initial direction of velocities.

(ii) Time would be least for particle E which was thrown vertically downward.

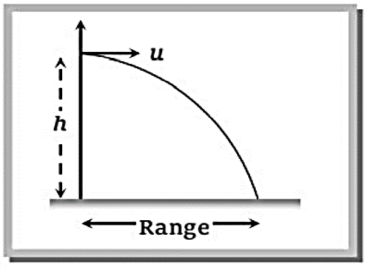
(iii) Time would be maximum for particle A which was thrown vertically upward.

Horizontal Range: - "The horizontal distance covered by the projectile from the foot of the power to the point where the projectile hits the ground is called horizontal range."

range : Let R is the horizontal distance travelled by

$$R = uT + \frac{1}{2}\,0\,T^2 \text{ (for horizontal motion)}$$

$$R = u\sqrt{\frac{2h}{g}}$$



Displacement of Projectile $(\vec{r})$

displacement $y = \frac{1}{2}gt^2$.

So, the position vector $\vec{r} = ut\hat{i} - \frac{1}{2}gt^2\hat{j}$

Therefore $r = ut\sqrt{1 + \left(\frac{gt}{2u}\right)^2}$ and $\alpha = \tan^{-1}\left(\frac{gt}{2u}\right)$

$\alpha = \tan^{-1}\left(\sqrt{\frac{gy}{2}}\Big/u\right)$ $\left(\text{as } t = \sqrt{\frac{2y}{g}}\right)$

Velocity of the projectile at any instant: At the instant t (when the body is at point P), let the velocity of the projectile be v. The velocity v has two rectangular components:

Instantaneous velocity

Horizontal component of velocity, $v_x = u$

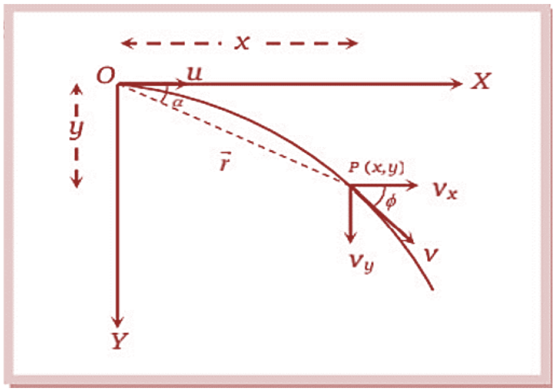
Vertical component of velocity, $v_y = 0 + gt = gt$

We know that, For motion along x-axis

$v_x = v_{0x} + a_x t$ [here, $v_{0x} = u$ & $a_x = 0$]

∴ $v_x = u + 0t$

∴ $v_x = u$ ....(i)



For vertical motion $v_y = v_{0y} + a_y t$

Here, $v_{0y} = 0$ and $a_y = g$

$v_y = 0 + gt$

∴ $v_y = gt$ .... (ii)

Both v_x and v_y comp. of velocity are ⊥ar to each other.

∴ Resultant velo. of the particle at time 't'

$$v = \sqrt{v_x^2 + v_y^2}$$

Direction: $\tan\phi = \frac{v_y}{v_x} = \frac{gt}{u}$

∴ $\phi = \tan^{-1}\frac{gt}{u}$

Examples based on *Projectile Fired Horizontally*

◆ **Formulae Used** 1. Position of the principle after time t: ---------------- $x = ut,\ y = \frac{1}{2} gt^2$

2. Equation of trajectory: $y = \frac{g}{2u^2} \cdot x^2$

3. Velocity after time t: $v = \sqrt{u^2 + g^2 t^2}$

$\beta = \tan^{-1} \frac{gt}{u}$

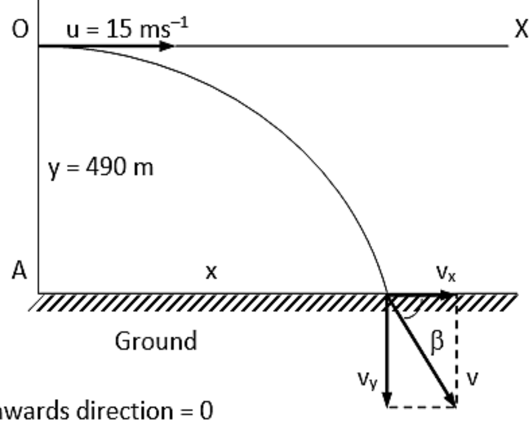
4. Time of flight: $T = \sqrt{\frac{2h}{g}}$

5. Horizontal range: $R = u \times T = u\sqrt{\frac{2h}{g}}$

◆ **Units Used** Distances x, y, h and R in metres, velocities, u and v in ms^{-1}, acceleration due to gravity g in ms^{-2}, and times t and T in second.

Q. 1. A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 ms^{-1}. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take g = 9.8 ms^{-2}).

Sol. Suppose the stone is thrown from the edge of a cliff with speed u = 15 ms^{-1} along the horizontal OX. It hits the ground at point P after time t.



Initial velocity in the downwards direction = 0

Vertical distance, OA = y = 490 m

As $y = \frac{1}{2} gt^2$

∴ $490 = \frac{1}{2} \times 9.8\ t^2$ or $t = \sqrt{100} = 10$ s

The horizontal and vertical components of speed v of the stone at point P are

$v_x = u = 15\ ms^{-1}$

$v_y = u_y + gt = 0 + 9.8 \times 10 = 98\ ms^{-1}$ ∴ $v = \sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 98^2} = 99.1\ ms^{-1}$

Q. 2. A projectile is fired horizontally with a velocity of 98 ms^{-1} from the top of a hill 490 m high. Find (i) the time taken to reach the ground (ii) the distance of the target from the hill and (iii) and velocity with which the projectile hits the ground.

Sol. (i) As shown in Fig., the projectile is fired from the top O of a hill with velocity. u = 98 ms^{-1} along the horizontal OX. It reaches the target P in time t.

Initial velocity in the downward direction = 0 ∴ Vertical distance, OA = y = 490 m

As $y = \frac{1}{2} gt^2$ ∴ $490 = \frac{1}{2} \times 9.8\ t^2$

or $t = \sqrt{100} = 10$ s

(ii) Distance of the target from the hill,

AP = x = Horizontal velocity × time = 98 × 10 = 980 m

(iii) The horizontal and vertical components of velocity v of the projectile at point P are

$v_x = u = 98\ ms^{-1}$

$v_y = u_x + gt = 0 + 9.8 \times 10 = 98\ ms^{-1}$

∴ $v = \sqrt{v_x^2 + v_y^2} = \sqrt{98^2 + 98^2} = 98\sqrt{2} = 138.59\ ms^{-1}$

If the resultant velocity v makes angle β with the horizontal, then

$\tan\beta = \frac{v_y}{v_u} = \frac{98}{98} = 1$ ∴ $\beta = 45°$

Q. 3. A body is thrown horizontally from the top of a tower and strikes the ground after three seconds at an angle of 45° with the horizontal. Find the height of the tower and the speed with which the body was projected. Take g = 9.8 ms^{-2}.

Sol. As shown in Fig., suppose the body is thrown horizontally from the top O of the tower of height y with velocity u. The body hits and ground after 3 s. Considering vertically downward motion of the body,

$y = u_y t + \frac{1}{2} gt^2 = 0 \times 3 + \frac{1}{2} \times 9.8 \times (3)^2 = 44.1\ m$ [∵ Initial vertical velocity, $u_y = 0$]

Final vertical velocity, $u_y = u_y + gt = 0 + 9.8 \times 3 = 29.4\ ms^{-1}$

Final horizontal velocity, $v_x = u$

As the resultant velocity v makes an angle of 45° with the horizontal, so

$\tan 45° = \frac{v_y}{v_x}$ or $1 = \frac{29.4}{u}$ or $u = 29.4\ ms^{-1}$

Q. 4. A bomb is dropped from an aeroplane when it is directly above a target at a height of 1000 m. The aeroplane is moving horizontally with a speed of 500 kmh^{-1}. By how much distance will the bomb miss the target?

Sol. As the aeroplane is moving horizontally, the initial downward velocity of the bomb, $u_y = 0$

Also, $y = 1000$ m, $g = 9.8\ ms^{-2}$, t =?

Now $y = u_y t + \frac{1}{2} gt^2$ ∴ $1000 = 0 + \frac{1}{2} \times 9.8\ t^2$

or $t = \sqrt{\frac{1000}{4.9}} = \frac{100}{7}$ s

Horizontal velocity of the aeroplane $= 500\ kmh^{-1} = 500 \times \frac{5}{18}\ ms^{-1} = \frac{1250}{9}\ ms^{-1}$

Distance by which the bomb misses the target = Horizontal distance covered by which it hits the ground

= Horizontal velocity × time = $1250 \times 100 = 1984.13$ m

Q. 5. A body is projected horizontally from the top of a cliff with a velocity of 9.8 ms^{-1}. What time elapses before horizontal and vertical velocities become equal? Take g = 9.8 ms^{-2}.

Sol. Horizontal velocity at any instant, $v_x = u = 9.8\ ms^{-2}$.

Vertical velocity at any instant, $v_y = 0 + gt = 9.8\ t$ ∴ $9.8 = 9.8\ t$ or $t = 1$ s

Q. 6. A marksman wishes to hit a target just in the same level as the line of sight. How high from the 1600 m and the muzzle velocity of the gun is 800 ms^{-1}? Take g = 9.8 ms^{-2}.

Sol. Let u be the speed of the bullet. In time t, it covers a horizontal distance,

$x = 1600$ m

But $x = ut$ ∴ $1600 = 800 \times t$ or $t = 2$ s

Distance through which the bullet is pulled down by the force gravity in 2s is $y = \frac{1}{2} gt^2 = \frac{1}{2} \times 9.8 \times (2)^2 = 19.6$ m

∴ Height of the gun from the target = 19.6 m

Q. 7. Two tall buildings face each other and are at a distance of 180 m from each other. With what velocity must a ball be thrown horizontally from a window 55 m above the ground in one building, so that it enters a window 10.9 m above the ground in the second building?

Sol. In Fig. A and B are two tall buildings which are 180 m apart. W_1 and W_2 are the two windows in A and B respectively.

Vertical downward distance to be covered by the ball

= Height of W_1 – Height of W_2

= 55 – 10.9 = 44.1 m

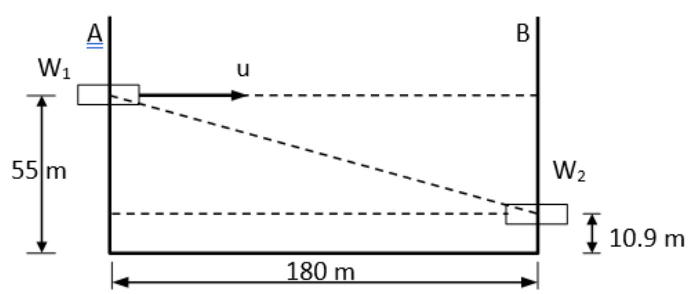
Initial vertical velocity of ball, u_y = C

As $y = u_y t + \frac{1}{2} gt^2$

∴ $44.1 = 0 + \frac{1}{2} \times 9.8\ t^2$ or $t^2 = \frac{44.1 \times 2}{9.8} = 9$

or $t = 3$s

Required horizontal velocity

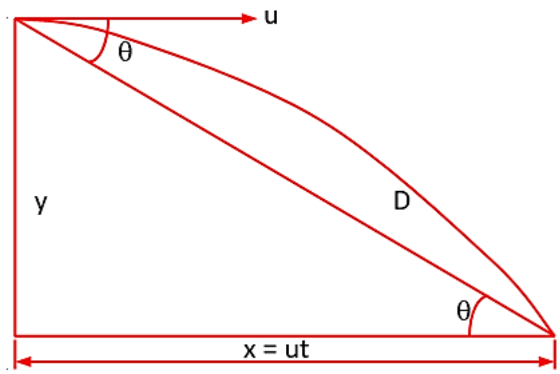


$= \frac{\text{Horizontal distance}}{\text{Time}} = \frac{180 \text{ m}}{3 \text{ s}} = 60 \text{ ms}^{-1}$

Q. 8. A particle is projected horizontally with a speed u from the top of plane inclined at an angle θ with the horizontal. How far from the point of projection will the particle strike the plane?

Sol. The horizontal distance covered in time t,

$x = ut$ or $t = \frac{x}{u}$ ... (1)



The vertical distance covered in time t,

$y = 0 + ½ gt^2 = ½ g \times x^2$ [using (1)]

Also, $\frac{y}{x} = \tan\theta$ or $y = x \tan\theta$

∴ $\frac{g x^2}{2 u^2} = x \tan\theta$

or $x\left(\frac{g x}{2 u^2}\right) = \tan\theta = 0$

As x = 0 is not possible, so $x = \frac{2u^2 \tan\theta}{g}$

The distance of the point of strike from the point of projection is

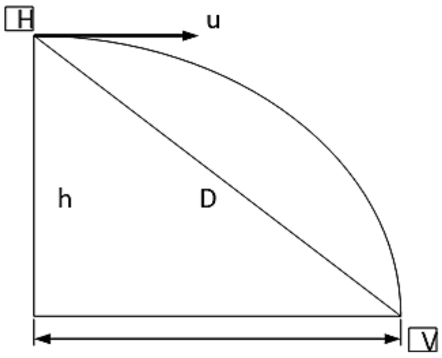
$D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x \tan\theta)^2} = x\sqrt{1 + \tan^2\theta} = x \sec\theta$

or $D = \frac{2u^2}{g} \tan\theta \sec\theta$

Q. 9. A helicopter on a flood relief mission flying horizontally with a speed u at an altitude h, has to drop a food packet for a victim standing on the ground. At what distance from the victim should the food packet be dropped?

Sol. In Fig., H represents position of the helicopter and V that of the victim. For vertical motion of the packet

$h = 0 + ½ gt^2$ or $t = \sqrt{\frac{2h}{g}}$



Horizontal distance covered by the food packet in time t, $x = ut = u\sqrt{\frac{2h}{g}}$

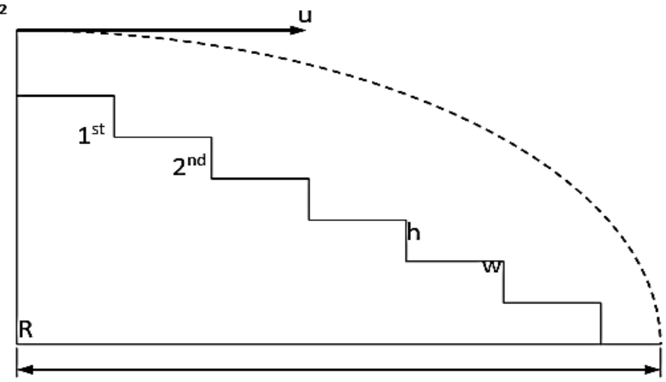
The distance of the point of projection from the food packet is $D = \sqrt{h^2 + x^2} = \sqrt{h^2 + \frac{2u^2 h}{g}}$

Q. 10. A ball rolls off the top of a stairway with a constant horizontal velocity u. If the steps are h metre high and w metre wide, show that the ball will just hit the edge of nth if $n = \frac{2hu^2}{gw^2}$

Sol. Refer to Fig., For n th step,

net vertical displacement = nh

net horizontal displacement = nw



Let t be the time taken by the ball to reach the nth step. Then

$R = ut$

or $n\omega = ut$ or $t = \frac{n\omega}{u}$

Also, $y = u_y t + \frac{1}{2} gt^2$

or $nh = 0 + \frac{1}{2} gt^2 = \frac{1}{2} g \left(\frac{n\omega}{u}\right)^2$ or $n = \frac{2hu^2}{g\omega^2}$

Problems For Practice

Q. 1. A plane is flying horizontally at a height of 1000 m with a velocity of 100 ms^{-1} when a bomb is released from it. Find (i) the time taken by it to reach the ground (ii) the velocity with which the bomb hits the target and (iii) the distance of the target.

Sol. (i) For vertical motion:

$y = \frac{1}{2} gt^2$ ∴ $1000 = \frac{1}{2} \times 9.8\, t^2$

or $t^2 = \frac{10000}{49}$ ∴ $t = \frac{100}{7}$ s = 14.28 s

(ii) Velocity with which the bomb hits the target,

$v = \sqrt{u^2 + g^2 t^2} = \sqrt{(100)^2 + \left(9.8 \times \frac{100}{7}\right)^2}$

$= \sqrt{100^2 + 140^2} = 175.05\ ms^{-1}$

$\tan \beta = \frac{gt}{u} = \frac{9.8 \times 100}{100 \times 7} = 1.4$ ∴ $\beta = 54° 28'$

(iii) Distance of the target, $x = ut = 1000 \times \frac{100}{7} = 1428.51$ m

Q. 2. A mailbag is to be dropped into a post office from an aeroplane flying horizontally with a velocity of 270 kmh^{-1} at a height of 176.4 m above the ground. How far must the aeroplane be from the post office at the time of dropping the bag so that it directly falls into the post office?

Sol. For vertical motion: $y = \frac{1}{2} gt^2$ ∴ $176.4 = \frac{1}{2} \times 9.8\, t^2$

or $t^2 = \frac{176.4}{4.9} = 36$ ∴ $t = 6s$

Also, $u = 270\ kmh^{-1} = \frac{270 \times 5}{18} = 75\ ms^{-1}$ ∴ $x = ut = 75 \times 6 = 450$ m

An aeroplane is flying at a constant horizontal velocity of 600 *km/hr* at an elevation of 6 *km* towards a point directly above the target on the earth's surface. At an appropriate time, the pilot releases a ball so that it strikes the target at the earth. The ball will appear to be falling **[MP PET 1993]**

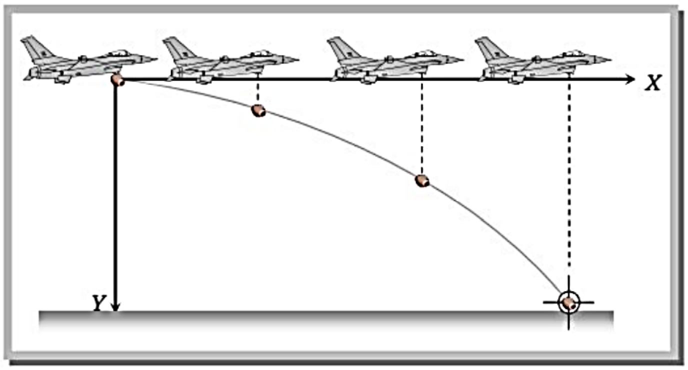
(a) On a parabolic path as seen by pilot in the plane

(b) Vertically along a straight path as seen by an observer on the ground near the target

(c) On a parabolic path as seen by an observer on the ground near the target

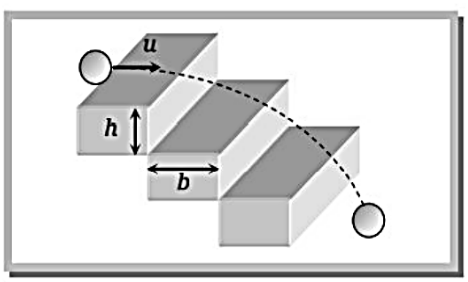
(d) On a zig-zag path as seen by pilot in the plane

(c)



The path of the ball appears parabolic to a observer near the target because it is at rest. But to a Pilot the path appears straight line because the horizontal velocity of aeroplane and the ball are equal, so the relative horizontal displacement is zero.

71. A staircase contains three steps each 10 *cm* high and 20 *cm* wide. What should be the minimum horizontal velocity of a ball rolling off the uppermost plane so as to hit directly the lowest plane



(a) 0.5 *m/s* (b) 1 *m/s* (c) 2 *m/s* (d) 4 *m/s*

(c) Formula for this condition is given by $n = \frac{2hu^2}{gb^2}$ where h = height of each step, b = width of step, u = horizontal velocity of projection, n = number of step.

$\Rightarrow 3 = \frac{2 \times 10 \times u^2}{10 \times 20^2} \Rightarrow u^2 = 200\ cm/sec = 2\ m/sec$

Two bullets are fired simultaneously, horizontally and with different speeds from the same place. Which bullet will hit the ground first

(a) The faster one
(b) Depends on their mass
(c) The slower one
(d) Both will reach simultaneously

Solution : (d)

An aeroplane is flying at a height of 1960 *m* in horizontal direction with a velocity of 360 *km/hr*. When it is vertically above the point. *A* on the ground, it drops a bomb. The bomb strikes a point *B* on the ground, then the time taken by the bomb to reach the ground is

(a) $20\sqrt{2}$ *sec*
(b) 20 *sec*
(c) $10\sqrt{2}$ *sec*
(d) 10 *sec*

Solution : (b) $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1960}{9.8}}$ = 20 *sec*

An aeroplane travelling horizontally at 78 ms^{-1} at a height of 210 m drops a bomb to hit the target.

(*i*) At what horizontal distance from the target should the bomb be dropped ?

(*ii*) Calculate the velocity of the bomb as it reaches the ground.

Solution. Fig. 9.10 shows the conditions of the problem. We have placed the origin of the coordinate system at the ground directly under the position of the plane when it drops the bomb.

(*i*) The time *t* taken by the bomb to reach the ground will be determined from the vertical motion of the bomb.

$$y - y_0 = v_{y0}\, t + \frac{1}{2} a_y\, t^2$$

Here $y - y_0 = -\,210$ m ; $v_{y0} = 0$;

$$a_y = -\,g = -\,9.8 \text{ ms}^{-2}$$

$$\therefore \quad -\,210 = -\,\frac{1}{2} \times 9.8 \times t^2$$

or $$t = \sqrt{\frac{210 \times 2}{9.8}} = 6.5 \text{ s}$$

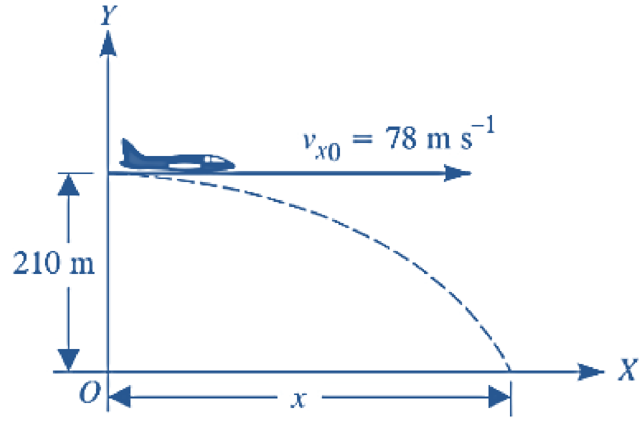


Fig. 9.10

Horizontal range, $R = x = {}^{*}v_x \times t = 78 \times 6.5 = 507$ m

(*ii*) The initial horizontal component (v_{x0}) of bomb's velocity remains 78 ms^{-1}. The initial vertical component (v_{y0}) of bomb's velocity changes with time. At any time *t*, its value is

$$v_y = v_{y0} - gt = 0 - 9.8 \times 6.5 = -\,64 \text{ ms}^{-1}$$

$\therefore$ Magnitude of bomb's velocity (v) when it reaches the ground is

$$v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(78)^2 + (-64)^2} = 100 \text{ ms}^{-1}$$

A rifle shoots a bullet with a muzzle velocity of 500 ms^{-1} at a small target 50 m away. How high above the target must the rifle be aimed at so that the bullet will hit the target? (Take g = 10 ms^{-2})

Solution. Time taken by the bullet to reach the target is $t = \frac{\text{distance}}{\text{velocity}} = \frac{50}{500} = 0.1$ s

Vertical fall of bullet due to gravity is $y = \frac{1}{2}gt^2 = \frac{1}{2} \times 10 \times (0.1)^2 = 0.05 \text{ m} = 5 \text{ cm}$

Therefore, the rifle must be aimed 5 cm above the target for the bullet to hit the target.

A bomb is dropped on an enemy post by an aeroplane flying with a horizontal velocity of 60 km/h and at a height of 490 m. How far the aeroplane must be from the enemy post at the time of dropping the bomb so that it may directly hit the target? What is the trajectory of the bomb as seen by an observer on earth and as seen by person sitting in the plane?

Solution. Horizontal speed of bomb, v_{x0} = 60 km/h

The time taken by the bomb to hit the target depends upon vertical motion alone. Let this time be t.

Now $$h = \frac{1}{2} g t^2$$

$\therefore$ $$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 490}{9.8}} = \sqrt{100} = 10 \text{s}$$

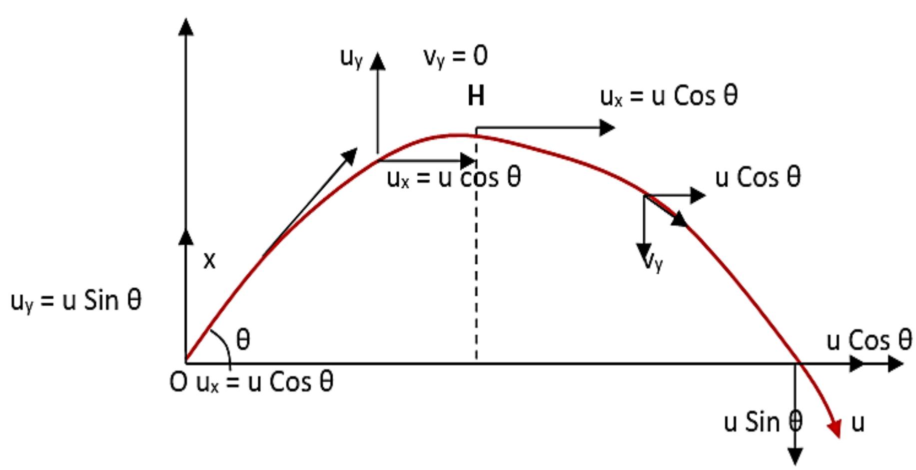
$\therefore$ Distance of aeroplane from enemy post at the time of dropping the bomb is

$$R = v_{x0} \times t = 60 \times \frac{10}{60 \times 60} \qquad \left(\because 10\text{s} = \frac{10}{60 \times 60}\text{h}\right)$$

$$= \frac{1}{6}\text{ km} = \frac{1}{6} \times 1000 \text{ m} = \frac{500}{3} \text{ m}$$

OBLIQUE PROJECTILE: [Projectile fired at an angle with the horizontal]:

The variation of the components of the velocity of projection of the projectile during the flight of the projectile.



Let a projectile is fired from point O with **initial velocity** $\vec{u}$ such that is makes an angle θ with the Horizontal.

<u>Resolve velocity u into two comp</u>. - (i) $u_x = u \cos\theta$ (ii) $u_y = u \sin\theta$

- **The horizontal comp. (u Cos θ) remains the same throughout the journey of the projectile (because no force acts in the horizontal direction).**

- **The vertical component (u sin θ) goes on decreasing as the projectile goes up from the point of projection and ultimately becomes zero at the highest point. (This is because the earth attracts the projectile towards its centre)**

After attaining the highest point H, the projectile starts its downward journey. Now vertical component starts increasing and becomes "u sin θ" when it hits the ground.

At the highest point: - ------(i) $u_x = u \cos\theta$ (hori- comp.)

------(ii) $u_y = 0$ (vert-comp)

------(iii) Linear momentum of mass m = m u cos θ.

------(iv) Kinetic Energy (KE) = $\frac{1}{2} mu^2 \cos^2\theta$.

------(v) 'g' acting vertically downwards makes an angle of 90^0 with u cos θ (Hori. Comp.)

> In projectile motion, horizontal component of velocity ($u \cos\theta$), acceleration (g) and mechanical energy remains constant while, speed, velocity, vertical component of velocity ($u \sin\theta$), momentum, kinetic energy and potential energy all changes. Velocity, and KE are maximum at the point of projection while minimum (but not zero) at highest point.

Let a projectile be thrown with initial velocity **u** at an angle **'θ'** (with the horizontal.)

--- Horizontal component = u Cos θ

--- Vertical component = u Sin θ

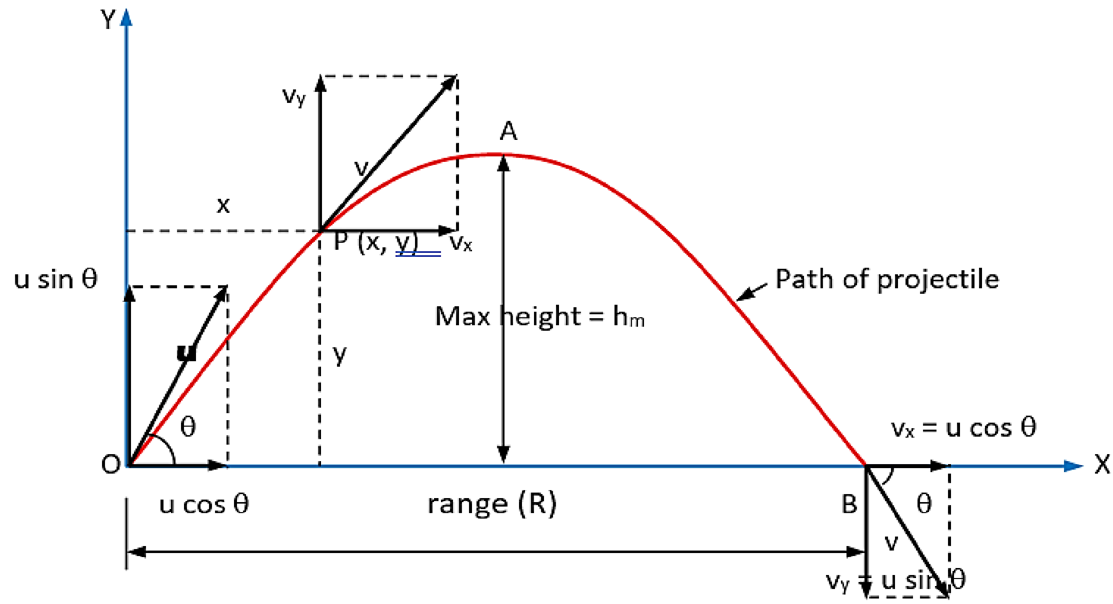
The vertical component of the velocity gradually reduces to zero at highest point (H) and then the projectile will move down ward to fall on the ground.

Let at an instant of time 't', the projectile be at a point P.

Let x & y = Horizontal distance & vertical distance travelled in time 't'

∴ $v_x = u_x + a_x t$ **(Horizontal motion) ....(i)**

$v_y = u_y + a_y t$ **(Vertical motion)........................(ii)**



MATHEMATICAL ANALYSIS OF OBLIQUE PROJECTILE MOTION

Suppose a projectile is projected from the origin O with an initial velocity $\vec{v}_0$ at an angle θ above the horizontal as shown in Fig. 9.4. At $t = 0$, the rectangular components of $\vec{v}_0$ are $v_{x0} = v_0 \cos\theta$ along the horizontal and $v_{y0} = v_0 \sin\theta$ along the vertical. Also at $t = 0$, $x_0 = 0$ and $y_0 = 0$ because the projectile is at the origin.

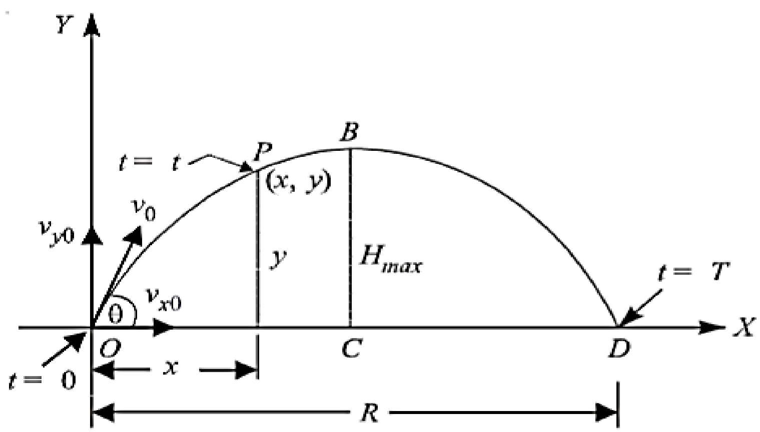


Fig. 9.4

Suppose at any time t, the projectile is at point $P(x, y)$. This means that at time t, the projectile has covered a horizontal distance x from the origin O and has attained a height y above the origin.

MOTION ALONG THE HORIZONTAL The horizontal component v_x $(= v_{x0} = v_0 \cos\theta)$ of the velocity is constant because there is no horizontal acceleration (*i.e.*, $a_x = 0$). At time t, the projectile is at point $P(x, y)$ and its position along X-axis is

$$x = x_0 + v_x t + \frac{1}{2} a_x t^2$$

or $$x = v_x t \qquad (\because\ x_0 = 0\ ,\ a_x = 0)$$

∴ $$x = (v_0 \cos\theta)\ t \qquad (\because\ v_x = v_{x0} = v_0 \cos\theta) \quad ...(i)$$

The horizontal velocity of the projectile at point P is v_x $(= v_0 \cos\theta)$; it remains the same throughout.

MOTION ALONG THE VERTICAL The initial vertical component of velocity (*i.e.*, at $t = 0$) is $v_0 \sin\theta$ and it changes at constant rate with time due to constant gravitational acceleration (g). Note that constant acceleration in the motion is always $-g$* whether the projectile goes up or comes down (*i.e.*, $a_y = -g$). At time t, the projectile is at point P (x, y) and its position along Y- axis is given by ;

$$y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2$$

* Here we choose y to be positive in the upward direction. Since g always acts downward whether motion of the projectile is upward or downward, $a_y = -g$.

or $$y = (v_0 \sin\theta) t - \frac{1}{2} g t^2 \qquad [\because v_{y0} = v_0 \sin\theta;\ y_0 = 0 \text{ and } a_y = -g]$$

∴ $$y = (v_0 \sin\theta) t - \frac{1}{2} g t^2 \quad ...(ii)$$

At time t, the projectile is at $P(x, y)$ and the vertical component of its velocity at this time is given by ;

$$v_y = v_{y0} - g t$$

or $$v_y = v_0 \sin\theta - g\ t \quad ...(iii)$$

Trajectory of projectile. The path or trajectory of the projectile (air friction neglected) is a parabola. To show this, we rewrite eq. (*ii*).

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

From eq. (*i*), $t = x/v_0\cos\theta$. Putting this value of t in the above equation, we have,

$$y = (v_0\sin\theta)\frac{x}{v_0\cos\theta} - \frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2$$

or $$y = x\tan\theta - \left(\frac{g}{2v_0^2\cos^2\theta}\right)x^2$$

From this equation, we see that y as a function of x has the form :

$$y = ax - bx^2 \qquad \text{... Equation of parabola}$$

where $a\ (=\tan\theta)$ and $b\left(=\frac{g}{2v_0^2\cos^2\theta}\right)$ are constants for any given projectile. *Therefore, the*

Therefore, the TRAJECTORY (path) of the PROJECTILE is PARABOLIC

PARAMETERS OF PROJECTILE MOTION

1. Time of flight (T)
2. Maximum height attained (H_{max})
3. Horizontal range (R)

The first two parameters (*i.e.*, T and H_{max}) are determined from the vertical motion of the projectile while the third (*i.e.*, R) is calculated from the horizontal motion of the projectile.

1. Time of flight (T). *It is the time elapsing from the launching to the time the projectile returns to the ground again.* Obviously, it is the time for which the projectile remains in air before coming to original height (ground). The position y of the projectile along Y-axis at any time t is

$$y = (v_0 \sin\theta)\, t - \frac{1}{2}g t^2$$

When projectile returns to ground, its vertical displacement is zero (*i.e.*, $y = 0$).

$$\therefore \quad 0 = (v_0\sin\theta)\, t - \frac{1}{2}g t^2$$

or $$0 = t\left(v_0\sin\theta - \frac{1}{2}gt\right)$$

$$\therefore \quad t = 0\ ; \quad t = \frac{2v_0\sin\theta}{g}$$

"Time of the flight is the total time for which the object is in flight (i.e from the point of projection to till it hit the horizontal plane."

- **Time of Ascent** - "Time taken by the object to go from the ORIGIN to the highest point (MAX VERTICAL HEIGHT).
- **Time of Descent**: - "Time taken by the object to go from highest point to the ground.

For vertical upward motion $0 = u\sin\theta - gt \Rightarrow t = (u\sin\theta/g)$

Now as time taken to go up is equal to the time taken to come down so

Time of flight $T = 2t = \frac{2u\sin\theta}{g}$

Horizontal Range: *It is the horizontal distance travelled by the projectile before returning to the ground (i.e., original height).* In other words, it is the horizontal distance travelled by the projectile during time T (= time of flight). The position of the projectile along X-axis at any time t is given by ;

$$x = (v_0 \cos\theta)\, t$$

$$\text{When} \quad t = T = \frac{2v_0 \sin\theta}{g} \;;\; x = R$$

$$\therefore \quad R = (v_0 \cos\theta)\left(\frac{2v_0 \sin\theta}{g}\right) = v_0^2\left(\frac{2\sin\theta\cos\theta}{g}\right)$$

$$\text{or} \quad R = \frac{v_0^2 \sin 2\theta}{g} \qquad (\because \sin 2\theta = 2\sin\theta\cos\theta)$$

"The maximum horizontal distance between the point of projection & the point on the horizontal plane where the projectile hits, is called Horizontal Range".

For Horizontal motion: - $x = u_x t + \frac{1}{2} a_x t^2$

$R = u \cos\theta\, T + \frac{1}{2}\; 0 \times T^2$ [Here, x = R and T = T)

=> $R = (u \cos\theta)\, T$

=> $R = u \cos\theta .\, 2u \sin\theta / g$

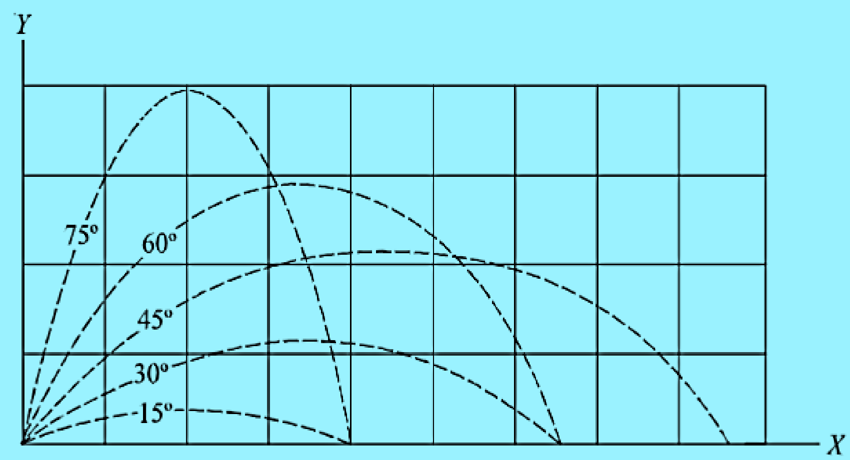
$R = u^2\, 2 \sin\theta \cos\theta / g$

$$R = \frac{u^2 \sin 2\theta}{g}$$ [$\because$ $2 \sin\theta \cos\theta = \sin 2\theta$]

Condition for maximum horizontal range. For a given initial speed v_0, the horizontal range will be maximum when $\sin 2\theta = 1$ or $\theta = 45°$.

$$\therefore \quad \text{Maximum range, } R_{max} = \frac{v_0^2}{g} \qquad \text{...when } \theta = 45°$$

Hence for a given initial speed, the horizontal range of the projectile will be maximum when the angle of projection is 45° (neglecting air friction).



Ranges of a projectile shot at the same speed at different projection angles

Maximum HEIGHT: (H_{max}). *It is the maximum height to which the projectile rises above the launching point (origin).* The position of the projectile along Y-axis at any time t is given by;

$$y = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2$$

When $t = \frac{T}{2} = \frac{v_0 \sin\theta}{g}$, then, $y = H_{max}$.

$$\therefore \quad H_{max} = (v_0 \sin\theta)\frac{(v_0 \sin\theta)}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\theta}{g}\right)^2$$

or

$$H_{max} = \frac{v_0^2 \sin^2\theta}{2g}$$

"The maximum vertical distance travelled by the projectile during its journey is called the maximum height attained by the projectile".

For vertical upward motion: - $v_y^2 - u_y^2 = 2a_y \times y$

$0^2 - x^2 \sin^2\theta = 2(-g) \times H$ (∴ at the highest point $v_y = 0$, $y = H$)

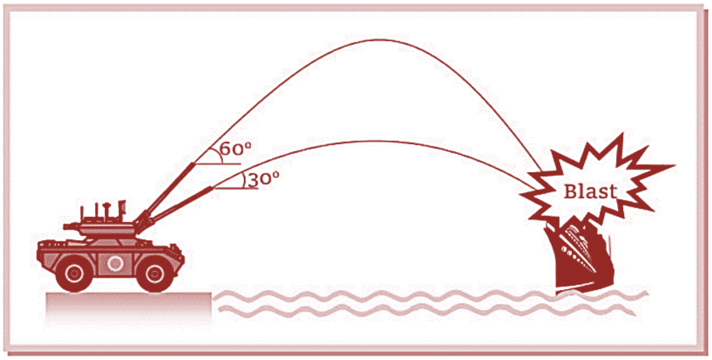
=> $\cancel{-}u^2 \sin^2\theta = \cancel{-}2gH$

=> $H = \frac{u^2 \sin^2\theta}{2g}$

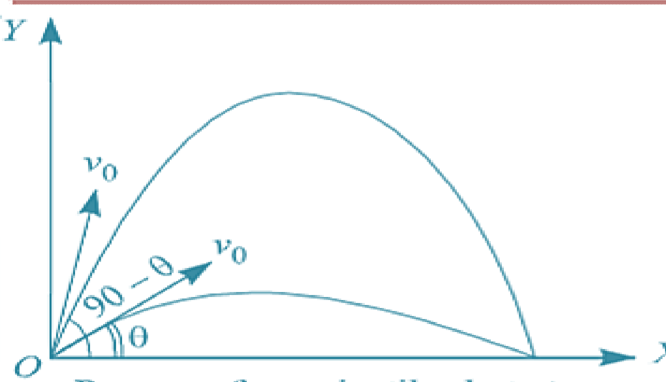
TWO ANGLES OF PROJECTION FOR SAME 'R'

If angle of projection is changed from θ to $\theta' = (90 - \theta)$ then range remains unchanged.

$$R' = \frac{u^2 \sin 2\theta'}{g} = \frac{u^2 \sin[2(90^o - \theta)]}{g} = \frac{u^2 \sin 2\theta}{g} = R$$



So a projectile has same range at angles of projection θ and $(90 - \theta)$, though time of flight, maximum height and trajectories are different.



Ranges of a projectile shot at the same speed at different projection angles.

FOR , COMPLEMENTARY ANGLE OF PROJECTIONS

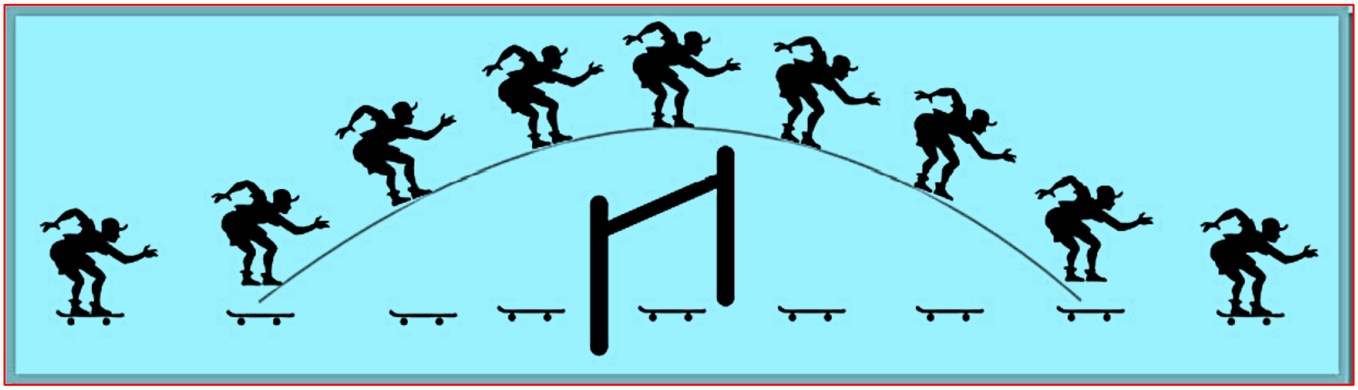
$$\frac{R_1}{R_2} = \frac{u^2 \sin 2\theta / g}{u^2 \sin[2(90^o - \theta)] / g} = 1 \Rightarrow \frac{R_1}{R_2} = 1$$

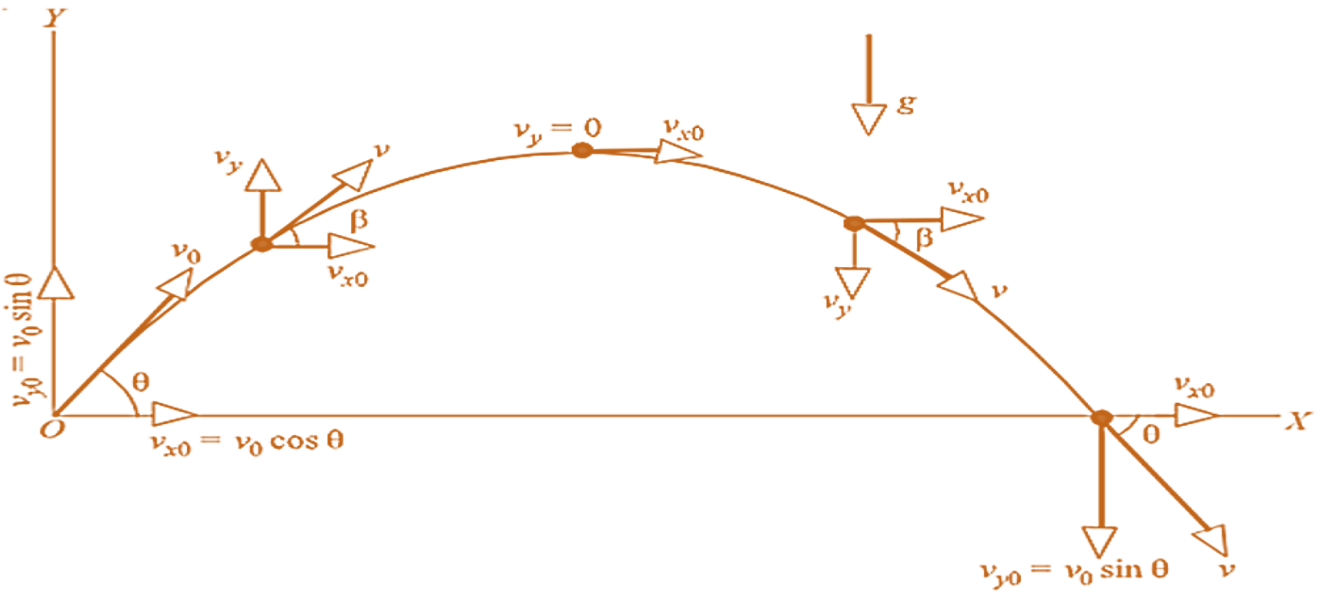
VELOCITY OF A PROJECTILE AT ANY INSTANT (INSTANTANEOUS VELOCITY)

(3) **Instantaneous velocity v :** In projectile motion, vertical component of velocity changes but horizontal component of velocity remains always constant.

Example : When a man jumps over the hurdle leaving behind its skateboard then vertical component of his velocity is changing, but not the horizontal component, which matches with the skateboard velocity.

As a result, the skateboard stays underneath him, allowing him to land on it.





During the flight of the projectile, it is under the action of two velocities

(*i*) a constant horizontal velocity v_{x0}.

(*ii*) uniformly changing velocity along the vertical or y-axis.

Therefore, the velocity v of the projectile changes with time. At $t = 0$, the vertical component of velocity is $v_0 \sin\theta$ and goes on decreasing uniformly till it becomes zero at the highest point. From now onward, the vertical component of velocity goes on increasing till it becomes $v_0 \sin\theta$ when projectile hits the ground [See Fig. 9.7].

At any time t, the vertical component of velocity is

$$v_y = v_0 \sin\theta - gt$$

Horizontal component of velocity, $v_x = v_{x0} = v_0 \cos\theta$

Since the velocities v_y and v_{x0} are always at right angles to each other, the projectile velocity is given by ;

$$v = \sqrt{(v_{x0})^2 + (v_y)^2} = \sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}$$

$$= \sqrt{v_0^2\cos^2\theta + v_0^2\sin^2\theta + g^2t^2 - 2v_0 gt\sin\theta}$$

$$= \sqrt{v_0^2(\cos^2\theta + \sin^2\theta) + g^2t^2 - 2v_0 gt\sin\theta}$$

$$\therefore \quad v = \sqrt{v_0^2 + g^2t^2 - 2v_0 gt\sin\theta}$$

Since v is tangent to the path at any instant, the angle β which it makes with the horizontal at the considered instant is

$$\tan\beta = \frac{v_y}{v_{x0}} = \frac{v_0\sin\theta - gt}{v_0\cos\theta} = \frac{v_0\sin\theta}{v_0\cos\theta} - \frac{gt}{v_0\cos\theta}$$

$$\tan\beta = \tan\theta - \frac{gt}{v_0\cos\theta}$$

angle β goes on changing with time t.

Let v_i be the instantaneous velocity of projectile at time t direction of this velocity is along the tangent to the trajectory at point P.

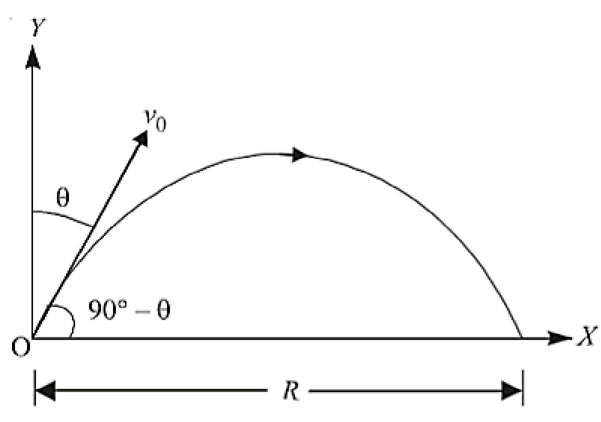
$$\vec{v}_i = v_x i + v_y \hat{j} \Rightarrow v_i = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$$

$$v_i = \sqrt{u^2 + g^2t^2 - 2u\,gt\sin\theta}$$

Direction of instantaneous velocity $\tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$ or $\alpha = \tan^{-1}\left[\tan\theta - \frac{gt}{u}\sec\theta\right]$

SPECIAL CASE: PROJECTILE FIRED AT AN ANGLE WITH THE VERTICAL

vertical. Fig. 9.8 shows a projectile fired with initial velocity v_0 at an angle θ with the vertical. Therefore, firing angle with the horizontal is $90° - \theta$. In order to find the various parameters of



(i) **Trajectory of projectile.** Changing θ to $90° - \theta$ in Art. 9.5, we have,

$$y = x\tan(90° - \theta) - \left(\frac{g}{2v_0^2\cos^2(90° - \theta)}\right)x^2$$

$$= x\cot\theta - \left(\frac{g}{2v_0^2\sin^2\theta}\right)x^2$$

$$\therefore \quad y = x\cot\theta - \frac{gx^2}{2v_0^2\sin^2\theta}$$

It is clear that the *trajectory of the projectile is parabolic.*

(*ii*) **Time of flight.** Changing θ to 90° – θ, time of flight is given by;

$$\text{Time of flight, } T = \frac{2v_0 \sin(90° - \theta)}{g} = \frac{2v_0 \cos\theta}{g}$$

(*iii*) **Maximum height attained.** Changing θ to 90° – θ, maximum height attained is given by ;

$$\text{Max. height attained, } H_{max} = \frac{v_0^2 \sin^2(90° - \theta)}{2g} = \frac{v_0^2 \cos^2\theta}{2g}$$

(*iv*) **Horizontal range.** Changing θ to 90° – θ, horizontal range is given by;

$$\text{Horizontal range, } R = \frac{v_0^2 \sin 2(90° - \theta)}{g} = \frac{v_0^2 \sin(180° - 2\theta)}{g} = \frac{v_0^2 \sin 2\theta}{g}$$

(*v*) **Velocity of projectile at any time *t*.** Changing θ to 90° – θ, the velocity (*v*) of the projectile is given by ;

$$v = \sqrt{v_0^2 + g^2t^2 - 2v_0 gt \sin(90° - \theta)}$$

NON – UNIFORM MOTION IN A PLANE

An object is said to have *non-uniform motion* in a plane if its velocity (and hence acceleration) changes from instant to instant. In such a case, we have to deal with *instantaneous velocity* and *instantaneous acceleration*. Obviously, the path of the object will not be a straight line (as for uniform motion) rather the object will follow a curved path.

Consider an object moving in the *XY* plane with a non-uniform velocity. Clearly, the object will follow a curved path as shown in Fig. 9.16. Suppose at time *t*, the object is at point *A*(*x*, *y*) and its position vector is $\vec{r}$. Let at time ($t + \Delta t$) its position be $B(x + \Delta x, y + \Delta y)$ and the position vector $\vec{r} + \vec{\Delta r}$. The displacement of the object in the small time interval Δ*t* is $\vec{\Delta r} = (\Delta x\,\hat{i} + \Delta y\,\hat{j})$.

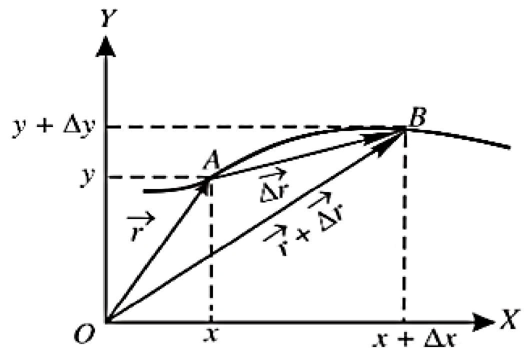


Fig. 9.16

1. **Instantaneous velocity vector.** The average velocity of the object between points *A* and *B* is

$$\vec{v}_{av} = \frac{\vec{\Delta r}}{\Delta t}$$

The instantaneous velocity $\vec{v}$ of the object at point *A* is the limit of the average velocity as the time interval Δ*t* approaches zero *i.e.*,

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\vec{\Delta r}}{\Delta t} = \frac{\vec{dr}}{dt}$$

$$\therefore \quad \vec{v} = \frac{\vec{dr}}{dt}$$

$$\vec{v} = \frac{\vec{dr}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j}) = \frac{*dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

or $$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

The coefficients are the scalar components of $\vec{v}$.

$$v_x = \frac{dx}{dt}; \quad v_y = \frac{dy}{dt}$$

The following points may be noted :

(*i*) The magnitude of the velocity is $v = \sqrt{v_x^2 + v_y^2}$.

(*ii*) The direction in which the object is heading at any time may be described in terms of angle θ between velocity vector and *X*-axis as :

$$\tan\theta = \frac{v_y}{v_x}$$

$\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = 0$ since these are unit vectors and are constant in magnitude and direction.

2. Instantaneous acceleration vector. If $\vec{v}$ and $\vec{v} + \overrightarrow{\Delta v}$ are the velocities of the object at times t and $t + \Delta t$, then its average acceleration between points *A* and *B* is

$$\vec{v}_{av} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

The instantaneous acceleration $\vec{a}$ is given by ;

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$\therefore \quad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v_x\hat{i} + v_y\hat{j}) = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

$$\therefore \quad \vec{a} = a_x\hat{i} + a_y\hat{j}$$

The two scalar components of $\vec{a}$ are :

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}; \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

.... END

Knowledge +

- A body is said to be projectile if it is projected into space with some initial velocity and then it continues to move in a vertical plane such that its horizontal acceleration is zero and vertical downward acceleration is equal to g.

- In projectile motion, the horizontal motion and the vertical motion are independent of each other i.e., neither motion affects the other.

- The horizontal range is maximum for $\theta = 45°$ and $R_m = u^2/g$.

- The horizontal range is same when the angle of projection is θ and $(90° - \theta)$.

- Again, the horizontal range is same for the angles of projection of $(45° - \theta)$.

- At the highest point of the parabolic path, the velocity and acceleration of a projectile are perpendicular each other.

- The velocity at the end of flight of an oblique projectile is the same in magnitude as at the beginning but the angle that it makes with the horizontal is negative of the angle of projection.

- In projectile motion, the kinetic energy is maximum at the point of projection or point of reaching the ground and is minimum at the highest point.

- There are two values of time for which the projectile is at the same height. The sum of these two times is equal to the time of flight.

- The maximum horizontal range is four times the maximum height attained by the projectile, when fired at $\theta = 45°$. Thus **$h_m = R_m/4 = u^2/4g$.**

- If a body is projected from a place above the surface of the earth, then for the maximum range the angle of projection should be slightly less than 45°. For javelin throw and discus throw, the athlete throws the projectile at an angle slightly less than 45° to the horizontal for achieving maximum range.

- The trajectory of a projectile is parabolic only when the acceleration of the projectile is constant and the direction of acceleration is different from the direction of projectile's initial velocity. The acceleration of a projectile thrown from
  the earth is equal to acceleration due to gravity (g) which remains constant if
  (i) The projectile does not go to a very larger height.
  (ii) The range of the projectile is not very large.
  (iii) The initial velocity of the projectile is not large.
  Thus, the trajectory of a bullet fired from a gun will be parabolic, but not so the trajectory of a missile.

- The shape of the trajectory of the motion of an object is not determined by position along but also depends on its initial position and initial velocity. Under the same acceleration due to gravity, the trajectory of an object can be a straight line
  or a parabola depending on the initial conditions.

Examples based on *Projectile Fired Horizontally*

- **Formulae Used**

1. For a projectile fired with velocity u at an angle θ with the horizontal: $u_x = u\cos\theta$, $u_y = u\sin\theta$, $a_x = 0$, $a_y = -g$
2. Position after time t:
   $x\ (u\cos\theta)\,t,\ y = (u\sin\theta)\,t - \frac{1}{2}gt^2$
3. Equation of trajectory:
   $y = x\tan\theta - \dfrac{g}{2u^2\cos^2\theta}\,.\,x^2$
4. Maximum height: $H = \dfrac{u^2\sin^2\theta}{2g}$
5. Time of flight, $T = \dfrac{2u\sin\theta}{g}$
6. Horizontal range, $R = \dfrac{u^2\sin 2\theta}{g}$
7. Maximum horizontal range is attained at θ = 45° and its value is $R_{max} = \dfrac{u^2}{g}$
8. Velocity after time t, $v_x = u\cos\theta$, $v_y = u\sin\theta - gt$
   $\therefore \quad v = \sqrt{v^2{}_x + v^2 x} \quad$ and $\quad \tan\beta = \dfrac{v_y}{v_x}$

- **Units Used**

Distances x, y, R and R_{max} are in metres, velocities u, v_x, v_y and v are in ms^{-1}, accelerations a_x, a_y and g are in ms^{-2} and times t and T in second.

Q. 1. A cricket ball is thrown at a speed of 28 ms^{-1} in a direction 30° above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball returns to the same level.

Sol. Here u = 28 ms^{-1}, θ = 30°

(a) Maximum height,

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{28^2\sin^2 30^\circ}{2\times 9.8} = 10.0\ \text{m}$$

(b) The time taken by the ball to return to the same level,

$$T = \frac{2u\sin\theta}{g} = \frac{2\times 28\times\sin 30^\circ}{9.8} = 2.9\ \text{s}$$

(c) The distance from the thrower to the point where the ball returns to the same level,

$$R = \frac{u^2\sin 2\theta}{g} = \frac{28\times 28\times\sin 60^\circ}{9.8} = 69.3\ \text{m}$$

Q. 2. A body is projected with a velocity of 30 ms^{-1} at an angle of 30° with the vertical. Find the maximum height, time of flight and the horizontal range.

Sol. Here u = 30 ms^{-1},

Angle of projection, θ = 90 – 30 = 60°

Maximum height,

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{30^2\sin^2 60^\circ}{2\times 9.8} = 34.44\ \text{m}$$

Time of flight,

$$T = \frac{2u\sin\theta}{g} = \frac{2\times 30\sin 60^\circ}{9.8} = 5.3\ \text{s}$$

Horizontal range,

$$R = \frac{u^2\sin 2\theta}{g} = \frac{30^2\sin 120^\circ}{9.8} = \frac{30^2\sin 60^\circ}{9.8} = 79.53\ \text{m}$$

Q. 3. **A cricketer can throw a ball to a maximum horizontal distance of 100 m. How high above the ground can the cricketer throw the same ball?**

Sol. Let u be the velocity of projection. Then

$$R_{max} = \frac{u^2}{g} = 100 \text{ m}$$

or $u^2 = 100\,g$ or $u = \sqrt{100\,g}$

For upward throw of the ball, we have

$u = \sqrt{100\,g}$, $v = 0$, $a = -g$, $s = ?$

As $v^2 - u^2 = 2as$

$\therefore$ $0 - 100\,g = 2(-g)\,s$

or $s = \frac{-100\,g}{-2g} = 50$ m

Thus the cricketer can throw the same ball to a height of 50 m.

Q. 4. **The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 ms^{-1} can go without hitting the ceiling of the ball?**

Sol. Here H = 25 m, u = 40 ms^{-1}

If the ball is thrown at an angle θ with the horizontal, then maximum height of flight,

$$H = \frac{u^2 \sin^2\theta}{2g} \quad \therefore \quad 25 = \frac{(40)^2 \sin^2\theta}{2 \times 9.8}$$

or $\sin^2\theta = \frac{25 \times 2 \times 9.8}{(40)^2} = 0.306$

or $\sin\theta = \sqrt{0.306} = 0.554$

and $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - 0.306}$

$= \sqrt{0.694} = 0.833$

The maximum horizontal distance is given by

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin\theta \cos\theta}{g} = \frac{2 \times (40)^2 \times 0.554 \times 0.833}{9.8} = 150.7 \text{ m}$$

Q. 5. **A bullet fired at an angle of 30° with the horizontal hits the ground 3 km away. By adjusting the angle of projection, can one hope to hit a target 5 km away? Assume the muzzle speed to be fixed and neglect air resistance.**

Sol. In the first case, R = 3 km = 3000 m, $\theta = 30°$

Horizontal range, $R = \frac{u^2 \sin 2\theta}{g}$ $\therefore$ $3000 = \frac{u^2 \sin 60°}{g}$

or $\frac{u^2}{g} = \frac{3000}{\sin 60°} = \frac{3000 \times 2}{\sqrt{3}} = 2000\sqrt{3}$

Maximum horizontal range,

$$R_{max} = \frac{u^2}{g} = 2000\sqrt{3} \text{ m} = 3464 \text{ m} = 3.64 \text{ km}$$

But distance of the target (5 km) is greater than the maximum horizontal range of 3.46 km, so the target cannot be hit by adjusting the angle of projection.

Q. 6. **A projectile has a range of 50 m and reaches a maximum height of 10 m. Calculate the angle at which the projectile is fired.**

Sol. Here R = 50 m, H = 10 m, $\theta = ?$

Horizontal range, $R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin\theta\cos\theta}{g}$ ... (1)

Maximum height, $H = \frac{u^2 \sin^2\theta}{2g}$ ... (2)

Dividing (2) by (1), we get

$$\frac{H}{R} = \frac{u^2 \sin^2\theta}{2g} \times \frac{g}{2u^2 \sin\theta\cos\theta} = \frac{1}{4}\tan\theta$$

or $\tan\theta = \frac{4H}{R} = \frac{4 \times 10}{50} = 0.8$

or $\theta = \tan^{-1}(0.8) = 38.66°$

Q. 7. A boy stands at 39.2 m from a building and throws a ball which just passes through a window 19.6 m above the ground. Calculate the velocity of projection of the ball.

Sol. Here H = 19.6 m

$R = 39.2 + 39.2 = 78.4$ m

As proved in the above example,

$$\frac{H}{R} = \frac{1}{4} = \tan\theta$$

or $\tan\theta = \frac{4H}{R} = \frac{4 \times 19.6}{78.4} = 1$

∴ $\theta = 45°$

As $\frac{u^2 \sin 2\theta}{g} = R$ ∴ $\frac{u^2 \sin 90°}{9.8} = 78.4$

or $u = \sqrt{78.4 \times 9.8} = 19.6\sqrt{2} = 27.72\ ms^{-1}$

Q. 8. Find the angle of projection for which the horizontal range and the maximum height are equal.

Sol. Given horizontal range = maximum height

or $\frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin^2\theta}{2g}$

or $2\sin\theta\cos\theta = \frac{\sin^2\theta}{2}$

or $\frac{\sin\theta}{\cos\theta} = 4$ or $\tan\theta = 4$ ∴ $\theta = 75°\ 58'$

Q. 9. Prove that the maximum horizontal range is four times the maximum height attained by the projectile, when fired at an inclination so as to have maximum horizontal range.

Sol. For θ = 45°, the horizontal range is maximum and is given by

$$R_{max} = \frac{u^2}{g}$$

Maximum height attained,

$$H_{max} = \frac{u^2 \sin^2 45°}{2g} = \frac{u^2}{4g} = \frac{R_{max}}{4}$$

or $R_{max} = 4H_{max}$.

Q. 10. A ball is kicked at an angle of 30° with the vertical. If the horizontal component of its velocity is 19.6 ms^{-1}, find the maximum height and horizontal range.

Sol. Here $\theta = 90° - 30° = 60°$

Horizontal velocity = $u\cos 60° = 19.6\ ms^{-1}$

∴ $u = \frac{19.6}{\cos 60°} = \frac{19.6}{0.5} = 39.2\ ms^{-1}$

∴ Maximum height,

$$H = \frac{u^2 \sin^2 60°}{2g} = \frac{(39.2)^2}{2 \times 9.8} \times \left(\frac{\sqrt{3}}{2}\right)^2 = 58.8$$

Horizontal range, $R = \frac{u^2 \sin\theta}{g} = \frac{(39.2)^2 \times \sin 120°}{9.8}$

$$= \frac{(39.2)^2}{9.8} \times \left(\frac{\sqrt{3}}{2}\right) = 135.8 \text{ m}$$

When $\theta = 30°$,

$$H_2 = \frac{u^2 \sin^2 30°}{2g} = \frac{u^2}{2g}\left(\frac{1}{2}\right)^2 = \frac{u^2}{8g}$$

$\therefore \quad H_1 : H_2 = \frac{3u^2}{8g} \times \frac{8g}{u^2} = 3 : 1$

Thus, the gun will shoot three times as high when elevated at an angle of 60° as when fired at an angle of 30°.

Horizontal range of a projectile, $R = \frac{u^2 \sin 2\theta}{g}$

When $\theta = 60°$, $R_1 = \frac{u^2 \sin 120°}{g} = \frac{\sqrt{3}\, u^2}{2g}$

When $\theta = 30°$, $R_2 = \frac{u^2 \sin 60°}{g} = \frac{\sqrt{3}\, u^2}{2g}$

Thus $R_1 = R_2$, i.e., the horizontal distance covered will be same in both cases.

Q. 12. A ball is thrown at angle θ and another ball is thrown at an angle $(90° - \theta)$ with the horizontal direction from the same point with velocity 39.2 ms^{-1}. The second ball reaches 50 m higher than the first ball. Find their individual heights.
Take g = 9.8 ms^{-2}.

Sol. **For the first ball:** Angle of projection = θ, Velocity of projection, u = 39.2 ms^{-1}

As maximum height, $H = \frac{u^2 \sin^2 \theta}{2g}$

$\therefore \quad H = \frac{(39.2)^2 \sin^2 \theta}{2 \times 9.8}$ ... (1)

For the second ball: Angle of projection = $90° - \theta$, velocity of projection, u = 39.2 ms^{-1}, maximum height reached = (H + 50) m 13

$\therefore \quad H + 50 = \frac{(39.2)^2 \sin^2 (90° - \theta)}{2 \times 9.8}$

or $\quad H + 50 = \frac{(39.2)^2 \cos^2 \theta}{2 \times 9.8}$ ... (2)

Adding (1) and (2), we get

$$2H + 50 = \frac{(39.2)^2}{2 \times 9.8}(\sin^2 \theta + \cos^2 \theta) = \frac{(39.2)^2}{2 \times 9.8} = 78.4$$

or $\quad 2H = 78.4 - 50 = 28.4$ or $H = 14.2$ m

$\therefore \quad$ Height of first ball = H = 14.2 m $\qquad$ Height of second ball = H + 50 = 14.2 + 50 = 64.2 m

Q. 13. If R is the horizontal range for θ inclination and h is the maximum height reached by the projectile, show that the maximum range is given by $\frac{R^2}{8h} + 2h$.

Sol. Horizontal range, $R = \frac{u^2 \sin 2\theta}{2g}$

$$\therefore \quad \frac{R^2}{8h} + 2h = \frac{u^4 (\sin 2\theta)^2}{g^2} \times \frac{2g}{8u^2 \sin^2 \theta} + \frac{2u^2 \sin^2 \theta}{2g}$$

$$= \frac{u^2 (2 \sin \theta \cos \theta)^2}{4 g \sin^2 \theta} + \frac{u^2 \sin^2 \theta}{g}$$

$$= \frac{u^2 \cos^2 \theta}{g} + \frac{u^2 \sin^2 \theta}{g}$$

$$= \frac{u^2}{g}(\cos^2 \theta + \sin^2 \theta) = \frac{u^2}{g}$$

or $\quad \frac{R^2}{8h} + 2h = R_{max}$

Q. 14. Show that there are two angles of projection for which the horizontal range is the same. Also show that the sum of the maximum heights for these two angles is independent of the angle of projection.

Sol. When a projectile is fired with velocity u at an angle θ with the horizontal, its horizontal range is

$$R = \frac{u^2 \sin 2\theta}{g}$$

Replacing θ by (90° – θ), we get

$$R' = \frac{u^2 \sin 2(90° - \theta)}{g} = \frac{u^2 \sin(180° - 2\theta)}{g}$$

$$= \frac{u^2 \sin 2\theta}{g}$$

i.e., $R' = R$

Hence there are two angles of projection θ and (90° – θ) for which the horizontal range R is same.

Now $H = \dfrac{u^2 \sin^2 \theta}{2g}$ ... (1)

and $H' = \dfrac{u^2 \sin^2 (90° - \theta)}{2g} = \dfrac{u^2 \cos^2 \theta}{2g}$ ... (2)

Adding equations (1) and (2), we get

$$H + H' = \frac{u^2}{2g}(\sin^2 \theta + \cos^2 \theta)$$

or $H + H' = \dfrac{u^2}{2g}$

Clearly, the sum of the heights for the two angles of projection is independent of the angle of projection.

Q. 15 Show that there are two values of time for which a projectile is at the same height. Also show that the sum of these two heights of equal to the time of flight.

Sol. For vertically upward motion of a projectile,

$$y = u \sin \theta \,.\, t - \tfrac{1}{2} gt^2$$

or $\tfrac{1}{2} gt^2 - u \sin \theta \,.\, t + y = 0$

This is a quadratic equation in t. Its roots are

$t_1 = \dfrac{u \sin \theta - \sqrt{u^2 \sin^2 \theta - 2 g y}}{g}$ (Lower value)

and $t_2 = \dfrac{u \sin \theta + \sqrt{u^2 \sin^2 \theta - 2g y}}{g}$ (Higher value)

These are the two values of time for which the vertical height y is same, first while going up and second while going down.

Now, $t_1 + t_2 = \dfrac{u \sin \theta - \sqrt{u^2 \sin^2 \theta - 2 gy}}{g} + \dfrac{u \sin \theta + \sqrt{u^2 \sin^2 \theta - 2 g y}}{g}$

or $t_1 + t_2 = \dfrac{2u \sin \theta}{g} = T$, the time of flight.

Q. 16. Two projectiles are thrown with different velocities and at different angles so as to cover the same maximum height. Show that the sum of the times taken by each to reach the highest point is equal to the total time taken by either of the projectiles.

Sol. If the two projectiles are thrown with velocities u_1 and u_2 at angle θ_1 and θ_2 with horizontal, then their maximum heights

will be $H_1 = \dfrac{u_1^2 \sin^2 \theta_1}{2g}$ and $H_2 = \dfrac{u_2^2 \sin^2 \theta_2}{2g}$

But $H_1 = H_2$

∴ $\dfrac{u_1^2 \sin^2 \theta_1}{2g} = \dfrac{u_2^2 \sin^2 \theta_2}{2g}$

or $u_1 \sin \theta_1 = u_2 \sin \theta_2$ ... (1)

Times of flight for the two projectiles are

$$T_1 = \frac{2u_1 \sin \theta_1}{g} \quad \text{and} \quad T_2 = \frac{2u_2 \sin \theta_2}{g}$$

Making use of equation (1), we get

$$T_1 = T_2 = \frac{2u_1 \sin \theta_1}{g} = \frac{2u_2 \sin \theta_2}{g}$$

Times taken to reach the highest point in the two cases will be

$$t_1 = \frac{u_1 \sin \theta_1}{g} \quad \text{and} \quad t_2 = \frac{u_2 \sin \theta_2}{g}$$

$$\therefore \quad t_1 + t_2 = \frac{u_1 \sin \theta_1}{g} + \frac{u_2 \sin \theta_2}{g}$$

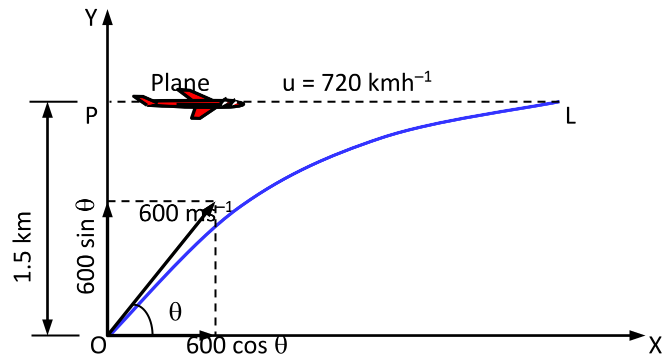
$$= \frac{2u_1 \sin \theta_1}{g} \quad \text{or} \quad \frac{2u_2 \sin \theta_2}{g} \qquad \text{[Using (1)]}$$

or $t_1 + t_2$ = Time of flight of either projectile.

Q. 17. A fighter plane flying horizontally at an altitude of 1.5 km with a speed 720 kmh^{-1} passes directly overhead an antiaircraft gun. At what angle from the vertical should the gun be fired for the shell muzzle speed 600 ms^{-1} to hit the plane? At what maximum altitude should the pilot fly the plane to avoid being hit? Take g = 10 ms^{-2}.

Sol. Speed of plane = 720 kmh^{-1} = 200 ms^{-1}

The shell moves along curve OL. The plane moves along PL.



Let them hit after a time t.

For hitting, horizontal distance travelled by the plane = Horizontal distance travelled by the shell.

Horizontal velocity of plane × t = Horizontal velocity of shell × t

$$200 \times t = 600 \cos \theta \times t$$

$$\cos \theta = \frac{200}{600} = \frac{1}{3} \quad \text{or} \quad \theta = 70° \, 30'$$

The shell should be fired at angle of 70° 30′ with the horizontal or 19°30′ with the vertical.

The maximum height of flight of the shell is

$$h = \frac{u^2 \sin^2 \theta}{2g} = \frac{u^2 (1 - \cos^2 \theta)}{2g}$$

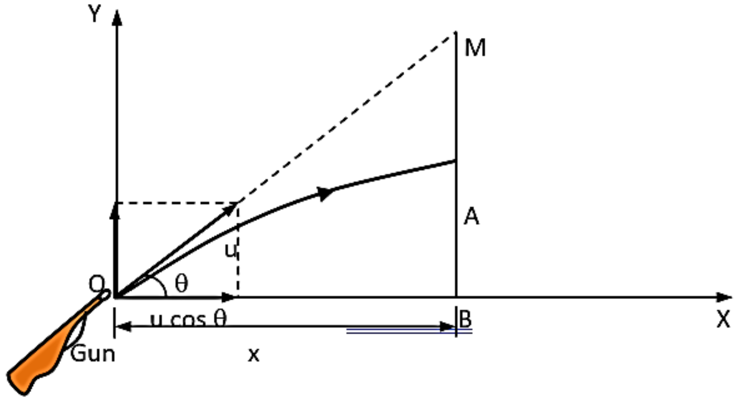
$$= \frac{(600)^2 \times \left(1 - \frac{1}{9}\right)}{2 \times 10} = 16000 \text{ m} = 16 \text{ km}$$

Thus, the pilot should fly the plane at a minimum altitude of 16 km to avoid being hit by the shell.

Q. 18. A hunter aims his gun and fires a bullet directly at a monkey on a free. At the instant the bullet leaves the barrel of the gun, the monkey drops. Will the bullet hit the monkey? Substantiate your answer with proper reasoning.

Sol. As shown in Fig., the gun at O is directed towards the monkey at position M. Suppose the bullet leaves the barrel of the gun with velocity u at an angle θ with the horizontal. Let the bullet cross the vertical line MB at A after tiem t.

Horizontal distance travelled. OB = x = u cos θ. T or $t = \dfrac{x}{u \cos \theta}$ ... (1)



For motion of the bullet from O to B, the vertical range is 15

$AB = u \sin \theta \,.\, t - \frac{1}{2} gt^2 = u \sin \theta \,.\, \dfrac{x}{u \cos \theta} - \frac{1}{2} gt^2$

$= x \tan \theta - \frac{1}{2} gt^2$ [using (1)]

Also, $MB = x \tan \theta$

∴ $MA = MB - AB$

$= x \tan \theta - [x \tan \theta - \frac{1}{2} gt^2] = \frac{1}{2} gt^2$

Thus, in a time t the bullet passes through A a vertical distance ½ gt^2 below M.

The vertical distance through which the monkey falls in time t

$= 0 + \frac{1}{2} gt^2 = \frac{1}{2} gt^2$

Thus, the bullet and the monkey will always reach the point A at the same time. Hence the bullet will always hit the monkey whatever be the velocity of the bullet.

Q. 19. A machine gun is mounted on the top of a tower 100 m high. At what angle should the gun be inclined to cover a maximum range of firing on the ground below? The muzzle speed of the bullet is 500 ms^{-1}, take g = 10 ms^{-2}.

Sol. Let u be the muzzle speed of the bullet fired from the gun (on the top of the tower) at an angle θ with the horizontal, as shown in Fig.

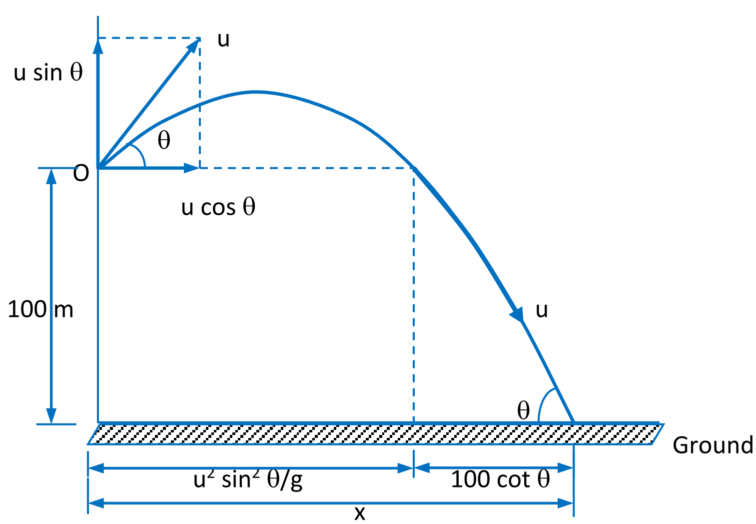
Clearly, the total range of firing on the ground is

$x = u^2 \sin 2\theta + 100 \cot \theta$

∴ $\dfrac{dx}{d\theta} = \dfrac{u^2 \times 2 \cos 2\theta}{g} + 100 \times (-\text{cosec}^2 \theta)$

$= \dfrac{2u^2}{g}(1 - 2\sin^2 \theta) - \dfrac{100}{\sin^2 \theta}$

$= 4500 - 9000 \sin^2 \theta - \dfrac{100}{\sin^2 \theta}$



For x to be maximum,

$\dfrac{dx}{d\theta} = 0$

or $4500 - 9000 \sin^2 \theta - \dfrac{100}{\sin^2 \theta} = \theta$

or $90 \sin^4 \theta - 45 \sin^2 \theta + 1 = 0$

or $\sin^2 \theta = \dfrac{45 \pm \sqrt{(-45)^2 - 4 \times 90 \times 1}}{2 \times 90}$

$= \dfrac{45 \pm 40.80}{180}$

Taking only positive sign,

$\sin^2 \theta = 0.4767$

or $\sin \theta = 0.6904$ Or $\theta = 43.7°$

Q. 20. At what angle should a body be projected with a velocity 24 ms^{-1} just to pass over the obstacle 16 m high at a horizontal distance of 32 m? Take g = 10 ms^{-2}.

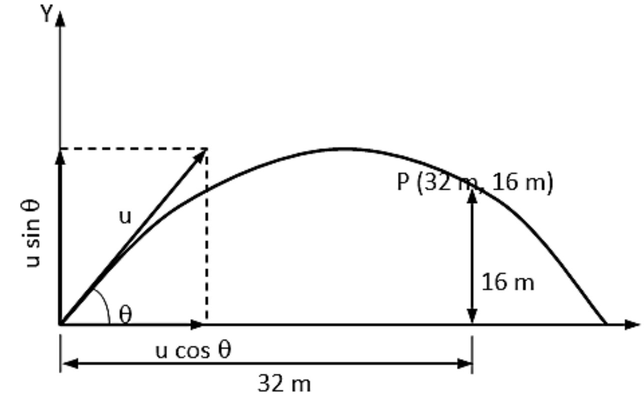
Sol. As shown in Fig., if point of projection is taken as the origin of the coordinate system, the projected body must pass through

A point having coordinates (32 m, 16 m). If u be the initial velocity of the projectile and θ the angle of projection, then

Horizontal component of initial velocity, $u_x = u \cos\theta$

Vertical component of initial velocity, $u_y = u \sin\theta$

If the body passes through point P after time t, then horizontal distance covered,



$$x = (u\cos\theta)\,t$$

Or $32 = (24\cos\theta)\,t$ ... (1)

Similarly, vertical distance covered,

$$y = (u\sin\theta)\,t - \tfrac{1}{2}gt^2$$

Or $16 = (24\sin\theta)\,t - \tfrac{1}{2}\times 10\times t^2$ ... (2)

From equation (1), $t = \dfrac{32}{24\cos\theta}$

Putting this value of t in equation (2), we get

$$16 = (24\sin\theta)\frac{32}{24\cos\theta} - 1\times 10\times\left(\frac{32}{24\cos\theta}\right)^2$$

or $16 = 32\tan\theta - 5\times\dfrac{16}{9\cos^2\theta}$

or $1 = 2\tan\theta - \dfrac{5}{9}\sec^2\theta$

or $9 = 18\tan\theta - 5(1+\tan^2\theta)$

or $5\tan^2\theta - 18\tan\theta + 14 = 0$

∴ $\tan\theta = \dfrac{18\pm\sqrt{(18)^2 - 4\times5\times14}}{10} = 2.462$ or 1.137 Hence $\theta = 67°\,54'$ or $45°\,40'$

Q. 21. A target is fixed on the top of a pole 13 metre high. A person standing at a distance of 50 metre from the pole of capable of projectile a stone with a velocity 10 $\sqrt{g}$ ms^{-1}. If he wants to strike the target in shortest possible time, at what angle should he project the stone?

Sol. The trajectory of a projectile is given by

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

As the target lies on the path of projectile, so

$$13 = 50\tan\theta - \frac{g\times(50)^2}{2(10\sqrt{g})^2}\sec^2\theta \quad [\because u = 10\sqrt{g}\ ms^{-1}]$$

or $13 = 50\tan\theta - \dfrac{25}{2}\sec^2\theta$

$= 50\tan\theta - \dfrac{25}{2}(1+\tan^2\theta)$

or $25\tan^2\theta - 100\tan\theta + 51 = 0$

∴ $\tan\theta = \dfrac{100\pm\sqrt{4900}}{50} = \dfrac{17}{5}$ or $\dfrac{3}{5}$

The horizontal distance covered in time t is given by

$x = (u\cos\theta)\,t$ or $50 = (10\sqrt{g})(\cos\theta)\,t$

or $t = \dfrac{5}{10\sqrt{g}\cos\theta}$

For t to be minimum, cos θ should be maximum, cos θ is greatest if tan θ is least, i.e.,

$\tan\theta = \left(\dfrac{3}{5}\right)$ or $\theta = \tan^{-1}\left(\dfrac{3}{5}\right) = 30°\,58'$

Q. 22. Show that the motion of one projectile as seen from another projectile will always be a straight-line motion.

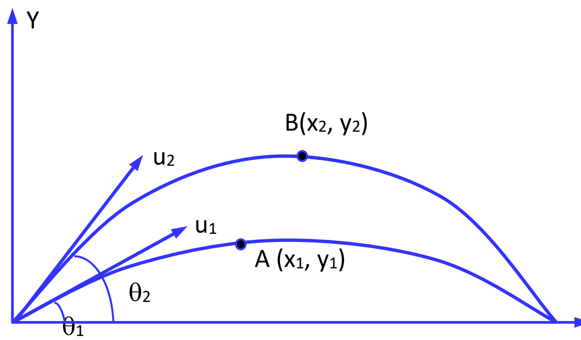
Sol. As shown in Fig. Suppose two projectiles are thrown from the origin O of the XY-plane with velocities u_1 and u_2, making angles θ_1 and θ_2 respectively with X-axis. After time t, let the two projectiles occupy position A (x_1, x_1) and B (x_2, y_2). Then

$$x_1 = u_1 \cos\theta_1 . t$$

and $y_1 = u_1 \sin\theta_1 . t - \frac{1}{2} gt^2$

Also, $x_2 = u_2 \cos\theta_2 . t$

and $y_2 = u_2 \sin\theta_2 . t - \frac{1}{2} gt^2$



$\therefore$ $x_2 - x_1 = (u_2 \cos\theta_2 - u_1 \cos\theta_1)\, t$

$y_2 - y_1 = (u_2 \sin\theta_2 - u_1 \sin\theta_1)\, t$

or $\frac{y_2 - y_1}{x_2 - x_1} = \frac{u_2 \sin\theta_2 - u_1 \sin\theta_1}{u_2 \cos\theta_2 - u_1 \cos\theta_1} = m$ (a constant)

if (x, y) be the coordinates of point B relative to the point A, then

$x_2 - x_1 = x$ and $y_2 - y_1 = y$

$\therefore$ $\frac{y}{x} = m$ or $y = mx$

This is the equation of a straight line. Hence the motion of a projectile as seen from another projectile is a straight-line motion.

Q. 23 A particle is projected over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If α and β be the base angles and θ the angle of projection, prove that $\tan\theta = \tan\alpha + \tan\beta$.

Sol. The situation is shown in Fig.

If R is the range of the particle, then from the figurewe have

$$\tan\alpha + \tan\beta = \frac{y}{x} + \frac{y}{R - x} = \frac{y(R - x) + xy}{x(R - x)}$$

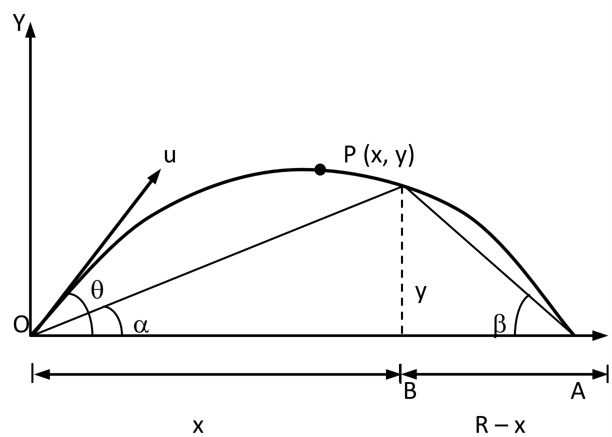
or $\tan\alpha + \tan\beta = \frac{y}{x} \times \frac{R}{(R - x)}$ ... (1)

Also, the trajectory of the particle is

$$y = x \tan\theta - \frac{1}{2} g \frac{x^2}{u^2 \cos^2\theta}$$

$$= x \tan\theta \left(1 - \frac{gx}{2u^2 \cos^2\theta \tan\theta}\right)$$

$$= x \tan\theta \left(1 - \frac{g}{u^2 \sin 2\theta}\, x\right) = x \tan\theta \left(1 - \frac{x}{R}\right)$$



or $\tan\theta = \frac{y}{x} \times \frac{R}{(R - x)}$ ... (2)

From equations (1) and (2), we get $\tan\theta = \tan\alpha + \tan\beta$

From (1) and (3), $s \cos\alpha = u \cos\beta \cdot t$

or $t = \dfrac{s\cos\alpha}{u\cos\beta}$

From (2) and (4),

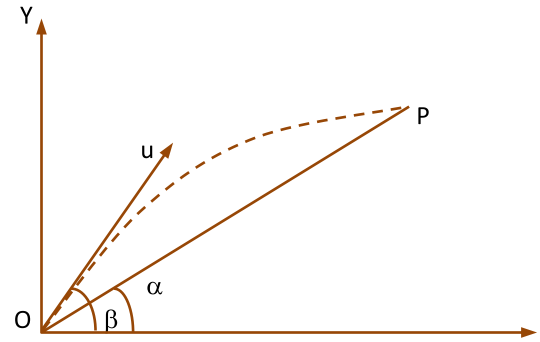
$s \sin\beta = u \sin\beta \cdot t - \frac{1}{2} gt^2$

$= u\sin\beta \cdot \dfrac{s\cos\alpha}{U\cos\beta} - \frac{1}{2} g \cdot \dfrac{s^2\cos^2\alpha}{u^2\cos^2\beta}$

$\therefore \quad g = \dfrac{2u^2}{g} = \left(\dfrac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\cos^2\alpha}\right)\cos\beta$

$= \dfrac{2u^2\sin(\beta-\alpha)\cos\beta}{g\cos^2\alpha}$

$= \dfrac{2\times(21)^2\sin(60^\circ - 30^\circ)\cos 60^\circ}{9.8\times\cos^2 30^\circ} = 30$ m



Problems For Practice

Q. 1. ***A football player kicks a ball at an angle of 37° to the horizontal with an initial speed of 15 ms^{-1}. Assuming that the ball travels in a vertical plane, calculate (i) the time at which the ball reaches the highest point (ii) the maximum height reached (iii) the horizontal range of the projectile and (iv) the time for which the ball is in air.***

Sol. (i) Time of ascent $= \dfrac{u\sin\theta}{g} = \dfrac{15\times\sin 37^\circ}{9.8} = 0.923$ s

(ii) $H = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{(15)^2\sin^2 37^\circ}{9.8} = 4.16$ m

(iii) $R = \dfrac{u^2\sin 2\theta}{g} = \dfrac{(15)^2\sin 74^\circ}{9.8} = 21.2$ m

(iv) $T = \dfrac{2u\sin\theta}{g} = \dfrac{2\times 15\sin 37^\circ}{9.8} = 1.84$ s

Q. 2. ***A body is projected with a velocity of 20 ms^{-1} in a direction making an angle of 60° with the horizontal. Calculate its (i) position after 0.5 s and (ii) velocity after 0.5 s.***

Sol. Here $u = 20\ ms^{-1}$, $\theta = 60^\circ$, $t = 0.5$ s

(i) $x = (u\cos\theta)\,t = (20\cos 60^\circ)\times 0.5 = 5$ m

$y = (u\sin\theta) - \frac{1}{2}gt^2 = (20\times\sin 60^\circ)\times 0.5 - \frac{1}{2}\times 9.8\times(0.5)^2 = 7.43$ m

(ii) $v_x = u\cos\theta = 20\cos 60^\circ = 10\ ms^{-1}$

$v_y = u\sin\theta - gt = 20\sin 60^\circ - 9.8\times 0.5$

$= 12.42\ ms^{-1}$

$\therefore \quad v = \sqrt{v^2_x + v^2_y} = \sqrt{(10)^2 + (12.42)^2} = 15.95\ ms^{-1}$

$\tan\beta = \dfrac{v_y}{v_x} = \dfrac{12.42}{10} = 1.242$

$\therefore \quad \beta = \tan^{-1} 1.242 = 51.16^\circ$

Q. 3. ***A bullet is fired at an angle of 15° with the horizontal and hits the ground 6 km away. Is it possible to hit a target 10 km away by adjusting the angle of projection assuming the initial speed to be the same?***

Sol. As $R = \dfrac{u^2\sin 2\theta}{g} \quad \therefore \quad 6 = \dfrac{u^2\sin 30^\circ}{g}$

or $\dfrac{u^2}{g} = \dfrac{6}{\sin 30^\circ} = 12$

Also, $R_{max} = \dfrac{u^2}{g} = 12$ km

As $R_{max} > 10$ km, the bullet can hit the target at a distance of 10 km by adjusting the angle of projection.

Q. 4. ***A football is kicked 20 ms^{-1} at a projection angle of 45°. A receiver on the goal line 25 metres away in the direction of the kick runs the same instant to meet the ball. What must be his speed, if he is to catch the ball before it hits the ground?***

Sol. Horizontal range,

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(20)^2 \times \sin 90°}{9.8} = 40.82 \text{ m}$$

Distance required to be covered by the receiver to meet the football = 40.82 – 25 = 15.82 m. This distance must be covered during the time of flight of the ball (2.866 s).

∴ Velocity with which player should run to catch the ball

$$= \frac{15.82}{2.886} = 5.483 \text{ ms}^{-1}$$

Q. 5. ***A bullet fired at an angle of 60° with the vertical hits the ground at a distance of 2 km. Calculate the distance at which the bullet will hit the ground when fired at an angle of 45°, assuming the speed to be the same.***

Sol. In the first case: $\theta = 90° - 60°$, R = 2 km

As $R = \frac{u^2 \sin 2\theta}{g}$ ∴ $2 = \frac{u^2 \sin 60°}{g}$ ... (1)

In the second case: $\theta = 45°$

∴ $R' = \frac{u^2 \sin 90°}{g}$ ... (2)

From (1) and (2), $\frac{R'}{2} = \frac{\sin 90°}{\sin 60°} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

∴ $R' = \frac{4 \times 1.732}{3} = 2.31$ km

Q. 6. **A person observes a bird on a tree 39.6 m high and at a distance of 59.2 m. With what velocity the person should throw an arrow at an angle of 45° so that it may hit the bird?**

Sol. For horizontal motion of the arrow:

$u \cos 45° \,.\, t = 59.2$ or $u = \frac{59.2\sqrt{2}}{t}$

For vertical motion of the arrow:

$$39.6 = u \sin 45° \,.\, t - \tfrac{1}{2} \times 9.8 \times t^2$$

∴ $39.6 = \frac{59.2\sqrt{2}}{t} \,.\, \frac{1}{\sqrt{2}}\, t - \tfrac{1}{2} \times 9.8\, t^2$

or $t^2 = \frac{59.2 - 39.6}{4.9} = \frac{19.6}{4.9} = 4$ or $t = 2$ s

Hence $u = \frac{59.2\sqrt{2}}{2} = 29.6\sqrt{2} = 41.86$ ms

Q. 7. ***A ball is thrown from the top of a tower with an initial velocity of 10 ms^{-1} at an angle of 30° with the horizontal. If it hits the ground at a distance of 17.3 m from the base of the tower, calculate the height of the tower. Given g = 10 ms^{-2}.***

Sol. Here $\theta = 30°$, u = 10 ms^{-1}, R = 17.3 m, g = 10 ms^{-2}

For horizontal motion, $R = u \cos \theta \,.\, t$

or $t = \frac{R}{u \cos \theta} = \frac{17.3}{10 \cos 30°} = \frac{17.3 \times 2}{10 \times \sqrt{3}}$

$= \frac{17.3 \times 2}{10 \times 1.73} = 2$ s

For vertical motion, $y = u \sin \theta \,.\, t - \tfrac{1}{2} gt^2 = 10 \sin 30° \times 2 - \tfrac{1}{2} \times 10 \times 2^2 = 10 - 20 = -10$ m; Height of tower = 10 m

Q. 8. Prove that the time-of-flight T and the horizontal range R of a projectile are connected by the equation: $gT^2 = 2R \tan\theta$, where θ is the angle of projection.

Sol. As $R = \dfrac{u^2 \sin 2\theta}{g}$ and $T = \dfrac{2u \sin\theta}{g}$

∴ $gT^2 = \dfrac{g \times 4u^2 \sin^2\theta}{g^2} = \dfrac{2u^2 \times 2\sin\theta\cos\theta}{g} \times \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{2u^2 \sin 2\theta}{g} \times \tan\theta = 2R\tan\theta$

Q. 9. A body is projected with velocity of 40 ms^{-1}. After 2s it crosses a vertical pole of height 20.4 m. Calculate the angle of projection and horizontal range.

Sol. Here $u = 40\ ms^{-1}$, $t = 2$ s, $y = 20.4$ m

As $y = u \sin\theta \times t - ½ gt^2$

∴ $20.4 = 40 \times \sin\theta \times 2 - ½\ 9.8 \times 2^2$

or $20.4 + 19.6 = 80 \sin\theta$ or $40 = 80\sin\theta$

∴ $\sin\theta = ½$ and $\theta = 30°$

$R = \dfrac{u^2 \sin 2\theta}{g} = \dfrac{(40)^2 \sin 60°}{9.8} = 141.39$ m

Q. 10. From the top of a tower 156.8 m high, a projectile is thrown up with velocity of 39.2 ms^{-1} making an angle of 30° with the horizontal direction. Find the distance from the foot of the tower where it strikes the ground and the time taken by it to do so.

Sol. As $y = u \sin\theta \times t - ½ gt^2$

∴ $-156.8 = 39.2 \times ½ \times t - 4.9\ t^2$

or $t^2 - 4t - 32 = 0$

∴ $t = 8, -4$

But $t = -4$ is meaningless, so $t = 8$ s

Hence $R = u\cos\theta \times t = 39.2 \cos 30° \times 8 = 271.6$ m

Q. 11. As shown in Fig., a body is projected with velocity u_1 from the point A. At the same time another body is projected vertically upwards with the velocity u_2 from the point B. What should be the value of u_1/u_2 for both the bodies to collide?

Sol. For two bodies to collide in air, $y_1 = y_2$

$u_1 = \sin 60° \times t - ½ gt^2 = u_2 t - ½ gt^2$

$u_1 \sin 60° \times t = u_2 t$

or $\dfrac{u_1}{u_2} = \dfrac{1}{\sin 60°} = \dfrac{2}{\sqrt{3}}$

Q. 12. A body is projected such that its kinetic energy at the top is 3/4th of its initial kinetic energy. What is the initial angle of projection of the projectile with the horizontal?

Sol. Initial K.E. $= ½ mu^2$

K.E. at the top $= ½ mu^2 \cos^2\theta$

∴ $\dfrac{½ mu^2 \cos^2\theta}{½ mu^2} = ¾$ or $\cos^2\theta = ¾$

or $\cos\theta = \dfrac{\sqrt{3}}{2}$ or $\theta = 30°$