AEP STUDY CIRCLE

SINCE 2001...

IIT-NEET-CBSE

ACCENTS EDUCATIONAL PROMOTERS

ONLINE-OFFLINE LEARNING ACADEMY

24 YEARS OF LEGACY

XII IIT-JEE

DETERMINANTS

MATHEMATICS

YOUR GATEWAY TO EXCELLENCE IN

IIT-JEE, NEET AND CBSE EXAMS

IIT-JEE

NEET

CBSE

DETERMINANTS

CONTACT US:

+91-9939586130

+91-9955930311

www.aepstudycircle.com

aepstudycircle@gmail.com

2ND FLOOR, SATKOUDI COMPLEX, THANA CHOWK, RAMGARH - 829122-JH

DETERMINANTS

DETERMINANTS

Development of determinants took place when mathematicians were trying to solve a system of simultaneous linear equations.

E.g. $\left.\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}\right] \Rightarrow \quad x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$ and $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

Mathematicians defined the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ as a determinant of order 2 and the four numbers arranged in row and column were called its elements. Its value was taken as $(a_1b_2 - a_2b_1)$ which is the same as the denominator. If we write the coefficients of the equations in the following form $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then such an arrangement is called a determinant. In a determinant, horizontal lines are known as rows and vertical lines are known as columns. The shape of every determinant is a square. If a determinant is of order n then it contains n rows and n columns.

E.g. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ are determinants of second and third order respectively.

Note: (i) No. of elements in a determinant of order n are n^2. (ii) A determinant of order 1 is the number itself.

1.1 Evaluation of the Determinant using SARRUS Diagram

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, the below diagram is a Sarrus Diagram obtained by adjoining the first two columns on the right and draw dark and dotted lines as shown.

The value of the determinant is $(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$.

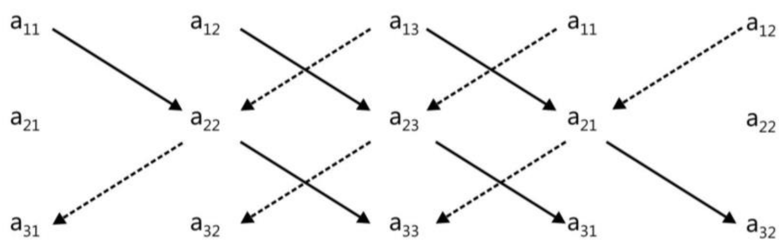


Figure 17.1

Illustration 1: Expand $\begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$ by Sarrus rules. **(JEE MAIN)**

Sol: By using Sarrus rule i.e. $\Delta = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$ we can expand the given determinant.

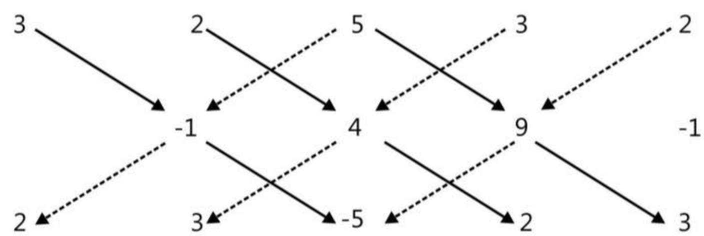


Figure 17.2

Here, $\Delta = \begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix} \Rightarrow \Delta = 15 - 36 + 90 + 16 + 135 + 10 = 230$

Illustration 2: Evaluate the determinant : $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$ **(JEE MAIN)**

Sol: By using determinant expansion formula we can get the result.

we have, $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1) = x^3 + x^2 - x^2 - x + x + 1 - x^2 + 1 = x^3 - x^2 + 2$

2. COFACTOR AND MINOR OF AN ELEMENT

Minor: Minor of an element is defined as the determinant obtained by deleting the row and column in which that element lies. e.g. in the determinant $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, minor of a_{12} is denoted as $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

Cofactor: Cofactor of an element a_{ij} is related to its minor as $C_{ij} = (-1)^{i+j} M_{ij}$, where 'i' denotes the i^{th} row and 'j' denotes the j^{th} column to which the element a_{ij} belongs.

Now we define the value of the determinant of order three in terms of 'Minor' and 'Cofactor' as

$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ or $D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Note:

(a) A determinant of order 3 will have 9 minors and each minor will be a determinant of order 2 and a determinant of order 4 will have 16 minors and each minor will be determinant of order 3.

(b) $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, i.e. cofactor multiplied to different row/column elements results in zero value.

Row and Column Operations

(a) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$, when $i \neq j$; This notation is used when we interchange i^{th} row (or column) and j^{th} row (or column).

(b) $R_i \leftrightarrow C_i$; This converts the row into the corresponding column.

(c) $R_i \to Rk_i$ or $C_i \to kC_i$; $k \in R$; This represents multiplication of i^{th} row (or column) by k.

(d) $R_i \to R_ik + R_j$ or $Ci \to C_ik + C_j$; $(i \neq j)$; This symbol is used to multiply i^{th} row (or column) by k and adding the j^{th} row (or column) to it.

Illustration 3: Find the cofactor of a_{12} in the following $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ **(JEE MAIN)**

Sol: In this problem we have to find the cofactor of a_{12}, therefore eliminate all the elements of the first row and the second column and by obtaining the determinant of remaining elements we can calculate the cofactor of a_{12}.

Here a_{12} = Element of first row and second column = –3

$$M_{12} = \text{Minor of } a_{12}(-3) = \begin{vmatrix} 2\cdots & -3 & \cdots 5 \\ & \vdots & \\ 6 & 0 & 4 \\ & \vdots & \\ 1 & 5 & -7 \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = 6(-7) - 4(1) = -42 - 4 = -46.$$

Cofactor of $(-3) = (-1)^{1+2}(-46) = -(-46) = 46$

Illustration 4: Write the minors and cofactors of the elements of the following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ **(JEE MAIN)**

Sol: By eliminating row and column of an element, the remaining is the minor of the element.

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$; M_{11} = Minor of element (2) = $\begin{vmatrix} 2\cdots & -4 \\ \vdots & \\ 0 & 3 \end{vmatrix} = 3$; Cofactor of $(2) = (-1)^{1+1}M_{11} = +3$

M_{12} = Minor of element (–4) = $\begin{vmatrix} 2\cdots & -4 \\ & \vdots \\ 0 & 3 \end{vmatrix} = 0$; Cofactor of $(-4) = (-1)^{1+2}M_{12} = (-1)0 = 0$

M_{21} = Minor of element (0) = $\begin{vmatrix} 2 & -4 \\ \vdots & \\ 0\cdots & 3 \end{vmatrix} = -4$; Cofactor of $(0) = (-1)^{2+1}M_{21} = (-1)(-4) = 4$

M_{22} = Minor of element (3) = $\begin{vmatrix} 2 & -4 \\ & \vdots \\ 0\cdots & 3 \end{vmatrix} = 2$; Cofactor of $(3) = (-1)^{2+2}M_{22} = +2$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$; M_{11} = Minor of element (a) = $\begin{vmatrix} a\cdots c \\ \quad\ \vdots \\ b \quad d \end{vmatrix} = d$; Cofactor of (a) $= (-1)^{1+1}M_{11} = (-1)^2 d = d$

M_{12} = Minor of element (c) = $\begin{vmatrix} a\cdots c \\ \quad\ \vdots \\ b \quad d \end{vmatrix} = b$; Cofactor of (c) $= (-1)^{1+2}M_{12} = (-1)^3 b = -b$

M_{21} = Minor of element (b) = $\begin{vmatrix} a \quad c \\ \vdots \quad\ \\ b\cdots d \end{vmatrix} = c$; Cofactor of (b) $= (-1)^{2+1}M_{21} = (-1)^3 c = -c$

M_{22} = Minor of element (d) = $\begin{vmatrix} a \quad c \\ \quad\ \vdots \\ b\cdots d \end{vmatrix} = a$; Cofactor of (d) $= (-1)^{2+2}M_{22} = (-1)^4 a = a$

Illustration 5: Find the minor and cofactor of each element of the determinant $\begin{vmatrix} 2 & -2 & 3 \\ 1 & 4 & 5 \\ 2 & 1 & -3 \end{vmatrix}$. **(JEE ADVANCED)**

Sol: By eliminating the row and column of an element, the determinant of remaining elements is the minor of the element. i.e. $M_{i\times j}$ and by using formula $(-1)^{i+j}M_{i\times j}$ we will get the cofactor of the element.

The minors are $M_{11} = \begin{vmatrix} 4 & 5 \\ 1 & -3 \end{vmatrix} = -17$, $M_{12} = \begin{vmatrix} 1 & 5 \\ 2 & -3 \end{vmatrix} = -13$, $M_{13} = \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = -7$

$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix} = 3$, $M_{22} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12$, $M_{23} = \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -6$

$M_{31} = \begin{vmatrix} -2 & 3 \\ 4 & 5 \end{vmatrix} = -22$, $M_{32} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7$, $M_{33} = \begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} = 10$

The cofactors are:

$A_{11} = (-1)^{1+1}M_{11} = M_{11} = -17$, $A_{12} = (-1)^{1+2}M_{12} = -M_{12} = 13$, $A_{13} = (-1)^{1+3}M_{13} = M_{13} = -7$

$A_{21} = (-1)^{2+1}M_{21} = -M_{21} = -3$, $A_{22} = (-1)^{2+2}M_{22} = M_{22} = -12$, $A_{23} = (-1)^{2+3}M_{23} = -M_{23} = 6$

$A_{31} = (-1)^{3+1}M_{31} = M_{31} = -22$, $A_{32} = (-1)^{3+2}M_{32} = -M_{32} = -7$, $A_{33} = (-1)^{3+3}M_{33} = M_{33} = 10$

3. PROPERTIES OF DETERMINANTS

Determinants have some properties that are useful as they permit us to generate the same results with different and simpler configurations of entries (elements).

(a) **Reflection Property:** The determinant remains unaltered if its rows are changed into columns and the columns into rows.

(b) **All-zero Property:** If all the elements of a row (or column) are zero, then the determinant is zero.

(c) **Proportionality (Repetition) Property:** If the all elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then the determinant is zero.

(d) **Switching Property:** The interchange of any two rows (or columns) of the determinant changes its sign.

(e) Scalar Multiple Property: If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

(f) Sum Property: $\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$

(g) Property of Invariance: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha b_1+\beta c_1 & b_1 & c_1 \\ a_2+\alpha b_2+\beta c_2 & b_2 & c_2 \\ a_3+\alpha b_3+\beta c_3 & b_3 & c_3 \end{vmatrix}$

That is, a determinant remains unaltered under an operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$, where $j, k \neq i$, or an operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$, where $j, k \neq i$

(h) Factor Property: If a determinant Δ becomes zero when we put $x = \alpha$, then $(x - \alpha)$ is a factor of Δ.

(i) Triangle Property: If all the elements of a determinant above or below the main diagonal consist of zeros, then the determinant is equal to the product of diagonal elements. That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

(j) Determinant of cofactor matrix: $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then $\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$

where C_{ij} denotes the cofactor of the element a_{ij} in Δ.

CONCEPTS

By interchanging two rows (or columns), the value of the determinant differs by a –ve sign.

If Δ' is the determinant formed by replacing the elements of a determinant Δ by their corresponding cofactors, then if $\Delta = 0$, then $\Delta^1 = 0$, else $\Delta' = \Delta^{n-1}$, where n is the order of the determinant.

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 6: Using properties of determinants, prove that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$ **(JEE MAIN)**

Sol: By using invariance and scalar multiple property of determinant we can prove the given problem.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} \quad [\text{Operating } C_1 \to C_1 + C_2 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \; [\text{Operating } (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)]$$

$$= (a+b+c)\{(c-b)(b-c)-(a-b)(a-c)\} = (a+b+c)(bc-b^2-c^2+bc-(a^2-ab-ac+bc)) = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

Illustration 7: Prove the following identity $\begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix} = 4\alpha^2\beta^2\gamma^2$ **(JEE MAIN)**

Sol: Take α, β, γ common from the L.H.S. and then by using scalar multiple property and invariance property of determinant we can prove the given problem.

$$\Delta = \begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix}$$

Taking α, β, γ common from C_1, C_2, C_3 respectively $\Delta = \alpha\beta\gamma \begin{vmatrix} -\alpha & \alpha & \alpha \\ \beta & -\beta & \beta \\ \gamma & \gamma & -\gamma \end{vmatrix}$

Now taking α, β, γ common from R_1, R_2, R_3 respectively $\Delta = \alpha^2\beta^2\gamma^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

Now applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$ we have $\Delta = \alpha^2\beta^2\gamma^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$

Now expanding along C_1, $\Delta = \alpha^2 \times \beta^2(-1) \times \gamma^2(-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = \alpha^2\beta^2(-1)\gamma^2(0-4) = 4\alpha^2\beta^2\gamma^2$

Hence proved.

Illustration 8: Show that $\begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \psi \\ \lambda & \mu & \nu \end{vmatrix} = \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix}$ **(JEE ADVANCED)**

Sol: Interchange the rows and columns across the diagonal using reflection property and then using the switching property of determinant we can obtain the required result.

L.H.S. $= \begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \psi \\ \lambda & \mu & \nu \end{vmatrix} = \begin{vmatrix} \alpha & \theta & \lambda \\ \beta & \phi & \mu \\ \gamma & \psi & \nu \end{vmatrix}$ (Interchanging rows and columns across the diagonal)

$$= (-1)\begin{vmatrix} \alpha & \lambda & \theta \\ \beta & \mu & \phi \\ \gamma & \nu & \psi \end{vmatrix} = (-1)^2\begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix} = \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix} = \text{R.H.S.}$$

Illustration 9: If a, b, c are all different and if $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, prove that abc = –1. **(JEE ADVANCED)**

Sol: Split the given determinant using sum property. Then by using scalar multiple, switching and invariance properties of determinants, we can prove the given equation.

$$D = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)^1\begin{vmatrix}1 & a^2 & a\\1 & b^2 & b\\1 & c^2 & c\end{vmatrix} + abc\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix} \quad [C_1 \leftrightarrow C_3 \text{ in 1st det.}]$$

$$= (-1)^2\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix} + abc\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix} \quad [C_2 \leftrightarrow C_3 \text{ in 1st det.}]$$

$$= \begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix} + abc\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix} = (1 + abc)\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}$$

$$= (1 + abc)\begin{vmatrix}1 & a & a^2\\0 & b-a & b^2-a^2\\0 & c-a & c^2-a^2\end{vmatrix} \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= (1 + abc)\begin{vmatrix}b-a & b^2-a^2\\c-a & c^2-a^2\end{vmatrix} \quad \text{(expanding along 1}^{st}\text{ row)} = (1 + abc)\,(b-a)\,(c-a)\begin{vmatrix}1 & b+a\\1 & c+a\end{vmatrix}$$

$= (1 + abc)\,(b - c)\,(c - a)\,(c + a - b - a) = (1 + abc)\,(b - a)\,(c - a)\,(c - b)$

$\Rightarrow D = (1 + abc)\,(a - b)\,(b - c)\,(c - a)$; But given $D = 0$

$\Rightarrow (1 + abc)\,(a - b)\,(b - c)\,(c - a) = 0$; $\therefore\ (1 + abc) = 0$

[since a, b, c are different $a \neq b, b \neq c, c \neq a$]; Hence, $abc = -1$

Illustration 10: Prove that $\begin{vmatrix}a+b+2c & a & b\\c & b+c+2a & b\\c & a & c+a+2b\end{vmatrix} = 2(a+b+c)^3$ **(JEE ADVANCED)**

Sol: Simply by using switching and scalar multiple property we can expand the L.H.S.

Given determinant $= \begin{vmatrix}a+b+2c & a & b\\c & b+c+2a & b\\c & a & c+a+2b\end{vmatrix}$

Applying $C_1 \to C_1 + (C_2 + C_3)$, we obtain

$$\begin{vmatrix}2(a+b+c) & a & b\\2(a+b+c) & b+c+2a & b\\2(a+b+c) & a & c+a+2b\end{vmatrix} = 2(a+b+c)\begin{vmatrix}1 & a & b\\1 & b+c+2a & b\\1 & a & c+a+2b\end{vmatrix}$$

$R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ given

$$2(a+b+c)\begin{vmatrix}1 & a & b\\0 & b+c+a & 0\\0 & 0 & c+a+b\end{vmatrix} = 2(a+b+c)\cdot 1\{(b+c+a)(c+a+b) - (0\times 0)\} = 2(a+b+c)^3$$

Hence proved.

Illustration 11: Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$ **(JEE ADVANCED)**

Sol: Expand the determinant $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$ by using scalar multiple and invariance property.

L.H.S.= $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$; Multiplying C_1, C_2, C_3 by a, b, c respectively

$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$; Now taking a, b, c common from R_1, R_2, R_3 respectively

$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} = \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$ $[C_1 \to C_1+C_2+C_3]$

$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix} = (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ $[R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1]$

$= (1+a^2+b^2+c^2)\ (1.1.1) = 1+a^2+b^2+c^2$ = R.HS.

Hence proved.

CONCEPTS

|AB| = |A||B|

The value of the determinant is the same when expanded by any row or any column. Using this property it is easier to expand determinant using a row or column in which most zeroes are involved.

Vaibhav Gupta (JEE 2009 AIR 54)

4. SYMMETRIC AND SKEW SYMMETRIC DETERMINANTS

4.1 Symmetric Determinant

A determinant is called Symmetric Determinant if $a_{ij} = a_{ji}, \forall$ i, j e.g. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

4.2 Skew Symmetric Determinant

A determinant is called a skew symmetric determinant if $a_{ij} = -a_{ji}\ \forall$ i, j for every element.

E.g. $\begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$

Note: (i) det $|A| = 0 \Rightarrow$ A is singular matrix (ii) det $|A| \neq 0 \Rightarrow$ A is non-singular matrix

CONCEPTS

The value of a skew symmetric determinant of an even order is always a perfect square and that of an odd order is always zero.

Vaibhav Krishnan (JEE 2009 AIR 22)

5. MULTIPLICATION OF TWO DETERMINANTS

(a) Multiplication of two second order determinants as follows: (as R to C method)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1 + b_1l_2 & a_1m_1 + b_1m_2 \\ a_2l_1 + b_2l_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

(b) Multiplication of two third order determinants is defined.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \quad \text{(as R to C method)}$$

$$= \begin{vmatrix} a_1l_1 + b_1l_2 + c_1l_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2l_1 + b_2l_2 + c_2l_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3l_1 + b_3l_2 + c_3l_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

Note:

(i) The two determinants to be multiplied must be of the same order.

(ii) To get the T_{mn} (term in the mth row nth column) in the product, Take the mth row of the 1st determinant and multiply it by the corresponding terms of the nth column of the 2nd determinant and add.

(iii) This method is the row by column multiplication rule for the product of 2 determinants of the nrd order determinant.

(iv) If Δ' is the determinant formed by replacing the elements of a Δ of order n by their corresponding co-factors then $\Delta' = \Delta^{n-1}$. (Δ' is called the reciprocal determinant).

Illustration 12: Reduce the power of the determinant $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$ to 1. **(JEE MAIN)**

Sol: By multiplying the given determinant two times we get the determinant as required.

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} b^2 + c^2 & ab & ac \\ ba & c^2 + a^2 & bc \\ ca & cb & a^2 + b^2 \end{vmatrix}$$

Illustration 13: Show that $\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$. **(JEE ADVANCED)**

Sol: By replacing all elements of L.H.S. to their respective cofactors and using determinant property we will obtain the required result.

Let $D = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Co-factors of 1st row of D are x^2+a^2, $ab+cx$, $ac-bx$. Co-factors of 2nd row of D are $ab - cx$, $x^2 + b^2$, $ax + bc$ and co-factors of 3rd row of D are $ac+bx$, $bc-ax$, x^2+c^2

$\therefore$ Determinant of cofactors of D is

$$D^c = \begin{vmatrix} x^2+a^2 & ab+cx & ac-bx \\ ab-cx & x^2+b^2 & ax+bc \\ ac+bx & bc-ax & x^2+c^2 \end{vmatrix} = \begin{vmatrix} a^2+x^2 & ab-cx & ac-bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & ax+bc & x^2+c^2 \end{vmatrix} = D^2$$

(Row interchanging into columns) $= \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$ ($D^c = D^2$, D is third order determinant)

Hence $\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$

6. SOME STANDARD DETERMINANTS

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ (ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(iii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc\,(a-b)(b-c)(c-a)$; (iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)\,(a+b+c)$

(v) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -a^3-b^3-c^3+3abc$

Illustration 14: Evaluate the determinant $\Delta = \begin{vmatrix} \sqrt{p}+\sqrt{q} & 2\sqrt{r} & \sqrt{r} \\ \sqrt{qr}+\sqrt{2p} & r & \sqrt{2r} \\ q+\sqrt{pr} & \sqrt{qr} & r \end{vmatrix}$, where p, q and r are positive real numbers. **(JEE MAIN)**

Sol: Taking $\sqrt{r}$ common from C_2 and C_3 of the given determinant using scalar multiple property and then expanding it using the invariance property we can evaluate the given problem.

We get $\Delta = r\begin{vmatrix} \sqrt{p}+\sqrt{q} & 2 & 1 \\ \sqrt{qr}+\sqrt{2p} & \sqrt{r} & \sqrt{2} \\ q+\sqrt{pr} & \sqrt{q} & \sqrt{r} \end{vmatrix}$

Applying $C_1 \to C_1 - \sqrt{q}C_2 - \sqrt{p}C_3$

We get $D = r\begin{vmatrix} -\sqrt{q} & 2 & 1 \\ 0 & \sqrt{r} & \sqrt{2} \\ 0 & \sqrt{q} & \sqrt{r} \end{vmatrix} = -r\sqrt{q}(r-\sqrt{2q}) = r\left(q\sqrt{2} - r\sqrt{q}\right)$.

Illustration 15: Let a, b, c be positive and not equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

(JEE ADVANCED)

Sol: By applying invariance and scalar multiple properties to the given determinant we can get the required result.

$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$; then $D = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$ $[C_1 \to C_1 + C_2 + C_3]$

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ [Taking (a + b+ c) common from the first column]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$ $[R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1]$

$= (a+b+c+)\,[(c-b)(b-c)-(a-b)(a-c)] = (a+b+c+)\,[bc+ca+ab-a^2-b^2-c^2]$

$= -(a+b+c)\,(a^2+b^2+c^2-bc-ca-ab) = -\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2bc-2ca-2ab)$

$= -\frac{1}{2}(a+b+c)\,[(a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ac)]$

$= -\frac{1}{2}(a+b+c)\,[(a-b)^2+(b-c)^2+(c-a)^2]$ ... ... (i)

$\because$ a, b, c, are positive $\Rightarrow a + b + c > 0$

$\because$ a, b, c are unequal $\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2 > 0$ ... ...(ii)

$\therefore$ From (i) and (ii), $\Delta < 0$.

Illustration 16: Show that $\Delta = \begin{vmatrix} 1 & \cos^2(\alpha-\beta) & \cos^2(\alpha-\gamma) \\ \cos^2(\beta-\alpha) & 1 & \cos^2(\beta-\gamma) \\ \cos^2(\gamma-\alpha) & \cos^2(\gamma-\beta) & 1 \end{vmatrix} = 2\sin^2(\beta-\gamma)\sin^2(\gamma-\alpha)\sin^2(\alpha-\beta)$ **(JEE ADVANCED)**

Sol:.By Putting $\beta-\gamma = A$, $\gamma-\alpha = B$, $\alpha-\beta = C$ and then by using switching and invariance properties we can prove the above problem.

We can write Δ as, $\Delta = \begin{vmatrix} 1 & \cos^2 C & \cos^2 B \\ \cos^2 C & 1 & \cos^2 A \\ \cos^2 B & \cos^2 A & 1 \end{vmatrix}$ (Note that A + B + C = 0.)

Using $C_2 \to C_2 - C_1$, $C_1 \to C_3 - C_1$ we get

$$\Delta = \begin{vmatrix} 1 & -\sin^2 C & -\sin^2 B \\ \cos^2 C & \sin^2 C & \cos^2 A - \cos^2 C \\ \cos^2 B & \cos^2 A - \cos^2 B & \sin^2 B \end{vmatrix} \begin{vmatrix} 1 & -\sin^2 C & -\sin^2 B \\ \cos^2 C & \sin^2 C & \sin B \sin(C-A) \\ \cos^2 B & \sin C \sin(B-A) & \sin^2 B \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & \sin^2 C & \sin^2 B \\ \cos^2 C & -\sin^2 C & \sin B \sin(-A) \\ \cos^2 B & \sin C \sin(B-A) & -\sin^2 B \end{vmatrix}$$

$[\because \ \cos^2 A - \cos^2 B = \sin(A+B)\sin(B-A),\ A+B=-C, C+A=-B\ ]$; $\quad = \sin C \sin B\ [\Delta_1]$

where $\Delta_1 = \begin{vmatrix} 1 & \sin^2 C & \sin B \\ \cos^2 C & -\sin^2 C & \sin(C-A) \\ \cos^2 B & \sin(B-A) & -\sin B \end{vmatrix}$ Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$\Delta_1 = \begin{vmatrix} 1 & \sin C & \sin B \\ -\sin^2 C & -2\sin^2 C & \sin(C-A) - \sin B \\ -\sin^2 B & \sin(B-A) - \sin C & -2\sin^2 B \end{vmatrix}$$

But sin (C – A) – sin B = sin (C – A) + sin (C + A) = 2 sin C cos A and sin (B – A) – sin C = 2 sin B cos A

Therefore, $\Delta_1 = \sin C \sin B \Delta_2$ where $\Delta_2 = \begin{vmatrix} 1 & \sin C & \sin B \\ \sin C & 2 & -2\cos A \\ \sin B & -2\cos A & 2 \end{vmatrix}$

Applying $R_2 \to R_2 - \sin C\ R_1$ and $R_3 \to -\sin B\ R_1$ we get

$$\Delta_2 = \begin{vmatrix} 1 & \sin C & \sin B \\ 0 & 2 - \sin^2 C & -2\cos A - \sin B \sin C \\ 0 & -2\cos A - \sin B \sin C & 2 - \sin^2 B \end{vmatrix} = (2 - \sin^2 B)(2 - \sin C) - (2\cos A + \sin B \sin C)^2$$

$$= 4 - 2\sin^2 B - 2\sin^2 C + \sin^2 B \sin^2 C - [4\cos^2 A + 4\cos A \sin B \sin C + \sin^2 B \sin^2 C]$$

$$= 4\sin^2 A - 2\sin^2 B - 2\sin^2 C - 4\cos A \sin B \sin C$$

$$= 2\sin^2 A - 2\,[\sin^2 B + \sin^2 C - \sin^2 A + 2\cos A \sin B \sin C]$$

But A + B + C = 0 implies; $\quad \sin^2 B + \sin^2 C - \sin^2 A = -2\cos A \sin B \sin C$

$\therefore \quad \Delta_2 = 2\sin^2 A$; $\quad$ Hence, $\quad D = \sin C \sin B \Delta_1 = \sin^2 C \sin^2 B \Delta_2$

$= 2\sin^2 A \sin^2 B \sin^2 C = 2\sin^2(\alpha - \beta)\sin^2(\beta - \gamma)\sin^2(\gamma - \alpha)$.

Illustration 17: Prove that the following determinant vanishes if any two of x; y; z are equal

(JEE ADVANCED)

$$\Delta = \begin{vmatrix} \sin x & \sin y & \sin z \\ \cos x & \cos y & \cos z \\ \cos^3 x & \cos^3 y & \cos^3 z \end{vmatrix}$$

Sol: Taking $\cos x$, $\cos y$, and $\cos z$ common from first, second and third column using scalar multiple and then using the invariance property we can prove the given statement.

Here, $\Delta = \cos x \cos y \cos z \begin{vmatrix} \tan x & \tan y & \tan z \\ 1 & 1 & 1 \\ \cos^2 x & \cos^2 y & \cos^2 z \end{vmatrix}$

$$= \cos x \cos y \cos z \begin{vmatrix} \tan x & \tan y - \tan x & \tan z - \tan y \\ 1 & 0 & 0 \\ \cos^2 x & \cos^2 y - \cos^2 x & \cos^2 z - \cos^2 y \end{vmatrix} (C_3 \to C_3 - C_2,\ C_2 \to C_2 - C_1)$$

Expanding along R_2; $\Delta = -\cos x \cos y \cos z \begin{vmatrix} \tan y - \tan x & \tan z - \tan y \\ \cos^2 y - \cos^2 x & \cos^2 z - \cos^2 y \end{vmatrix}$

$$= -\cos x \cos y \cos z \begin{vmatrix} \dfrac{\sin(y-x)}{\cos x \cos y} & \dfrac{\sin(z-y)}{\cos y . \cos z} \\ \sin^2 x - \sin^2 y & \sin^2 y - \sin^2 z \end{vmatrix} = \begin{vmatrix} \cos z . \sin(x-y) & \cos x . \sin(y-z) \\ \sin(x+y) . \sin(x-y) & \sin(y+z) . \sin(y-z) \end{vmatrix} \quad \ldots (i)$$

$$= \sin(x-y)\sin(y-z) \begin{vmatrix} \cos z & \cos x \\ \sin(x+y) & \sin(y+z) \end{vmatrix} = \sin(x-y)\sin(y-z)[\sin(y+z)\cos z - \sin(x+y)\cos x]$$

$$= \frac{1}{2}\sin(x-y)\sin(y-z)[\{\sin(y+2z) + \sin y\} - \{\sin(y+2x\} + \sin y\}]$$

$$= \frac{1}{2}\sin(x-y)\sin(y-z)[\sin(y+2z) - \sin(y+2x)] = \frac{1}{2}\sin(x-y)\sin(y-z)2\cos(x+y+z)\sin(z-x)$$

$$= \sin(x-y)\sin(y-z)\sin(z-x)\cos(x+y+z)$$

Clearly, Δ is zero when any two of x, y, z are equal or $x + y + z = \frac{\pi}{2}$.

Hence proved.

7. SYSTEM OF EQUATIONS

7.1 Involving Two Variables

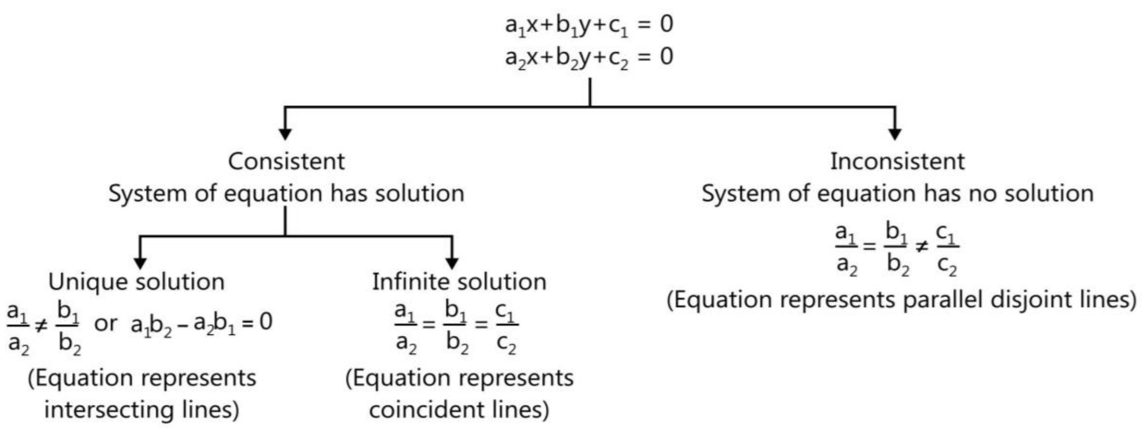


Figure 17.3

Solution to this system of equations is given by $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}$ or $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$; $y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$

where $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$ and $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

7.2 Involving Three Variables

$a_1x + b_1y + c_1z = d_1 \qquad a_2x + b_2y + c_2z = d_2 \qquad a_3x + b_3y + c_3z = d_3$

To solve this system we first define the following determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now following algorithm is followed to solve the system (**CRITERION FOR CONSISTENCY**)

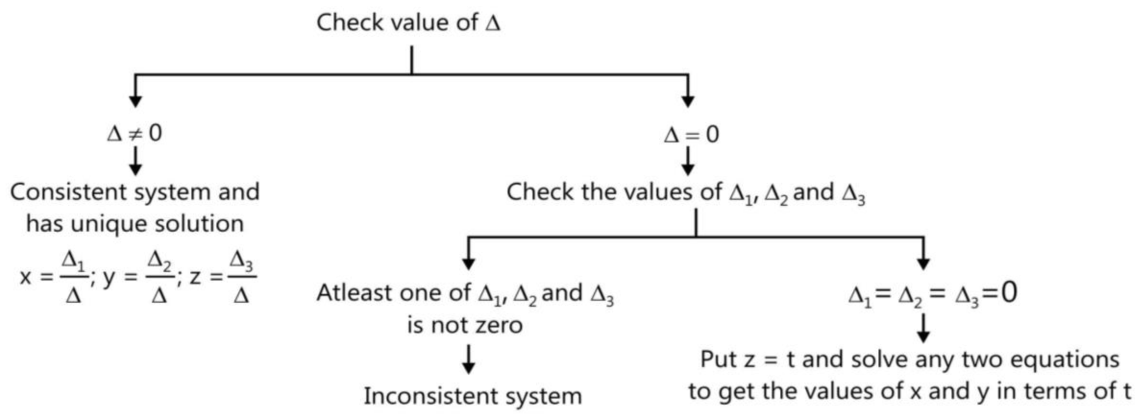


Figure 17.4

This method of finding solution to a system of equations is called Cramer's rule.

Note:

(a) If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then system of equation may or may not be consistent:

- **(i)** If the value of x, y and z in terms of t satisfy the third equation then system is said to be consistent and will have infinite solutions.
- **(ii)** If the values of x, y, z don't satisfy the third equation, then system is said to be inconsistent and will have no solution.

(b) If $d_1 = d_2 = d_3 = 0$, then system of linear equations is known as Homogeneous linear equations, which always possess at least one solution i.e. (0, 0, 0). This is called trivial solution for homogeneous linear equations.

(c) If the system of homogeneous linear equations possess non-zero/nontrivial solutions, and $\Delta = 0$. In such case given system has infinite solutions.

We can also solve these solutions using the matrix inversion method.

We can write the linear equations in the matrix form as AX = B where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Now, solution set is obtained by solving $X = A^{-1}B$. Hence the solution set exists only if the inverse of A exists.

Illustration 18: Solve the following equations by Cramer's rule x + y + z = 9, 2x + 5y + 7z = 52, 2x+ y – z = 0. **(JEE MAIN)**

Sol: Here in this problem define the determinants $\Delta, \Delta_1, \Delta_2$ and Δ_3 and find out their value by using the invariance property and then by using Cramer's rule, we can get the values of x, y and z.

Here $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$ (Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$\therefore \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 2 & -1 & -3 \end{vmatrix} = 1\,(-9 + 5) = -4; \quad \Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$ (Applying $C_2 \to C_2 + C_3$)

$\therefore \quad \Delta_1 = \begin{vmatrix} 9 & 2 & 1 \\ 52 & 12 & 7 \\ 0 & 0 & -1 \end{vmatrix} = -1\,(108 - 104) = -4; \Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$ (Applying $C_1 \to C_1 + 2C_3$)

$\therefore \quad \Delta_2 = \begin{vmatrix} 3 & 9 & 1 \\ 16 & 52 & 7 \\ 0 & 0 & -1 \end{vmatrix} = -1\,(156 - 144) = -12$ and $\Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$ (Applying $C_1 \to C_1 - 2C_2$)

$\therefore \quad \Delta_3 = \begin{vmatrix} -1 & 1 & 9 \\ -8 & 5 & 52 \\ 0 & 1 & 0 \end{vmatrix}$ (Applying $C_1 \to C_1 - 2C_2$) $= -1(-52 + 72) = -20$

$\therefore$ By Cramer's rule $x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{-12}{-4} = 3$ and $z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5$

$\therefore$ x = 1, y = 3, z = 5

Illustration 19: Solve the following linear equations: $\frac{4}{x+5} + \frac{3}{y+7} = -1$ and $\frac{6}{x+5} - \frac{6}{y+7} = -5$ **(JEE MAIN)**

Sol: Here in this problem first put $\frac{1}{x+5}$ = a and $\frac{1}{y+7}$ = b and then define the determinants Δ, Δ_1 and Δ_2. Then by using Cramer's rule we can get the values of x and y.

Let us put $\frac{1}{x+5}$ = a and $\frac{1}{y+7}$ = b then the 2 linear equations become

4a + 3b = – 1 ...(i)

and 6a – 6b = –5 ...(ii);

Using Cramer's Rule, we get,

$\frac{x}{\begin{vmatrix} -1 & 3 \\ -5 & -6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 4 & -1 \\ 6 & -5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 4 & 3 \\ 6 & -6 \end{vmatrix}} \quad \Rightarrow \frac{a}{6+15} = \frac{b}{-20+6} = \frac{1}{-24-18}$

$\therefore \quad \frac{a}{21} = \frac{b}{-14} = \frac{1}{-42} \quad \Rightarrow a = \frac{-1}{2}$ and $b = \frac{1}{3}$

$\therefore \quad a = -\frac{1}{2} \quad \Rightarrow \frac{1}{x+5} = -\frac{1}{2} \quad \Rightarrow 2 = -x - 5 \Rightarrow x = -7$

$b = \frac{1}{3} \quad \Rightarrow \frac{1}{y+7} = \frac{1}{3} \quad \Rightarrow 3 = y + 7 \quad \Rightarrow y = -4$

Illustration 20: For what value of k will the following system of equations possess nontrivial solutions. Also find all the solutions of the system for that value of k.

x + y – kz = 0; 3x – y – 2z = 0; x – y + 2z = 0. **(JEE ADVANCED)**

Sol: Here in this problem first define Δ. As we know that, for non-trivial solution $\Delta = 0$.

So by using the invariance property we can solve $\Delta = 0$ and will get the value of k.

For non-trivial solution, $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -k \\ 3 & -1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 0 \quad \Rightarrow \begin{vmatrix} 2 & 0 & -k+2 \\ 2 & 0 & -4 \\ 1 & -1 & 2 \end{vmatrix} = 0 \quad [R_1 \to R_1 + R_3, R_2 \to R_2 - R_3]$$

Expanding along C_2. $\Rightarrow$ $-(-1)[-8-2(2-k)] = 0$ $\Rightarrow$ $2k - 12 = 0$ $\Rightarrow$ $k = 6$

Putting the value of k in the given equation, we get,

$x + y - 6z = 0$ ... (i)

$3x - y - 2z = 0$ ... (ii)

$x - y + 2z = 0$ ... (iii)

(i) + (ii) $\Rightarrow 4x - 8z = 0$ $\quad \therefore z = \dfrac{x}{2}$

Putting the value of z in (i), we get $x + y - 3x = 0$ $\quad \therefore y = 2x$

Thus when k = 6, solution of the given system of equations will be $x = t$, $y = 2t$, $z = \dfrac{t}{2}$, when t is an arbitrary number.

Illustration 21: Solve the following equations by matrix inversion.

$2x + y + 2z = 0 \qquad 2x - y + z = 10 \qquad x + 3y - z = 5$ **(JEE ADVANCED)**

Sol: By writing the given equations into the form of AX = D and then multiplying both side by A^{-1} we will get the required value of x, y and z.

In the matrix form, the equations can be written as $\begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

$$\therefore \quad AX = D \text{ where } A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

$\Rightarrow \quad A^{-1}(AX) = A^{-1}D \quad \Rightarrow \quad X = A^{-1}D$ ... (i)

$$\text{Now } A^{-1} = \frac{\text{adj}A}{|A|}; \quad |A| = \begin{vmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2(1-3) - 1(-2-1) + 2(6+1) = 13$$

$$\text{The matrix of cofactors of } |A| \text{ is } \begin{bmatrix} -2 & 3 & 7 \\ 7 & -4 & -5 \\ 3 & 2 & 4 \end{bmatrix}. \text{ So, adj } A = \begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}; \ A^{-1} = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}.$$

$$\therefore \quad \text{from (1), } X = \frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 0+70+15 \\ 0-40+10 \\ 0-50-20 \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix}; \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 85/13 \\ -30/13 \\ -70/13 \end{bmatrix}.$$

$$\Rightarrow \quad x = \frac{85}{13}, y = \frac{-30}{13}, z = \frac{-70}{13}$$

CONCEPTS

In general if r rows (or columns) become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of the given determinant.

Anvit Tawar (JEE 2009 AIR 9)

7.3 Some Important Results

The lines: $a_1x+b_1y+c_1=0$ ... (i)

$a_2x+b_2y+c_2=0$ ... (ii)

$a_3x+b_3y+c_3=0$ ... (iii)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$

This is the condition for the consistency of three simultaneous linear equations in 2 variables.

(a) $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines if

$abc+2fgh-af^2-bg^2-ch^2=0=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(b) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is : $D=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. If D = 0 then the three points are collinear.

(c) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$.

(d) If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the determinants.

E.g., $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

It should be noted that while applying operations on determinants at least one row (or column) must remain unchanged i.e.

Maximum number of simultaneous operations = order of determinant – 1

CONCEPTS

Always expand a determinant along a row or a column with maximum zeros.

To find the value of the determinant, the following steps are taken.

Take any row (or column); the value of the determinant is the sum of products of the elements of the row (or column) and the corresponding determinant obtained by omitting the row and the column of the elements with a proper sign, given by $(-1)^{p+q}$ where p and q are the no. of row and the no. of column respectively.

Vaibhav Krishnan (JEE 2009 AIR 22)

8. DIFFERENTIATION AND INTEGRATION OF DETERMINANTS

Let $\Delta(x)=\begin{vmatrix} f_1(x) & g_1(x)\\ f_2(x) & g_2(x)\end{vmatrix}$, where $f_1(x), f_2(x), g_1(x)$ and $g_2(x)$ are functions of x. Then,

$$\Delta'(x)=\begin{vmatrix} f_1'(x) & g_1'(x)\\ f_2(x) & g_2(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & g_1(x)\\ f_2'(x) & g_2'(x)\end{vmatrix} \quad \text{Also, } \Delta'(x)=\begin{vmatrix} f_1'(x) & g_1(x)\\ f_2'(x) & g_2(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & g_1'(x)\\ f_2(x) & g_2'(x)\end{vmatrix}$$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged. If we write $\Delta(x)=[C_1 \quad C_2]$, where C_i denotes the i^{th} column, then $\Delta'(x)=[C_1' \quad C_2]+[C_1 \quad C_2']$, where C_i' denotes the column obtained by differentiating functions in the i^{th} column C_i. Also, if $\Delta(x)=\begin{bmatrix} R_1\\ R_2\end{bmatrix}$, then $\Delta'(x)=\begin{bmatrix} R_1'\\ R_2\end{bmatrix}+\begin{bmatrix} R_1\\ R_2'\end{bmatrix}$ Similarly, we can differentiate determinants of higher order.

Note: Differentiation can also be done column wise by taking one column at a time.

If f(x), g(x) and h(x) are functions of x and a, b, c, α, β and γ are constants such that

$$\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x)\\ a & b & c\\ \alpha & \beta & \gamma\end{vmatrix}, \text{ then the integration of } \Delta(x) \text{ is given by } \int \Delta(x)dx=\begin{vmatrix} \int f(x)dx & \int g(x)dx & \int h(x)dx\\ a & b & c\\ \alpha & \beta & \gamma\end{vmatrix}$$

Illustration 22: If $\Delta(x)=\begin{vmatrix} \sin^2 x & \log\cos x & \log\tan x\\ n^2 & 2n-1 & 2n+1\\ 1 & -2\log 2 & 0\end{vmatrix}$, then evaluate $\int_0^{\pi/2}\Delta(x)\,dx$. **(JEE MAIN)**

Sol: By applying integration on variable elements of determinant we will solve the given problem.

We have, $\Delta(x)=\begin{vmatrix} \sin^2 x & \log\cos x & \log\tan x\\ n^2 & 2n-1 & 2n+1\\ 1 & -2\log 2 & 0\end{vmatrix}$; $\int_0^{\pi/2}\Delta(x)\,dx=\begin{vmatrix} \int_0^{\pi/2}\sin^2 x\,dx & \int_0^{\pi/2}\log\cos x\,dx & \int_0^{\pi/2}\log\tan x\,dx\\ n^2 & 2n-1 & 2n+1\\ 1 & -2\log 2 & 0\end{vmatrix}$

$$=\begin{vmatrix} \frac{\pi}{4} & -\frac{\pi}{2}\log 2 & 0\\ n^2 & 2n-1 & 2n+1\\ 1 & -2\log 2 & 0\end{vmatrix}=\frac{\pi}{4}\begin{vmatrix} 1 & -2\log 2 & 0\\ n^2 & 2n-1 & 2n+1\\ 1 & -2\log 2 & 0\end{vmatrix}=\frac{\pi}{4}\times 0=0$$

Illustration 23: If $f(x)=\begin{vmatrix} x^n & \sin x & \cos x\\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2}\\ a & a^2 & a^3\end{vmatrix}$, then show that $\frac{d^n}{dx^n}\{f(x)\}=0$ at x = 0. **(JEE ADVANCED)**

Sol: By applying integration on variable elements of the determinant we will solve the given problem.

We have, $f(x)=\begin{vmatrix} x^n & \sin x & \cos x\\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2}\\ a & a^2 & a^3\end{vmatrix}$; $\frac{d^n}{dx^n}\{f(x)\}=\begin{vmatrix} \frac{d^n}{dx^n}(x^n) & \frac{d^n}{dx^n}(\sin x) & \frac{d^n}{dx^n}(\cos x)\\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2}\\ a & a^2 & a^3\end{vmatrix}$

$$= \begin{vmatrix} n! & \sin\left(x+\frac{n\pi}{2}\right) & \cos\left(x+\frac{n\pi}{2}\right) \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}; \qquad \left(\frac{d^n}{dx^n}\{f(x)\}\right)_{x=0} = \begin{vmatrix} n! & \sin\frac{n\pi}{2} & \cos+\frac{n\pi}{2} \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix} = 0$$

PROBLEM-SOLVING TACTICS

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and in general

$\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ where n is any positive integer and $f^n(x)$ denotes the n^{th} derivative of f(x).

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$, where a, b, c, l, m and n are constants.

$$\Rightarrow \int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a & b & c \\ l & m & n \end{vmatrix}$$

If the elements of more than one column or rows are functions of x then the integration can be done only after evaluation/expansion of the determinant.

FORMULAE SHEET

(a) Determinant of order 3 × 3 $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ b_3 & b_3 \end{vmatrix}$

(b) In the determinant $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, minor of a_{12} is denoted as $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

(c) Cofactor of an element $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

(d) Properties of determinants:

(i) Reflection property: $\left|A_{i\times j}\right| = \left|A_{j\times i}\right|$

(ii) All-zero property: If all the elements of a row (or column) are zero, then the determinant is zero.

(iii) Proportionality (Repetition) Property: If all the elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then the determinant is zero.

(iv) Switching Property: The interchange of any two rows (or columns) of the determinant changes its sign.

(v) Scalar Multiple Property: If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

(vi) Sum Property: $\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$

(vii) Property of Invariance: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha b_1+\beta c_1 & b_1 & c_1 \\ a_2+\alpha b_2+\beta c_2 & b_2 & c_2 \\ a_3+\alpha b_3+\beta c_3 & b_3 & c_3 \end{vmatrix}$

That is, a determinant remains unaltered under an operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$, where $j, k \neq i$, or an operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$, where $j, k \neq i$.

(viii) Triangle Property: $\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3$

(e) Cramer's rule : if $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$ and $a_3x+b_3y+c_3z=d_3$ then $x=\frac{\Delta_1}{\Delta}, y=\frac{\Delta_2}{\Delta}, z=\frac{\Delta_3}{\Delta}$ where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

And if $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ then $x=\frac{\Delta_1}{\Delta}\ y=\frac{\Delta_2}{\Delta}$.

Where $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$ and $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

(f) **(i)** lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(ii) $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(iii) area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is : $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

(iv) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(g) If $\Delta(x)=\begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x)\end{vmatrix}$ then $\Delta'(x)=\begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x)\end{vmatrix}$ or $\begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x)\end{vmatrix}$

(h) If $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ \alpha & \beta & \gamma\end{vmatrix}$ then $\int\Delta(x)dx=\begin{vmatrix} \int f(x)dx & \int g(x)dx & \int h(x)dx \\ a & b & c \\ \alpha & \beta & \gamma\end{vmatrix}$

Solved Examples

JEE Main/Boards

Example 1: Prove that

$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa\end{vmatrix}=pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a\end{vmatrix}$. Use p + q + r = 0.

Sol: By using the expansion formula of determinants we can prove this.

L.H.S. $=\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa\end{vmatrix}=$

$pa\begin{vmatrix} ra & pb \\ pc & qa\end{vmatrix}-qb\begin{vmatrix} qc & pb \\ rb & qa\end{vmatrix}+rc\begin{vmatrix} qc & ra \\ rb & pc\end{vmatrix}$

$=pa(a^2qr-p^2bc)-qb(q^2ac-prb^2)+rc(pqc^2-r^2ab)$

$=a^3pqr-p^3abc-q^3abc+b^3pqr-r^3abc$

$=pqr(a^3+b^3+c^3)-abc(p^3+q^3+r^3)$

$\because\ p+q+r=0$ ... (given)

$(p+q+r)^3=0$

$\Rightarrow p^3+q^3+r^3-pqr=0 \Rightarrow p^3+q^3+r^3=3pqr$

$\Rightarrow$ L.H.S. $=pqr(a^3+b^3+c^3)-abc(3pqr)$

$\Rightarrow$ L.H.S. $=pqr(a^3+b^3+c^3-3abc)$ .... (i)

R.H.S. $=pqr\left[a\begin{vmatrix} a & b \\ c & a\end{vmatrix}-b\begin{vmatrix} c & b \\ b & a\end{vmatrix}+c\begin{vmatrix} c & a \\ b & c\end{vmatrix}\right]$

$=pqr[a(a^2-bc)-b(ca-b^2)+c(c^2-ab)]$

$=pqr[a^3-abc-abc+b^3+c^3-abc]$

$\Rightarrow$ R.H.S. $=pqr(a^3+b^3+c^3-3abc)$ .... (ii)

From eq. (i) and (ii), we get

$\therefore$ L.H.S. = R.H.S.

Example 2: Prove that the determinant

$\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x\end{vmatrix}$ is independent of θ.

Sol: Simply by expanding the given determinant we can prove it.

We have, $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x\end{vmatrix}$

$=x\begin{vmatrix} -x & 1 \\ 1 & x\end{vmatrix}-\sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x\end{vmatrix}+\cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1\end{vmatrix}$

$x(-x^2-1)-\sin\theta(-x\sin\theta-\cos\theta)+\cos\theta(-\sin\theta+x\cos\theta)$

$=-x^3-x+x\sin^2\theta+\sin\theta\cos\theta-\sin\theta\cos\theta+x\cos^2\theta$

$=-x^3-x+x(\sin^2\theta+\cos^2\theta)=-x^3-x+x$

Thus, the determinant is independent of θ.

Example 3: Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a\end{vmatrix}=0$, $a\neq 0$.

Sol: We can expand the above determinant by applying the invariance and scalar multiple properties, and hence we can easily solve this problem.

We have, $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

Operation: $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0 \Rightarrow (3x+a)\begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} = 0$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

We get $(3x+a)\begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0$

$$\Rightarrow \quad (3x+a)\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = 0 \qquad \Rightarrow \quad a^2(3x+a) = 0$$

$$\Rightarrow \; 3x + a = 0, \; [\because a \neq 0] \Rightarrow \; x = -\frac{a}{3}$$

Hence Proved.

Example 4: Solve, using Cramer's rule 3x – 2y + 4z = 5; x + y + 3z = 2; –x + 2y – z = 1

Sol: By defining D, D_1, D_2, D_3 and by using Cramer's Rule we will get required result.

$$D = \begin{vmatrix} 3 & -2 & 4 \\ 1 & 1 & 3 \\ -1 & 2 & -1 \end{vmatrix} = -5$$

$$D_1 = \begin{vmatrix} 5 & -2 & 4 \\ 2 & 1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = -33, \; D_2 = \begin{vmatrix} 3 & 5 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & -1 \end{vmatrix} = -13$$

$$D_3 = \begin{vmatrix} 3 & -2 & 5 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{vmatrix} = 12$$

By Cramer's Rule, $x = \frac{D_1}{D} = \frac{-33}{-5} = \frac{33}{5}$,

$$y = \frac{D_2}{D} = \frac{-13}{-5} = \frac{13}{5}; \; z = \frac{D_3}{D} = \frac{12}{-5} = \frac{-12}{5}$$

Example 5: Solve the following system of equations by Cramer's Rule

$$2x - y + 3z = 9; \quad x + y + z = 6; \quad x - y + z = 2$$

Sol: By defining $\Delta, \Delta_x, \Delta_y, \Delta_z$ and by using Cramer's Rule we will get the required result.

Here, $\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

=2(1 + 1) + 1 (1 – 1) + 3 (–1 – 1) = – 2,

$$\Delta_x = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 9(1+1) + 1(6-2) + 3(-6-2) = -2$$

$$\Delta_y = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2(6-2) - 9(1-1) + 3(2-6) = -4$$

$$\Delta_z = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = 2(2+6) + 1(2-6) + 9(-1-1) = -6$$

By Cramer's Rule

$$x = \frac{\Delta_x}{\Delta} = 1, \; y = \frac{\Delta_y}{\Delta} = 2, \; z = \frac{\Delta_z}{\Delta} = 3$$

Example 6: Show that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Sol: By using invariance and scalar multiple property we can expand given determinant and can prove it.

$$\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$[C_1 \to C_1 + C_2 + C_3]$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & a & c+a+b \end{vmatrix}$$

[by $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= 2(a+b+c)[1\{(b+c+a)^2 - 0\}]$$

$$= 2(a+b+c)(a+b+c)^2 = 2(a+b+c)^3$$

Example 7: Using determinants, show that the points (11, 7), (5, 5) and (–1, 3) are collinear.

Sol: If these points are collinear then the area of a triangle made by joining these points will be zero.

The area of the triangle formed by the given points

$$=\frac{1}{2}\begin{vmatrix}11 & 7 & 1\\ 5 & 5 & 1\\ -1 & 3 & 1\end{vmatrix}$$

Operate: $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$

$$=\frac{1}{2}\begin{vmatrix}6 & 2 & 0\\ 6 & 2 & 0\\ -1 & 3 & 1\end{vmatrix}=\frac{1}{2}\cdot 0=0$$

($\because$ R_1 and R_2 are identical)

Hence, the given points are collinear.

Example 8: If A and B are two matrices such that AB = B and BA = A, then $A^2 + B^2$.

Sol: By using the multiplication property of matrices we can solve given problem.

$A^2 + B^2 = AA + BB$

= A(BA) + B(AB) [Given AB = B and BA = A]

= (AB)A + (BA)B

[Matrix multiplication is associatively]=BA + AB

[Given AB = B and BA = A] =A+B

[Given AB = B and BA = A]

Example 9: Find the value of $\begin{vmatrix}1^2 & 2^2 & 3^2 & 4^2\\ 2^2 & 3^2 & 4^2 & 5^2\\ 3^2 & 4^2 & 5^2 & 6^2\\ 4^2 & 5^2 & 6^2 & 7^2\end{vmatrix}$

Sol: By applying the invariance property we can find the value of the given determinant.

$$\begin{vmatrix}1^2 & 2^2 & 3^2 & 4^2\\ 2^2 & 3^2 & 4^2 & 5^2\\ 3^2 & 4^2 & 5^2 & 6^2\\ 4^2 & 5^2 & 6^2 & 7^2\end{vmatrix}=\begin{vmatrix}1 & 4 & 9 & 16\\ 4 & 9 & 16 & 25\\ 9 & 16 & 25 & 36\\ 16 & 25 & 36 & 49\end{vmatrix}=\begin{vmatrix}1 & 4 & 9 & 16\\ 3 & 5 & 7 & 9\\ 5 & 7 & 9 & 11\\ 7 & 9 & 11 & 13\end{vmatrix}$$

[Applying $R_4 \to R_4 - R_3$, $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$]

$$=\begin{vmatrix}1 & 4 & 9 & 16\\ 3 & 5 & 7 & 9\\ 2 & 2 & 2 & 2\\ 2 & 2 & 2 & 2\end{vmatrix}$$

[Applying $R_4 \to R_4 - R_3$, $R_3 \to R_3 - R_2$]

=0

JEE Advanced/Boards

Example 1: Without expanding, evaluate the determinant

$$\begin{vmatrix}\sin\alpha & \cos\alpha & \sin(\alpha+\delta)\\ \sin\beta & \cos\beta & \sin(\beta+\delta)\\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta)\end{vmatrix}$$

Sol: By using the formula sin(A+ B) = sin A cos B + cos A sin B and invariance property of determinants we can expand the given determinant.

$$\text{Let } \Delta=\begin{vmatrix}\sin\alpha & \cos\alpha & \sin(\alpha+\delta)\\ \sin\beta & \cos\beta & \sin(\beta+\delta)\\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta)\end{vmatrix}$$

$$\Rightarrow \Delta=\begin{vmatrix}\sin\alpha & \cos\alpha & \sin\alpha\cos\delta+\cos\alpha\sin\delta\\ \sin\beta & \cos\beta & \sin\beta\cos\delta+\cos\beta\sin\delta\\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta+\cos\gamma\sin\delta\end{vmatrix}$$

$$\Rightarrow \Delta=\begin{vmatrix}\sin\alpha & \cos\alpha & 0\\ \sin\beta & \cos\beta & 0\\ \sin\gamma & \cos\gamma & 0\end{vmatrix}$$

[Applying $C_3 \to C_3 - \cos\delta\,.\,C_1 - \sin\delta\,.\,C_2$]

$\Rightarrow \Delta = 0$ [$\because$ C_3 consists of all zeroes]

Example 2: By using properties of determinants prove that

$$\begin{vmatrix}1 & x & x^2\\ x^2 & 1 & x\\ x & x^2 & 1\end{vmatrix}=(1-x^3)^2$$

Here in this problem by using invariance and scalar multiple properties we will expand the given determinant and we will prove it.

Sol: L.H.S. = $\begin{vmatrix}1 & x & x^2\\ x^2 & 1 & x\\ x & x^2 & 1\end{vmatrix}=\begin{vmatrix}1+x+x^2 & x & x^2\\ 1+x+x^2 & 1 & x\\ 1+x+x^2 & x^2 & 1\end{vmatrix}$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$$=(1+x+x^2)\begin{vmatrix}1 & x & x^2\\ 1 & 1 & x\\ 1 & x^2 & 1\end{vmatrix}$$

$$=(1+x+x^2)\begin{vmatrix}1 & x & x^2\\ 0 & 1-x & x-x^2\\ 0 & x^2-x & 1-x^2\end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (1+x+x^2)(1)\{(1-x)(1-x^2)-(x^2-x)(x-x^2)\}$

$= (1+x+x^2)(1-x)^2\{1+x+x^2\}$

$= \{(1-x)(1+x+x^2)\}^2 = (1-x^3)^2$ = R.H.S.

Example 3: Show that $x = -(a+b+c)$ is one root of the equation: $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$ and solve the equation completely.

Sol: We can expand given determinant using the invariance and scalar multiple properties and by solving we will find out required result.

By $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+c & a \\ x+a+b+c & a & x+b \end{vmatrix} = 0$$

$$\Rightarrow (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+c & a \\ 1 & a & x+b \end{vmatrix} = 0$$

$$\Rightarrow (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x-b+c & a-c \\ 0 & a-b & x+b-c \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1;\ R_3 \to R_3 - R_1$

On expanding by first column, we get

$(x+a+b+c)[(x-b+c)(x+b-c)-(a-b)(a-c)] = 0$

$\Rightarrow (x+a+b+c)[x^2-(b-c)^2-(a^2-ac-ab+bc)] = 0$

$\Rightarrow (x+a+b+c)(x^2-b^2-c^2+2bc-a^2+ac+ab-bc] = 0$

$\Rightarrow (x+a+b+c)(x^2-a^2-b^2-c^2+ab+bc+ca) = 0$

Either $x + a + b + c = 0 \Rightarrow x = -(a + b + c)$

or $\quad x^2-a^2-b^2-c^2+ab+bc+ca = 0$

$\Rightarrow \quad x = \pm\sqrt{a^2+b^2+c^2-ab-bc-ca}$

Example 4: If the area of a triangle is 35 sq. units with vertices (2, –6), (5, 4) and (k, 4), then find k.

Sol: As we know that the area of the triangle = $\frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$ where (a, b) (c, d) (e, f) are the vertices of triangle. Therefore by substituting the value of vertices we will get required result.

Let the vertices of triangle be A(2, –6), B(5, 4) and C(k, 4). Since the area of the triangle ABC is 35 sq. units, we have, $\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = \pm 35 \Rightarrow \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 3 & 10 & 0 \\ k-2 & 10 & 0 \end{vmatrix} = \pm 35$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$\Rightarrow \frac{1}{2}\begin{vmatrix} 3 & 10 \\ k-2 & 10 \end{vmatrix} = \pm 35$ [Expanding along C_3]

$\Rightarrow \quad \frac{1}{2}\{30-10(k-2)\} = \pm 35$

$\Rightarrow \quad 30-10k+20 = \pm 70 \Rightarrow \quad 10k = 50 \mp 70$

$\Rightarrow \quad k = +12$ or $k = -2$

Example 5: Solve the following system of equations by using determinants: $x+y+z=1$,

$ax+by+cz=k;\ a^2x+b^2y+c^2z=k^2$

Sol: Here in this problem first define D, D_1, D_2 and D_3. then by using Cramer's rule we can solve it.

We have, $D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a).1.\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

[Expanding along R_1]

$= (b-a)(c-a)(c+a-b-a)$

$= (b-a)(c-a)(a-b)$ ... (i)

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-k)(k-b)$$

[Replacing a by k in (i)]

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix} = (k-c)(c-a)(a-k)$$

[Replacing b by k in (i)]

and $$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix} = (a-b)(b-k)(k-a)$$

[Replacing c by k in (i)]

$$\therefore \; x = \frac{D_1}{D} = \frac{(b-c)(c-k)(k-b)}{(b-c)(c-a)(a-b)} = \frac{(c-k)(k-b)}{(c-a)(a-b)},$$

$$y = \frac{D_2}{D} = \frac{(k-c)(c-a)(a-k)}{(b-c)(c-a)(a-b)} = \frac{(k-c)(a-k)}{(b-c)(a-b)}$$

$$z = \frac{D_3}{D} = \frac{(a-b)(b-k)(k-a)}{(b-c)(c-a)(a-b)} = \frac{(k-a)(b-k)}{(c-a)(b-c)}$$

Example 6: Show that

$$\begin{vmatrix} 1+a_1+b_1 & a_1+b_2 & a_1+b_3 \\ a_2+b_1 & 1+a_2+b_2 & a_2+b_3 \\ a_3+b_1 & a_3+b_2 & 1+a_3+b_3 \end{vmatrix}$$

$$= 1 + \sum_{i=1}^{3}(a_i+b_i) + \sum_{1\le i<j\le 3}(a_i-a_j)(b_j-b_i)$$

Sol: By putting $\alpha = a_1 - a_2$, $\beta = a_2 - a_3$, then $\alpha + \beta = a_1 - a_3$, $u = b_1 - b_2$, $v = b_2 - b_3$, then $u + v = b_1 - b_3$. Using the invariance property expand the given determinant, and then comparing it to the R.H.S. of the given problem we can prove it.

Let Now R.H.S.

$$= 1 + \sum_{i=1}^{3}(a_i+b_i) + \sum_{1\le i<j\le 3}(a_i-a_j)(b_j-b_i)$$

$$= 1 + (a_1+b_1+a_2+b_2+a_3+b_3) + (a_1-a_2)(b_2-b_1)$$

$$+ (a_2-a_3)(b_3+b_2) + (a_1-a_3)(b_3-b_1)$$

$$= 1 + (a_1+b_1+a_2+b_2+a_3+b_3) - \alpha u - \beta v - (\alpha+\beta)(u+v)$$

$$= 1 + (a_1+b_1+a_2+b_2+a_3+b_3) - 2\alpha u - 2\beta v - \beta u - \alpha v \quad .... (i)$$

Now L.H.S. $$= \begin{vmatrix} 1+\alpha & \alpha-1 & \alpha \\ \beta & 1+\beta & \beta-1 \\ a_3+b_1 & a_3+b_2 & 1+a_3+b_3 \end{vmatrix}$$

$[R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$

$$= \begin{vmatrix} 2 & -1 & \alpha \\ -1 & 2 & \beta-1 \\ u & v-1 & 1+a_3+b_3 \end{vmatrix}$$

$[C_1 \to C_1 - C_2, C_2 \to C_2 - C_3]$

$$= \begin{vmatrix} 0 & -1 & \alpha \\ 3 & 2 & \beta-1 \\ u+2v-2 & v-1 & 1+a_3+b_3 \end{vmatrix} \quad [C_1 \to C_1 + 2C_2]$$

$$= [3(1+a_3+b_3) - (u+2v-2)(\beta-1)]$$

$$+ a_2[3(v-1) - 2(u+2v-2)]$$

$$= 3 + 3(a_3+\beta_3) - u\beta - 2v\beta + 2\beta + u + 2v$$

$$-2 + \alpha(-v+1-2u)$$

$$= 1 + 3(a_3+b_3) + 2\beta + u + 2v + \alpha - u\beta \;\; -2v\beta - \alpha v - 2u\alpha$$

$$= 1 + 3(a_3+\beta_3) + 2(a_2-a_3) + b_1 - b_2 + 2(\beta_2 - b_3)$$

$$+ a - a - 2\alpha u - 2\beta v - u\beta - v\alpha$$

$$= 1 + (a_1+b_1+a_2+b_2+a_3+b_3) - 2\alpha u \;\; -2\beta v - u\beta - v\alpha$$

= RHS [From (i)]

Example 7: Find values of c for which the equations $2x + 3y = 3$; $(c+2)x + (c+4)y = c+6$ $(c+2)^2 x + (c+4)^2 y = (c+6)^2$ are consistent and hence solve the equation.

Sol: Here in this problem first define given equations as Δ and solve it as $\Delta = 0$ by using the invariance method.

The equation will be consistent, if

$$\begin{vmatrix} 2 & 3 & 3 \\ c+2 & c+4 & c+6 \\ (c+2)^2 & (c+4)^2 & (c+6)^2 \end{vmatrix} = 0$$

Applying $C_3 \to C_3 - C_2$, we get

$$\begin{vmatrix} 2 & 3 & 0 \\ c+2 & c+4 & 2 \\ (c+2)^2 & (c+4)^2 & 4(c+5) \end{vmatrix} = 0$$

Solving, we get $c^2 + 10c = 0$

or $c = 0, -10$ ... (i)

If c = 0, the system of equations becomes

$$\begin{cases} 2x + 3y = 3 \\ 2x + 4y = 6 \end{cases} \Rightarrow x = -3,\; y = 3 \quad .... (ii)$$

If c = –10, then system of equations becomes

$$\begin{cases} 2x+3y=3 \\ -8x-6y=-4 \end{cases} \Rightarrow x = -\frac{1}{2},\ y=\frac{4}{3} \quad \text{.... (iii)}$$

16x + 9y = 4

Hence the solutions are given by (ii) and (iii).

Example 8: If (a_r, b_r), r = 1, 2, 3 be the vertices of a triangle, prove that

$$\Delta = \begin{vmatrix} a_2-a_3 & b_2-b_3 & a_1(a_2-a_3)+b_1(b_2-b_3) \\ a_3-a_1 & b_3-b_1 & a_2(a_3-a_1)+b_2(b_3-b_1) \\ a_1-a_2 & b_1-b_2 & a_3(a_1-a_2)+b_3(b_1-b_2) \end{vmatrix} = 0 \quad \cdots \text{(i)}$$

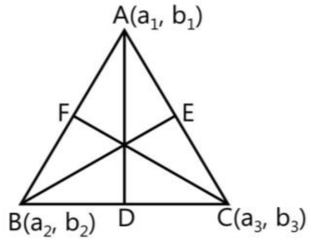
and hence show that the altitudes of a triangle are concurrent.

Sol: Using the invariance method we can expand the given determinant and using the equations of altitude we can prove it

$$\Delta = \begin{vmatrix} a_2-a_3 & b_2-b_3 & a_1(a_2-a_3)+b_1(b_2-b_3) \\ a_3-a_1 & b_3-b_1 & a_2(a_3-a_1)+b_2(b_3-b_1) \\ a_1-a_2 & b_1-b_2 & a_3(a_1-a_2)+b_3(b_1-b_2) \end{vmatrix} = 0$$

[Applying $R_1 \to R_1 + R_2 + R_3$]

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ a_3-a_1 & b_3-b_1 & a_2(a_3-a_1)+b_2(b_3-b_1) \\ a_1-a_2 & b_1-b_2 & a_3(a_1-a_2)+b_3(b_1-b_2) \end{vmatrix}$$



$\therefore$ Equation of altitude AD is:

$$y-b_1 = -\frac{a_2-a_3}{b_2-b_3}(x-a_1)$$

or $x(a_2-a_3)+y(b_2-b_3) = a_1(a_2-a_3)+b_1(b_2-b_3)$ ... (ii)

Similarly equation of altitudes BE and CF are

$$x(a_3-a_1)+y(b_3-b_1) = a_2(a_3-a_1)+b_2(b_3-b_1) \quad \text{.... (iii)}$$

$$x(a_1-a_2)+y(b_1-b_2) = a_3(a_1-a_2)+b_3(b_1-b_2) \quad \text{.... (iv)}$$

Altitudes (ii), (iii), (iv) are concurrent, since the determinant given by L.H.S. of (i) is zero.

Example 9: Let λ and α be real. Find the set of all values of λ and α for which the system of linear equations $\lambda x+(\sin\alpha)y+(\cos\alpha)z=0$

$x+(\cos\alpha)y+(\sin\alpha)z=0$ $-x+(\sin\alpha)y-(\cos\alpha)z=0$ has a non-trivial solution. If $\lambda=1$, find all values of α.

Sol: Here in this problem first define the given equations as Δ and as we know that for non-trivial solution $\Delta=0$.

For non-trivial solution, condition is $\Delta=0$.

$$\Delta = \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

or $\lambda[-\cos^2\alpha-\sin^2\alpha] - \sin\alpha\,[-\cos\alpha+\sin\alpha]$

$+\cos\alpha\,[\sin\alpha+\cos\alpha]=0$

or $\lambda=\sin 2\alpha+\cos 2\alpha \quad \therefore \quad \alpha\in R;\ |\lambda|\le\sqrt{2}$

If $\lambda=1$, then $1=\sin 2\alpha+\cos 2\alpha$

$$\cos\left(2\alpha-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$$

$$\Rightarrow 2\alpha-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4} : n\in I \Rightarrow \alpha = n\pi\pm\frac{\pi}{8}+\frac{\pi}{8} : n\in I$$

Example 10: For a fixed positive integer n, if

$$\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2!) & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

then show that $\left[\frac{\Delta}{(n!)^3}-4\right]$ is divisible by n.

Sol: By using the scalar multiple property of determinants we can take $(n!)^3$,(n+1) and (n + 2) common and using the invariance property we can solve the given problem.

$$\Delta = (n!)^3\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ n+1 & (n+2)(n+1) & (n+3)(n+2)(n+1) \\ (n+2)(n+1) & (n+3)(n+2)(n+1) & (n+4)(n+3)(n+2)(n+1) \end{vmatrix}$$

Taking (n+1) and (n + 2) (n + 1) common from C_2 and C_3 respectively, we get

$\Delta=(n!)^3(n+2)(n+1)$

$$\begin{vmatrix} 1 & 1 & 1 \\ n+1 & n+2 & n+3 \\ (n+2)(n+1) & (n+3)(n+2) & (n+4)(n+3) \end{vmatrix}$$

[Apply $C_3 \to C_3 - C_1$ and $C_2 \to C_2 - C_1$ then

$$\begin{vmatrix} 1 & 0 & 0 \\ n+1 & 1 & 2 \\ (n+2)(n+1) & 2(n+2) & 4n+10 \end{vmatrix}$$

$= (n!)^3 (n+1)^2 (n+2)[4n+10-4(n+2)]$

$\Delta = (n!)^3 (n^2+2n+1)(2n+4) = (n!)^3 (2n^3+8n^2+10n+4)$

$\therefore \left[\dfrac{\Delta}{(n!)^3} - 4\right] = 2n^3+8n^2+10n$,

$2n(n^2+4n+5)$, which is divisible by n.

JEE Main/Boards

Exercise 1

Q.1 Find x, if $\begin{vmatrix} -1 & 2 \\ 4 & 8 \end{vmatrix} = \begin{vmatrix} 2 & x \\ x & -4 \end{vmatrix}$.

Q.2 It matrix A = $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, find [A].

Q.3 Given $\begin{vmatrix} 4 & -1 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix}$, find (i) M_{23} (ii) C_{32}.

Q.4 Area of a triangle with vertices (k, 0), (1, 1) and (0, 3) is 5 sq. units. Find the value(s) of k.

Q.5 Find the area of a triangle, whose vertices are (0, 3) (–1, 4), (2, 6).

Q.6 Given determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Find the value of $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{33}$.

Q.7 Find the value of p, such that the matrix $\begin{bmatrix} -1 & 24 \\ 4 & p \end{bmatrix}$ is singular.

Q.8 Given I_2. Find $|I_2|$. Also find $|3I_2|$.

Q.9 Find the value of x, such that the points (0,2), (1, x), (3, 1) are collinear.

Q.10 For two given square matrices A and B of the same order, such that |A| = 20 and |B| = –20, find |AB|.

Q.11 Find the adjoint of matrix A = $\begin{bmatrix} 3 & 1 \\ -5 & 4 \end{bmatrix}$.

Q.12 Find the inverse of matrix $\begin{bmatrix} 1 & 3 \\ -6 & -18 \end{bmatrix}$, if possible.

Q.13 Without expanding, find the value of $\begin{vmatrix} 3 & 1 & 8 \\ -4 & 2 & 16 \\ -5 & 3 & 24 \end{vmatrix}$.

Q.14 If a = $\begin{pmatrix} x & 0 & 1 \\ 2 & -1 & 4 \\ 1 & 2 & 0 \end{pmatrix}$ is a singular matrix, find x.

Q.15 Find the area of the triangle whose vertices are (3, 1) (4, 3) and (–5, 4).

Q.16 Find the value of x, if area of triangle is 35 square cms with vertices (x, 4), (2, –6) and (5, 4).

Q.17 Show that the following determinant vanishes:

$$\begin{vmatrix} 5 & 15 & -25 \\ 7 & 21 & 30 \\ 8 & 24 & 42 \end{vmatrix}.$$

Q.18 Using properties of determinants, prove that :

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$$

Q.19 If points (2, 0), (0, 5) and (x, y) are collinear, then show that $\dfrac{x}{2} + \dfrac{y}{5} = 1$.

Q.20 If for matrix A, |A| = 3 find |5A|, where matrix A is of order 2 × 2.

Q.21 Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, such that |A| = –10. Find $a_{11}C_{11} + a_{12}C_{12}$.

Q.22 Without expanding prove that, the value of determinant

$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ is zero.

Q.23 A is a non-singular matrix of order 3 and |A| = –4. Find |adj.A|.

Q.24 Is it possible to find the inverse of a matrix $\begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \end{bmatrix}$? Given reasons.

Q.25 Given a square matrix A of order 3 × 3, such that |A| = 12, find the value of |A.adj. A|.

Q.26 Compute A^{-1} for the matrix $\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}$ and show that $A^{-1} = \frac{1}{19}A$.

Q.27 Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Verify that (i) $(\text{adj } A)^{-1}$ (ii) $(A^{-1})^{-1} = A$.

Q.28 Using matrix method, examine the system of equations: 2x + 5y = 7, 6x + 15y = 13 for consistency.

Q.29 Find the inverse of matrix $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$ and show that $aA^{-1} = (a^2 + bc + 1)I - aA$.

Q.30 If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$,

show that $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Q.31 If $A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$, prove that $A^{-1} = A^2 - 6A + 11\,I$.

Q.32 If $A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & -1 \end{bmatrix}$,

verify that $(AB)^{-1} = B^{-1}A^{-1}$.

(33-38) Using properties of determinant, prove that

Q.33 $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a + b + c)(ab + bc + ca - a^2 - b^2 - c^2)$

$= 3abc - a^3 - b^3 - c^3$.

Q.34 $\begin{vmatrix} y+z & x & y \\ z+x & y & x \\ x+y & z & z \end{vmatrix} = (x + y + z)(x - z)^2$.

Q.35 $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$.

Q.36 $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ac \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)(a^2 + b^2 + c^2)$.

Q.37 $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$.

Q.38 $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2 - ac)(ax^2+2bxy+cy^2)$

Q.39 Write the minors and cofactors of the elements of second row of the following determinant:

$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$.

Q.40 Find the quadratic function defined by the equation $f(x) = ax^2 + bx + c$, if f(0) = 6(2) = 11 and f(–3) = 6, using determinants.

Q.41 Examine whether the system of equations: 2x – y = 5, 4x – 2y = 10 is consistent or inconsistent.

Q.42 Verify, whether the system of equations: 3x – y – 2z = 2, 2y – z = –1, 3x – 5y = 3 is consistent or inconsistent.

Without expanding the determinants, show that

Q.43 $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

Q.44 $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Q.45 $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} = 0$

Q.46 Solve for x, $\begin{vmatrix} x^2 & 0 & 3 \\ x & 1 & -4 \\ 1 & 2 & 0 \end{vmatrix} = 11$.

Q.47 If $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, verify that $A^2 - 4A - I = O = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and hence find A^{-1}.

Q.48 Evaluate : $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix}$.

Q.49 Show that {–(a + b + c)} is root of the following equation:

$$\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0.$$

Q.50 Using properties of determinants, prove that : $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$.

Q.51 Using properties of determinants, prove that:

$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3.$$

Q.52 Show that $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ satisfies the equation $x^2 - 6x + 17 = 0$.

Hence find A^{-1}.

Q.53 Find matrix A if, $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} A \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 3 & -1 \end{bmatrix}$.

Q.54 Given $A\begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B^{-1}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Compute $(AB)^{-1}$.

Q.55 For the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, verify that $A^3 - 6A^2 + 9A - 4I = O$, hence find A^{-1}.

Exercise 2

Single Correct Choice Type

Q.1 If a, b, c are all different from zero and $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$, then the value of $a^{-1} + b^{-1} + c^{-1}$ is:

(A) abc (B) $a^{-1}\, b^{-1}\, c^{-1}$

(C) –a – b – c (D) –1

Q.2 If a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then

(A) abc (ab + bc + ca) = a + b + c

(B) (a + b + c) (ab + bc + ca) = abc

(C) abc (a + b + c) = ab + bc + ca

(D) None of these

Q.3 If $(\sin^{-1} x + \sin^{-1} w)(\sin^{-1} y + \sin^{-1} z) = p^2$, then $\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix}$; $(N_1, N_2, N_3, N_4 \in N)$

(A) Has a maximum value 2.

(B) Has a minimum value 2.

(C) In independent of N_1, N_2, N_3, N_4

(D) None of these

Q.4 If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$ then $\begin{vmatrix} a_{n-3} & a_{n-1} & a_{n+1} \\ a_{n-6} & a_{n-3} & a_{n+3} \\ a_{n-14} & a_{n-7} & a_{n+7} \end{vmatrix}$ is

(A) 1 (B) 2 (C) 0 (D) –1

Q.5 The absolute value of the determinant $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ is

(A) $16\sqrt{2}$ (B) $8\sqrt{2}$ (C) 8 (D) None of these

Q.6 $D_1 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, $D_2 = \begin{vmatrix} bc-a^2 & ac-b^2 & ab-c^2 \\ ac-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$,

$D_3 = \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{vmatrix}$ then

(A) $D_1 \le 0$, if a+b+c>0 (B) $D_2^2 = D_3$

(C) $D_1^2 = D_2 = D_3$ (D) $D_2 \ne D_3 = D_1^2$

Q.7 $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$, where a, b, c are distinct positive reals, then abc is always less than

(A) $\frac{1}{243}$ (B) $\frac{1}{729}$ (C) $\frac{1}{27}$ (D) $\frac{1}{81}$

Q.8 The value of 'a' for which the system of equations, $(a+1)^3x + (a+2)^3y = (a+3)^3$,

$\begin{vmatrix} \log_x xyz & \log_x y & \log_x z \\ \log_y xyz & 1 & \log_y z \\ \log_z xyz & \log_z y & 1 \end{vmatrix}$ and x + y = 1 are consistent is

(A) –2 (B) 1 (C) 0 (D) None

Q.9 The following system of equations 3x – 7y + 5z = 3; 3x + y + 5z = 7 and 2x + 3y + 5z = 5 are

(A) Consistent with trivial solution

(B) Consistent with unique non-trivial solution

(C) Consistent with infinite solution

(D) Inconsistent with no solution

Q.10 The system of equations $(\sin\theta)x + 2z = 0$, $(\cos\theta)x + (\sin\theta)y = 0$, $(\cos\theta)y + 2z = a$ has

(A) Non unique solution

(B) A unique solution which is a function of a and θ

(C) A unique solution which is independent of a and θ

(D) A unique solution which is independent of θ only

Q.11 The equation

$\begin{vmatrix} (1+x)^2 & (1-x)^2 & -(2+x^2) \\ 2x+1 & 3x & 1-5x \\ x+1 & 2x & 2-3x \end{vmatrix} +$

$\begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^3 & 3x & 2x \\ 1-2x & 3x-2 & 2x-3 \end{vmatrix} = 0$

(A) Has no real solution

(B) Has 4 real solutions

(C) Has two real and two non-real solutions

(D) Has infinite number of solutions, real or non-real

Q.12 The system of equation :

$2x\cos^2\theta + y\sin 2\theta - 2\sin\theta = 0$;

$x\sin 2\theta + 2y\sin^2\theta = -2\cos\theta$;

$x\sin\theta - y\cos\theta = 0$, for all values of θ, can

(A) Have a unique nontrivial solution

(B) Not have a solution

(C) Have infinite solutions

(D) Have a trivial solution

Q.13 If x, y, z are not all simultaneously equal to zero, satisfying the system of equations $(\sin 3\theta)x - y + z = 0$; $(\cos 2\theta)x + 4y + 3z = 0$; 2x + 7y + 7z = 0, then the number of principal values of θ is

(A) 2 (B) 4 (C) 5 (D) 6

Q.14 For a non-zero, real a, b and c

$$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = \alpha\, abc$$

then the value of α is

(A) –4 (B) 0 (C) 2 (D) 4

Q.15 Number of value of 'a' for which the system of equations, $a^2x+(2-a)y=4+a^2$; $ax+(2a-1)y=a^5-2$ possess no solution is

(A) 0 (B) 1 (C) 2 (D) Infinite

Previous Years' Questions

Q.1 The determinant

$$\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$$ ***(1997)***

(A) x, y, z are in AP (B) x, y, z are in GP

(C) x, y, z are in HP (D) xy, yz, zx are in AP

Q.2 If $f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$,

then f(100) is equal to ***(1999)***

(A) 0 (B) 1 (C) 100 (D) –100

Q.3 If the system of equations x – ky – z = 0, kx – y – z = 0, x + y – z = 0 has a non-zero solution, then possible values of k are ***(2000)***

(A) –1, 2 (B) 1, 2 (C) 0, 1 (D) –1, 1

Q.4 The number of distinct real roots of ***(2001)***

$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(A) 0 (B)2 (C) 1 (D) 3

Q.5 If the system of equations x + ay = 0, az + y = 0 and ax + z = 0 has infinite solutions, then the value of a is ***(2002)***

(A) –1 (B) 1

(C) 0 (D) No real values

Q.6 The number of values of k for which the system of equations (k + 1)x + 8y = 4k, kx+ (k + 3)y = 3k – 1 has infinite solutions.

Assertion Reasoning Type

(A) Both assertion and reason are true and reason is the correct explanation of Assertion.

(B) Both assertion and reason are true and reason is not the correct explanation of assertion.

(C) Assertion is true but reason is false

(D) Assertion is false but reason is true.

Q.7 Consider the system of equations x – 2y + 3z = –1, x – 3y + 4z = 1 and –x + y – 2z = k

Statement-I: The system of equation has no solution for $k \neq 3$. and

Statement-II: The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 0$. ***(2008)***

Q.8 Given, x = cy + bz, y = az + cx, z = bx + ay, where x, y, z are not all zero, prove that $a^2+b^2+c^2+2ab=1$ ***(1978)***

Q.9 If α be a repeated root of a quadratic equation f(x) = 0 and A(x), B(x) and C(x) be polynomials of degree 3, 4 and 5 respectively, then show that

$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by f(x), where prime denotes the derivatives. ***(1984)***

Q.10 If matrix $A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, where a, b, c are real positive number, abc = 1 and $A^TA=I$, then find the value of $a^3+b^3+c^3$. ***(2003)***

Q.11 The number of values of k, for which the system of equations: $(k+1)x+8y=4k$ $kx+(k+3)=3k-1$ Has no solution, is: ***(2013)***

(A) Infinite (B) 1 (C) 2 (D) 3

Q.12 If $\alpha, \beta \neq 0$, and $f(n)=\alpha^n+\beta^n$

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+g(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+g(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)(\alpha-\beta)^2,$$

then K is equal to: ***(2014)***

(A) $\alpha\beta$ (B) $\frac{\alpha}{\beta}$ (C) 1 (D) -1

Q.13 The set of all values of λ for which the system of linear equations:

$2x_1 - 2x_2 + x_3 = \lambda x_1$

$2x_1 - 3x_2 + 2x_3 = \lambda x_2$

$-x_1 + 2x_2 = \lambda x_3$

Has a non – trivial solution. ***(2015)***

(A) Is an empty set

(B) Is a singleton

(C) Contains two elements

(D) Contains more than two elements

Q.14 The system of linear equations ***(2016)***

$x + \lambda y - z = 0$

$\lambda x - y - z = 0$

$x + y - \lambda z = 0$

has a non-trivial solution for:

(A) Infinitely many values of λ

(B) Exactly one value of λ

(C) Exactly two values of λ

(D) Exactly three values of λ

JEE Advanced/Boards

Exercise 1

Q.1 Solve the following using Cramer's rule and state whether consistent or not.

(a) $x + 2y + z = 1$
$3x + y + z = 6$
$x + 2y = 0$

(b) $x + y + z - 6 = 0$
$2x + y - z - 1 = 0$
$x + y - 2z + 3 = 0$

(c) $7x - 7y + 5z = 3$
$3x + y + 5z = 7$
$2x + 3y + 3z = 5$

Q.2 For what value of K do the following system of equations possess a non-trivial (i.e. not all zero) solution over the set of rational Q? x + Ky + 3z = 0, 3x + Ky – 2z = 0, 2x + 3y – 4z = 0. For that value of K, find all the solutions of the system.

Q.3 The system of equations $\alpha x + y + z = \alpha - 1$ $x + \alpha y + z = \alpha - 1$; $x + y + \alpha z = \alpha - 1$ has no solution. Find α.

Q.4 If the equations a(y + z) = x, b(z + x) = y, c(x + y) = z have non-trivial solutions, then find the value of $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}$.

Q.5 Given x = cy + bz; y = az + cx; z = bx + ay where x, y, z are not all zero, prove that $a^2 + b^2 + c^2 + 2abc = 1$.

Q.6 Given $a = \frac{x}{y-z}$; $b = \frac{y}{z-x}$; $c = \frac{z}{x-y}$ where x, y, z are not all zero, prove that 1 + ab + bc + ca = 0.

Q.7 If $\sin q \neq \cos q$ and x, y, z satisfy the equations $x\cos p - y\sin p + z = \cos q + 1$

$x\sin p + y\cos p + z = 1 - \sin q$

$x\cos(p+q) - y\sin(p+q) + z = 2$

Then find the value of $x^2 + y^2 + z^2$.

Q.8 Investigate for what values of λ, μ the simultaneous equations x + y + z = 6; x+ 2y + 3z = 10 and $x + 2y + \lambda z = \mu$ have;

(a) A unique solution

(b) An infinite number of solutions

(c) No solution

Q.9 For what values of p, the equations: x + y + z = 1; x + 2y + 4z = p and x + 4y + 10z = p^2 have a solution? Solve them completely in each case.

Q.10 Solve the equations : Kx + 2y – 2z = 1; 4x + 2Ky – z = 2; 6x + 6y + Kz = 3 considering specially the case when K = 2.

Q.11 (a) Let a, b, c, d are distinct numbers to be chosen from the set {1, 2, 3, 4, 5}. If the least possible positive solution for x to the system of equation $\left.\begin{matrix} ax+by=1 \\ cx+dy=2 \end{matrix}\right\}$ can be expressed in the form $\frac{p}{q}$ where p and q are relatively prime, then find the value of (p + q).

(b) Find the sum of all positive integral values of a for which every solution to the system of equations x + ay = 3 and ax + 4y = 6 satisfy the inequalities x > 1, y > 0.

Q.12 If the following system of equations (a – t)x + by + cz = 0, bx + (c – t)y + az = 0 and cx + ay + (b – t)z = 0 has non-trivial solutions for different values of t, then show that we can express product of these values of t in the form of determinant.

Q.13 Show that the system of equations3x – y + 4z = 3, x + 2y – 3z = –2 and 6x + 5y + λz = –3 has atleast one solution for any real number λ. Find the set of solutions of $\lambda = -5$.

Q.14 Solve the system of equations:

$$\left.\begin{matrix} z+ay+a^2x+a^3=0 \\ z+by+b^2x+b^3=0 \\ z+cy+c^2x+c^3=0 \end{matrix}\right]$$

Q.15 (a) Consider the system of equations

$$\alpha x - y + z = \alpha$$
$$x - \alpha y + z = 1$$
$$x - y + \alpha z = 1$$

If L, M and N denotes the number of integral values of α in interval [–10, 10] for which the system of the equations has unique solution, no solution and infinite solutions respectively, then find the value of (L – M + N).

(b) If the system of equations is

$$2x+3y-z=0$$
$$3x+2y+kz=0$$
$$4x+y+z=0$$

have a set of non-zero integral solutions then, find the smallest positive value of z.

(c) Given a, b ∈ {0, 1, 2, 3, 4,......, 9, 10}.

Consider the system of equations

$$x+y+z=4$$
$$2x+y+3z=6$$
$$x+2y+az=b$$

Let L: denotes number of ordered pairs (a, b) so that the system of equations has unique solution,

M: denotes number of ordered pairs (a, b) so that the system of equations has no solution and

N: denotes number of ordered pairs (a, b) so that the system of equations has infinite solutions. Find (L + M – N).

Q.16 (a) Prove that the value of the determinant

$$\begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix}$$ is real.

(b) On which one of the parameter out of a, p, d or x value of the determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend.

(c) If $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = 0$ and x, y, z are all different then, prove that xyz = –1.

Q.17 Prove that (a) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

(b) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = [(x-y)(y-z)(z-x)(x+y+z)]$

Q.18 (a) Let $f(x) = \begin{vmatrix} x & 1 & \frac{-3}{2} \\ 2 & 2 & 1 \\ \frac{1}{x-1} & 0 & \frac{1}{2} \end{vmatrix}$ Find the minimum value of f(x) (given x > 1).

(b) If $a^2+b^2+c^2+ab+bc+ca \le 0 \;\; \forall$ a, b, c ∈ R, then find the value of the determinant

$$\begin{vmatrix} (a+b+2)^2 & a^2+b^2 & 1 \\ 1 & (b+c+2)^2 & b^2+c^2 \\ c^2+a^2 & 1 & (c+a+2)^2 \end{vmatrix}$$

Q.19 If $D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ and

$D' = \begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix}$ then prove that D′ = 2D.

Q.20 Prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Q.21 Let

$$f(x) = \begin{vmatrix} \sin x & \sin(x+h) & \sin(x+2h) \\ \sin(x+2h) & \sin x & \sin(x+h) \\ \sin(x+h) & \sin(x+2h) & \sin x \end{vmatrix}.$$

If $\lim_{h\to 0}\frac{f(x)}{h^2}$ has the value equation to k(sin3x + $\sin^3$x) find $k \in N$.

Q.22 Prove that

$$\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha+\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$$

$$= -64(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$$

Q.23 If a, b and c are the roots of the cubic $x^3-3x^2+2=0$ then find the value of the determinant.

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Q.24 Solve for x

(a) $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$

(b) $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

Q.25 If a + b + c = 0, solve for x :

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

Q.26 Let a, b, c are the solutions of the cubic $x^3-5x^2+3x-1=0$, then find the value of the determinant $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

Q.27 Show that, $\begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix}$ is divisible by λ^2 and find the other factor.

Q.28 Prove that

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Q.29 In a ΔABC, determine condition under which

$$\begin{vmatrix} \cos\frac{A}{2} & \cot\frac{B}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2}+\tan\frac{C}{2} & \tan\frac{C}{2}+\tan\frac{A}{2} & \tan\frac{A}{2}+\tan\frac{B}{2} \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Exercise 2

Single Correct Choice Type

Q.1 Let m be a positive integer &

$$D_r = \begin{vmatrix} 2r-1 & {}^mC_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} \ (0 \le t \le m),$$

then the value of $\sum_{r=0}^{m} D_r$ is given by

(A) 0 (B) m^2-1 (C) 2^m (D) $2^m\sin^2(2^m)$

Q.2 If α, β and γ are real numbers, then

$$D = \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & \cos(\beta-\gamma) & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & 1 & 1 \end{vmatrix} =$$

(A) –1 (B) $\cos\alpha\cos\beta\cos\gamma$

(C) $\cos\alpha + \cos\beta + \cos\gamma$ (D) Zero

Q.3 If a, b and c are non-zero real numbers, then

$$D = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$$

(A) abc (B) $a^2b^2c^2$

(C) bc + ca + ab (D) Zero

Q.4 If $f'(x) = \begin{vmatrix} mx & mx-p & mx+p \\ n & n+p & n-p \\ mx+2n & mx+2n+p & mx+2n-p \end{vmatrix}$,

then y=f(x) represents

(A) A straight line parallel to x-axis

(B) A straight line parallel to y-axis

(C) Parabola

(D) A straight line with negative slope

Q.5 If $D(x) = \begin{vmatrix} x-1 & (x-1)^2 & x^3 \\ x-1 & x^2 & (x+1)^3 \\ x & (x+1)^2 & (x+1)^3 \end{vmatrix}$,

then the coefficient of x in D(x) is

(A) 5 (B) –2 (C) 6 (D) 0

Q.6 The number of integral solutions of |D|=8, where

$$D = \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} \text{ is}$$

(A) 3 (B) 8 (C) 16 (D) 24

Q.7 Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$,

then the maximum value of f(x) is equal to

(A) 2 (B) 4 (C) 6 (D) 8

Q.8 If $px^4 + qx^3 + rx^2 + sx + t$

$$\begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & 2-x & x-3 \\ x-3 & x+4 & 3x \end{vmatrix}$$ then t is equal to

(A) 33 (B) 0 (C) 21 (D) None

Q.9 If $D = \begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$, then D is equal to

(A) $1+a^2+b^2+c^2$ (B) $a^2+b^2+c^2$

(C) $(a+b+c)^2$ (D) None of these

Q.10 If $\alpha+\beta+\gamma=\pi$, then the value of

$$\begin{vmatrix} \sin(\alpha+\beta+\gamma) & \sin\beta & \cos\gamma \\ -\sin\beta & 0 & \tan\alpha \\ \cos(\alpha+\beta) & -\tan\alpha & 0 \end{vmatrix} \text{ is}$$

(A) 0 (B) 1

(C) 2 (D) $2.\sin\beta.\cos\gamma.\tan\alpha$

Q.11 If the entries of 3 × 3 determinant are zero or one then the value of the determinant

(A) Cannot be 3 (B) Cannot be 2

(C) Can be –2 (D) Is essentially zero

Q.12 In a third order determinant, each element of the first column consists of sum of two terms, each element of the second column consists of sum of three terms and each element of the third column consists of sum of four terms. Then it can be decomposed into n determinants, where n has the value

(A) 1 (B) 9 (C) 16 (D) 24

Q.13 If the system of equations x + 2y + 3z = 4, x + py + 2z =3, μx + 4 y + z = 3 has an infinite number of solutions, then

(A) p = 2, $\mu = 3$ (B) p = 2, $\mu = 4$

(C) 3p = 2μ (D) None of these

Q.14 Number of triplets of a, b and c for which the system of equations, ax – by = 2a – b and (c + 1)x + cy = 10 – a + 3b has infinitely many solutions and x = 1, y = 3 is one of the solutions, is

(A) Exactly one (B) Exactly two

(C) Exactly three (D) Infinitely many

Q.15 If the system of equations ax + y + z = 0, x + by + z = 0 & x + y + cz = 0 (a , b, $c \neq 1$) has a non-trivial solution , then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ is

(A) –1 (B) 0 (C) 1 (D) None of these

Q.16 The determinant

$$\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix} \text{ is}$$

(A) 0

(B) Independent of θ

(C) Independent of ϕ

(D) Independent of θ and ϕ both

Q.17 The values θ, λ for which the following equations $x\sin\theta - y\cos\theta + (\lambda+1)z = 0$; $x\cos\theta + y\sin\theta - \lambda z = 0$; $\lambda x + (\lambda+1)y + z\cos\theta = 0$ are consistent with infinite solution, are

(A) $\theta = n\pi, \lambda \in, \lambda \in R - \{0\}$

(B) $\theta = 2n\pi$, λ is any rational number

(C) $\theta = (2n+1)\pi, \lambda \in R^{+}, n \in I$

(D) $\theta = (2n+1)\frac{\pi}{2}, \lambda \in R, n \in I$

Q.18 If the system of equations, $a^2x - ay = 1 - a$ and $bx + (3-2b)y = 3 + a$ possess a unique solution x = 1, y = 1 then

(A) a = 1; b = –1 (B) a = –1, b = 1

(C) a = 0, b = 0 (D) None of these

Q.19 Let $D = \begin{vmatrix} {}^{n+2}C_n & {}^{n+3}C_{n+1} & {}^{n+4}C_{n+2} \\ {}^{n+3}C_{n+1} & {}^{n+4}C_{n+2} & {}^{n+5}C_{n+3} \\ {}^{n+4}C_{n+2} & {}^{n+5}C_{n+3} & {}^{n+6}C_{n+6} \end{vmatrix}$ and $n \in N$

then the value of D is equal to

(A) –1 (B) 0 (C) 1

(D) (n + 2) (n + 3) (n + 4) (n + 5) (n + 6)

Q.20 The set of equations $\lambda x - y + (\cos\theta)z = 0$; $3x + y + 2z = 0$; $(\cos\theta)x + y + 2z = 0$, $0 \le \theta < 2\pi$, has nontrivial solution(s)

(A) For no value of λ and θ

(B) For all values of λ and θ

(C) For all values of λ and only two values of θ

(D) For only one value of λ and all values of θ

Multiple Correct Choice Type

Q.21 The determinant

$$\begin{vmatrix} \cos(x-y) & \cos(y-z) & \cos(z-x) \\ \cos(x+y) & \cos(y+z) & \cos(z+x) \\ \sin(x+y) & \sin(y+z) & \sin(z+x) \end{vmatrix} =$$

(A) $2\sin(x-y)\sin(y-z)\sin(z-x)$

(B) $-2\sin(x-y)\sin(y-z)\sin(z-x)$

(C) $2\cos(x-y)\cos(y-z)\cos(z-x)$

(D) $-2\cos(x-y)\cos(y-z)\cos(z-x)$

Q.22 The value of θ lying between $-\frac{\pi}{4}$ and $\frac{\pi}{2}$ and $-0 \le A \le \frac{\pi}{2}$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} \text{ are}$$

(A) $A = \frac{\pi}{4}, \theta = -\frac{\pi}{8}$ (B) $A = \frac{3\pi}{8} = \theta$

(C) $A = \frac{\pi}{5} 9\ \theta = -\frac{\pi}{8}$ (D) $A = \frac{\pi}{6} = \theta = \frac{3\pi}{8}$

Q.23 If $\begin{vmatrix} 1 & a & a^2 \\ 1 & x & x^2 \\ b^2 & ab & a^2 \end{vmatrix} = 0$

(A) x = a (B) x = b (C) $x = \frac{1}{a}$ (D) $x = \frac{a}{b}$

Q.24 The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero, if

(A) a, b, c are in AP

(B) a, b, c are in GP

(C) α is a root of the equation $ax^2 + bx + c = 0$

(D) $(x-\alpha)$ is a factor of $ax^2 + 2bx + c$

Q.25 The set of equations $x - y + 3z = 2$, $2x - y + z = 4$, $x - 2y + \alpha z = 3$ has

(A) Unique solution only for $\alpha = 0$

(B) Unique solution for $\alpha \neq 8$

(C) Infinite number of solution of $\alpha = 8$

(D) No solution for $\alpha = 8$

Q.26 Which of the following determinant(s) vanish(es)?

(A) $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$ (B) $\begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix}$

(C) $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$ (D) $\begin{vmatrix} \log_x xyz & \log_x y & \log_x z \\ \log_y xyz & 1 & \log_y z \\ \log_z xyz & \log_z y & 1 \end{vmatrix}$

Q.27 If the system of equation $a^2x - by = a^2 - b$ and $bx - b^2y = 2 + 4b$ possess an infinite number of solutions then the possible values of 'a' and 'b' are

(A) a = 1, b = –1 (B) a = 1, b = –2

(C) a = –1, b = –1 (D) a = –1, b = –2

Q.28 If p, q, r, s are in A.P, and

$$f(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$ such that

$\int_0^2 f(x)dx = -4$, then the common difference of the A.P. can be

(A) –1 (B) ½ (C) 1 (D) None of these

Q.29 If the system of equations x+ y – 3 =0, (1 + K)x + (2 + K)y – 8 = 0 and x – (1 + K) y + (2 + K) are consistent then the value of K is

(A) 1 (B) 3/5 (C) –5/3 (D) 2

Q.30 If $D = \begin{vmatrix} \frac{1}{z} & \frac{1}{z} & -\frac{(x+y)}{z^2} \\ -\frac{(y+z)}{x^2} & \frac{1}{x} & \frac{1}{x} \\ -\frac{y(y+z)}{x^2z} & -\frac{x+2y+z}{xz} & -\frac{y(x+y)}{xz^2} \end{vmatrix}$ then

(A) D is independent of x (B) D is independent of y

(C) D is independent of z (D) D is dependent of x, y, z

Previous Years' Questions

Q.1 The parameter, on which the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$$ does not depend upon, is ***(1997)***

(A) a (B) p (C) d (D) x

Q.2 Let λ and α be real. Find the set of all values of λ for which the system of linear equations $\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$

$x + (\cos\alpha)y + (\sin\alpha)z = 0$ and

$-x + (\sin\alpha)y - (\cos\alpha)z = 0$ has a non-trivial solution.

For $\lambda = 1$, find all value of α. ***(1993)***

Q.3 Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$

represents a straight line.

Q.4 For what value of k does the following system of equations possess a non-trivial solution over the set of rationals x + y – 2z = 0; 2x – 3y + z = 0 and x – 5y + 4z = k. Find all the solution. ***(1979)***

Q.5 For what value of m does the system of equations 3x + my = m and 2x – 5y = 20 has a solution satisfying the conditions x > 0, y > 0. ***(1979)***

Q.6 Prove that for all values of θ

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$ ***(2000)***

Q.7 The total number of ways in which 5 balls of different colours can be distributed among 5 persons so that each person gets at least one ball is ***(2012)***

(A) 75 (B) 150 (C) 120 (D) 243

Q.8 Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha ?$$ ***(2015)***

(A) –4 (B) 9 (C) –9 (D) 4

Q.9 The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1=27x^3 \end{vmatrix} = 10$ is **(2016)**

Essential Questions

JEE Main/Boards

Exercise 1

Q.15 Q.29 Q.38
Q.48 Q.52 Q.54
Q.55

Exercise 2

Q.1 Q.6 Q.13

Previous Years' Questions

Q.2 Q.6 Q.7
Q.10

JEE Advanced/Boards

Exercise 1

Q.1 Q.3 Q.8
Q.11 Q.15 Q.23

Exercise 2

Q.1 Q.11 Q.14
Q.20 Q.25 Q.30

Previous Years' Questions

Q.2 Q.5 Q.7
Q.10

Answer Key

JEE Main/Boards

Exercise 1

Single Correct Choice Type

Q.1 $x = \pm 2\sqrt{2}$ **Q.2** 1 **Q.3** (i) 1 (ii) –16 **Q.4** $-\frac{7}{2}, \frac{13}{2}$ **Q.5** $\frac{5}{2}$ sq. units

Q.6 0 **Q.7** –96 **Q.8** 1; 9 **Q.9** $\frac{5}{3}$ **Q.10** –400

Q.11 $\begin{bmatrix} 4 & -1 \\ 5 & 3 \end{bmatrix}$ **Q.12** Not possible **Q.13** 0 **Q.14** $x = \frac{5}{8}$ **Q.15** $\frac{19}{2}$ sq. units

Q.16 -2 **Q.20** 75 **Q.21** –10 **Q.23** 16 **Q.24** No

Q.25 1728 **Q.26** $A^{-1} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ **Q.28** Inconsistent

Q.29 $\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$ **Q.39** $M_{21}=39$, $M_{22}=3$, $M_{23}=-11$, $C_{21}=-39$, $C_{22}=3$, $C_{23}=11$

Q.40 $f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + 6$ **Q.41** Consistent **Q.42** Inconsistent

Q.46 $x = -\frac{7}{4}$ or 1 **Q.47** $A^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$ **Q.48** 0

Q.52 $A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$ **Q.53** $\frac{1}{4}\begin{bmatrix} 49 & -18 \\ -23 & 10 \end{bmatrix}$ **Q.54** $\frac{-1}{11}\begin{bmatrix} 3 & -19 & 12 \\ 4 & -18 & 5 \\ 4 & -29 & 27 \end{bmatrix}$

Q. 55 $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Exercise 2

Single Correct Choice Type

Q.1 D **Q.2** A **Q.3** A **Q.4** C **Q.5** A **Q.6** C

Q.7 C **Q.8** A **Q.9** B **Q.10** B **Q.11** D **Q.12** B

Q.13 C **Q.14** D **Q.15** C

Previous Years' Questions

Q.1 B **Q.2** A **Q.3** D **Q.4** C **Q.5** A **Q.6** 1

Q.7 A **Q.10** 4 **Q.11** B **Q.12** C **Q.13** C **Q.14** D

JEE Advanced/Boards

Exercise 1

Q.1 (a) x = 1, y = 2, z = 3; consistent (b) x = 2, y = –1, z = 1; consistent (c) Inconsistent

Q.2 $K=\frac{33}{2}, x:y:z=-\frac{15}{2}:1:-3$ **Q.3** –2 **Q.4** 2 **Q.7** 2

Q.8 (a) $\lambda \neq 3$ (b) $\lambda = 3,\ \mu = 10$ (c) $\lambda = 3, \mu \neq 10$

Q.9 x = 1 + 2K, y = –3K, z = K, when p = 1; x = 2K, y = 1 – 3K, z = K when p = 2; where $K \in R$

Q.10 If $K \neq 2$, $\frac{x}{2(K+6)}=\frac{y}{2K+3}=\frac{z}{6(K-2)}=\frac{1}{2(K^2+2K+15)}$, If K = 2, then $x=\lambda, y=\frac{1-2\lambda}{2}$ and z = 0 where $\lambda \in R$

Q.11 (a) 19 (b) 4 **Q.12** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Q.13 If $\lambda \neq -5$, then $x=\frac{4}{7}; y=-\frac{9}{7}$ and z = 0; If $\lambda = 5$ then $x=\frac{4-5K}{7}; y=\frac{13K-9}{7}$ and z = K where $K \in R$.

Q.14 x = –(a + b + c), y = ab + bc + ca, z = – abc

Q.15 (a) 21 (b) 5 (c) 119 **Q.16** (b) p **Q.18** (a) 4 (b) 65

Q.21 3 **Q.23** –108 **Q.24** (a) x = –1 or x = –2; (b) x = 4

Q.25 X = 0 or $x \pm \sqrt{\frac{3}{2}(a^2+b^2+c^2)}$ **Q.26** 80

Q.27 $\lambda^2(a^2+b^2+c^2+\lambda)$ **Q.29** Triangle ABC is isosceles.

Exercise 2

Single Correct Choice Type

Q.1 A **Q.2** D **Q.3** A **Q.4** A **Q.5** A **Q.6** D

Q.7 C **Q.8** C **Q.9** A **Q.10** A **Q.11** C **Q.12** D

Q.13 D **Q.14** B **Q.15** C **Q.16** B **Q.17** D **Q.18** A

Q.19 A **Q.20** A

Multiple Correct Choice Type

Q.21 A,D **Q.22** A, B, C, D **Q.23** A, D **Q.24** B, D **Q.25** B, D **Q.26** A, B, C, D

Q.27 A, B, C, D **Q.28** A, C **Q.29** A, C **Q.30** A, B, C

Previous Years' Questions

Q.1 B **Q.2** Zero **Q.4** k = 0, the given system has infinitely many solutions

Q.5 $m < -\frac{15}{2}$ or m > 30 **Q.7** B **Q.8 B** C **Q.9** 2

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\begin{vmatrix} -1 & 2 \\ 4 & 8 \end{vmatrix} = \begin{vmatrix} 2 & x \\ x & -4 \end{vmatrix}$

$-8 - 8 = 2(-4) - x^2 = -8 - x^2$

$\Rightarrow x = \pm\sqrt{8} = \pm 2\sqrt{2}$

Sol 2: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $|A| = 1[1] - 2(0) = 1$

Sol 3: $\begin{vmatrix} 4 & -1 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix}$,

(i) $M_{23} = \begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix} = 0 - (-1) = 1$

$C_{23} = (-1)^{2+3} = -1$

(ii) $C_{32} = (-1)^{2+3} = \begin{vmatrix} 4 & 0 \\ 2 & 4 \end{vmatrix} = -16$

Sol 4: Area of triangle, [(k, 0), (1, 1), (0, 3)] = 5 unit^2

$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & k & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = \frac{1}{2}\left|1[3-0] + k(1-3)\right| = 5$

$\Rightarrow |-2k + 3| = 10$

$\Rightarrow -2k + 3 = 10$ or $2k - 3 = 10$

$\Rightarrow k = -\frac{7}{2}$ or $k = \frac{13}{2}$

Sol 5: Vertices of triangle (0, 3) (–1, 4) (2, 6)

$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 3 \\ 1 & -1 & 4 \\ 1 & 2 & 6 \end{vmatrix} = \frac{1}{2}\left|\left[-1(6) - 8\right] + 3[2+1]\right|$

$= \frac{1}{2}|-14 + 9| = \frac{1}{2}|5| = \frac{5}{2}$ Sq. Unit

Sol 6: $D \Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$

$= a_{11}\begin{vmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \end{vmatrix} + a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}(-1)^{2+3}$

$= a_{11}[a_{13}a_{32} - a_{33}a_{12}] + a_{12}[a_{11}a_{33} - a_{31}a_{13}] - a_{13}[a_{11}a_{32} - a_{31}a_{12}]$

$= 0$

It can be directly said as it is a property

Sol 7: $\begin{vmatrix} -1 & 24 \\ 4 & P \end{vmatrix} = A$ (assume)

$|A| = \begin{vmatrix} -1 & 24 \\ 4 & P \end{vmatrix} = -P - 4(24) = -(96 + P) = 0$

$\Rightarrow P = -96$

Sol 8: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $|I_2| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

$3I_2 = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$,

$|3I_2| = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 9$

Sol 9: (0, 2), (1, x) and (3, 1) points are collinear

So $\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & x \\ 1 & 3 & 1 \end{vmatrix} = 0$

$\Rightarrow [1 - 3x] + 2[3 - 1] = 0 \Rightarrow 1 - 3x + 4 = 0$

$\Rightarrow 3x = 5 \Rightarrow x = \frac{5}{3}$

Sol 10: $|A| = 20$, $|B| = -20$

$|AB| = |A|\,|B| = 20(-20) = -400$

Sol 11: $A = \begin{bmatrix} 3 & 1 \\ -5 & 4 \end{bmatrix}$

$C_{11} = 4, C_{12} = 5, C_{21} = -1, C_{22} = 3$

$\text{adj}A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} 4 & -1 \\ 5 & 3 \end{bmatrix}$

Sol 12: $A = \begin{bmatrix} 1 & 3 \\ -6 & -18 \end{bmatrix}$

$|A| = -18 - [3][-6] = -18 + 18 = 0$

So, A^{-1} does not exist

Sol 13: $\begin{vmatrix} 3 & 1 & 8 \\ -4 & 2 & 16 \\ -5 & 3 & 24 \end{vmatrix} = (8)\begin{vmatrix} 3 & 1 & 1 \\ -4 & 2 & 2 \\ -5 & 3 & 3 \end{vmatrix}$

Two columns are same, so determinant is 0

Sol 14: $a = \begin{pmatrix} x & 0 & 1 \\ 2 & -1 & 4 \\ 1 & 2 & 0 \end{pmatrix}$ is singular

So $|a| = 0$

$\Rightarrow x[-8] + 1\{2(2) - (-1)(1)\}] = -8x + 4 + 1 = 0$

$\Rightarrow 8x = 5 \to x = \frac{5}{8}$

Sol 15: Vertices $\to$ (3, 1) (4, 3) and (–5, 4)

$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & -5 & 4 \end{vmatrix}$

$= \frac{1}{2}[16 + 15 + 3[3 - 4] + 1[-5 - 4]]$

$= \frac{1}{2}[31 - 3 - 9] = \frac{19}{2}$ sq. unit

Sol 16: Vertices (x, y) (2, –6), (5, 4)

$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & x & 4 \\ 1 & 2 & -6 \\ 1 & 5 & 4 \end{vmatrix} = \frac{1}{2}[8 + 30 + x[-6 - 4] + 4[5 - 2]]$

$38 - 10x + 12 = 70 \quad \Rightarrow 10x = 50 - 70 = -20$

$\Rightarrow x = \frac{-20}{10} = -2$

Sol 17: $\begin{vmatrix} 5 & 15 & -25 \\ 7 & 21 & 30 \\ 8 & 24 & 42 \end{vmatrix} = (3)\begin{vmatrix} 5 & 5 & -25 \\ 7 & 7 & 30 \\ 8 & 8 & 42 \end{vmatrix}$

Two column are same so Determinants is 0

Sol18: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ $C_2 \to C_2 + C_3$

$\Rightarrow \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix}$

$= (a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$ $C_1 \to C_2 - C_1$

$= (a + b + c)\begin{vmatrix} 0 & 1 & b+c \\ 0 & 1 & a+c \\ 0 & 1 & a+b \end{vmatrix} = 0$

Sol 19: (2, 0), (0, 5) and (x, y) are collinear

$\Rightarrow \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 5 \\ 1 & x & y \end{vmatrix} = 0$

$1[-5x] + 2[5 - y] = 0$

$-5x + 10 - 2y = 0$

$5x + 2y = 10 \rightarrow \frac{x}{2} + y - 5 = 1$

Sol 20: $|A| = 3$, A's order $\rightarrow 2 \times 2$

$|5A| = (5)^2 |A| = 25 \times 3 = 75$

Sol 21: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $|A| = -10$

$|A| = a_{11}C_{11} + a_{12}C_{12}$ (along first row) $= |A| = -10$

Sol 22: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} \xrightarrow{R_2 - R_1,\ R_3 - R_1} \begin{vmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{vmatrix}$

$= (a-b)(a-c)\begin{vmatrix} 1 & a & b+c \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$

Since the columns are linearly dependent, hence the value of determinant is zero.

Sol 23: $|A| = -4$

Order of A = 3

$|adjA| = |A|^{3-1} = (-4)^2 = 16$

Sol 24: $\begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \end{bmatrix}$

It is not a square matrix, so inverse not exist

Sol 25: $|A| = 12$, A's order 3×3

$|A.\ adjA| = |A|\ |A|^{3-1} = |12|^3 = 1728$

Sol 26: $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, $A^{-1} = \frac{1}{19}A$

$C_{11} = -2,\ C_{12} = -5,\ C_{21} = -3,\ C_{22} = 2$

$A^{-1} = \frac{adjA}{|A|} = \frac{1}{(-4-15)}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19}A$

Sol 27: $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$C_{11} = 15 - 1 = 14$

$C_{12} = 1 + 10 = 11$

$C_{13} = -2 - 3 = -5,$

$C_{21} = 1 + 10 = 11$

$C_{22} = 5 - 1 = 4$

$C_{23} = -2 - 1 = -3$

$C_{31} = -2 - 3 = -5,$

$C_{32} = -2 - 1 = -3$

$C_{33} = 3 - 4 = -1$

$|A| = 1[14] - 2[11] + 1[-5]$

$= 14 - 5 - 22 = -13$

$AdjA = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$,

$A^{-1} = \frac{1}{|A|}adjA$

$A^{-1} = \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$,

$|A^{-1}| = \frac{1}{|A|} = \frac{1}{-13}$

For adjA,

$C_{11} = \begin{vmatrix} 4 & -3 \\ -3 & -1 \end{vmatrix} = -4 - 9 = -13,$

$C_{12} = 15 + 11 = 26$

$C_{13} = -33 + 20 = -13,$

$C_{21} = 15 + 11 = 26,$

$C_{22} = -14 - 25 = -39$

$C_{21} = C_{12},\ C_{31} = C_{13},$

$C_{23} = -55 + 42 = -13,$

$C_{38} = 56 - 121 = -65$

So $|A^{-1}|^{-1} = \frac{1}{|A^{-1}|}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$

$= \frac{1}{-13}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$

Sol 28: $2x + 5y = 7$, $6x + 15y = 13$

$$D = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 30 - 30 = 0$$

D = 0. So system is inconsistent

Sol 29: $A = \begin{vmatrix} a & b \\ c & (1+bc)/a \end{vmatrix}$

$$|A| = a\left(\frac{1+bc}{a}\right) - bc = 1 + bc - bc = 1$$

$$\text{adj}A = \begin{bmatrix} (1+bc)/a & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}A}{|A|} = \begin{bmatrix} (1+bc)/a & -b \\ -c & a \end{bmatrix}$$

$(a^2 + bc + 1)I - aA$

$$= \begin{bmatrix} a^2+bc+1 & 0 \\ 0 & a^2+bc+1 \end{bmatrix} - a\begin{bmatrix} a & b \\ c & (1+bc)/a \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc+1-a^2 & -ab \\ -ac & a^2+bc+1-(1+bc) \end{bmatrix}$$

$$= a\begin{bmatrix} (1+bc)/a & -b \\ -c & a \end{bmatrix} = aA^{-1}$$

R.H.L. = L.H.S.

Sol 30: $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$,

$$A^1 = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$|A| = 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\text{adj}A = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}A}{|A|} = \cos^2 x\begin{bmatrix} 1 & -\frac{\sin x}{\cos x} \\ \frac{\sin x}{\cos x} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x & -\sin x\cos x \\ \sin x\cos x & \cos^2 x \end{bmatrix}$$

$$A^1A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} \cos^2 x & -\sin x\cos x \\ \sin x\cos x & \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \sin^2 x & -\sin x\cos x - \sin x\cos x \\ \sin x\cos x + \sin x\cos x & -\sin^2 x + \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix} = \text{R.H.S.}$$

Sol 31: $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$,

Assume $A - xI = 0$ ..... (i)

$|A_1 - xI| = 0$

$$\text{So} \begin{vmatrix} 2-x & 0 & -1 \\ 5 & 1-x & 0 \\ 0 & 1 & 3-x \end{vmatrix} = 0$$

$\Rightarrow (2 - x)[x^2 + 3 - 4x] -1[5] = 0$

$\Rightarrow -x^3 - 3x + 4x^2 + 2x^2 + 6 - 8x = 5$

$\Rightarrow x^3 - 6x^2 + 11x = 1$

$\Rightarrow x^2 - 6x + 11 = \frac{1}{x} = x^{-1}$

$(A - xI) = 0$

$\Rightarrow A^2 - 6A + 11I = A^{-1}$

Sol 32: $A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 2+2+2 & 6+2-3 & 4+2-1 \\ -2+1+4 & -6+1-6 & -4+1-2 \\ 1-2+4 & 3-2-6 & 2-2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 & 5 \\ 3 & -11 & -5 \\ 3 & -5 & -2 \end{bmatrix}$$

For A

$C_{11} = 2 + 4 = 6,$

$C_{12} = 2 + 4 = 6,$

$C_{13} = 4 - 1 = 3,$

$C_{21} = -2 - 4 = -6,$

$C_{22} = 4 - 1 = 3,$

$C_{23} = 2 + 4 = 6,$

$C_{31} = 4 - 1 = 3,$

$C_{32} = -2 - 4 = -6,$

$C_{33} = 2 + 4 = 6$

$|A| = 2(6) + 2(6) + 1(3) = 27$

$$A^{-1} = \frac{\text{adj}A}{|A|} = \frac{1}{27}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

For B

$C_{11} = -1 + 3 = 2,$

$C_{12} = 2 + 1 = 3,$

$C_{13} = -3 - 2 = -5,$

$C_{21} = -6 + 3 = -3,$

$C_{22} = -1 - 4 = -5,$

$C_{23} = -3 - 6 = -9,$

$C_{31} = 3 - 2 = 1,$

$C_{32} = 2 - 1 = 1,$

$C_{33} = 1 - 3 = -2$

$|B| = 1[2] + 3[3] + 2[-5] = 11 - 10 = 1$

$$B^{-1} = \frac{\text{adj}B}{|B|} = \begin{bmatrix} 2 & -3 & 1 \\ 3 & -5 & 1 \\ -5 & -9 & -2 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{9}\begin{bmatrix} 4-6+1 & -4-3+2 & 2+6+2 \\ 6-10+11 & -6-5+2 & 3+10+2 \\ -10-18-2 & 16-3-4 & -5+18-4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -1 & -5 & 10 \\ -3 & -9 & 15 \\ -30 & 3 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 5 & 5 \\ 3 & -11 & -5 \\ 3 & -5 & -2 \end{bmatrix},$$

$|AB| = 6(-3) + 5(-9) + 5(18) = -18 - 45 + 90 = +27$

$C_{11} = 22 - 25 = -3$

$C_{12} = -9$

$C_{13} = -15 + 33 = 18$

$C_{21} = -25 + 10 = -15$

$C_{22} = -12 - 15 = -27$

$C_{23} = 30 + 15 = 45$

$C_{31} = -25 + 55 = 30$

$C_{32} = -30 + 15 = 15,$

$C_{33} = -66 - 15 = -81$

$$\text{So } (AB)^{-1} = \begin{bmatrix} -3 & -15 & 30 \\ -9 & -27 & 15 \\ 18 & 45 & -81 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -1 & -5 & 10 \\ -3 & -9 & 15 \\ -30 & 3 & 9 \end{bmatrix} = B^{-1}A^{-1}$$

Sol 33: $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= (a+b+c)(ab + bc + ca - a^2 - b^2 - c^2)$

$= 3abc - a^3 - b^3 - c^3$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$R_1 \to R_1 - R_3,\ R_2 \to R_2 - R_3$

$$(a+b+c)\begin{vmatrix} 0 & b-c & c-b \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}$$

$= (a+b+c)[(b-a)(a-b) - (c-a)(c-b)]$

$= (a+b+c)(ab + bc + ca - a^2 - b^2 - c^2)$

$= a^2b - a^2b + \ldots\ldots + 3abc - a^3 - b^3 - c^3$

$= 3abc - a^3 - b^3 - c^3$

Sol 34: $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x + y + z)(x - z)^2$

$R \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 2(x+y+z) & x+y+z & x+y+z \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$C_1 \to C_1 - C_2 - C_3$

$$= (x+y+z)\begin{vmatrix} 2-1-1 & 1 & 1 \\ z+x-z-x & z & x \\ x+y-y-z & y & z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 0 & 1 & 1 \\ 0 & z & x \\ x-z & y & z \end{vmatrix}$$

$= (x + y + z)(x - z)(x - z)$

Sol 35: $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab + b + ca)^3$

$C_1 \to C_1 + C_3,\ C_2 \to C_2 + C_3$

$$\begin{vmatrix} c^2 & (b+c)^2 & c^2+bc \\ (a+c)^2 & c^2 & c^2+ac \\ a^2 & b^2 & -ab \end{vmatrix}$$

$= c^2[-abc^2 - b^2(c^2 + ac)] +$

$(b+c)^2 [a^2(c^2+ac) + ab(a+c)^2]$

$(c^2 + b) [b^2(a + c)^2 - c^2a^2]$

$= [-abc^4 - c^4b^2 - c^3b^2a + (b+c)^2 (a^2c^2 + a^3c + a^3b + abc^2 + 2a^2bc] + (c^2 + bc)[b^2a^2 + b^2c^2 + 2acb^2 - c^2a^2]$

This on simplification comes out to be equal to

$(ab + bc + ca)^3$

Sol 36: $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

$R_1 \to R_1 - R_3,\ R_2 \to R_2 - R_3$

$$= \begin{vmatrix} (b-c)^2-(a+b)^2 & a^2-c^2 & bc-ab \\ (c+a)^2-(a+b)^2 & b^2-c^2 & ac-ab \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} (c-a)[c+a+2b] & (c-a)(-a-c) & b(c-a) \\ (b-c)[-c-b-2a] & (b-c)(b+c) & -a(b+c) \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix} a+2b+c & -a-c & b \\ -(2a+b+c) & b+c & -a \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$= (c - a)(b - c)$

$$\begin{vmatrix} a+2b+(-2a-b+c) & -a-c+b+c & b-a \\ -(2a+b+c) & b+c & -a \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix} b-a & b-a & b-a \\ -2a-b-c & b+c & -a \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & b+c & -a \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix}$$

$C_1 \to C_1 + C_2 - 2C_3$

$= (c-a)(b-c)(b-c)$

$$\begin{vmatrix} z-z & 1 & 1 \\ -2a-b-c+b+c+2a & b+c & -a \\ a^2+b^2+2ab+c^2-2ab & c^2 & ab \end{vmatrix}$$

$= -(a-b)(b-c)(c-a)$

$$\begin{vmatrix} z-z & 1 & 1 \\ -2a-b-c+b+c+2a & b+c & -a \\ a^2+b^2+2ab+c^2-2ab & c^2 & ab \end{vmatrix}$$

$= -(a-b)(b-c)(c-a)(a^2+b^2+c^2)(-a-b-c)$

$= (a-b)(b-c)(c-a)(a^2+b^2+c^2)(a+b+c)$

Sol 37: $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix}$

$= (a\ b + c)(a^2 + b^2 + c^2)$

$$= \begin{vmatrix} a & b-c & c+b \\ a & b & c-a \\ a & a+b & c \end{vmatrix} + \begin{vmatrix} 0 & b-c & c+b \\ c & b & c-a \\ -b & a+b & c \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & a+b & c \end{vmatrix} + \begin{vmatrix} 0 & 1 & c+b \\ c & 1 & c-a \\ -b & b & c \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & -c & c+b \\ c & 0 & c-a \\ -b & a & c \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & a+b & c \end{vmatrix} + b\begin{vmatrix} 0 & 1 & c+b \\ c & 1 & c-a \\ -b & 1 & c \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{vmatrix} + \begin{vmatrix} 0 & -c & c \\ c & 0 & c \\ -b & a & c \end{vmatrix}$$

Using, $C_2 \to C_2 - C_1$ and $C_3 \to C_3\text{-}C_1$

in (i), (ii), (iii)

$$a\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix} + \begin{vmatrix} 0 & 1 & c+b \\ c & 0 & -a-b \\ -b & 0 & -b \end{vmatrix}$$

$$+ c\begin{vmatrix} 0 & -c & 1 \\ c & c & 0 \\ -b & a+c & 0 \end{vmatrix}$$

$= a^2[a + b + c] + b^2[a + b + c] + c^2[a + b + c]$

$= (a^2 + b^2 + c^2) (a + b + c)$

Sol 38: $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}$

$R_3 \to R_3 - xR_1 - yR_2$

$$0 = \begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ 0 & 0 & (ax^2+byx+byx+cy^2) \end{vmatrix}$$

$0 = -(ax^2 + 2bxy + cy^2) (ac - b^2)$

$0 = (b^2 - ac) (ax^2 + 2bxy + cy^2)$

$$a\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix} + b\begin{vmatrix} 0 & 1 & c+b \\ c & 0 & -a-b \\ -b & 0 & -b \end{vmatrix}$$

$$+ c\begin{vmatrix} 0 & -c & 1 \\ c & c & 0 \\ -b & a+c & 0 \end{vmatrix}$$

Sol 39: $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$

$\Rightarrow M_{21} = \begin{vmatrix} 2 & 3 \\ -7 & 9 \end{vmatrix} = 18 + 21 = 39$

$M_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix} = 9 - 6 = 3$

$M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & -7 \end{vmatrix} = -7 - 4 = -11$

$C_{21} = -39, C_{22} = 3, C_{23} = 11$

Sol 40: $f(x) = ax^2 + bx + c, f(0) = 6$

$f(2) = 11, f(-3) = 6$

$$\begin{bmatrix} 0^2 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & -3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 6 \end{bmatrix}$$

$D = -12 - 18 = -30$

$$D_a = \left|\begin{bmatrix} 6 & 0 & 1 \\ 11 & 2 & 1 \\ 6 & -3 & 1 \end{bmatrix}\right|$$

$= 6[2+3]+1[-33 - 12] = 30 - 33 - 12 = -15$

$a = \dfrac{D_a}{D} = \dfrac{-15}{-30} = \dfrac{1}{2}$

$$D_b = \begin{vmatrix} 0 & 6 & 1 \\ 4 & 11 & 1 \\ 9 & 6 & 1 \end{vmatrix}$$

$= 6[9 - 4] + 1[24 - 99] = 30 + 24 - 99 = -45$

$b = \dfrac{-45}{-30} = \dfrac{3}{2}$

$$D_c = \begin{vmatrix} 0 & 0 & 6 \\ 4 & 2 & 17 \\ 9 & -3 & 6 \end{vmatrix} = 6[-12 - 18] = -30 \times 6$$

$C = \dfrac{D_C}{D} = \dfrac{-30\times 6}{-30} = 6$

Equation $\to ax^2+bx+c = \dfrac{x^2}{2} + \dfrac{3}{2}x + 6$

Sol 41: $2x - y = 5$

$4x - 2y = 10$

$D = \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = 0$

$D_x = \begin{vmatrix} 5 & -1 \\ 10 & -2 \end{vmatrix} = 0, D_y = \begin{vmatrix} 2 & 5 \\ 4 & 10 \end{vmatrix} = 0$

So system has infinite solution (consistent).

Sol 42: $3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

$$D = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$= 3[-5] - 1[-3] - 2[-6] = -15 + 3 + 12 = 0$

$$D_x = \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$= 3[1 + 4] - 5[2 + 2] = 15 - 20 = -5 \neq 0$

So system is inconsistent.

Sol 43: $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$$\text{L.H.S.} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \quad C_2 \leftrightarrow C_3$$

$$= -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} \quad C_1 \leftrightarrow C_2$$

$$= (-1)^2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \text{R.H.S.}$$

Sol 44: $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

$$\text{L.H.S.} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} \times \frac{abc}{abc}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \text{R.H.S.}$$

$C_2 \leftrightarrow C_3$ and then $C_1 \leftrightarrow C_2$

Sol 45: $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$

$$= -(p-q)\begin{vmatrix} q-p & q-r \\ r-p & 0 \end{vmatrix} + (p-r)\begin{vmatrix} q-p & 0 \\ r-p & r-q \end{vmatrix}$$

$= +(p - q\ (q - r)\ (r - p) - (p - q)\ (q - r)\ (r - p) = 0$

Sol 46: $\begin{vmatrix} x^2 & 0 & 3 \\ x & 1 & -4 \\ 1 & 2 & 0 \end{vmatrix} = 11$

$\Rightarrow x^2[8] + 3[2x - 1] = 11$

$\Rightarrow 4x^2 + 3x - 7 = 0$

$\Rightarrow (x - 1)\ (4x + 7) = 0$

$\Rightarrow (x - 1)(4x + 7) = 0$

$\therefore x = 1 \text{ or } -\frac{7}{4}$

Sol 47: $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, $A^2 - 4A - I = 0$

Assume $A - xI = 0 \rightarrow$

$$\begin{bmatrix} 3-x & 2 \\ 2 & 1-x \end{bmatrix} = 0$$

$(3 - x)(1 - x) - 4 = 0$

$3 + x^2 - x - 3x - 4 = 0$

$\Rightarrow x^2 - 4x - 1 = 0$

$A - xI = 0$

$\Rightarrow A^2 - 4A - I = 0$ Hence proved.

$\Rightarrow A^{-1}[A^2 - 4A - I] = 0$

$A - 4I - A^{-1} = 0$

$$\Rightarrow A - 4I = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3-4 & 2 \\ 2 & 1-4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$

Sol 48: $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix} = \frac{1}{(abc)}\begin{vmatrix} a/a & a^3 & abc \\ b/b & b^3 & abc \\ c/c & c^3 & abc \end{vmatrix}$

$= \frac{abc}{(abc)}\begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} = 0$

Sol 49: $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$

Have to show that $x = -(a + b + c)$

$R_1 \to R_1 + R_2 + R_3$

$\begin{vmatrix} x+a+b+c & x+a+b+c & x+a+b+c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$

$(x+a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$

$x + a + b + c = 0 \Rightarrow x = -(a + b + c)$

Sol 50 : $\begin{vmatrix} x+4 & x & 2 \\ 2 & x+4 & x \\ x & x & x+4 \end{vmatrix}$

$C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$

$\begin{vmatrix} 4 & 0 & x \\ 0 & 4 & x \\ -4 & -4 & x+4 \end{vmatrix}$

$= (4x + 16 + 4x) + 16x = 48x + 64$

Sol 51: $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

$C_1 \to C_1 + C_3$

$= \begin{vmatrix} a+b+c & a-b & a \\ a+b+c & b-c & b \\ a+b+c & c-a & c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a-b & a \\ 1 & b-c & b \\ 1 & c-a & c \end{vmatrix}$

$R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$

$= (a + b + c)\begin{vmatrix} 0 & a+c-2b & a-b \\ 0 & b-2c+a & b-c \\ 1 & c-a & c \end{vmatrix}$

$= (a + b + c) [(b - c)(a + c - 2b) - (a - b)(b + a - 2c)]$

$= a^2b + b^2a - b^2a - a^2b + \ldots\ldots + 3abc - a^3 - b^3 - c^3$

$= 3abc - a^3 - b^3 - c^3$

Sol 52: $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$

Assume $|A - xI| = 0$

$\Rightarrow \left|\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$

$\Rightarrow \left|\begin{bmatrix} 2-x- & -3 \\ 3 & 4-x \end{bmatrix}\right| = |0| = 0$

$\Rightarrow (2 - x)(4 - x) + 9 = 0$

$\Rightarrow 8 + x^2 - 4x - 2x + 9 = 0$

$\Rightarrow x^2 - 6x + 17 = 0$ and $|A - xI| = 0$

So, A satisfied this equation

$\Rightarrow A^2 - 6A + 17\,I = 0$

$A^{-1}[A^2 - 6A + 17I] = 0$

$\Rightarrow A - 6I + 17A^{-1} = 0$

$-17A^{-1} = (A - 6I)$

$A^{-1} = \frac{-1}{17}(A - 6I) = \frac{1}{17}\left[\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$

$A^{-1} = -\frac{1}{17}\begin{bmatrix} 2-6 & -3 \\ 3 & 4-6 \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -4 & -3 \\ 3 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$

Sol 53: $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} A \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 3 & -1 \end{bmatrix}$

Assume BAC = D

$|B| = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 6 - 4 = 2$

$\text{Adj } B = \begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix}$

$B^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

$B^{-1}BAC = B^{-1}D$

$AC = B^{-1}D = \frac{1}{2}\begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix}\begin{vmatrix} 1 & 6 \\ 3 & -1 \end{vmatrix}$

$AC = \frac{1}{2}\begin{bmatrix} 3-12 & 18+4 \\ -1+6 & -6-2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -9 & 22 \\ 5 & -8 \end{bmatrix}$

$|C| = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = -2$

$adjC = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}$, $C^{-1} = \frac{1}{2}\begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix}$

$ACC^{-1} = B^{-1}DC^{-1}$

$A = \frac{1}{2}\times\frac{1}{2}\begin{bmatrix} -9 & 22 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix}$

$A = \frac{1}{4}\begin{bmatrix} 27+22 & -18 \\ -15-8 & 10 \end{bmatrix} = +\frac{1}{4}\begin{bmatrix} 49 & -18 \\ -23 & 10 \end{bmatrix}$

Sol 54: $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$

$C_{11} = 3 - 6 = -3$,

$C_{12} = 3 - 2 = 1$,

$C_{13} = 4 - 3 = 1$,

$C_{21} = 8$,

$C_{22} = 5 - 4 = 1$,

$C_{23} = -10$,

$C_{31} = -12$,

$C_{32} = 8 - 15 = -7$,

$C_{33} = 15$

$|A| = 5(-3) + 4(1) = -15 + 4 = -11$

$A^{-1} = \frac{1}{|A|}adjA = \frac{-1}{11}\begin{bmatrix} -3 & 8 & -12 \\ 1 & 1 & -7 \\ 1 & -10 & 15 \end{bmatrix}$

$B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} -3 & 8 & 12 \\ 1 & 1 & -7 \\ 1 & -10 & 15 \end{bmatrix}$

$\frac{-1}{11}\begin{bmatrix} -3+3+3 & 8+3-30 & -12-21+45 \\ -3+4+3 & 8+4-30 & -12-28+45 \\ -3+3+4 & 8+3-40 & -12-21+60 \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} 3 & -19 & 12 \\ 4 & -18 & 5 \\ 4 & -29 & 27 \end{bmatrix}$

Sol 55: $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$,

Assume $(A - XI) = 0$

$\Rightarrow \begin{bmatrix} 2-x & -1 & 1 \\ -1 & 2-x & -1 \\ 1 & -1 & 2-x \end{bmatrix} = 0$

$(2 - x)\,[(2 - x)^2 - 1[-1 + 2 - x] + 1[1 - 2 + x] = 0$

$\Rightarrow (2 - x)[4 + x^2 - 4x - 1] - 1 + x - 1 + x = 0$

$\Rightarrow 6 - x^3 + 2x^2 - 8x - 3x + 4x^2 - 2 + 2x = 0$

$\Rightarrow -x^3 + 6x^2 - 9x + 4 = 0$

$\Rightarrow x^3 - 6x^2 + 9x - 4 = 0$

$|A - xI| = 0$, so this equation satisfied A

$\Rightarrow A^3 - 6A^2 + 9A - 4I = 0 \Rightarrow A^{-1}[A^3 - 6A^2 + 9A - 4I] = A^{-1}0 = 0$

$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$

$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & +1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$

$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$,

$4A^{-1} = A^2 - 6A + 9I$

$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$4A^{-1} = \begin{bmatrix} 6-12+9 & -5+6 & 5-6 \\ -5+6 & 6-12+9 & -5+6 \\ 5-6 & -5+6 & 6-12+9 \end{bmatrix}$

$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Exercise 2

Single Correct Choice Type

Sol 1: (D) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$

$$abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix} = 0$$

$$\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix} = 0$$

$C_2 \to c_2 - c_1, C_3 \to c_3 - c_1$

$$\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix} = 0$$

$\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 0 \Rightarrow a^{-1} + b^{-1} + c^{-1} = -1$

Sol 2: (A) $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} + \begin{vmatrix} a & a^3 & -1 \\ b & b^3 & -1 \\ c & c^3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix}$$

$R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$

$$\Rightarrow abc\begin{vmatrix} 0 & a^2-c^2 & a^3-c^3 \\ 0 & b^2-c^2 & b^3-c^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a-c & a^3-c^3 & 0 \\ b-c & b^3-c^3 & 0 \\ c & c^3 & 1 \end{vmatrix}$$

$\Rightarrow abc\ [(a^2–c^2)(b^3 - c^3) - (b^2 - c^2)(a^3 - c^3)$

$= [(a - c)(b^3 - c^3) - (b - c)(a^3 - c^3)$

$\Rightarrow abc\ (a - c)(b - c)[(a + c)(b^2 + c^2 + bc)$

$- (b + c)(a^2 + c^2 + ac)$

$= (a - c)(b - c)\ [b^2 + c^2 + bc - (a^2 + c^2 + ac)]$

$abc\ [ab^2 + ac^2 + abc + cb^2 + c^3 + bc^2$

$- ba^2 - bc^2 - abc - ca^2 - c^3 - ac^2]$

$= b^2 + c^2 + bc - a^2 - c^2 - ac$

$= (b - a)(b + a + c)$

$\Rightarrow abc\ (b - a)\ [ab + c(b + a)]$

$= (b - a)(a + b + c)$

$\Rightarrow abc\ [ab + bc + ca] = [a + b + c]$

Sol 3: (A) $(\sin^{-1}x + \sin^{-1}w)(\sin^{-1}y + \sin^{-1}z) = \pi^2$

$D \to \begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix}$

$-1 \le (x, y, w, z) \le 1$

$x^{N_1}w^{N_4} - z^{N_3}y^{N_4}$

If x = y = 7 = w = – 1

$\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix} \to (-1)^{N_2+N_4} - (-1)^{N_2+N_3}$

For max value

$N_1 + N_4 = 2n.\ N_2 + N_3 = 2m + 1$

$\Rightarrow n,m \in N$

Value $(-1)2n - (-1)^{2n} - (-1)^{2m+1}$

$\Rightarrow 1 - (-1) = 2$

Min value $\to -1 - 1 = -2$

Dependent of N_1, N_2, N_3, N_4

Sol 4: (C) $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + \ldots a_{2n}x^{2n}$

$$\begin{vmatrix} a_{n-3} & a_{n-1} & a_{n+1} \\ a_{n-6} & a_{n-3} & a_{n+3} \\ a_{n-14} & a_{n-7} & a_{n+7} \end{vmatrix}$$

$(1+x+x^2)^n = (x^2+x+1)^n$

$a_{n-1} = a_{n+1}$

$a_0 = a_n$

$a_{n-r} = a_{n+r}, O \le r \le n$

So determinate $\rightarrow \begin{vmatrix} a_{n-3} & a_{n-1} & a_{n-1} \\ a_{n-6} & a_{n-3} & a_{n-3} \\ a_{n-14} & a_{n-7} & a_{n-7} \end{vmatrix}$

$C_2 \rightarrow C_2 - C_3$

$$\begin{vmatrix} a_{n-3} & 0 & a_{n-1} \\ a_{n-6} & 0 & a_{n-3} \\ a_{n-14} & 0 & a_{n-7} \end{vmatrix} = 0$$

Sol 5: (A) $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ $C_1 \rightarrow C_1 - C_2 - C_3$

$$(-1)^3 \begin{vmatrix} -1-2-1 & 2 & 1 \\ 3+2\sqrt{2}-2-2\sqrt{2}-1 & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2}-2+2\sqrt{2}-1 & 2-2\sqrt{2} & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} -4 & 2 & 1 \\ 0 & 2+2\sqrt{2} & 1 \\ 0 & 2-2\sqrt{2} & 1 \end{vmatrix} = +4\,[2 + 2\sqrt{2} - (2 - 2\sqrt{2})]$$

$= +4\,[4\sqrt{2}] = 16\sqrt{2}$

Sol 6: (C) $D_1 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$,

$$D_2 = \begin{vmatrix} bc-a^2 & ac-b^2 & ab-c^2 \\ ac-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$D_1^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{vmatrix}$$

$= D_3$ (given)

in $D_3 \rightarrow C_1 \rightarrow C_1 - C_3, C_2 \rightarrow C_2 - C_3$

and assume $T = a^2+b^2+c^2-ab-bc-ca$

$$\begin{vmatrix} T & 0 & ab+bc+ca \\ +0 & T & ab+bc+ca \\ -T & -T & a^2+b^2+c^2 \end{vmatrix}$$

$$= T^2\begin{vmatrix} 1 & 0 & ab+bc+ca \\ 0 & 1 & ab+bc+ca \\ -1 & -1 & a^2+b^2+c^2 \end{vmatrix}$$

$T^2[a^2+b^2+c^2+ab+bc+ca+ab+bc+ca(1)]$

$= T^2[a^2+b^2+c^2+2(ab+bc+ca)]$

$$D_2 = \begin{vmatrix} bc-a^2 & ac-b^2 & ab-c^2 \\ ac-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$= C_1 \rightarrow C_1 + C_2 + C_3$

$$D_2 = \begin{vmatrix} -T & ac-b^2 & ab-c^2 \\ -T & ab-c^2 & bc-a^2 \\ -T & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$D_2 = T\begin{vmatrix} -1 & ac-b^2 & ab-c^2 \\ -1 & ab-c^2 & bc-a^2 \\ -1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$R_1 \rightarrow R_1 - R_3, R_2 \rightarrow R_2 - R_3$

$$D_2 = T\begin{vmatrix} 0 & ac-b^2+a^2-bc & ab-c^2+b^2-ac \\ 0 & ab-c^2+a^2-bc & bc-a^2+b^2-ac \\ -1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$D_2 = -T\,[(ac-b^2+a^2-bc)(bc-a^2+b^2$

$-ac) - (ab-c^2+b^2-ac)(ab-c^2+a^2-bc)]$

$D_2 = (-T)(-T)\,[T + 3(ab+bc+ca)]$

$D_2 = T^2[a^2+b^2+c^2+2(ab+bc+ca)]$

$\therefore D_1^2 = D_2 = D_3$

Sol 7: (C) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

L.H. S. = $(a-c)(b-c)(b-a)$

in L.H.S. $C_1 \rightarrow C_1 - C_3, C_2 \rightarrow C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+c^2+ac & b^2+c^2+bc & c^3 \end{vmatrix}$$

$\Rightarrow (a-c)(b-c)[b^2 + c^2 + bc - a^2 - c^2 - ac]$

$\Rightarrow (a-c)(b-c)(b-a)(b+a+c)$

$\to a + b + c = 1$

$= (a-c)(b-c)(b-a)$

$abc = a + b + c$

A.M. $\geq$ G.M

$\frac{a+b+c}{3} \geq (abc)^{1/3}$; $\quad \frac{1}{3} \geq (abc)^{1/3}$

$\frac{1}{27} \geq abc \to abc$ is always less than 1/27

Sol 8: (A) $(a+1)^3x + (a+2)^3y = (a+3)^3$

$(4+1)x + (a+2)y = (a+3)$

$x + y = 1$

Here for two variable thus equation

So $D = D_x = D_{y=0}$ for consistent

$$D = \begin{vmatrix} (a+1)^3 & (a+2)^3 \\ (a+1) & (a+2) \end{vmatrix}$$

$$= (a+1)(a+2)\begin{vmatrix} (a+1)^2 & (a+2)^2 \\ 1 & 1 \end{vmatrix}$$

$= (a+1)(a+2)[a^2 + 1 + 2a - a^2 - 4 - 4a] = (a+1)(a+2)(-2a-3)$ ... (i)

$$D_x = \begin{vmatrix} (a+3)^3 & (a+2)^2 \\ (a+3) & (a+2) \end{vmatrix}$$

$$= (a+2)(a+3)\begin{vmatrix} (a+3)^2 & (a+2)^2 \\ 1 & 1 \end{vmatrix}$$

$= (a+2)(a+3)(a^2 + 9 + 6a - a^2 - 4 - 4a)$

$= (a+2)(a+3)(5+2a)$ ... (ii)

$$D_y = \begin{vmatrix} (a+1)^3 & (a+2)^3 \\ (a+1) & (a+3) \end{vmatrix}$$

$$= (a+1)(a+3)\begin{vmatrix} (a+1)^2 & (a+3)^2 \\ 1 & 1 \end{vmatrix}$$

$= (a+1)(a+3)[a^2 + 1 + 2a - a^2 - 9 - 6a]$

$= (a+1)(a+3)(-4a-8)$

$= -4(a+2)(a+1)(a+3)$

$D = D_x = D_y = 0$

$\Rightarrow a = -2$ (common solution in all)

Sol 9: (B) $3x - 7y + 5z = 3$, $3x + y + 5z = 7$

$2x + 3y + 5z = 5$

$$D = \begin{vmatrix} 3 & -7 & 5 \\ 3 & 1 & 5 \\ 2 & 3 & 5 \end{vmatrix}$$

$= 3[5-15] - 7[10-15] + 5[9-2]$

$= -30 + 35 + 35 = 40 \neq 0$

So system is consistent with unique non trivial solution.

Sol 10: (B) $(\sin\theta)x + 27 = 0$

$(\cos\theta)x + \sin\theta\, y = 0$

$(\cos\theta)y + 2z = 0$

$$D = \begin{vmatrix} \sin\theta & 0 & 2 \\ \cos\theta & \sin\theta & 0 \\ 0 & \cos\theta & 2 \end{vmatrix}$$

$D = \sin\theta(\sin\theta.2) + 2(\cos^2\theta)$

$= 2(\sin^2\theta + \cos^2\theta) = 2$ Constant

$$C = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix},\ D_x = \begin{vmatrix} 0 & 0 & 2 \\ 0 & \sin\theta & 0 \\ a & \cos\theta & 2 \end{vmatrix}$$

So system has a unique solution which is a function of a and θ

Sol 11: (D) $\begin{vmatrix} (1+x)^2 & (1-x)^2 & -(2+x^2) \\ 2x+1 & 3x & 1-5x \\ x+1 & 2x & 2-3x \end{vmatrix}$

$$+\begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ 1-2x & 3x-2 & 2x-3 \end{vmatrix} = 0$$

In 2nd determinate $R_3 \to + R_3 + R_1$

$\begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ +(2+x^2) & -(1-5x) & 3x-2 \end{vmatrix} = -|B| \ \because 1 - 2x + (1+x)^2$

$= 2 + x^2 + 2x - 2x = 2 + x^2$

$R_3 \to -R_3$

$$\Rightarrow |A| - \begin{vmatrix} (1+x)^2 & (2x+1) & x+1 \\ (1-x)^2 & 3x & 2x \\ -(2+x^2) & 1-5x & 2-3x \end{vmatrix} = |B|$$

Now all rows of A is equal to columns of B

$\Rightarrow |B| = -|A|$

$|A| - |A| = 0$ (always)

For every valued of x

$|A| + |B|$ is zero

Therefore infinite solutions

Sol 12: (B) $2x \cos^2\theta + y \sin 2\theta - 2\sin\theta = 0$ ... (i)

$x \sin 2\theta + 2y \sin^2\theta = -2\cos\theta$ ...(ii)

$x \sin\theta - y\cos\theta = 0$ ...(iii)

for (i) & (ii)

$$D = \begin{vmatrix} 2\cos^2\theta & \sin 2\theta \\ \sin 2\theta & 2\sin^2\theta \end{vmatrix}$$

$= 4\sin^2\theta\cos^2\theta - 4\sin^2\theta\cos^2\theta = 0$

$= (\sin 2\theta = 2\sin\theta\cos\theta)$

$$D_x = \begin{vmatrix} 2\sin\theta & \sin 2\theta \\ -2\cos\theta & 2\sin^2\theta \end{vmatrix}$$

$= 4\sin^3\theta + 4\sin\theta\cos^2\theta$

$= 4\sin\theta(\sin^2\theta + \cos^2\theta) = 4\sin x$

for consistent $D_x = 0 \to 4\sin\theta = 0$

$\theta \in n\pi, n \in I$

$$D_y = \begin{vmatrix} 2\sin^2\theta & 2\sin\theta \\ \sin 2\theta & -\sin\theta \end{vmatrix} = 4\cos^3\theta - 4\sin^2\theta\cos\theta$$

$= -4\cos\theta[\sin^2\theta + \cos^2\theta) = -4\cos\theta$

$D_y = 0\ \theta = (2n+1)\pi/2, n \in I$

Sin θ and cos θ both are not zero for same θ, so for every value of θ system has not a solution

Sol 13: (C) $(\sin 3\theta)x - y + z = 0$

$(\cos 2\theta)x + 4y + 3z = 0$

$2x + 7y + 7z = 0$

$$(\because\ c = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

So $D_x = D_y = D_z = 0$

$$D = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix}$$

x, y, z are not all simultaneously equal to zero so for solution (not-trivial), D = 0

$\sin 3\theta\,[28 - 21] - 1[6 - 7\cos 2\theta]$

$+ 1\,[7\cos 2\theta - 8] = 0$

$\Rightarrow 7\sin 3\theta - 6 + 7\cos^2\theta + 7\cos 2\theta - 8 = 0$

$\Rightarrow 7\sin 3\theta + 14\cos 2\theta = 14$

$\Rightarrow \text{Sin } 3\theta + 2\cos 2\theta = 2$

$\Rightarrow \text{Sin } 3\theta + 2\cos 2\theta = 2$

$\Rightarrow 3\sin\theta - 4\sin^3\theta + 2[1 - 2\sin^2\theta] = 2$

$\Rightarrow 3\sin\theta - 4\sin^3\theta + 2 - 4\sin^2\theta = 2$

assume $\sin\theta = x$

$\Rightarrow 4x^3 + 4x^2 - 3x = 0$

$\Rightarrow x[4x^2 + 4x - 3] = 0$

$\Rightarrow x[4x^2 + 6x - 2x - 3] = 0$

$x = 0$ or $2x(2x + 3) - 1(2x + 3) = 0$

$(2x + 3)(2x - 1) = 0 \Rightarrow x = 1/2$ or $-3/2$

$x = 0, ½, -3/2$ but $-1 \le \sin\theta \le 1$

$\sin\theta \ne -3/2$

$\sin\theta \in \{0, ½\}, x = 0, \pi/6, 5\pi/6, \pi, 12\pi$

between $[0, 2\pi]$

No. of principle value = 5

Sol 14: (D) $$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{a^2+c^2}{b} \end{vmatrix} = \alpha\, abc$$

$$abc \begin{vmatrix} \frac{a^2+b^2}{c^2} & 1 & 1 \\ 1 & \frac{b^2+c^2}{a^2} & 1 \\ 1 & 1 & \frac{a^2+c^2}{b^2} \end{vmatrix}$$

$$\Rightarrow abc\left[\left(\frac{a^2+b^2}{c^2}\right)\left(\frac{(b^2+c^2)(a^2+c^2)}{a^2b^2} - 1\right)\right.$$

$$-\left[\frac{a^2+c^2}{b^2}-1\right]+1\left[1-\left(\frac{b^2+c^2}{a^2}\right)\right]\Bigg]$$

$$\Rightarrow abc\left[\frac{(a^2b^2+a^2c^2+b^4+b^2c^2)(a^2+c^2)-b^4a^2-a^4b^2}{a^2b^2c^2}\right.$$

$$\left.-\left(\frac{a^2+c^2-b^2}{b^2}\right)+\frac{a^2-b^2-c^2}{a^2}\right]$$

$$\Rightarrow \frac{abc}{a^2b^2c^2}\begin{bmatrix} a^4b^2+a^4c^2+a^2b^4+a^2b^2c^2+a^2b^2c^2+a^2c^4 \\ +b^4c^2+b^2c^4-a^2b^2c^2-a^4c^2-a^2c^4+a^2c^2b^2 \\ +b^2a^2c^2-b^4c^2-b^2c^4-a^4b^2-a^2b^4 \end{bmatrix}$$

$$= \frac{1}{abc}\left[4a^2b^2c^2+a^4b^2+a^2b^4\right] = 4abc = 2abc$$

$\Rightarrow \alpha = 4$

Sol 15: (C) $a^2x + (2-a)y = 4 + a^2$

$ax + (2a-1)y = a^5 - 2$

$D = \begin{vmatrix} a^2 & 2-a \\ a & 2a-1 \end{vmatrix} = a^2(2a-1) + (a-2)\,a$

$= 2a^3 - a^2 + a^2 - 2a$

For $D = 0 = 2a(a^2-1) \rightarrow +1, -1, 0$

$D_x = \begin{vmatrix} 4+a^2 & 2-a \\ a^5-2 & 2a-1 \end{vmatrix}$

$= (4+a^2)(2a-1) + (a-2)(a^5-2)$

$= 8a - 4 + 2a^3 - a^2 + a^6 - 2a^5 - 2a + 4$

at $a = 0$ $D_x = 0$

So $D_y = \begin{vmatrix} a^2 & 4+a^2 \\ a & a^3-2 \end{vmatrix} = \begin{vmatrix} 0 & 4+0 \\ 0 & 0-2 \end{vmatrix}$

So at $a = 0$, system has infinite solution

At $a = -1, +1$, $D = 0$, and $D_x, D_y \neq 0$

$\Rightarrow$ No solution, no. of values = 2

Previous Years Questions

Sol 1: (B) Given $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$

Applying $C_1 \rightarrow C_1 - (pC_2 + C_3)$

$$\Rightarrow \begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2+yp+yp+z) & xp+y & yp+z \end{vmatrix} = 0$$

$\Rightarrow -(xp^2+2yp+z)(xz-y^2) = 0$

$\therefore$ Either $xp^2+2yp+z = 0$ or $y^2 = xz$

$\Rightarrow$ x, y, z are in GP.

Sol 2: (A) Given

$$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - (C_1 + C_2)$

$$= \begin{vmatrix} 1 & x & 0 \\ 2x & x(x-1) & 0 \\ 3x(x-1) & x(x-1)(x-2) & 0 \end{vmatrix} = 0$$

$\therefore$ $f(x) = 0$ $\Rightarrow f(100) = 0$

Sol 3: (D) Since, the given system has non-zero solution.

$$\therefore \begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 - C_2$, $C_2 \rightarrow C_2 + C_3$

$$\Rightarrow \begin{vmatrix} 1+k & -k-1 & -1 \\ 1+k & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 0$$

$\Rightarrow 2(k+1) - (k+1)^2 = 0$

$\Rightarrow (k+1)(2-k-1) = 0$

$\Rightarrow k = \pm 1$

Note: There is a golden rule in determinant that n one's $\Rightarrow$ (n – 1) zero's or n(constant) $\Rightarrow$ (n – 1) zero's for all constant should be in a single row or a single column.

Sol 4: (C) Given $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix}$$

$$=(2\cos x+\sin x)\begin{vmatrix}1 & \cos x & \cos x\\ 1 & \sin x & \cos x\\ 1 & \cos x & \sin x\end{vmatrix}=0$$

Applying $R_2 \to C_2 - R_1, R_3 \to C_3 - R_1$

$$\Rightarrow (2\cos x+\sin x)\begin{vmatrix}1 & \cos x & \cos x\\ 0 & \sin x-\cos x & 0\\ 0 & 0 & \sin x-\cos x\end{vmatrix}=0$$

$\Rightarrow\ (2\cos x+\sin x)(\sin x-\cos x)^2=0$

$\Rightarrow\ 2\cos x+\sin x=0$ or $\sin x-\cos x=0$

$\Rightarrow\ 2\cos x=-\sin x$ or $\sin x=\cos x$

$\Rightarrow\ \cot x=-\dfrac{1}{2}$ gives no solution in $-\dfrac{\pi}{4}\le x\le\dfrac{\pi}{4}$ and sin x = cos x $\Rightarrow$ tan x = 1

$\Rightarrow\ x=\dfrac{\pi}{4}$

Sol 5: (A) Given equations

x + ay = 0, az + y = 0 and ax + z = 0

has infinite solutions.

$$\therefore \begin{vmatrix}1 & a & 0\\ 0 & 1 & a\\ a & 0 & 1\end{vmatrix}=0 \Rightarrow 1+a^3=0 \text{ or } a=-1$$

Sol 6: (1) For infinitely many solution, we must have

$$\frac{k+1}{k}=\frac{8}{k+3}=\frac{4k}{3k-1}\Rightarrow k=1$$

Sol 7: (A) The given system of equation can be expressed as

$$\begin{bmatrix}1 & -2 & 3\\ 1 & -3 & 4\\ -1 & 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ 1\\ k\end{bmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 + R_1$

$$\Rightarrow \sim\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ k-1\end{bmatrix}$$

$$\Rightarrow \sim\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ k-3\end{bmatrix}$$

$\Rightarrow R_3 \to R_3 - R_2$

When $k\ne 3$, the given system of equation has no solution.

$\Rightarrow$ Statement I is true. Clearly, Statement II is also true as it is rearrangement of rows and columns of

$$\begin{bmatrix}1 & -2 & 3\\ 1 & -3 & 4\\ -1 & 1 & -2\end{bmatrix}$$

Sol 8: Given systems of equation can be rewritten as

$-x + cy + bz = 0$

$cx - y + az = 0$ and $bx + ay - z = 0$

Above system of equations are homogeneous equation. Since, x, y and z are not all zero, so it has non-trivial solution.

Therefore, the coefficient of determinant must be zero

$$\therefore \begin{vmatrix}-1 & c & b\\ c & -1 & a\\ b & a & -1\end{vmatrix}=0$$

$\Rightarrow -1(1-a^2)-c(-c-ab)+b(ca+b)=0$

$\Rightarrow a^2+b^2+c^2+2abc-1=0$

$\Rightarrow a^2+b^2+c^2+2abc=1$

Sol 9: Since α is repeated root of f(x) = 0.

$\therefore\ f(x)=a(x-\alpha)^2$, $a\in$ constant $(\ne 0)$

Let $\phi(x)=\begin{vmatrix}A(x) & B(x) & C(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$

[To show $\phi(x)$ is divisible by $(x-\alpha)^2$, it is sufficient to show that $\phi(\alpha)$ and $\phi'(\alpha)=0$].

$$\therefore\ \phi(\alpha)=\begin{vmatrix}A(\alpha) & B(\alpha) & C(\alpha)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}=0$$

[$\because$ R_1 and R_2 are identical]

Again, $\phi'(x)=\begin{vmatrix}A'(x) & B'(x) & C'(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$

$$\phi'(\alpha)=\begin{vmatrix}A'(\alpha) & B'(\alpha) & C'(\alpha)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}=0$$

[$\because$ R_1 and R_3 are identical]

Thus, α is repeated root of $\phi(x)=0$

Hence, $\phi(x)$ is divisible by f(x).

Sol 10: (4) Given $A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, abc = 1

and $A^T A = 1$ ... (i)

Now, $A^T A = 1$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow a^2+b^2+c^2=1$ and ab + bc + ca = 0 ... (ii)

We know, $a^3+b^3+c^3-3abc$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$\Rightarrow a^3+b^3+c^3=(a+b+c)(1-0)+3$

[from equation (i) and (ii)]

$\therefore a^3+b^3+c^3=(a+b+c)+3$ ... (iii)

Now, $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)=1$... (iv)

$\therefore$ From equation (iii), $a^3+b^3+c^3=1+3$

$\Rightarrow a^3+b^3+c^3=4$

Sol 11: (B) $\Delta=\begin{vmatrix} k+1 & 8 \\ k & k+3 \end{vmatrix}=k^2+4k+3-8k$

$=k^2-4k+3$

$=(k-3)(k-1)$

$\Delta_1=\begin{vmatrix} 4k & 8 \\ 3k-1 & k+3 \end{vmatrix}=4k^2+12k-24k+8$

$=4k^2-12k+8 \;=4(k^2-3k+2)=4(k-2)(k-1)$

$\Delta_2=\begin{vmatrix} k+1 & 4k \\ k & 3k-1 \end{vmatrix}=3k^2+2k-1-4k^2$

$=-k^2+2k-1=-(k-1)^2$

As given no solution $\Rightarrow \Delta_1 \;\&\; \Delta_2 \neq 0$

$\Delta=0$

$\Rightarrow k=3$

Sol 12: (C) $\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^3 \end{vmatrix}$

$$=\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \beta \end{vmatrix}\times\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$$

$=(1-\alpha)^2(\alpha-\beta)^2(\beta-1)^2$

$K=1$

Sol 13: (C) $(2-\lambda)x_1-2x_1+x_3=0$

$2x_1-(3+\lambda)x_2+2x_3=0$

$-x_1+2x_2-\lambda x_3=0$

Non-trivial solution

$\Delta=0$

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix}=0$$

$(1-\lambda)\{3\lambda+\lambda^2-4\}+2.\{-2\lambda+2\}+(4-3-\lambda)=0$

$\Rightarrow (6\lambda+2\lambda^2-8-3\lambda^2-\lambda^3+4\lambda)-4\lambda+4+1-\lambda=0$

$\Rightarrow -\lambda^2-\lambda^2-5\lambda+3=0$

$x^3-\lambda^2+2\lambda^2-2\lambda-3\lambda+3=0$

$\Rightarrow \lambda^2(\lambda^2-1)+2\lambda(\lambda-1)-3(\lambda-1)=0$

$\Rightarrow (\lambda-1)(\lambda^2+2\lambda-3)=0$

$\Rightarrow (\lambda-1)(\lambda+3)(\lambda-1)=0$

$\lambda=1,1,-3$

Sol 14: (D) $x+\lambda y-z=0$

$\lambda x-y-z=0$

$x+y-\lambda z=0$

For non-trivial solution $\Rightarrow \Delta=0$

$$\Rightarrow \begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix}=0$$

$\Rightarrow \lambda+1-\lambda\{-\lambda^2+1\}-(\lambda+1)=0$

$\Rightarrow \lambda(\lambda^2-1)=0$

$\lambda=0,\pm1$

JEE Advanced/Boards

Exercise 1

Sol 1: (a) x + y + z = 6

2x + y – z = 1

x + y – 2z = –3

$$D = \begin{vmatrix} 1 & 1 & +1 \\ 2 & 1 & -1 \\ 1 & 1 & -2 \end{vmatrix} \quad C_1 \to C_1 - C_2$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & +1 \\ 1 & 1 & -1 \\ 0 & 1 & -2 \end{vmatrix} = 1[+2 + 1] = 3$$

$$D_x = \begin{vmatrix} 6 & 1 & +1 \\ 1 & 1 & -1 \\ -3 & 1 & -2 \end{vmatrix} \quad C_3 \to C_3 + C_2$$

$$\Rightarrow \begin{vmatrix} 6 & 1 & 2 \\ 1 & 1 & 0 \\ -3 & 1 & -1 \end{vmatrix} = 1[6 - 1] + 2(1 + 3) = -5 + 8 = 3$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 1 & -1 \\ 1 & -3 & -2 \end{vmatrix}$$

$= 1[-2 - 3] + 6[-1 + 4] + 1[-6 - 1] = -5 + 18 - 7 = 6$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 1 \\ 1 & 1 & -3 \end{vmatrix}$$

$= 1[-3 - 1] + 1[1 + 6] + 6[2 - 1] = -4 + 7 + 6 = 9$

$x = \frac{Dx}{D} = \frac{3}{3} = 1$, $y = \frac{Dy}{D} = \frac{6}{3} = 2$, $z = \frac{Dz}{D} = \frac{9}{3} = 3$

Here, it is consistent

(b) x + 2y + z = 1

3x + y + z = 6

x + 2y = 0

$$D = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 1[6 - 1] + 1[0] = 5$$

$$D_x = \begin{vmatrix} 1 & 2 & 1 \\ 6 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} = 2[6 - 1] = 10,$$

$y = \frac{D_z}{D} = \frac{10}{5} = 2$

$$D_y = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 6 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1[1 - 6] = -5,$$

$y = \frac{D_y}{D} = \frac{-5}{5} = -1$

$$D_z = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 6 \\ 1 & 2 & 0 \end{vmatrix} = 1[6 - 1] + 6[0] = 5m,$$

$Z = \frac{D_z}{D} = \frac{5}{5} = 1$

(c) 7x – 7y + 5z = 3

3x + y + 5z = 7

2x + 3y + 5z = 5

$$D = \begin{vmatrix} 7 & -7 & 5 \\ 3 & 1 & 5 \\ 2 & 3 & 5 \end{vmatrix} \quad R_1 \to R_1 - R_3; R_2 \to R_2 - R_3$$

$$\begin{vmatrix} 5 & -10 & 0 \\ 1 & -2 & 0 \\ 2 & 3 & 5 \end{vmatrix} = 5[-10 + 10] = 0$$

$$D_x = \begin{vmatrix} 3 & -7 & 5 \\ 7 & 1 & 5 \\ 5 & 3 & 5 \end{vmatrix} \quad R_1 \to R_1 - R_3; R_2 \to R_2 - R_3$$

$$\begin{vmatrix} -2 & -10 & 0 \\ 2 & -2 & 0 \\ 5 & 3 & 5 \end{vmatrix} = 5[4 + 20] = 120 \neq 0$$

D = 0 but Dx ≠ 0, so, system is inconsistent

Sol 2: x + ky + 3z = 0 ... (i)

3x + ky – 2z = 0 ... (ii)

2x + 3y – 4z = 0 ... (iii)

Equation has non-trivial solution.

So, $D = D_x = D_y = D_z = 0$

$$D = \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix}$$

$= 1[-4k + 6] + k[-4 + 12] + 3[9 - 2k]$

$= -4k + 6 + 8k + 27 - 6k = 33 - 2k = 0$

$K = \frac{33}{2}$, assuming x = t

From equation (ii) – (i)

$2x - 5z = 0$

$z = \frac{2x}{5} = \frac{2t}{5}$, (x = t)

In (iii)

$\Rightarrow 2t + 3y - 4z = 0 \rightarrow 3z = 4z - 2t$

$\Rightarrow 3y = 4\left(\frac{2t}{5}\right) - 2t = \frac{8t - 10t}{5} = \frac{-2t}{5}$

$\Rightarrow y = \frac{-2t}{15}$

$(x, y, z) \Rightarrow \left(t, \frac{-2t}{15}, \frac{2t}{5}\right)$ $t \in R$

Sol 3: $\alpha x + y + z = \alpha - 1$

$x + \alpha y + z = \alpha - 1$

$x + y + \alpha z = \alpha - 1$

$$D = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix}$$

$D = \alpha[\alpha^2 - 1] + 1[1 - \alpha] + 1[1 - \alpha]$

$= \alpha^3 - \alpha + 2 - 2\alpha = \alpha^3 - 3\alpha + 2$

$\alpha^3 - 3\alpha + 2$,

At $\alpha = 1 \Rightarrow 1 - 3 + 2 = 0$

So $(\alpha - 1)$ is a factor of $\alpha^3 - 3\alpha + 2$

Now, $\alpha^3 - 3\alpha + 2$ can be witten as

$\Rightarrow \alpha^3 - \alpha^2 + \alpha^2 - \alpha - 2\alpha + 2$

$\Rightarrow \alpha^2(\alpha - 1) + \alpha(\alpha - 1) - 2(\alpha - 1)$

$\Rightarrow (\alpha - 1)(\alpha^2 + \alpha - 2)$

$D = (\alpha - 1)(\alpha^2 + \alpha - 2)$

$D = (\alpha - 1)(\alpha^2 + 2\alpha - \alpha - 2)(\alpha - 1)$

$D = (\alpha - 1)[\alpha(\alpha + 2) - 1(\alpha + 2)]$

$D = (\alpha - 1)(\alpha + 2)(\alpha - 1)$

For D = 0, $\alpha = 1$ or -2

For $\alpha = 1$,

$$D_x = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0, \text{ so consistent}$$

So on D_y and $D_t = 0$ $\quad \therefore \alpha \neq 1 \Rightarrow \alpha = -2$

Sol 4: $a(y + z) = x \rightarrow x - ay - az = 0$

$b(z + x) = y \rightarrow bx - y + bz = 0$

$c(x + y) = z \rightarrow cx + cy - z = 0$

$c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $D_x = D_y = D_z = 0$

So for non-trivial solution, D = 0

$$D = \begin{vmatrix} 1 & -a & -a \\ b & -1 & b \\ c & c & -1 \end{vmatrix} \quad C_1 \rightarrow C_1 - C_3;\ C_2 \rightarrow C_2 - C_3$$

$$D = \begin{vmatrix} 1+a & 0 & -a \\ 0 & -(1+b) & b \\ 1+c & 1+c & -1 \end{vmatrix}$$

$$\rightarrow \left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right) = 0$$

or from equation

$a = \frac{x}{y+z}$, $b = \frac{y}{x+z}$, $c = \frac{z}{x+y}$

$1 + a = \frac{x+y+z}{y+z}$; $\quad 1 + b = \frac{x+y+z}{x+z}$;

$1 + c = \frac{x+y+z}{x+y}$

$$\frac{1}{1+a} + \frac{1}{a+b} + \frac{1}{1+c} = \frac{x+y+y+z+z+x}{x+y+z}$$

$$= \frac{2(x+y+z)}{(x+y+z)} = 2$$

Sol 5: $x = cy + bz \rightarrow x - cy - bz = 0$

$y = az + cx \rightarrow cx - y + az = 0$

$z = bx + ay \rightarrow bx + ay - z = 0$

$c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$

$D_x = D_y = D_z = 0$,

But system has solution. So D = 0

$$D = \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 1[1 - a^2] + c[-c - ab) - b[ac + b] = 0$$

$1 - a^2 - c^2 - abc - abc - b^2 = 0$

$a^2 + b^2 + c^2 + 2ab = 1$

Sol 6: $a = \frac{x}{y-z} \to x - ay + az = 0$

$b = \frac{y}{z-x} \to bx + y - bz = 0$

$c = \frac{z}{x-y} \to cx - cy - z = 0$

$c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $D_x = D_y = D_z = 0$,

For solution $\to D = 0$

$$D = \begin{vmatrix} 1 & -a & a \\ b & 1 & -b \\ c & -c & -1 \end{vmatrix}$$

$= 1[-1 - bc] - a[-bc + b] + a[-bc-c]$

$= -1 - bc + abc - ab - abc - ac$

$= -1(ab + bc + ca + 1) = 0$

$= ab + bc + ca + 1 = 0$

Sol 7: sinq $\neq$ cosq

Xcosp – ysinp + z = cosq + 1 ... (i)

xsinp + ycosp + z = 1 – sinq ... (ii)

xcos(p + q) – ysin(p + q) + z = 2 ... (iii)

cos(A + B) = cosA cosB – sinAsinB

sin(p + q) = sinpcosq + cosp sinq

equation $(i)^2$ + equation $(ii)^2$

$\Rightarrow x^2(\sin^2p+\cos^2p) + y^2(\cos^2p+\sin^2p)$

–2xy cospsinp + 2xcospz – 2yzsinp + 2xysinpcosp

+ 2xzsinp + $2z^2$ + 2yzcosp

= z + 1 – 2sinq + 2cosq

$\Rightarrow x^2 + y^2 + z^2$ + 2xycosp – 2yzsinp

+ 2xzsinp + z^2 + 2yzcosp

= 2 + 1 – 2sinq + 2cosq

From equation (iii) and (i)

= $x^2 + y^2 + z^2$ + 2z (1 + cosq – z) + 2q(1 – sinq – z) z^2

= 3 – 2(sinq – cosq)

= $x^2+y^2+z^2$ + 2z(2 + cosq – sinq – 2z)

= 3 + 2(cosq – sinq)

For equation (iii)

$\Rightarrow$ 2z(2 + cosq – sinq – 2z) = 1 + 2 |cosq – sinq|

$\therefore x^2 + y^2 + z^2 = 2$

Sol 8: $x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

(a) A unique solution, $D \neq 0$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix}$$

$= 1[2\lambda - 6] + 1[-\lambda + 3] + 0$

$= 2\lambda - 6 + 3 - \lambda = \lambda - 3 \neq 0$

$\lambda \neq 3$

(b) Infinite solution

So D = 0, Dx = Dy = Dz = 0

$D = 0 \to \lambda = 3$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & 3 \end{vmatrix}$$

$\Rightarrow 6[0] + 1[3\mu - 30] + [20 - 2\mu]$

$\Rightarrow (\mu-10)5 = 0$

$\mu = 10$

(c) No solution $\to D = 0, D_x \neq 0$

$\lambda = 3, \mu \neq 10$

Sol 9: $x + y + z = 1$

$x + 2y + 4z = p$

$x + 4y + 10z = p^2$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix}$$

$D = 1[20 - 16] + 1[4 - 10] + 1[4 - 2] = 4 - 6 + 2 = 0$

So for solution, $D_x = D_y = D_z = 0$

$$D_x = \begin{vmatrix} 1 & 1 & 1 \\ p & 2 & 4 \\ p^2 & 4 & 10 \end{vmatrix}$$

$= 1[20 - 16] + 1[4p^2 - 10p] + 1[4p - 2p^2] = 0$

$= 4 + 4p^2 - 10p + 4p - 2p^2 = 0$

$2p^2 - 6p + 4 = 0$

$p^2 - 3p + 2 = 0$

$p^2 - 2p - p + 2 = 0$

$(p-2)(p-1) = 0 \Rightarrow p = 1$ or 2

For $p = 1$

$\Rightarrow x + y + z = 1$ ... (i)

$x + 2y + 4z = 1$ ... (ii)

$x + 4y + 10z = 1$ ... (iii)

Assume that $x = k$

Equation (ii) – ii(i)

$-x + 2z = -1$

$\Rightarrow 2z = x - 1 \Rightarrow z = \frac{k-1}{2}$

So $y = 1 - z - x = 1 - k - \frac{(k-1)}{2}$

$y = \frac{2-2k-k+1}{2} = \frac{3-3k}{2}$

$(x,y,z) = \left(k, \frac{3-3k}{2}, \frac{k-1}{2}\right)$

At $p = 2$

$x + y + z = 1$ ... (1)

$x + 2y + 4z = 2$ ... (2)

$x + 4y + 10z = 4$ ... (3)

Assume $x = k$

Equation (2) – 2(1)

$-x + 2z = 0$

$\Rightarrow x = 2z = k \Rightarrow z = \frac{k}{2}$

$y = 1 - x - z = 1 - k - \frac{k}{2}$

$= 1 - \frac{3k}{2} = \frac{2-3k}{2}$

Sol 10: $Kx + 2y - 2z = 1$

$4x + 2Ky - z = 2$

$6x + 6y + Kz = 3$

$D = \begin{bmatrix} K & 2 & -2 \\ 4 & 2K & -1 \\ 6 & 6 & K \end{bmatrix}$ at $K = 2$ (given)

$= \begin{vmatrix} 2 & 2 & -2 \\ 4 & 4 & -1 \\ 6 & 6 & 2 \end{vmatrix} = 0$

$\Rightarrow 2x + 2y - 2z = 1$ ... (i)

$4x + 4y - z = 2$ ... (ii)

$6x + 6y + 2z = 3$ ... (iii)

Assume $x = l$

Equation (iii), (ii) – (iii).(ii)

$7z = 0 \rightarrow z = 0$

$2y = 1 + 2z - 2x = 1 - 2\lambda$

$(x, y, z) = (\lambda, 1-2\lambda, \lambda)$

If $K \neq 2$

$D = \begin{bmatrix} K & 2 & -2 \\ 4 & 2K & -1 \\ 6 & 6 & K \end{bmatrix}$

$= K[2K^2 + 6] + 2[-6 - 4K] - 2[24 - 12K]$

$= 2K^3 + 6K - 12 - 8K - 48 + 24K$

$= 2K^3 + 22K - 60 = 2(K^3 + 11K - 30)$

At $K = 2$

$\Rightarrow 2(8 + 11(2) - 30) = 0$

So $(K - 2)$ is a factor

$\frac{k^3 + 11k - 30}{k-2} = K^2 + 2K + 15$

$D = 2(K - 2)(K^2 + 2K + 15)$

$D_x = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 2K & -1 \\ 3 & 6 & K \end{bmatrix}$

$= 2K^2 + 6 + 2[-3 - 2K] - 2[12 - 6K]$

$= 2K^2 + 6 - 6 - 4K - 24 + 12K$

$= 2K^2 + 8K - 24 = 2[K^2 + 4K - 12]$

$= 2[K^2 + 6K - 2K - 12] = 2[K(K+6) - 2(K + 6)]$

$= 2(K - 2)(K + 6)$

Similarly, $D_y = (K - 2)(2K + 3)$ and $D_z = 6(K - 2)^2$

if $K \neq 2$,

$\frac{x}{2(K+6)} = \frac{y}{2K+3} = \frac{z}{6(K-2)} = \frac{1}{2(K^2+2K+15)}$

Sol 11: (a) a, b, c, d are distinct no.

$a, b, c, d \in \{1, 2, 3, 4, 5\}$

$ax + by = 1$

$cx + dy = 2$

$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ ... (i)

$D_x = \begin{vmatrix} 1 & b \\ 2 & d \end{vmatrix} = d - 2b,$

$x = \frac{D_x}{D} = \frac{d-2b}{ad-bc}$

for least possible +ve value of x

d – 2b = 1 (least natural number)

(d, b) → (3, 1) or (5, 2)

ad – bc should be maximum for least x

(a, b) → (3, 1) (ad – bc) → (3a – c)

a,c ∈ {7, 4, 5}

Max. → 3(5) – 2 = 15 – 2

$x = \frac{1}{13}$

If a, b → (5, 2),

ad – bc → 5a – 2c,

a, c ∈ {1, 3, 4}

Max. 5a – 2c → 5(4) – 2(1) = 18

$\rightarrow x = \frac{1}{18} = \frac{p}{q}$ (min.)

p + q = 1 + 18 = 19

(b) x + ay = 3 and ax + 4y = 6 → x > 1, y > 0

$D = \begin{vmatrix} 1 & a \\ a & 4 \end{vmatrix} = 4 - a^2,$

$D_x = \begin{vmatrix} 3 & a \\ 6 & 4 \end{vmatrix} = 12 - 6a$

$D_y = \begin{vmatrix} 1 & 3 \\ a & 6 \end{vmatrix} = 6 - 3a,$

$x > 0, \frac{D_x}{D} > 1 \rightarrow \frac{6(2-a)}{(2-a)(2+a)} > 1$

$\frac{6}{(2-a)} > 1,$

2 + a < 6 → a = 1, 3

$y = \frac{D_y}{D} = \frac{3(2-a)}{6(2-a)} = \frac{3}{6} = \frac{1}{2}$ + ve

So a is 1 and 3

1 + 3 = 4

Sol 12: (a – t)x + by + cz = 0

bx + (c – t)y + az = 0

cx + ay + (b – t)z = 0

Has non-trivial solution,

So D = 0

$D = \begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & b-t \end{vmatrix} = 0$

Assume $D = a_0t^3 + b_0t^2 + c_0t + d_0 = 0$

So $t_1t_2t_3 = \frac{-d_0}{a_0}$

At t = 0, $D = d_0$

So $d_0 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

And a_0 is coefficient of t^3 = (–1)(–1)(–1) = –1

$t_1t_2t_3 = \frac{-d_0}{-1} = d_0 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Sol 13: 3x – y + 4z = 3

X + 2y – 3z = –2

6x + 5y + λz = –3,

$D = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix}$

⇒ 3(2λ + 15) –1[–18 – λ) + 4[5 – 12]

⇒ 6λ + 45 + 18 + λ – 28 = 7λ + 35 = 7(λ + 5)

D = 7(λ + 5)

$D_x = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & \lambda \end{vmatrix}$

= 3[2λ + 15] + 1[–2λ – 9] + 4[–10 + 6]

= 6λ + 45 – 2λ – 9 – 16

= 4λ + 20 = 4(λ + 5);

$x = \frac{D_x}{D} = \frac{4(\lambda+5)}{7(\lambda+5)} = \frac{4}{7}$

$D_y = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & \lambda \end{vmatrix}$

$= 3[-2\lambda - 9] + 3[-18 - \lambda] + 4[-3 + 12]$

$= -6\lambda - 27 - 54 - 3\lambda + 36$

$= -9\lambda - 45 = -(\lambda + 5)$

$\Rightarrow y = \dfrac{D_y}{D} = \dfrac{-9(\lambda+5)}{7(\lambda+5)} = \dfrac{-9}{7}$

$D_z = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix}$

$= 3[-6 + 10] + 1[-3 + 12] + 3[5 - 12]$

$= 12 + 9 - 21 = 0,$

$z = \dfrac{D_z}{D} = 0$

So x, y, z is not dependent on λ

(if $\lambda \neq -5$)

At $\lambda = -5$

$3x - y + 4z = 3$ ... (i)

$x + 2y - 3z = -2$ ... (ii)

$6x + 5y - 5z = -3$ ... (iii)

Assume z = k, (iii) – (ii)(i)

$7y - 13z = -9$

$\Rightarrow y = \dfrac{13k - 9}{7}$

So, $x = 3z - 2y - z = 3k - \dfrac{2}{7}(13k - 9) - z = \dfrac{4 - 5k}{7}$

$(x, y, z)\left(\dfrac{4-5k}{7}, \dfrac{13k-9}{7}, k\right)$

Sol 14: $z + ay + a^2x + a^3 = 0$

$z + by + b^2x + b^3 = 0$

$z + cy + c^2x + c^3 = 0$

Now, $c = \begin{bmatrix} a^3 \\ b^3 \\ c^3 \end{bmatrix}$

$D = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

$x = \dfrac{D_x}{D}, y = \dfrac{D_y}{D}, z = \dfrac{D_z}{D}$

$D_x = \begin{vmatrix} 1 & a & -a^3 \\ 1 & b & -b^3 \\ 1 & c & -c^3 \end{vmatrix}$

$= -(a + b + c)(a - b)(b - c)(c - a);$

$\therefore x = -(a + b + c)$

$D_y = \begin{vmatrix} 1 & -a^3 & a^2 \\ 1 & -b^3 & b^2 \\ 1 & -c^3 & c^2 \end{vmatrix}$

$= (ab + bc + ca)(a - b)(b - c)(c - a)$

$\therefore y = [ab + bc + ca]$

$D_z = \begin{vmatrix} -a^3 & a & a^2 \\ -b^3 & b & b^2 \\ -c^3 & c & c^2 \end{vmatrix}$

$= -abc(a - b)(b - c)(c - a)$

$\therefore z = -abc$

Sol 15: (a) $\alpha x - y + z = \alpha$

$x - \alpha y + z = 1$

$x - y + \alpha z = 1$

$D = \alpha[-\alpha^2 + 1] - 1[1 - \alpha] + [-1 + \alpha]$

$= -\alpha^3 + \alpha - 2 + 2a$

$= (-\alpha^3 + 3\alpha - 2) = -(a^3 - 3\alpha + 2)$

At $\alpha = 1$

$D = -(1 - 3 + 2) = 0$

So $(\alpha - 1)$ is a factor

$\dfrac{\alpha^3 - 3\alpha + 2}{\alpha - 1} = \alpha^2 + \alpha - 2$

So $D = -(\alpha - 1)(\alpha^2 + \alpha - 2)$

$= -(\alpha - 1)(\alpha^2 + 2\alpha - \alpha - 2)$

$= -(\alpha - 1)[\alpha(\alpha + 2) - 1(\alpha + 2)]$

$= -(\alpha - 1)(\alpha - 1)(\alpha + 2)$

$\alpha \in [-10, 10]$

So, α has an integral value

$D_x = \begin{vmatrix} \alpha & -1 & 1 \\ 1 & -\alpha & 1 \\ 1 & -1 & \alpha \end{vmatrix}$

So x = 1,

$D_x = -(\alpha - 1)^2(\alpha + 2)$

$D_y = \begin{vmatrix} \alpha & \alpha & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0,$

$D_z = \begin{vmatrix} \alpha & -1 & \alpha \\ 1 & -\alpha & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$

(a) Unique solution,

So $D \neq 0 \rightarrow \alpha \neq 1, -2$

Number of values for α in

$[-10, 10] = 21 - 2 = 19 = L$

(b) Number solution is not possible for every value of α, system has atleast one solution. So M = 0

(c) Infinite solution $\rightarrow D = 0$

$\alpha = 1, -2 \rightarrow N = 2$

$L - M + N = 19 + 2 = 21$

(b) $2x + 3y - z = 0$

$3x + 2y + kz = 0$

$4x + y + z = 0$

Has non-trivial solution

So, $D = 0 \Rightarrow \begin{vmatrix} 2 & 3 & -1 \\ 3 & 2 & k \\ 4 & 1 & 1 \end{vmatrix}$

$= 2[2 - k] + 3[4k - 3] - 1[3 - 8] = 0$

$4 - 2k + 12k - 9 + 5 = 10k = 0$

$\Rightarrow k = 0$

$\Rightarrow 2x + 3y - z = 0$ ... (i)

$3x + 7y = 0$ ... (ii)

$4x + y + z = 0$ ... (iii)

(iii) – (ii) (i) $\rightarrow$

$-5y + 32 = 0 \rightarrow 3z = 5y$

$3x = -2y \rightarrow x = \frac{-2y}{3}$,

$z = \frac{5}{3}y, y = y$

x, y, z are integer , so at for x and z to be integer x = n

$= -\frac{2}{3}y$

$\rightarrow y = \frac{3n}{-2}$ (also an integer)

So at n = –2, –7y = 3, z = 5 (minimum +ve value)

(c) a, b $\in$ {0, 1, 2, ..., 10}

$x + y + z = 4$

$2x + y + 3z = 6$

$x + 2y + az = b$

$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & a \end{vmatrix}$

$= 1(a - 6) + 1(3 - 2a) + 1(4 - 1)$

$= a - 6 + 3 - 2a + 3 = -9$

$D_x = \begin{vmatrix} 4 & 1 & 1 \\ 6 & 1 & 3 \\ b & 2 & a \end{vmatrix}$

$= 4(a-6) + 1(3b-6a) + 1(12-b)$

$= 4a - 24 + 3b - 6a + 12 - b$

$= -2a + 2b - 12$

$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 6 & 3 \\ 1 & b & a \end{vmatrix}$

$= 6a - 3b + 4[3 - 2a] + b - 6$

$= 6a - 3b + 12 - 8a + b - 6$

$= -2a - 2b + 6$

$D_z = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 1 & 6 \\ 1 & 2 & b \end{vmatrix}$

$= b - 12 + 6 - 2b + 4[4 - 1]$

$= b - 12 + 6 - 2b + 12$

$= -b + 6$

(i) Unique solution so $D \neq 0$

$\rightarrow a \neq 0$

$\therefore$ a $\in$ {1, 2, ... , 10},

b $\in$ {0, 1, ... , 10}

$L = 10 \times 11 = 110$

(ii) Number solution D = 0, a = 0

$D_x \neq 0 \rightarrow 2b \neq 12 \rightarrow b \neq 6$,

and $D_y \neq 0 \rightarrow b \neq 3$

$M = 1(11 - 2) = 9$

(iii) Infinite solution $D=0 \rightarrow a=0$,

$D_x = D_y = D_z = 0$

But D_x and D_z can't be zero at same times, so no possible common solution N = 0

L + M – N = 110 + 9 – 0 = 119

Sol 16: $\begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix}$

(a) Assume $z_1 = 5 + 3i$, $z_2 = \frac{2}{3} + 4i$

$z_3 = 4 + 5i$

$(z^3)^2 = 4^2 + 5^2 = 41$

$\Rightarrow \begin{vmatrix} -7 & z_1 & \bar{z}_2 \\ \bar{z}_1 & 8 & z_3 \\ z_2 & \bar{z}_3 & 9 \end{vmatrix}$

$= -7[72 - z_3\bar{z}_3] + z_1[z_2z_3 - 9\bar{z}_2] + \bar{z}_2[\bar{z}_1\bar{z}_3 - 8z_2]$

$= -7[72 - 41] + (5 + 3i)$

$\left[\left(\frac{2}{3}+4i\right)(4+5i) - 9\left(\frac{2}{3}-4i\right)\right]$

$+ \left(\frac{2}{3}-4i\right)\left[(5-3i)(4-5i) - 8\left(\frac{2}{3}+4i\right)\right]$

$= -7(31) + (5 + 3i)$

$\left[\frac{8}{3} + 16i + \frac{10}{3}i - 20 - 6 + 36i\right]$

$+ \left(\frac{2}{3}-4i\right)\left[20 - 15 - 12i - 25i - \frac{16}{3} - 32i\right]$

$= -217 + (5 + 3i)\left[\frac{-70}{3} + \frac{160i}{3}\right] + \left(\frac{2}{3}-4i\right)\left[-\frac{1}{3} - 69i\right]$

Coefficient of i

$= -70 + \frac{800}{3} - 46 + \frac{4}{3} = -106 + \frac{804}{3}$

(b) $\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

$= 1[\cos px \sin(p+d)x - \cos(p+d)x \sin px]$

$+ a[\cos(p+d)x \sin(p-d)x - ws(p-d)x \sin(p+d)x]$

$+ a^2[\cos(p-d)x \sin px - \cos px \sin(p-d)x]$

$= [\sin x\,(p+d-p) + a[\sin x\,(p-d-p-d)]$

$+ a^2[\sin x\,(p-p+d)]$

$= \sin x\, d + a \sin(-2d) + a^2 \sin dx$

It dose not depend upon p

(c) $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = \begin{vmatrix} x^3 & x^2 & x \\ y^3 & y^2 & y \\ z^3 & z^2 & z \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$

$= xyz\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$

$= (xyz + 1)(x - y)(y - z)(z - x) = 0$

(given) x, y, z are all different

So $(xyz + 1) = 0 \Rightarrow xyz = -1$

Sol 17: (a) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

$R_1 \to R_1 - R_3,\ R_2 \to R_2 - R_3$

$\begin{vmatrix} a^2+2a-3 & 2a+1 & 0 \\ 2a-2 & a-2 & 0 \\ 3 & 3 & 1 \end{vmatrix}$

$= (a-1)(a^2 + 2a - 3) - 4(a-1)^2$

$= (a-1)[(a^2 + 3a - a - 3) - 4(a-1)]$

$(a-1)[(a-1)(a+3) - 4(a-1)]$

$= (a-1)^2[a + 3 - 4] = (a-1)^3$

(b) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^2 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ x-2 & y-2 & 2 \\ x^3-z^3 & y^3-z^3 & z^3 \end{vmatrix}$

$= (x-z)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ x^2+y^2+xz & y^2+z^2+yz & z^3 \end{vmatrix}$

$\therefore a^2 - b^3 = (a-b)(a^2 + b^2 + ab)$

$= (x-z)(y-z)(y^2 + z^2 + yz - x^2 - z^2 - xz)$

$= (x-z)(y-z)[z(y-x) + (y^2 - x^2)]$

$= (x-y)(y-z)(z-x)(x+y+z)$

Sol 18: (a) f(x) = $\begin{vmatrix} x & 1 & -3/2 \\ 2 & 2 & 1 \\ \frac{1}{x-1} & 0 & 1/2 \end{vmatrix}$ x > 1

f(x) = x [1 – 0] + 1$\left[\frac{1}{x-1}-1\right]$ + $\frac{3}{2}\left[\frac{2}{x-1}\right]$

= x + $\frac{1}{x-1}$ – 1 + $\frac{3}{x-1}$

= $(x-1)+\frac{3}{x-1}$ = $\frac{(x-1)^2+3}{x-1}$

= $\frac{x^2+1+3-2x}{x-1}$ = $\frac{x^2-2x+4}{x-1}$

F′(x) = 1 – $\frac{3}{(x-1)^2}$ ⇒ 01 = $\frac{3}{(x-1)^2}$

⇒ $(x-1)^2 = 3 \Rightarrow x = 1 \pm \sqrt{3}$

But x . so x = $1 \pm \sqrt{3}$

f″(x) = $\frac{-6}{(x-z)^3}$, at x = 1 + $\sqrt{3}$

f″(x) = $\frac{-6}{3\sqrt{3}}$ > 0 so minima

f(1 + $\sqrt{3}$) = $\sqrt{3}$ + $\frac{3}{\sqrt{3}}$ = $2\sqrt{3}$

But if x is integer for min. volue of f(x)

⇒ x = [1 + $\sqrt{3}$] = 2

F(x) = f(2) = 1 + $\frac{3}{1}$ = 4

(b) $a^2 + b^2 + c^2 + ab + bc + ca \le O\ \forall\ a,b,c \in R$

$$\begin{vmatrix} (a+b+2)^2 & a^2+b^2 & 1 \\ 1 & (b+c+2)^2 & b^2+c^2 \\ c^2+b^2 & 1 & (c+a+2)^2 \end{vmatrix}$$

$(a + b)^2 + (b + 1)^2 + (c + a)^2 \ge 0$

(always & square is +ve)

= 2 $(a^2 + b^2 + c^2 + bc + ca + ab)$

Its given that $a^2+b^2+c^2 + bc + ca + ab \le 0$

So $0 \le a^2 + b^2 + c^2 + ab + ca + ab \le 0$

⇒ $(a + b)^2 + (b + 1)^2 + (c + a)^2 = 0$

⇒ a = b = c = 0

$\begin{vmatrix} 2^2 & 0 & 1 \\ 1 & 2^2 & 0 \\ 0 & 1 & 2^2 \end{vmatrix}$ = 4[16] + [1] = 65

Sol 19: D = $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$, D′ = $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix}$

D′= $\begin{vmatrix} b & c+a & a+b \\ a & b+c & c+a \\ c & a+b & b+c \end{vmatrix}$ + $\begin{vmatrix} c & c+a & a+b \\ b & b+c & c+a \\ a & a+b & b+c \end{vmatrix}$

$C_2 \to C_2 + C_1 - C_3$, $C_2 \to C_2 - C_1$,

$C_3 \to C_3 - C_1C_3 \to C_3 - C_2$

D′ = $\begin{vmatrix} b & c & a \\ a & b & c \\ c & a & b \end{vmatrix}$ + $\begin{vmatrix} c & a & b \\ b & c & a \\ a & b & c \end{vmatrix}$

After swapping rows according to D

D′ = $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ + $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ = 2D

Sol 20: $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

$C_1 \to C_1 - BC_3$, $C_2 \to C_1 + AC_3$

$$\begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1-a^2+b^2 & 2a \\ b+b+a^2b+b^3 & -a+a^3-ab^2 & 1-a^2-b^2 \end{vmatrix}$$

$R_3 \to R_3 + aR_2 - bR_1$

= $\begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$

= $(1 + a^2 + b^2)^3$

Sol 21: f(x) = $\begin{vmatrix} \sin x & \sin(x+h) & \text{sain}(x+2h) \\ \sin(x+2h) & \sin x & \sin(x+h) \\ \sin(x+h) & \sin(x+2h) & \sin x \end{vmatrix}$

sin (A + B) = sin A cos B + cos A sin B

sin (x + nh) = sin x cos (nh) + sin nh cos x

$\Rightarrow \lim_{h\to 0}$ sin (x + nh) = (sin x) 1+ (nh) cos x

$$\Rightarrow f(x)\begin{vmatrix} \sin x & \sin x + n\cos x & \sin x + 2h\cos x \\ \sin x + 2h\cos x & \sin x & \sin x + h\cos x \\ \sin x + h\cos x & \sin x + 2h\cos x & \sin x \end{vmatrix}$$

$C_1 \to C_1 \to C_3$, $C_2 \to C_2 - C_3$

$$f(x) = \begin{vmatrix} -2h\cos x & -h\cos_2 x & \sin x + 2h\cos x \\ h\cos x & -h\cos x & \sin x + h\cos x \\ \cos x & 2h\cos x & \sin x \end{vmatrix}$$

$$f(x) = h^2\begin{vmatrix} -2\cos x & -\cos x & \sin x + 2h\cos x \\ \cos x & -\cos x & \sin x + h\cos x \\ \cos x & 2\cos x & \sin x \end{vmatrix}$$

$$\lim_{h\to 0}\frac{f(x)}{h^2} = \begin{vmatrix} -2\cos x & -\cos x & \sin x \\ \cos x & -\cos x & \sin x \\ \cos x & 2\cos x & \sin x \end{vmatrix}$$

$R_1 \to R_1, R_3, R_2 \to R_2 - R_3$

$$= \begin{vmatrix} -3\cos x & -3\cos x & 0 \\ 0 & -3\cos x & 0 \\ \cos x & 2\cos x & \sin x \end{vmatrix}$$

$= \sin x\ (9\cos^2 x) = \sin x\ (9 - 9\sin^2 x)$

$= 9\sin x - 9\sin^3 x = 3\ (3\sin x - 3\sin^3 x)$

$= 3\ [\sin 3x + \sin^3 x] = k(\sin 3x + \sin^3 x)$

$\Rightarrow K = 3$

Sol 22: $\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$

$R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4-(\alpha+\beta-\gamma-\delta)^4 & (\beta+\gamma-\alpha-\delta)^4-(\alpha+\beta-\gamma-\delta)^2 & 0 \\ (\gamma+\alpha-\beta-\delta)^4-(\alpha+\beta-\gamma-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2-(\alpha+\beta-\gamma-\delta)^2 & 0 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$$

$= ((\beta + \gamma - \alpha - \delta)^2 - (\alpha + \beta - \gamma - \delta)^2$

$- ((\gamma + \alpha - \beta - \delta)^2 - (\alpha + \beta - \gamma - \delta)^2)$

$$\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^2+(\alpha^2+\beta-\gamma-\delta)^2 & 1 & 0 \\ (\gamma+\alpha-\beta-\delta)^2+(\alpha+\beta-\gamma-\delta)^2 & 1 & 0 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$$

$= ((\beta - \gamma - \delta - \delta)^2 - (\alpha + \beta - \gamma - \delta)^2)$

$((\gamma + \alpha - \beta - \delta)^2 - (\alpha + \beta - \gamma - \delta)^2)$

$[(\beta + \gamma - \alpha - \delta)^2 + (\delta^2 + \beta^2 - \gamma - \delta)^2$

$(\gamma + \alpha - \beta - \delta)^2 - (\alpha + \beta - \rho - \delta)^2]$

$= -2\ (\alpha - \beta)2\ (\alpha - \gamma)2\ (\alpha - \gamma)2\ (\beta - \gamma)2$

$(\beta - \delta)2\ (\gamma - \delta)\ (-1)^6$

$= -64(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \gamma)(\gamma - \delta)$

Sol 23: $x^3 - 3x^2 + 2 = 0$

At x = 1 $\Rightarrow 1 - 3 + 2 = 0$.

So (x – 1) is a factor of $x^3 - 3x^2 + 2 = 0$

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$\Rightarrow x^3 - 3x^2 + 2 = (x - 1)\ (x^2 - 2x - 2)$

a = 1, and bc = – 2, b + c = 2

$\Rightarrow bc = 1 \pm \sqrt{3}$, $c^2b^2 = 4 \pm 2\sqrt{3}$

$$= \begin{vmatrix} 2^2 & 1 & 1 \\ 4+2\sqrt{3} & (2-\sqrt{3})^2 & 4+2\sqrt{3} \\ 4-2\sqrt{3} & 4-2\sqrt{3} & (2+\sqrt{3})^2 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 & 1 \\ 4+2\sqrt{3} & 7-4\sqrt{3} & 4+2\sqrt{3} \\ 4-2\sqrt{3} & 4-2\sqrt{3} & 7+4\sqrt{3} \end{vmatrix}$$

$= 4[49 - 48] - [16 - 12] - 1\ [16 + 12) + 28$

$+ 30\sqrt{3} + 24] + [16 - 12 - (28 + 24 - \sqrt{3}\ (30)\ = -108$

Sol 24: (a) $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$

$$= \frac{1}{2}\begin{vmatrix} x+2 & 4x+6 & 3x+4 \\ 2x+3 & 6x+8 & 4x+5 \\ 3x+5 & 10x+16 & 10x+17 \end{vmatrix}$$

$c_2 \to c_2 - c_1 - c_3$

$$= \frac{1}{2}\begin{vmatrix} x+2 & 0 & 3x+4 \\ 2x+3 & 0 & 4x+5 \\ 3x+5 & -3x-6 & 10x+17 \end{vmatrix}$$

$\Rightarrow \frac{[3x+6]}{2}$ [(3x + 4) (2x + 3)–(x + 2) (4x + 5)]= 0

$\Rightarrow (3x + 6)[6x^2 + 17x + 12 - 4x^2 - 13x - 10] = 0$

$\Rightarrow (3x + 6)[2x^2 + 4x + 2] = 0$

$\Rightarrow (x + 2)(x^2 + 2x + 1) = 0$

$\Rightarrow (x + 2)(x + 1)^2 = 0$

$\Rightarrow x = -2, -1$

(b) $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

$R_1 \to R_2 - R_3,\ R_2 \to R_2 - R_3$

$\Rightarrow \begin{vmatrix} 6 & 24 & 60 \\ 4 & 18 & 48 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 1 & 4 & 10 \\ 2 & 9 & 24 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

$= 9(3x - 64) - 24(2x - 27) + 4[24(x - 8)$

$-2(3x - 64)] + 10[2(2x - 27) - 9(x - 8)]$

$= (6 - 48 + 96 - 24 + 40 - 90)(x - 4) = 0$

$\Rightarrow x = 4$

Sol 25: $a + b + c = 0$

$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

$c_1 \to c_1 + c_2 + c_3$

$\begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix}$

$= (a + b + c - x)\begin{vmatrix} 1 & c & b \\ 1 & b.x & a \\ 1 & a & c-x \end{vmatrix} = 0$

$a + b + c = 0\ (a + b + c - x) = -x = 0$

$\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix}$

$= (b - x)(c - x) - a^2 + c(a - c + x) + b(c - b + x] = 0$

$bc - x(b + 1) + x^2 - a^2 + a(-c^2 + cx + ba - b^2 + bx = 0$

$x^2 + x(b + c - b - c) = a^2 + b^2 + c^2 - (ab + bc + ca)$

$x^2 = a^2 + b^2 + c^2 - (ab + bc + ca)$

$\therefore a + b + c = 0 \Rightarrow (a + b + c)^2 = 0$

$a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow ab + bc + ca = \dfrac{-(a^2 + b^2 + c^2)}{2}$

$x^2 = a^2 + b^2 + c^2 + \dfrac{a^2 + b^2 + c^2}{2} = \dfrac{3}{2}(a^2 + b^2 + c^2)$

$x = \pm\sqrt{\dfrac{3}{2}(a^2 + b^2 + c^2)}$

$x = 0\ \begin{vmatrix} a & c & b \\ c & b & a \\ b & c & a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc = 0$

$\therefore a + b + c = 0$

Sol 26: $x^3 - 5x^2 + 3x - 1 = 0$ ⟨ a, b, c

$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}\ C_1 \to C_1 + C_2 + C_3$

$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$

$= 5\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix} = 5\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & a+c & a+b \end{vmatrix}$

$= 5[(b - c)(a + b) + (a - c)(a + c) + 2(bc - ab - bc + c^2)]$

$= 5[ab - ac + b^2 - bc + a^2 - c^2 - 2ab + 2c^2]$

$= 5[a^2 + b^2 + c^2 - (ab + bc + ca)]$

$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$

$= 25 - 2(3) = 19$

$= 5[19 - 3] = 5.16 = 80$

Sol 27: $\begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & b^2 & c^2+\lambda \end{vmatrix}$

$= \dfrac{1}{abc}\begin{vmatrix} (a^2+\lambda) & ab^2 & ac^2 \\ a^2b & b(b^2+\lambda) & bc^2 \\ a^2c & b^2c & c(c^2+\lambda) \end{vmatrix}$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+\lambda & b^2 & c^2 \\ a^2 & b^2+\lambda & c^2 \\ a^2 & b^2 & c^2+\lambda \end{vmatrix}$$

$R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$

$$= \begin{vmatrix} \lambda & 0 & -\lambda \\ 0 & \lambda & -\lambda \\ a^2 & b^2 & c^2+\lambda \end{vmatrix} = \lambda^2\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ a^2 & b^2 & (c^2+\lambda) \end{vmatrix}$$

$= \lambda^2 (c^2 + \lambda + b^2 - 1[-a^2])$

$= \lambda^2 (a^2 + b^2 + c^2 + \lambda)$

Sol 28: $= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$4\begin{vmatrix} a^2 & b^2-a^2 & c^2-a^2 \\ a & b-a & c-a \\ 1 & 1 & 1 \end{vmatrix}$$

$= 4\left[\left(b^2-a^2\right)\left(c-a\right)-\left(c^2-a^2\right)\left(b-a\right)\right]$

$= -4\ (c-a)(b-a)(b-c)$

$= 4(c-b)(b-c)(c-a)$

Sol 29:

$$\begin{vmatrix} \cot\frac{A}{2} & \cot\frac{B}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2}+\tan\frac{C}{2} & \tan\frac{C}{2}+\tan\frac{A}{2} & \tan\frac{A}{2}+\tan\frac{B}{2} \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$

$$\begin{vmatrix} \cot\frac{A}{2}-\cot\frac{C}{2} & \cot\frac{B}{2}-\cot\frac{C}{2} & \cot\frac{C}{2} \\ \tan\frac{C}{2}-\tan\frac{A}{2} & \tan\frac{C}{2}-\tan\frac{B}{2} & \tan\frac{A}{2}+\tan\frac{B}{2} \\ 0 & 0 & 1 \end{vmatrix}$$

$= \left(\tan\frac{C}{2}-\tan\frac{B}{2}\right)\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)$

$- \left(\cot\frac{B}{2}-\cot\frac{C}{2}\right)\left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)$

$= \tan\frac{C}{2}\cot\frac{A}{2} - 1 - \tan\frac{B}{2}\cot\frac{A}{2} + \tan\frac{B}{2}\cot\frac{C}{2}$

$- \cot\frac{B}{2}\tan\frac{C}{2} + \tan\frac{A}{2}\cot\frac{B}{2} + 1 - \tan\frac{A}{2}\cot\frac{C}{2}$

We know that $\tan\frac{A}{2} = \frac{1}{\cot\frac{A}{2}}$

$$= \frac{\tan\frac{C}{2}}{\tan\frac{A}{2}} - \frac{\tan\frac{B}{2}}{\tan\frac{A}{2}} - \frac{\tan\frac{C}{2}}{\tan\frac{B}{2}} + \frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}$$

$$- \frac{\tan\frac{A}{2}}{\tan\frac{C}{2}} + 1 - 1 + \frac{\tan\frac{B}{2}}{\tan\frac{C}{2}} = 0$$

$$\Rightarrow \frac{1}{\tan\frac{A}{2}}\left[\tan\frac{C}{2}-\tan\frac{B}{2}\right] + \frac{1}{\tan\frac{B}{2}}\left[\tan\frac{A}{2}-\tan\frac{C}{2}\right]$$

$$+ \frac{1}{\tan\frac{C}{2}}\left[\tan\frac{B}{2}-\tan\frac{A}{2}\right] = 0$$

It can only happen when two angles are equal.

$\Rightarrow \Delta ABC$ is isosceles

Exercise 2

Single Correct Choice Type

Sol 1: (A) $D_r = \begin{vmatrix} 2r-1 & {}^mC_r & 1 \\ m^2-1 & 2^m & 1+m \\ \sin^2(m^2) & \sin^2 m & \sin^2(m+1) \end{vmatrix}$

$\sum_{r=0}^{m} D_r =$

$$\begin{vmatrix} \sum_{r=0}^{m}(2r-1) & \sum_{r=0}^{m}{}^mC_r & m+1 \\ (m+1)(m^2-1) & (m+1)2^m & (m+1)^2 \\ (m+1)\sin^2(m^2) & (m+1)\sin^2 m & (m+1)\sin^2(m+1) \end{vmatrix}$$

$$= \begin{vmatrix} (m+1)(m-1) & 2^m & (m+1) \\ (m+1)(m^2-1) & (m+1)2^m & (m+1)^2 \\ (m+1)\sin^2 m^2 & (m+1)\sin^2 m & (m+1)\sin^2(m+1) \end{vmatrix}$$

Common (m + 1) from C_1, C_3 and R_2

$$= (m+1)^3\begin{vmatrix} m-1 & 2^m & 1 \\ m-1 & 2^m & 1 \\ \sin^2 m^2 & (m+1)\sin^2 m & \sin^2(m+1) \end{vmatrix} = 0$$

Sol 2: (D) $D = \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$

$D = 1 - \cos(\beta-\gamma)\cos(\gamma-\beta) + \cos(\beta-\alpha)$

$[\cos(\gamma-\beta)\cos(\alpha-\gamma) - \cos(\alpha-\beta)]$

$+ \cos(\gamma-\alpha)[\cos(\alpha-\beta)\cos(\beta-\gamma)$

$- \cos(\alpha-\gamma)]$

$D = 1 - \cos^2(\beta-\gamma) + \cos(\beta-\alpha)\cos(\gamma-\beta)$

$\cos(\alpha-\gamma) - \cos^2(\beta-\alpha) + \cos(\gamma-\alpha)$

$\cos(\alpha-\beta)\cos(\beta-\gamma) - \cos^2(\gamma-\alpha)$

$D = 1 + 2\dfrac{\cos(\beta-\gamma)}{2}[\cos(\gamma-\beta) + \cos$

$(\gamma-\beta-2\alpha)] - \cos^2(\gamma-\alpha) - \cos^2(\beta-\alpha) - \cos^2(\beta-\gamma)$

$D = 1+\cos^2(\beta-\gamma)+\left[\dfrac{\cos 2(\beta-\alpha)+\cos 2(\gamma-\alpha)}{2}\right]$

$- \cos^2(\beta-\gamma) - \cos^2(\gamma-\alpha) - \cos^2(\beta-\alpha)$

$= 1 + \dfrac{1}{2}\left(2\cos^2(\beta-\alpha)-1+2\cos^2(\gamma-\alpha)-1\right)$

$- \cos^2(\beta-\alpha) - \cos^2(\gamma-\alpha)$

$= 1 - \left(\dfrac{2}{2}\right) = \cos^2(\beta-\alpha) + \cos^2(\gamma-\alpha)$

$- \cos^2(\beta-\alpha) - \cos^2(\gamma-\alpha) = 1 - 1 = 0$

Sol 3: (A) $D = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & a(b+c) \\ bc^2a^2 & abc & b(a+c) \\ ca^2b^2 & abc & c(a+b) \end{vmatrix}$$

$$= \frac{abc.abc}{abc}\begin{vmatrix} bc & 1 & a(b+c) \\ ac & 1 & b(a+c) \\ ab & 1 & c(a+b) \end{vmatrix}$$

$C_3 \to C_3 + C_1$

$$= abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ac & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix}$$

$$= (abc)(ab + bc + ca)\begin{vmatrix} bc & 1 & 1 \\ ac & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} = 0$$

Sol 4: (A)

$$f'(x) = \begin{vmatrix} mx & mx-p & mx+p \\ n & n+p & n-p \\ mx+2n & mx+2n+p & mx+2n-p \end{vmatrix}$$

$C_2 \to C_2 + C_3$

$$f'(x) = \begin{vmatrix} mx & 2mx & mx+p \\ n & 2n & n-p \\ mx+2n & 2(mx+2n) & mx+2n-p \end{vmatrix}$$

$C_2 \to C_2 - 2C_1$

$$f'(x) = \begin{vmatrix} mx & 0 & mx+p \\ n & 0 & n-p \\ mx+2n & 0 & mx+2n-p \end{vmatrix} = 0$$

$y = f(x)$

$y' = 0$

$y = K$

It is a straight line parallel to x–axis.

Sol 5: (A) $D(x) = \begin{vmatrix} x-1 & (x-1)^2 & x^3 \\ x-1 & x^2 & (x+1)^3 \\ x & (x+1)^2 & (x+1)^3 \end{vmatrix}$

Assume $D(x)\ a_0 + a_1x + \ldots$

$D'(x) = a_1 + 2a_2x$

At $x = 0$ $D'(0) = a_1$

$$D'(x) = \begin{vmatrix} +1 & (x-1)^2 & x_3 \\ +1 & x^2 & (x+1)^3 \\ 1 & (x+1)^2 & (x+1)^3 \end{vmatrix} + \begin{vmatrix} x-1 & 2(x-1) & x^3 \\ x-1 & 2x & (x+1)^3 \\ x & 2(x+1) & (x+1)^3 \end{vmatrix}$$

$$+ \begin{vmatrix} x-1 & (x-1)^2 & 3x^2 \\ x-1 & x^2 & 3(x+1)^2 \\ x & (x+1)^2 & 3(x+1)^2 \end{vmatrix} \text{ at } x = 0$$

$$D'(0) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 3 \end{vmatrix}$$

$= 1[-1] + 1\,[1 - 1] - 1[-2] -2\,(1) -1\,(-3) + 1[3]$

$= -1 + 0 + 2 - 2 + 3 + 3 = -1 + 6 = 5$

Sol 6: (D) $D = \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$, $|D| = 8$

$R_1 \to R_1 - R_2 - R_3$

$R_2 \to R_2 - R_3$

$$D = \begin{vmatrix} 0 & -2x & -2x \\ z-y & z & -y \\ y & x & x+y \end{vmatrix}$$

$$D = (-2x)\begin{vmatrix} 0 & 1 & 1 \\ z-y & z & -y \\ y & x & x+y \end{vmatrix}$$

$C_2 \to c_2 - c_3$

$$D = (-2x)\begin{vmatrix} 0 & 0 & 1 \\ z-y & z+y & -y \\ y & -y & x+y \end{vmatrix} = (-2x)\,[-y(z-y) - y\,(z+y)$$

$= -2x\,[-yz + y^2 - y^2 - yz] = \}4xyz| = 8$ given $|xyz| = 2$

For $\to 2 \to (2, 1, 1)\ (-2, 1, -1)\ (2, -1, -1)$

$\Rightarrow \frac{3!}{2!} + 3! + \frac{3!}{2!} = 12$

For $\to -2 \to (2, 1, -1), (-2, -1, -1), (-2, 1, 1) = 12$

$3 + 6 + 3 = 12$

Total solution = 12 + 12 = 24

Sol 7: (C) $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$

$R_1 \to R_1 - R_3,\ R_2 \to R_2 - R_3$

$$f(x) = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

$= 1 + 4\sin 2x + \cos^2 x - 1(-\sin^2 x)$

$= 1 + (\sin^2 x + \cos^2 x) + 4\sin 2x = 2 + 4\sin 2x$

For max value

$\sin 2x = 1$

$\Rightarrow 2 + 4\sin 2x = 2 + 4 = 6$

Sol 8: (C) $\begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & 2-x & x-3 \\ x-3 & x+4 & 3x \end{vmatrix} = px^4 + qx^3 + rx^2 + 5x + t$

At $x = 0$

$$\begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -3 \\ -3 & 4 & 0 \end{vmatrix} = t$$

$t = 1\,[-9] + 3\,[4 + 6] = 30 - 9 = 21$

Sol 9: (A) $D = \begin{vmatrix} a^2+1 & ab & ac \\ ba & 1+b^2 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} a^3+a & a^2b & a^2c \\ b^2a & b+b^3 & b^2c \\ c^2a & c^2b & c+c^3 \end{vmatrix}$$

$R_1 \to aR_1,\ R_2 \to bR_2,\ R_3 \to cR_3$

$$D = \frac{abc}{abc}\begin{vmatrix} 1+a^2 & a^2 & a^2 \\ b^2 & 1+b^2 & b^2 \\ c^2 & c^2 & 1+c^2 \end{vmatrix}$$

$C_1 \to C_1 - C_3,\ C_2 \to C_2 - C_3$

$$\begin{vmatrix} 1 & 0 & a^2 \\ 0 & 1 & b^2 \\ -1 & -1 & 1+c^2 \end{vmatrix} = 1[1 + c^2 + b^2] + a^2\,[+1]$$

$= 1 + a^2 + b^2 + c^2$

Sol 10: (A) $\alpha + \beta + \gamma = \pi$

$$\begin{vmatrix} \sin(\alpha+\beta+\gamma) & \sin\beta & \cos\gamma \\ \sin\beta & 0 & \tan\alpha \\ \cos(\alpha+\beta) & \tan\alpha & 0 \end{vmatrix}$$

$\sin \pi = 0$

$\alpha + \beta = \pi - \gamma,\ \cos(\pi - \gamma) = -\cos\gamma$

$$\begin{vmatrix} 0 & \sin\beta & \cos\gamma \\ -\sin\beta & 0 & \tan\alpha \\ -\cos\alpha & -\tan\alpha & 0 \end{vmatrix}$$

$$= \sin\beta\left[-\left(\frac{\sin\alpha}{\cos\alpha}\right)\cos\gamma\right] + \cos\gamma\left[\sin\beta\left(\frac{\sin\alpha}{\cos\alpha}\right)\right]$$

$$= \frac{\sin\beta\sin\alpha\cos\gamma}{\cos\gamma}$$

Sol 11: (C) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $a_{ij} \in \{0, 1\}$

$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \Rightarrow 1[-1] + 1[-1] = -2$

$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1[-1] + 1[1] = 2$

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0, \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= 1[-1] - 1[-1] + 1[-1]$

Cannot be $\rightarrow 3$

$D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{21}a_{12}a_{33} < 3$

As, for it to be 3, atleast one terms must be 0 but there sum would not be 3

Sol 12: (D) Order 3×3

First column consists of sum of 2 terms

2^{nd} "3"

3^{rd}"4"

Total no. of determinats = 2.3.4 = 24

Sol 13: (D) $x + 2y + 3z = 4$

$x + py + 2z = 3$

$\mu x + 4y + z = 3$

$D = \begin{vmatrix} 1 & 2 & 3 \\ 1 & p & 2 \\ \mu & 4 & 1 \end{vmatrix}, C = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$

$D = p - 8 + 2[2\mu 4 - 1] + 3[4 - p\mu)$

$D = p - 8 + 4\mu - 2 + 12 - 3p\mu = p + 4\mu - 3p\mu + 2$

For infinite solution D= 0, $D_x = D_y = D_z = 0$

$P + 4\mu - 3p\mu + 2 = 0$

$D_x = \begin{vmatrix} 4 & 3 & 3 \\ 3 & p & 2 \\ 3 & 4 & 1 \end{vmatrix} = 4[p-8] + 2[6-3] + 3[12 - 3p]$

$= 4p - 32 + 6 + 36 - 9p = 0$

$\rightarrow p = 10/5 = 2$

$D_y = \begin{vmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ \mu & 3 & 1 \end{vmatrix} = 1[3-6] + 4[2\mu-1] + 3[3-3\mu] = 0$

$-3 + 8\mu - 4 + 9 - 9\mu = 0$

$2 = \mu$

For equation (i) $p + 4(2) - 3p(2) + 2 = 0$

$P + 8 - 6p + 2 = 0$

$\rightarrow p = 10/5 = 2$

$D_z = \begin{vmatrix} 1 & 2 & 4 \\ 1 & p & 3 \\ \mu & 4 & 3 \end{vmatrix}$

$= 3p - 12 + 2(3\mu - 3) + 4(4 - p\mu)$

$= 3p - 12 + 6\mu - 6 + 16 - 4p\mu$

$= 3p + 6\mu - 4p\mu - 2$

At $p = 2, \mu = 2$

$\Rightarrow 3(2) + 6(2) - 4(2)(2) - 2$

$\Rightarrow 6 + 12 - 16 - 2 = 0$

At $p = 2, \mu = 2$, system has infinite solutions.

Sol 14: (B) $ax - by = 2a - b$

$(c + 1)x + cy = 10 - a + 3b$

For infinitely many solution

$D = \begin{vmatrix} a & -b \\ c+1 & c \end{vmatrix} = 0$ $ac + b(c + 1) = 0$

$ac + bc + b = 0$ ...(i)

$D_{x=0} \begin{vmatrix} 2a-b & -b \\ 10-a+3b & c \end{vmatrix}$

$= c(2a - b) + b(10 - a + 3b) = 0$

$2ac - bc + 10b - ba + 3b^2 = 0$ ... (ii)

$D_{y=0} = \begin{vmatrix} a & 2a-b \\ c+1 & 10-a+3b \end{vmatrix}$

$\Rightarrow a(10 - a + 3b) + (2a - b)(-1 - c) = 0$

$\Rightarrow 10a - a^2 + 3ba - 2a - 2ac + b + bc = 0$

At x = 1, y = 3

$a - 3b = 2a - b$

$o = a + 2b \Rightarrow a = -2b$

$c + 1 + 3c = 10 - a + 3b (\because -a = 2b)$

$4c = 9 + 2b + 3b = 9 + 5b$

$4c = 9 + 5b$

In equation (i) $ac + bc + b = 0$

$(-2b)\frac{(9+5b)}{4} + b\frac{(9+5b)}{4} + b = 0$

$-18b - 10b^2 + 9b + 5b^2 + 4b = 0$

$-5b^2 - 5b = 0$

$b^2 + b = 0$

$b = -1$ or o

$a = 2$ or 0

$c = 1$ or 9/4

$(a, b, c) \to$ exactly $\Rightarrow (-1, 2, 1)$ or $(0, 0, 9/4)$

Sol 15: (C) $ax + y + z = 0$

$x + by + z = 0$

$x + y + cz = 0$ a, b, c $\neq 1$

$$D = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}, C = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

So, $D_x = D_y = D_z = 0$

But system has nontrival solution

So, D = 0 and a, b, c $\neq 1$

$$D = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$$

$C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$

$$D = \begin{vmatrix} a-1 & 0 & 1 \\ 0 & b-1 & 1 \\ 1-c & 1-c & c \end{vmatrix}$$

$$= (1-a)(1-b)(1-c)\begin{vmatrix} -1 & 0 & \frac{1}{1-a} \\ 0 & -1 & \frac{1}{1-b} \\ 1 & 1 & \frac{c}{1-c} \end{vmatrix} = 0$$

And a, b, c $\neq 1$, So

$$\begin{vmatrix} -1 & 0 & \frac{1}{1-a} \\ 0 & -1 & \frac{1}{1-b} \\ 1 & 1 & \frac{c}{1-c} \end{vmatrix} = 0$$

$$\Rightarrow -1\left[\frac{-c}{1-c} - \frac{1}{1-b}\right] + \frac{1}{1-a}[1] = 0$$

$$\Rightarrow \frac{1}{1-a} + \frac{c+1-1}{1-c} + \frac{1}{1-b} = 0$$

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} - \frac{(1-c)}{(1-c)} = 0$$

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

Sol 16: (B) $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos^2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$

$\Rightarrow \cos(\theta + \phi)[\cos\theta\cos\phi - \sin\theta\sin\phi] +$

$\sin(\theta + \phi)[\sin\theta - \cos\phi] + \sin\phi\cos\theta]$

$+ \cos 2\phi(\sin^2\theta + \cos^2\theta)$

$\Rightarrow \cos^2(\theta + \theta) + \sin^2(\theta + \phi) + \cos 2\phi$

$= 1 + \cos 2\phi$

So determinant is only dependent of ϕ

Sol 17: (D) $x\sin\theta - y\cos\theta + (\lambda + 1)z = 0$

$x\cos\theta + y\sin\theta - \lambda . z = 0$

$\lambda x + (\lambda + 1)y + z\cos\theta = 0$

$$D = \begin{vmatrix} \sin\theta & -\cos\theta & \lambda+1 \\ \cos\theta & \sin\theta & -\lambda \\ \lambda & \lambda+1 & \cos\theta \end{vmatrix}$$

$D = \sin\theta[\sin\theta\cos\theta + \lambda^2 + \lambda]$

$+ \cos(\cos^2\theta + \lambda^2] + (\lambda + 1)(\lambda\cos\theta + \cos\theta - \lambda\sin\theta)$

$D = (\sin^2\theta + \cos^2\theta)\cos\theta + \sin\theta(\lambda^2 + \lambda$

$-\lambda^2 - \lambda) + \cos\theta(\lambda^2 + \lambda^2 + \lambda + \lambda + 1)$

$D = \cos\theta(2\lambda^2 + 2\lambda + 2) = \cos\theta[\lambda^2 + 1 + (\lambda + 1)^2]$

So for D = 0 ($\because$ System has infinite solution)

$\cos\theta = 0, \theta \in (2n + 1)\pi/2, \lambda \in R, n \in I$

Sol 18: (A) $a^2x - ay = 1 - a$

$bx + (3 - 2b)y = 3 + a$ $C = \begin{bmatrix} 1-a \\ 3+a \end{bmatrix}$

Unique solution x = 1, y = 1

$D = \begin{vmatrix} a^2 & -a \\ b & 3-2b \end{vmatrix}$, at (x, y) ⇒ (1, 1)

$a^2 - a = 1 - a$

$a^2 = 1 \Rightarrow a = \pm 1$

And b + 3 – 2b = 3 + a

3 – b = 3 + a

a = – b

So (a, b) ⇒ (1, – 1) or (–1, 1)

At (–1, 1) ⇒ x + y = 1 – (–1) = 2

x + y = 2

And x + (3 – 2)y = 3 – 1 = 2

x + y = 2

Both equations are same so, D =0 at (–1, 1)

So it is not unique solution

(a, b) ≠ (–1, 1) ∵ (a, b) = (1, – 1)

Sol 19: (A) $D = \begin{vmatrix} ^{n+2}C_n & ^{n+3}C_{n+1} & ^{n+4}C_{n+2} \\ ^{n+3}C_{n+1} & ^{n+4}C_{n+2} & ^{n+5}C_{n+3} \\ ^{(n+4)}C_{n+2} & ^{n+5}C_{n+3} & ^{n+6}C_{n+6} \end{vmatrix}$

$$D=\begin{vmatrix} \frac{(1+n)(n+2)}{2} & \frac{(n+3)(n+2)}{2} & \frac{(n+4)(n+3)}{2} \\ \frac{(n+3)(n+2)}{2} & \frac{(n+4)(n+3)}{2} & \frac{(n+5)(n+4)}{3} \\ \frac{(n+4)(n+3)}{2} & \frac{(n+5)(n+4)}{2} & \frac{(n+6)(n+5)}{2} \end{vmatrix}$$

At n = 1

$$D = \begin{vmatrix} 3 & 6 & 10 \\ 6 & 10 & 15 \\ 10 & 15 & 21 \end{vmatrix}$$

= 3 [210 – 225) + 6 (150 – 126) +10(90–100)

= – 45 + 144 – 100 = – 1

There is only one option (A)

Which satisfied the ans.

Using, $C_1 \to C_1 - C_2$, $C_3 \to C_3 - C_2$,

$R_3 \to R_3 - R_2$, we get

$$\begin{vmatrix} -(n+2) & \frac{(n+3)(n+2)}{2} & n+3 \\ -1 & n+3 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$$

Sol 20: (A) $\lambda x - y + \cos\theta\, z = 0$

3x + y + 2z = 0

cos x + y + 2z = 0

$0 \le \theta \le 2\pi$

$C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So $D_x = D_y = D_z = 0$

$$D = \begin{vmatrix} \lambda & -1 & \cos\theta \\ 3 & 1 & 2 \\ \cos\theta & 1 & 2 \end{vmatrix}$$

For non-trivial solution

$D = 0 \therefore D_x = D_y = D_z = 0$

$\lambda[2-2]+1[6 - 2\cos\theta]+\cos\theta\,[3 - \cos\theta] = 0$

$\Rightarrow 6 - 2\cos\theta + 3\cos\theta - \cos^2\theta = 0$

$\Rightarrow \cos^2\theta - \cos^2\theta - 6 = 0$

$\Rightarrow \cos^2\theta - 3\cos\theta + 2\cos\theta - 6 = 0$

$\Rightarrow \cos\theta(\cos\theta - 3) + 2(\cos\theta - 3) = 0$

$\Rightarrow (\cos\theta - 3)(\cos\theta + 2) = 0$

$\Rightarrow \cos\theta = 3$ or $\cos\theta = -2$

But $-1 \le \cos\theta \le 1$

So $\cos\theta \ne 3, -2$

There is no solution for non-trivial solution

Multiple Correct Choice Type

Sol 21: (A, D) $\begin{vmatrix} \cos(x-y) & \cos(y-z) & \cos(z-x) \\ \cos(x+y) & \cos(y+z) & \cos(z+x) \\ \sin(x+y) & \sin(y+z) & \sin(z+x) \end{vmatrix}$

cos (A – B) = cos A cos B + sin A sin B

sin (A + B) = cos B sin A + sin B cos A

cos (A + B) = cos A cos B – sin A sin B

= cos (x – y) [cos (y + z) sin (z + x) – cos (z + x)

sin (y + z)] + cos (y – z) [cos (z + x) sin (x + y) – sin (x + z) cos (x + y)] + cos (z – x) [cos (x + y) sin (y + z) – cos (y + z) cos (x + y)]

= cos (x – y) [sin (z + x – y – z)]+ cos (y – z) [sin (x + y – z – x)]+ cos (z – x) [sin (y + z – x – y)]

= cos (x – y) sin (x – y) + cos (y – z) sin (y – z) + cos (z – x) sin (z – x)

$= \frac{1}{2}$[sin 2 (x –y) + sin (2(y – z) + sin 2(z – x)]

$= \frac{1}{2}$[2 cos (x + z – zy) sin (x + z) – 2 sin (x – z) cos (x – z)]

$= \frac{1}{2}$sin (x – z) [cos (x + z – 2y) – cos (x – z)]

$= \sin(x-z)\left[2\sin\frac{(x-y)}{2}\sin\frac{(y-z)}{2}\right]$

= 2 sin (x – y) sin (y – z) sin (z – x)

Sol 22: (A, B, C, D) $\frac{-\pi}{4} < \theta < \frac{\pi}{2}, 0 \le A \le \pi/2$

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0$$

$R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2 A & \cos^2 A & 1+\sin^2\theta \end{vmatrix} = 0$$

$\Rightarrow 1 + 2\sin 4\theta + \cos^2 A - 1\,[-\sin^2 A]$

$\Rightarrow 1 + 2\sin 4\theta + \sin^2\theta + \cos^2\theta$

$\Rightarrow 2 + 2\sin 4\theta = 0$ (only depend on θ)

$\Rightarrow \sin 4\theta = -1 \Rightarrow 4\theta \in -\pi/2 + 2n\pi,\ n \in I$

$\theta \in -\pi/8 + n\pi/2,\ n \in I$

(A) $\theta \to -\pi/8$

(B) $\theta \to 3\pi/8$ $\sin 4\theta = -1$

(C) $\theta = -\pi/8$ $\sin 4\theta = -1$

(D) $\theta = 3\pi/8$

Sol 23: (A, D) $\begin{vmatrix} 1 & a & a^2 \\ 1 & x & x^2 \\ b^2 & ab & a^2 \end{vmatrix} = 0$

$\Rightarrow xa^2 - x^2ab + a\,[x^2b^2 - a^2] + a^2\,[ab - xb^2] = 0$

$\Rightarrow x\,[a^2 - a^2b^2] + x^2\,[ab^2 - ab] - a^3 + a^3b = 0$

$\Rightarrow x^2ab\,(b-1) + a^2x\,(1-b^2) + a^3\,(b-1) = 0$

$\Rightarrow x^2ab - a^2x\,(1+b) + a^3\,(+1) = 0$

$\Rightarrow x^2ab - x\,(a^2 + a^2b) + a^3 = 0$

$\Rightarrow x^2ab - a^2x - a^2\,(bx - a) = 0$

$\Rightarrow (bx - a)\,(ax - a^2) = 0$

$\Rightarrow bx - a = 0$ or $ax - a^2 = 0$

$\Rightarrow x = \frac{a}{b}$ or $x = \frac{a^2}{a} = a$

Sol 24: (B, D) $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$

$R_3 \to R_3 - \alpha R_1 - R_2$

$$\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & -(a\alpha^2+b\alpha+b\alpha+c) \end{vmatrix}$$

$= (a\alpha^2 + 2b\alpha + c)\,(b^2 - ac) = 0$

So $(b^2 - ac) = 0$

$b^2 = ac \to$ b is GM of a, c $\to$ ab, c are in GP

or $(a\alpha^2 + 2b\,\alpha + c) = 0$

$\Rightarrow x = \alpha \to (x - \alpha)$

$\Rightarrow ax^2 + 2bx + c$, $(x - \alpha)$ is a factor of this)

Sol 25: (B, D) $x - y + 3z = 2$

$2x - y + z = 4$

$x - 2y + \alpha z = 3$

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -1 & 1 \\ 1 & -2 & \alpha \end{vmatrix}$$

$= 1\,[-\alpha + 2] - 1\,[1 - 2\alpha] + 3\,[-4 + 1]$

$= -\alpha + 2 - 1 + 2\alpha - 9 = \alpha - 8$

$D \ne 0 \to \alpha \ne 8$

If $\alpha = 8$, $D = 0$

$$D_x = \begin{vmatrix} 2 & -1 & 3 \\ 4 & -1 & 1 \\ 3 & -2 & \alpha \end{vmatrix}$$

$= 2\,[-\alpha + 2] - 1\,[3 - 4\alpha] + 3\,[-8 + 3]$

$= -2\alpha + 4 - 3 + 4\alpha - 15 = 2\alpha - 14 = 2(\alpha - 7)$

= At $\alpha = 8$ (0 = 0), $D_x \ne 0$

So, at $\alpha = 8$, system has no solution.

Sol 26: (A,B,C,D) $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$

$= \frac{1}{abc}\begin{vmatrix} a & abc & abc(b+c) \\ b & abc & abc(c+a) \\ c & abc & abc(a+b) \end{vmatrix} = \frac{(abc)^2}{abc}\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix}$

$C_3 \to C_3 + C_1$

$= (abc)\begin{vmatrix} a & 1 & a+b+c \\ b & 1 & a+b+c \\ c & 1 & a+b+c \end{vmatrix} = (abc)\ (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$

$C_2 \to c_2 - c_3$

$= (a\ b\ c)\ (a+b+c)\begin{vmatrix} a & 0 & 1 \\ b & 0 & 1 \\ c & 0 & 1 \end{vmatrix} = 0$

(B) $\begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} c & abc & c\left(\frac{1}{a}+\frac{1}{b}\right) \\ a & abc & a\left(\frac{1}{b}+\frac{1}{c}\right) \\ b & abc & b\left(\frac{1}{c}+\frac{1}{a}\right) \end{vmatrix}$

$\frac{abc}{abc}\begin{vmatrix} c & \frac{c}{c} & \frac{c}{a}+\frac{c}{b} \\ a & \frac{a}{a} & \frac{a}{b}+\frac{a}{c} \\ b & \frac{b}{b} & \frac{b}{a}+\frac{b}{c} \end{vmatrix}$ $C_3 \to c_3 + c_2$

$\begin{vmatrix} c & 1 & c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \\ a & 1 & a\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \\ b & 1 & b\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \end{vmatrix} = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} c & 1 & C \\ a & 1 & a \\ b & 1 & b \end{vmatrix}$

$C_1 \to c_1 - c_3$

$= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 0 & 1 & c \\ 0 & 1 & a \\ 0 & 1 & b \end{vmatrix} = 0$

(C) $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix} = \begin{vmatrix} 0 & (a-b) & (a-c) \\ -(a-b) & 0 & (b-c) \\ -(a-c) & -(b-c) & 0 \end{vmatrix}$

This is stew symmetric matrix so value of determinate is zero.

(D) $\begin{vmatrix} \log_x xyz & \log_x y & \log_x z \\ \log_y xyz & 1 & \log_x z \\ \log_x xyz & \log_z y & 1 \end{vmatrix}$

$\begin{vmatrix} \frac{\log xyz}{\log x} & \frac{\log y}{\log x} & \frac{\log x}{\log x} \\ \frac{\log xyz}{\log y} & \frac{\log y}{\log y} & \frac{\log z}{\log y} \\ \frac{\log xyz}{\log z} & \frac{\log y}{\log z} & \frac{\log z}{\log z} \end{vmatrix}$

$= \frac{(\log xyz)(\log y)(\log z)}{(\log x)(\log y)(\log z)}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

$(\because\ C_1 = c_2 = c_3)$

Sol 27: (A, B, C, D) $a^2x - by = a^2 - b^2$posses an infinite no. of solution

$bx - b^2y = 2 + 4b$

So $D = 0 \Rightarrow \begin{vmatrix} a^2 & -b \\ b & -b^2 \end{vmatrix} = -b^2a^2 + b^2 = 0$ Þ

$\Rightarrow b^2(1 - a^2) = 0$

$\to b = 0$ or $a = \pm 1$ ... (i)

$D_x = 0 \Rightarrow \begin{vmatrix} a^2-b & -b \\ 2+4b & -b^2 \end{vmatrix}$

$= -b^2(a^2 - b) + b(2 + 4b) = 0$

$-a^2b^2 + b^3 + 2b + 4b^2 = 0$

$b(b^2 + 4b + 2 - a^2b) = 0$

$\Rightarrow b = 0$ or $b^2 + 4b + 2 = a^2b$ ... (ii)

$D_x = 0 \Rightarrow \begin{vmatrix} a^2 & a^2-b \\ b & 2+4b \end{vmatrix} = a^2(2 + 4b) + b(b - a^2) = 0$

$2a^2 + 4a^2b + b^2 - ba^2 = 0$ ...(iii)

All option are satisfied equation (i, ii, iii)

Sol 28: (A, C) p, q, r, s are in AP

P = p q = p + d r = p + 2d, s = p + 3d

D is common difference of the A.P.

$f(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$

$C_2 \to c_2 - c_1,\ C_3 \to c_3 - c_1$

$$\begin{vmatrix} 1 & x+y+z & -(x+y) \\ -1 & 0 & x \\ y(x-z) & x[(y-z)(x+y+z)-2yz] & -xy(x+y) \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$D = \frac{(x+y+z)}{x^4z^4}$$

$$\begin{vmatrix} 0 & x+y+z & -y \\ -1 & 0 & x \\ y(x-z) & x[(y-z)(x+y+z)(-xyz) & -xy(x+y) \end{vmatrix}$$

$$\frac{x+y+2}{x^4+4}$$

$$\left[(x+y+z)(xy+y)-xy(x-z)+yx(-xyz)(x+y+z)(y-z)\right]$$

Previous Years Questions

Sol 1: (B) Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 \to C_1 + C_3$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ \cos(p-d)x+\cos(p+d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x+\sin(p+d)x & \sin px & \sin(p+d)x \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ 2\cos px\cos dx & \cos px & \cos(p+d)x \\ 2\sin px\cos dx & \sin px & \sin(p+d)x \end{vmatrix}$$

Applying $C_1 \to C_1 - 2\cos dx\, C_2$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$$

$\Rightarrow \Delta = (1+a^2-2a\cos dx)$

$[\sin(p+d)x\cos px - \sin px\cos(p+d)x]$

$\Rightarrow \Delta = (1+a^2-2a\cos dx)\sin dx$

Which is independent of p.

Sol 2: Since, $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is linear equation in three variables and that could have only unique, no solution or infinitely many solution.

$\therefore$ It is not possible to have two solutions.

Hence, number of matrices A is zero.

Sol 3: Given $\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} a^2x-aby-ac & bx+ay & cx+a \\ abx+a^2y & -ax+by-c & cy+b \\ acx+a^2 & cy+b & -ax-by+c \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & ay+bx & cx+a \\ (a^2+b^2+c^2)y & by-c-ax & b+cy \\ a^2+b^2+c^2 & b+cy & c-ax-by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} x & ay+bx & cx+a \\ y & by-c-ax & b+cy \\ 1 & b+cy & c-ax-by \end{vmatrix} = 0$$

$(\because\ a^2+b^2+c^2=1)$

Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{ax}\begin{vmatrix} x^2 & axy & ax \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + yR_2 + R_3$

$$\Rightarrow \frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{ax}[(x^2+y^2+1)\{(-c-ax)(-ax-by)-b(cy)\}] = 0$$

$$\Rightarrow \frac{1}{ax}[(x^2+y^2+1)(acx+bcy+a^2x^2+abxy-bcy)] = 0$$

$$\Rightarrow \frac{1}{ax}[(x^2+y^2+1)(acx+a^2x^2+abxy)] = 0$$

$$\Rightarrow \frac{1}{ax}[ax(x^2+y^2+1)(c+ax+by)] = 0$$

$\Rightarrow (x^2+y^2+1)(ax+by+c)=0 \quad \Rightarrow \quad ax+by+c=0$

Which represents a straight line.

Sol 4: Since, the given system of equations posses non-trivial solution, if

$$\begin{vmatrix} 0 & 1 & -2 \\ 0 & -3 & 1 \\ k & -5 & 4 \end{vmatrix} = 0 \quad k = 0$$

On solving the equations $x = y = z = \lambda$ (say)

$\therefore$ For k = 0, the system has infinite solutions for $\lambda \in R$.

Sol 5: Given system of equations are

3x + my = m and 2x – 5y = 20

Here, $\Delta = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = -15-2m$

and $\Delta_x = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix} = -25m$; $\quad \Delta_y = \begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix} = 60-2m$

If $\Delta = 0$, then system inconsistent i.e. it has no solution.

If $\Delta \neq 0$ i.e. $m \neq \frac{15}{2}$, then system has a unique solution for any fixed value of m.

We have, $x = \frac{\Delta_x}{\Delta} = \frac{-25m}{-15-2m} = \frac{25m}{15+2m}$

and $y = \frac{\Delta_y}{\Delta} = \frac{60-2m}{-15-2m} = \frac{2m-60}{15+2m}$

For $x > 0$, $\frac{25m}{15+2m} > 0$

$\Rightarrow m > 0$ or $m < -\frac{15}{2}$ ... (i)

and $y > 0$, $\frac{2m-60}{2m+15} > 0$

$\Rightarrow m > 30$ or $m < -\frac{15}{2}$ ... (ii)

From equation (i) and (ii) we get

$m < -\frac{15}{2}$ or $m > 30$

Sol 6: Let

$$\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) + \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) + \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) + \sin\left(2\theta-\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

Now, $\sin\left(\theta+\frac{2\pi}{3}\right) + \sin\left(\theta-\frac{2\pi}{3}\right)$

$$= 2\sin\left(\frac{\theta+\frac{2\pi}{3}+\theta-\frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta+\frac{2\pi}{3}-\theta+\frac{2\pi}{3}}{2}\right)$$

$= 2\sin\theta\cos\frac{2\pi}{3} = 2\sin\theta\cos\left(\pi-\frac{\pi}{3}\right)$

$= -2\sin\theta\cos\frac{\pi}{3} = -\sin\theta$

and $\cos\left(\theta+\frac{2\pi}{3}\right) + \cos\left(\theta-\frac{2\pi}{3}\right)$

$$= 2\cos\left(\frac{\theta+\frac{2\pi}{3}+\theta-\frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta+\frac{2\pi}{3}-\theta+\frac{2\pi}{3}}{2}\right)$$

$= 2\cos\theta\cos\left(\frac{2\pi}{3}\right) = 2\cos\theta\left(-\frac{1}{2}\right) = -\cos\theta$

and $\sin\left(2\theta+\frac{4\pi}{3}\right) + \sin\left(2\theta-\frac{4\pi}{3}\right)$

$$= 2\sin\left(\frac{2\theta+\frac{4\pi}{3}+2\theta-\frac{4\pi}{3}}{2}\right)\cos\left(\frac{2\theta+\frac{4\pi}{3}-2\theta+\frac{4\pi}{3}}{2}\right)$$

$= 2\sin 2\theta\cos\frac{4\pi}{3} = 2\sin 2\theta\cos\left(\pi+\frac{\pi}{3}\right)$

$= -2\sin 2\theta\cos\frac{\pi}{3} = -\sin 2\theta$

$$\therefore \Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

= 0 (Since, R_1 and R_2 are proportional).

Sol 7: (B) Method I

Total no. of ways $= 3^5 - {}^3C_1 3 - 1^5 + {}^3C_2 3 - 1^5$

$= 243 - 3\times 32 + 3 = 246 - 96 = 150$

Alternative Method

System I

Boxes	I	II	III	
Balls	I	2	2	

For this system no. of ways

$$= \left(\frac{5!}{2!2!1!}\times\frac{1}{2!}\right)\times\left(\frac{5\times4\times3\times2}{2\times2\times2}\right)\times 6 = 90$$

System II

Boxes	I	II	III	
Balls	I	3	1	

For this system no. of ways

$$= \left(\frac{5!}{3!1!1!}\times\frac{1}{2!}\right)\times 3! = 10\times 6 = 60$$

Total no. of ways $= 90 + 60 = 150$

Sol 8: (B, C)

$$\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2\\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2\end{vmatrix} = -648\alpha$$

$$\begin{vmatrix}1+\alpha^2+2\alpha & 1+4\alpha^2+4\alpha & 1+9\alpha^2+6\alpha\\ 4+\alpha^2+4\alpha & 4+4\alpha^2+8\alpha & 4+9\alpha^2+12\\ 9+\alpha^2+6\alpha & 9+4\alpha^2+12\alpha & 9+9\alpha^2+18\alpha\end{vmatrix} R_2 \to R_2 - R_1$$

$R_3 \to R_3 - R_2 = -648\alpha$

$$\begin{vmatrix}1+\alpha^2+2\alpha & 1+4\alpha^2+4\alpha & 1+9\alpha^2+6\alpha\\ 3+2\alpha & 3+4\alpha & 3+6\alpha\\ 5+2\alpha & 5+4\alpha & 5+6\alpha\end{vmatrix} C_2 \to C_2 - C_3$$

$C_2 \to C_2 - C_2 = -648\alpha$

$$\begin{vmatrix}\alpha^2-4 & 4\alpha^2-4 & 9\alpha^2-4\\ -2 & -2 & -2\\ 5+2\alpha & 5+4\alpha & 5+6\alpha\end{vmatrix} = -648\alpha$$

$$-2\begin{vmatrix}\alpha^2-4 & 4\alpha^2-4 & 9\alpha^2-4\\ 1 & 1 & 1\\ 5+2\alpha & 5+4\alpha & 5+6\alpha\end{vmatrix} = 648\alpha$$

$$-2\begin{vmatrix}\alpha^2-4 & 3\alpha^2 & 8\alpha^2\\ 1 & 0 & 0\\ 5+2\alpha & 2\alpha & 4\alpha\end{vmatrix} = -648\alpha$$

$-2\left(12\alpha^3 - 16\alpha^3\right) = -648\alpha$

$\Rightarrow 2\left(-4\alpha^3\right) = -648\alpha$

$\Rightarrow \alpha\left(\alpha^2 - 81\right) = 0$

$\Rightarrow \alpha = 0, 9, -9$

Sol 9: (2) $\begin{vmatrix}1 & 1 & 1+x^3\\ 2 & 4 & 1+8x^3\\ 3 & 9 & 1+27x^3\end{vmatrix} = 10$

$$x^3\begin{vmatrix}1 & 1 & 1\\ 2 & 4 & 1\\ 3 & 9 & 1\end{vmatrix} + x^6\begin{vmatrix}1 & 1 & 1\\ 2 & 4 & 8\\ 3 & 9 & 27\end{vmatrix} = 10$$

$$x^3\begin{vmatrix}1 & 0 & 0\\ 2 & 2 & -1\\ 3 & 6 & -2\end{vmatrix} + x^6\begin{vmatrix}1 & 0 & 0\\ 2 & 2 & 6\\ 3 & 6 & 24\end{vmatrix} = 10$$

$6x^3 + x^3 - 5 = 0 \Rightarrow 6x^6 + 6x^3 - 5x^3 - 5 = 0$

$\left(6x^3 - 5\right)\left(x^3 + 1\right) = 0$

$x^3 = \frac{5}{6}$ or $x^3 = -1$ Two real distinct values of x.