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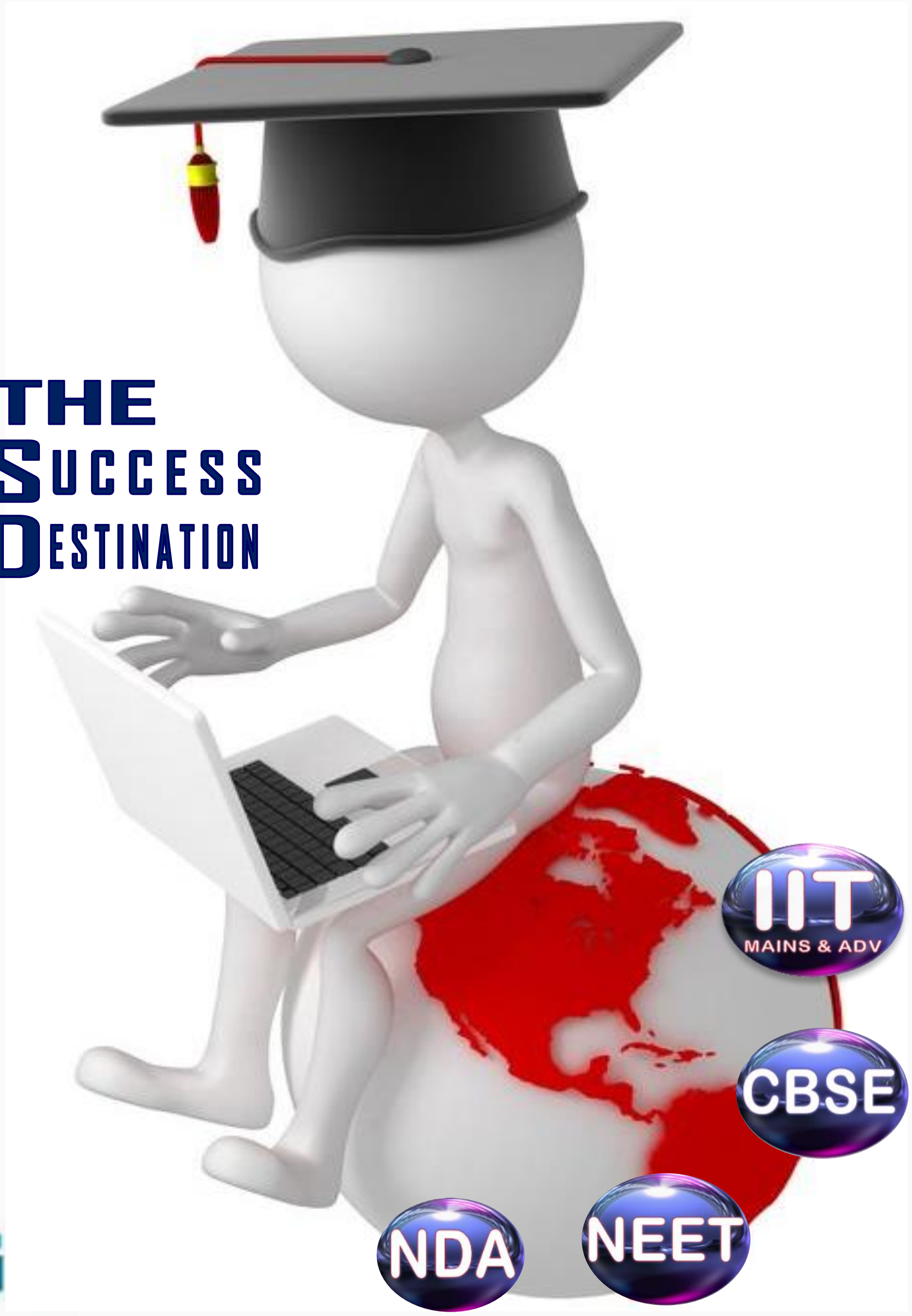

INVERSE T-FUNCTIONS

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2 INVERSE-TRIGONOMETRIC FUNCTIONS

Mathematics, in general, is fundamentally the science of self-evident things

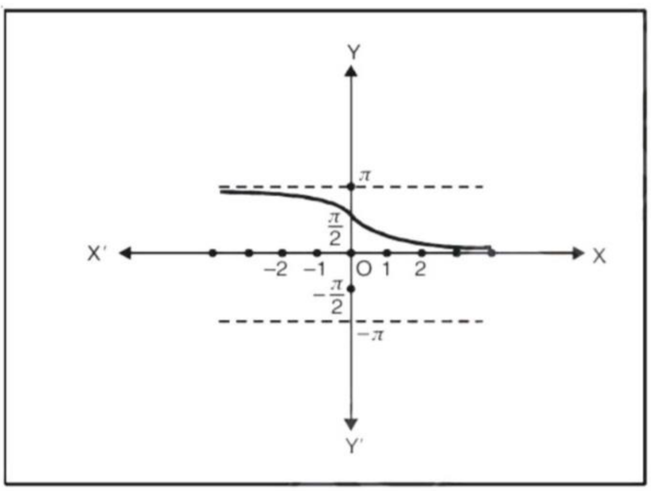
—Felix Klein

Objectives

After studying the material of this chapter you should be able to :

- Understand the definations of inverse-trigonometric functions.
- Understand the domains and ranges of inverse-trigonometric functions with the help of their graphs.
- Understand the principal values of inverse-trigonometric functions.
- Understand the properties of inverse trigonometric functions.
- Understand the application of inverse trigonometric functions.

2.0. INTRODUCTION



Inverse trigonometric functions play a vital role in Calculus and help us to define several integrals. This concept is widely used in Engineering, Geometry and Physics. The restrictions on domains of trigonometric functions ensure the existence of their inverses.

Inverse trigonometric functions are the inverse functions of the trigonometric functions. They are not the reciprocals of the trigonometric functions.

In this chapter we will study the following concepts :

- Inverse trigonometric functions and their graphical representations
- Domain and Range of Inverse trigonometric functions
- Properties of inverse trigonometric functions

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2.1. INVERSE-TRIGONOMETRIC FUNCTIONS

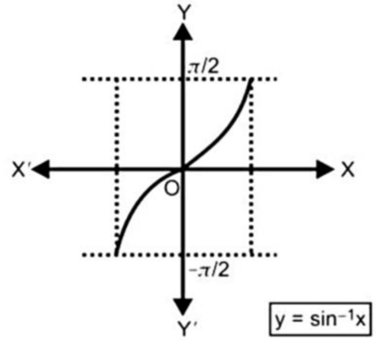
(i) Definition

The inverse-sine function is defined as :

$$y = f^{-1}(x) = \sin^{-1} x$$

iff $\quad x = \sin y \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$

The graph of $\sin^{-1} x$ is as shown.



KEY POINT

Domain of $\sin^{-1} x = [-1, 1]$ and Range of $\sin^{-1} x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

PREPARATION OF IMAGE

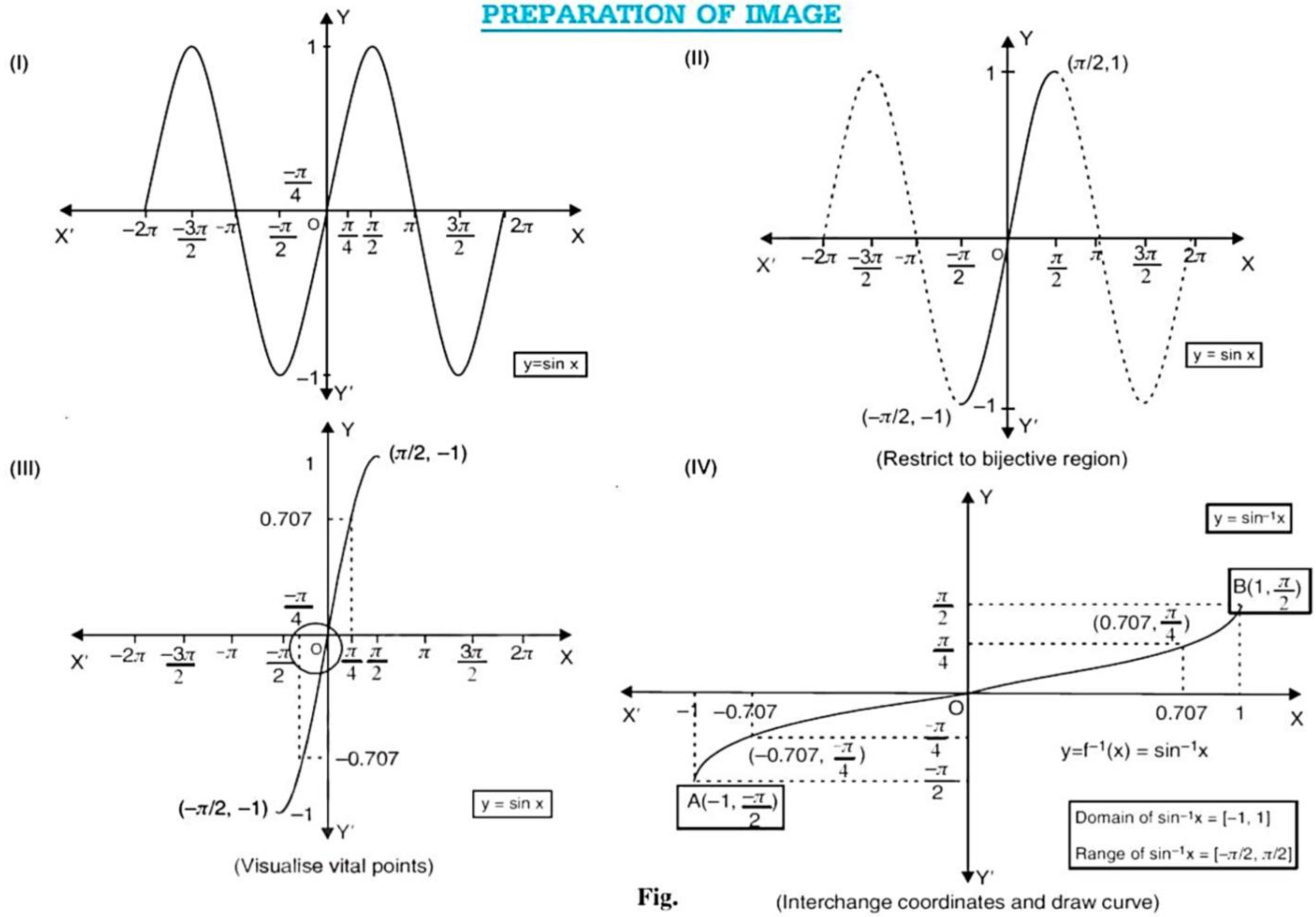


Fig.

(ii) Definition

The inverse-cosine function is defined as :

$$y = f^{-1}(x) = \cos^{-1} x$$

iff $\quad x = \cos y \text{ and } y \in [0, \pi].$

The graph of $\cos^{-1} x$ is as shown.

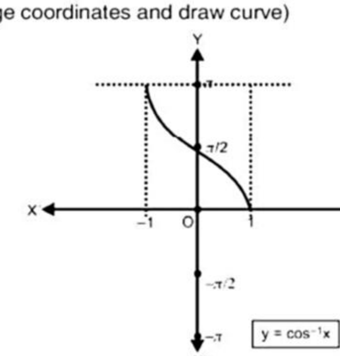
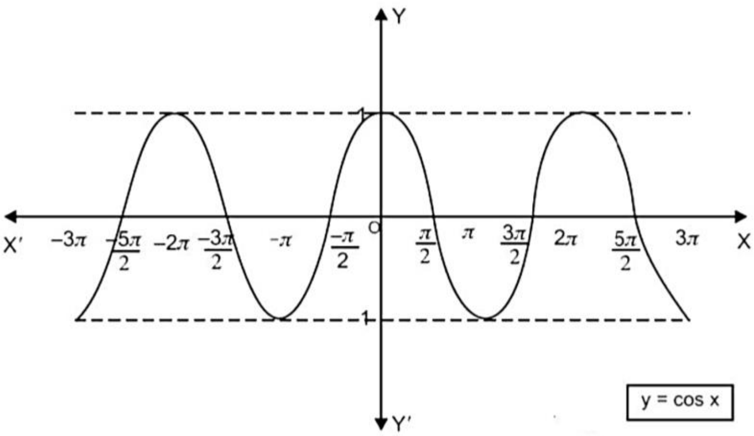


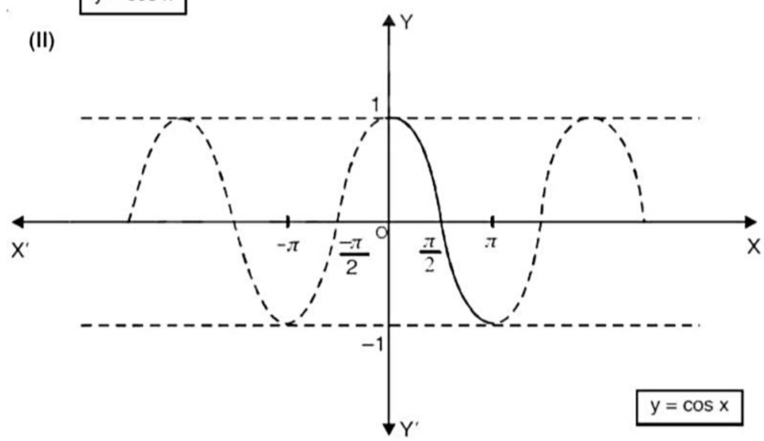
Fig.

PREPARATION OF IMAGE

(I)

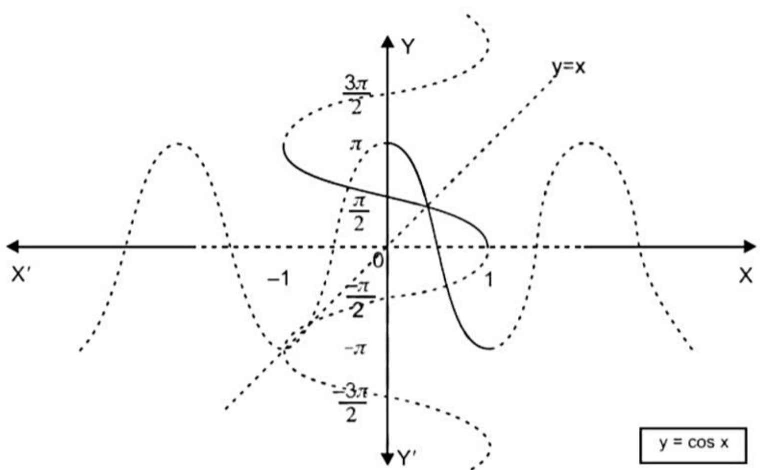


(II)



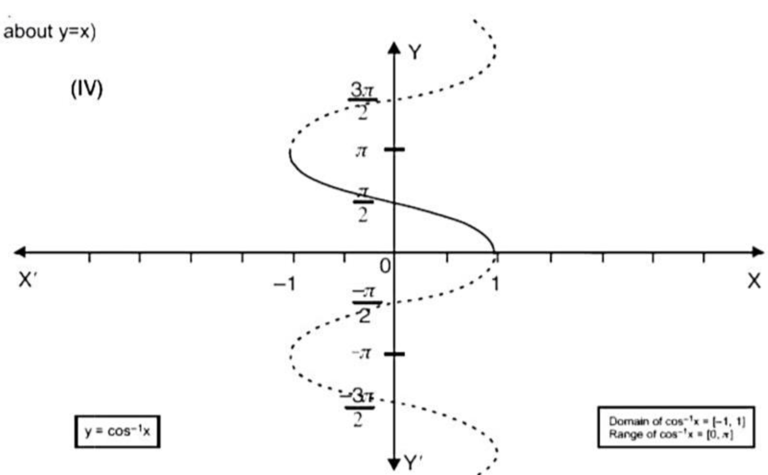
(Restrict to bijective region)

(III)



(Take mirror image of curve in 2nd step about y=x)

(IV)



KEY POINT

Domain of $\cos^{-1} x = [-1, 1]$ and Range of $\cos^{-1} x = [0, \pi]$.

(iii) **Definition**

The inverse-tangent function is defined as :

$$y = f^{-1}(x) = \tan^{-1} x$$

$$\text{iff} \quad x = \tan y \quad \text{and} \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

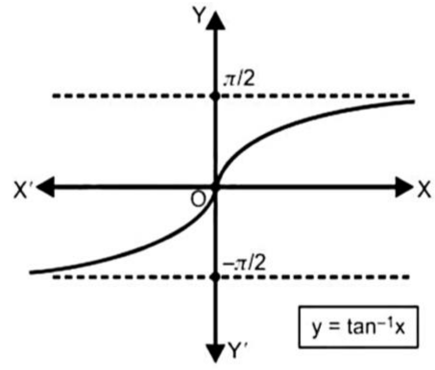


Fig.

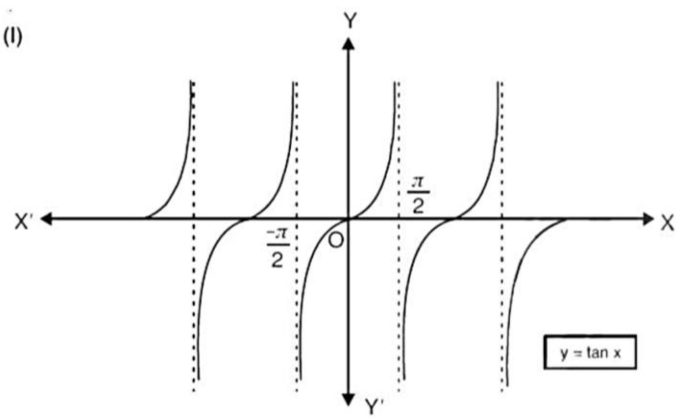
The graph of $\tan^{-1} x$ is as shown.

KEY POINT

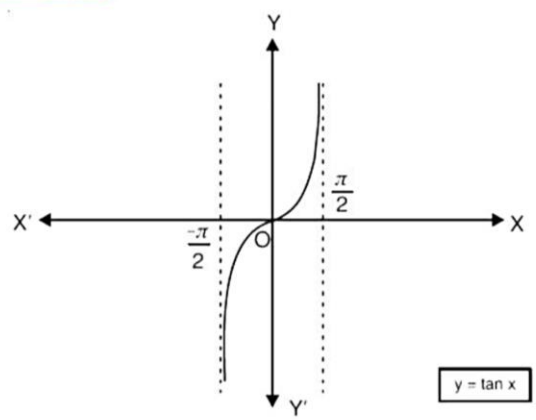
Domain of $\tan^{-1} x = \mathbf{R}$ and Range of $\tan^{-1} x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

PREPARATION OF IMAGE

(I)

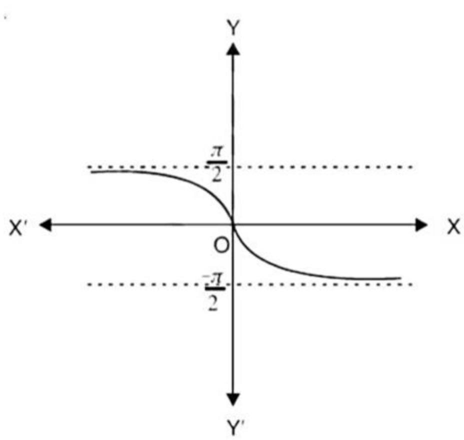


(II)



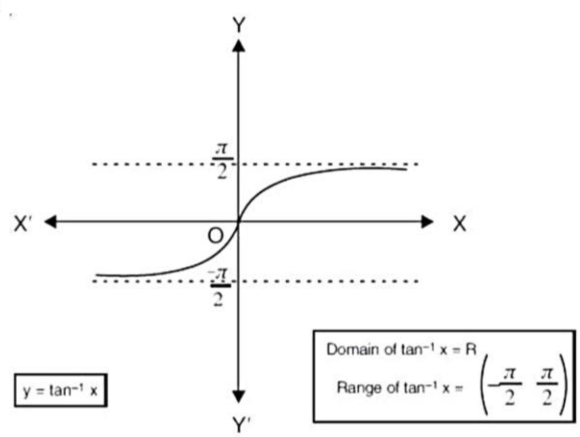
(Restrict to bijective region)

(III)



(Rotate 90° counter clockwise)

(IV)



(Mirror about Y-axis)

(iv) **Definition**

The inverse-cotangent function is defined as :

$$y = f^{-1}(x) = \cot^{-1} x$$

$$\text{iff } x = \cot y \text{ and } y \in (0, \pi).$$

The graph of $\cot^{-1} x$ is as shown.

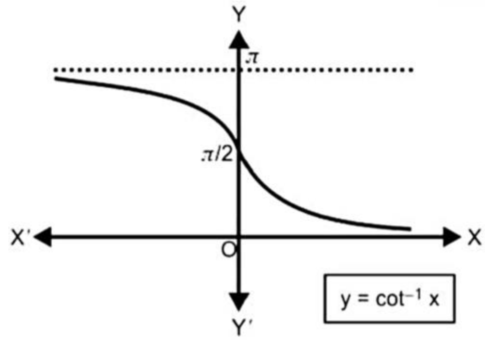


Fig.

KEY POINT

Domain of $\cot^{-1} x = \mathbf{R}$ and Range of $\cot^{-1} x = (0, \pi)$.

(v) **Definition**

The inverse-secant function is defined as :

$$y = f^{-1}(x) = \sec^{-1} x$$

$$\text{iff} \quad x = \sec y$$

$$\text{and} \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$$

The graph of $\sec^{-1} x$ is as shown.

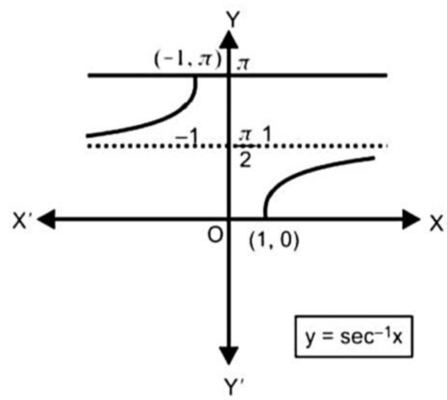


Fig.

KEY POINT

Domain of $\sec^{-1} x = (-\infty, -1] \cup [1, \infty)$.

Range of $\sec^{-1} x = \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. ***(Assam B. 2017)***

(vi) **Definition**

The inverse-cosecant function is defined as :

$$y = f^{-1}(x) = \operatorname{cosec}^{-1} x$$

$$\text{iff} \quad x = \operatorname{cosec} y$$

$$\text{and } y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right].$$

The graph of $\operatorname{cosec}^{-1} x$ is as shown.

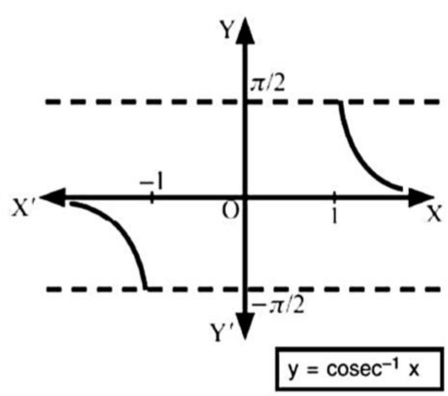


Fig.

KEY POINT

Domain of $\operatorname{cosec}^{-1} x = (-\infty, -1] \cup [1, \infty)$. ***(Assam B. 2018)***

Range of $\operatorname{cosec}^{-1} x = \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. ***(Assam B. 2017)***

Note : On similar lines, the readers and try the images for (*iv*) to (*vi*)

2.2. TABLE OF INVERSE-TRIGONOMETRIC FUNCTIONS

We give the table giving the inverse-trigonometric functions and their principal value branches :

TABLE

Function	*Principal Value Branch*	
	Domain	**Range**
(*i*) $y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(*ii*) $y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
(*iii*) $y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(*iv*) $y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
(*v*) $y = \sec^{-1} x$	$1 \le x < \infty$ $-\infty < x \le -1$	$0 \le y < \frac{\pi}{2}$ $\frac{\pi}{2} < y \le \pi$
(*vi*) $y = \text{cosec}^{-1} x$	$1 \le x < \infty$ $-\infty < x \le -1$	$0 < y \le \frac{\pi}{2}$ $-\frac{\pi}{2} \le y < 0$

Note : When y is + ve ($0 \le y \le 1$), there are two angles : one between 0 and $\pi/2$ and the other between $-\pi/2$ and 0 having their cosine equal to y ($\cos x$ is an even function of x).

Here we take smallest positive angle as the principal value of $\cos^{-1} y$.

For Example : $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$ and not $-30°$ even though $\cos(-30°) = \frac{\sqrt{3}}{2}$.

ILLUSTRATIVE EXAMPLES

Example 1. Find the value of

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}).$$ *(C.B.S.E. 2018)*

Solution. $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{6}\right)$$

$$= -\frac{\pi}{2}.$$

Example 2. Find the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$.

(Jammu B. 2017 ; Kerala B. 2013; H.B. 2012)

Solution. $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Hence, the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$.

Example 3. What is the principal value of :

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)?$$ *(A.I.C.B.S.E. 2011)*

Solution. π. $\left[\because \cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)\right.$

$$\left. = \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \frac{\pi}{3} = \pi.\right]$$

Example 4. Find the principal value of : $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$. *(A.I.C.B.S.E. 2012)*

Solution. $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\sec^{-1}(-2) = \frac{2\pi}{3} \in \left[0, \frac{\pi}{2}\right] - \left\{\frac{\pi}{2}\right\}.$$

$\therefore$ $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}.$$

Example 5. Evaluate : $\tan^{-1}\left(2\cos\left(2\sin^{-1}\left(\frac{1}{2}\right)\right)\right)$.

(N.C.E.R.T. ; H.P.B. 2013 ; Jammu B. 2012)

Solution. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

$= \tan^{-1}\left[2\cos\left(2.\frac{\pi}{6}\right)\right]$

$= \tan^{-1}\left[2\cos\frac{\pi}{3}\right] = \tan^{-1}\left(2.\frac{1}{2}\right)$

$= \tan^{-1}(1) = \frac{\pi}{4}$.

EXERCISE 2 (a)

Fast Track Answer Type Questions

FTATQ

1. (*a*) **Fill in the blanks :**

(*i*) Principal value of $\cot^{-1}(\sqrt{3})$ is ....

(Jammu B. 2018)

(*ii*) Principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is ....

(Jammu B. 2017)

(*iii*) Principal value of $\tan^{-1}(-1)$ is.................

(Jammu B. 2017)

(*iv*) Principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is.................

(Kashmir B. 2016)

(*v*) Principal value of $\tan^{-1}(-\sqrt{3})$ is.................

(Meghalaya B. 2018; Kashmir B. 2016)

(*vi*) Principal value of $\text{cosec}^{-1}(-\sqrt{2})$ is.................

(Kashmir B. 2016)

(*b*) Write the domain of $f(x) = \cos^{-1}(x)$.

(Karnataka B. 2014)

Find the principal values of the following (2–8) :

2. (*i*) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (Jammu B. 2012; C.B.S.E. 2010)

(*ii*) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$. (N.C.E.R.T. ; Uttarakhand B. 2013)

3. (*i*) $\cos^{-1}\left(-\frac{1}{2}\right)$ (Kerala B. 2014; H.P.B. 2010 S)

(*ii*) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (C.B.S.E. 2010)

(*iii*) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. (N.C.E.R.T.; Kashmir B. 2013)

4. (*i*) $\tan^{-1}(-\sqrt{3})$ (*ii*) $\tan^{-1}(-1)$. (N.C.E.R.T.; C.B.S.E. (F) 2011)

5. (*i*) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (N.C.E.R.T.)

(*ii*) $\cot^{-1}(-\sqrt{3})$. (A.I.C.B.S.E. 2010)

6. (*i*) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (N.C.E.R.T.)

(*ii*) $\sec^{-1}(-2)$. (A.I.C.B.S.E. 2010)

7. (*i*) $\text{cosec}^{-1}(2)$ (H.P.B. 2010 S) (*ii*) $\text{cosec}^{-1}(-\sqrt{2})$. (N.C.E.R.T. Karnataka B. 2017)

8. (*i*) $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ (Kashmir B. 2017)

(*ii*) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ (Tripura B. 2016; C.B.S.E. 2011)

(*iii*) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ (Jammu B. 2012)

(*iv*) $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ (Assam B. 2015)

(*v*) $\sin\left(\cos^{-1}\frac{1}{2}\right)$ (H.B. 2015, 12)

(*vi*) $\cos\left(\sin^{-1}\frac{5}{13}\right)$. (H.B. 2014)

Short Answer Type Questions

SATQ

9. Find the value of $\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$. *(C.B.S.E. 2010)*

10. Write the principal value of :

(i) $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$ *(C.B.S.E. 2013)*

(ii) $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)$ *(A.I.C.B.S.E. 2013)*

(iii) $\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right)$. *(C.B.S.E. 2012)*

11. Find the value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$. *(N.C.E.R.T.)*

12. Find the value of :

$\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\sin^{-1}\left(\frac{1}{2}\right)$. *(H.B. 2013)*

13. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$. *(A.I.C.B.S.E. 2013)*

Answers

1. (a) (i) $\frac{\pi}{6}$ (ii) $-\frac{\pi}{6}$ (iii) $-\frac{\pi}{4}$ (iv) $\frac{2\pi}{3}$ (v) $-\frac{\pi}{3}$ (vi) $-\frac{\pi}{4}$

(b) [–1, 1].

2. (i) $-\frac{\pi}{3}$ (ii) $\frac{\pi}{4}$.

3. (i) $\frac{2\pi}{3}$ (ii) $\frac{5\pi}{6}$ (iii) $\frac{3\pi}{4}$.

4. (i) $-\frac{\pi}{3}$ (ii) $-\frac{\pi}{4}$.

5. (i) $\frac{2\pi}{3}$ (ii) $\frac{5\pi}{6}$.

6. (i) $\frac{\pi}{6}$ (ii) $\frac{2\pi}{3}$.

7. (i) $\frac{\pi}{6}$ (ii) $-\frac{\pi}{4}$.

8. (i) $\frac{\pi}{6}$ (ii) $\frac{5\pi}{6}$ (iii) $-\frac{\pi}{4}$ (iv) $\frac{2\pi}{5}$ (v) $\frac{\sqrt{3}}{2}$ (vi) $\frac{12}{13}$.

9. $\frac{\pi}{2}$. 10. (i) $\frac{11\pi}{12}$ (ii) $-\frac{\pi}{2}$ (iii) $\frac{2\pi}{3}$.

11. $\frac{2\pi}{3}$. 12. $\frac{5\pi}{6}$.

13. $\frac{\pi}{3}$.

Hints to Selected Questions

9. $\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)+\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

$= -\frac{\pi}{6}+\frac{2\pi}{3}=\frac{\pi}{2}$.

10. (i) $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4}+\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

$= \frac{\pi}{4}+\frac{2\pi}{3}=\frac{11\pi}{12}$.

(ii) $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)$

$= \tan^{-1}\left(\tan\frac{\pi}{3}\right)-\cot^{-1}\left(-\cot\frac{\pi}{6}\right)$

$= \frac{\pi}{3}-\cot^{-1}\left[\cot\left(-\frac{\pi}{6}\right)\right]$

$= \frac{\pi}{3}-\cot^{-1}\left[\cot\left(\pi-\frac{\pi}{6}\right)\right]$

$= \frac{\pi}{3}-\cot^{-1}\left(\cot\frac{5\pi}{6}\right)=\frac{\pi}{3}-\frac{5\pi}{6}=-\frac{\pi}{2}$.

11. $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

$= \cos^{-1}\left(\cos\frac{\pi}{3}\right)+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)$

$= \frac{\pi}{3}+2\left(\frac{\pi}{6}\right)=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$.

13. $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]= \tan^{-1}\left[2\sin\left(2.\frac{\pi}{6}\right)\right]$

$=\tan^{-1}\left[2\sin\frac{\pi}{3}\right] = \tan^{-1}\left[2.\frac{\sqrt{3}}{2}\right]$

$=\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$.

2.3. PROPERTIES OF INVERSE-TRIGONOMETRIC FUNCTIONS

(*a*) $x = \sin^{-1}(\sin x) = \cos^{-1}(\cos x) = \tan^{-1}(\tan x)$
$= \cot^{-1}(\cot x) = \sec^{-1}(\sec x) = \text{cosec}^{-1}(\text{cosec } x)$.

Proof. Let $\sin x = y$.

Then $x = \sin^{-1} y = \sin^{-1}(\sin x)$.

Similarly other parts.

(*b*) (*i*) $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$, $x \geq 1$ or $x \leq -1$ (*ii*) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$, $x \geq 1$ or $x \leq -1$ (*iii*) $\cot^{-1}x = \tan^{-1}\frac{1}{x}$, $x > 0$.

Proof. (*i*) Let $\text{cosec}^{-1} x = y$. Then $x = \text{cosec } y$

$\Rightarrow \frac{1}{x} = \sin y \quad \Rightarrow \quad y = \sin^{-1}\left(\frac{1}{x}\right)$.

Hence, $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$.

(*ii*) and (*iii*) can be proved similarly.

(*c*) (*i*) $\sin^{-1}(-x) = -\sin^{-1} x$, $x \in [-1, 1]$
(*ii*) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $x \in [-1, 1]$
(*iii*) $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in \mathbf{R}$
(*iv*) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, $x \in \mathbf{R}$
(*v*) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \geq 1$
(*vi*) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, $|x| \geq 1$.

Proof. (*i*) Let $\sin^{-1}(-x) = y$.

$\therefore \quad -x = \sin y$

$\Rightarrow \quad x = -\sin y = \sin(-y) \quad \Rightarrow \quad -y = \sin^{-1}x$

$\Rightarrow \quad y = -\sin^{-1}x$.

Hence, $\sin^{-1}(-x) = -\sin^{-1}x$.

(*ii*) Let $\cos^{-1}(-x) = y$.

$\therefore \quad -x = \cos y$

$\Rightarrow \quad x = -\cos y = \cos(\pi - y)$

$\Rightarrow \quad \cos^{-1}x = \pi - y = \pi - \cos^{-1}(-x)$.

Hence, $\cos^{-1}(-x) = \pi - \cos^{-1} x$.

(*iii*)–(*vi*) can be proved similarly.

(*d*) (*i*) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $x \in [-1, 1]$ (H.B. 2018; Jammu B. 2017; Jharkhand B. 2013)

(*ii*) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in \mathbf{R}$ (H.B. 2018)

(*iii*) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$, $|x| \geq 1$ (H.B. 2018)

(*iv*) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, $xy < 1$ (H.B. 2018; Kerala B. 2016)

(*v*) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$, $xy > -1$ (Jammu B. 2017)

(*vi*) (I) $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2}$, $|x| \leq 1$

(II) $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$, $x \geq 0$

(III) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, $-1 < x < 1$.

Proof. (*i*) Let $\sin^{-1}x = y$.

$\therefore \quad x = \sin y \quad \Rightarrow \quad x = \cos\left(\frac{\pi}{2} - y\right)$

$$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - y \qquad \Rightarrow \quad \cos^{-1}x + y = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}. \qquad \text{Hence, } \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$

(*ii*) Let $\tan^{-1}x = y.$

$$\therefore \quad x = \tan y \qquad \Rightarrow \quad x = \cot\left(\frac{\pi}{2} - y\right)$$

$$\Rightarrow \quad \cot^{-1}x = \frac{\pi}{2} - y \qquad \Rightarrow \quad \cot^{-1}x + y = \frac{\pi}{2}$$

$$\Rightarrow \quad \cot^{-1}x + \tan^{-1}x = \frac{\pi}{2}. \qquad \text{Hence, } \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}.$$

(*iii*) can be proved similarly.

(*iv*) Let $\tan^{-1}x = \theta_1$ and $\tan^{-1}y = \theta_2$.

$\therefore$ $x = \tan\theta_1$ and $y = \tan\theta_2$.

Now $$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{x+y}{1-xy} \qquad \Rightarrow \quad \theta_1 + \theta_2 = \tan^{-1}\frac{x+y}{1-xy}.$$

Hence, $$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy},\ xy < 1^* \qquad \text{... (1)}$$

(*v*) can be proved similarly.

(*vi*) Let $\tan^{-1}x = y.$

$\therefore$ $x = \tan y.$

(I) $$\frac{2x}{1+x^2} = \frac{2\tan y}{1+\tan^2 y} = \frac{2\sin y\cos y}{\cos^2 y + \sin^2 y} = \frac{\sin 2y}{1} = \sin 2y.$$

Thus $$2y = \sin^{-1}\frac{2x}{1+x^2}.$$

Hence, $$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}.$$

(II) $$\frac{1-x^2}{1+x^2} = \frac{1-\tan^2 y}{1+\tan^2 y} = \cos 2y.$$

Thus $$2y = \cos^{-1}\frac{1-x^2}{1+x^2}.$$

Hence, $$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}.$$

(III) $$\frac{2x}{1-x^2} = \frac{2\tan y}{1-\tan^2 y} = \tan 2y.$$

Thus $$2y = \tan^{-1}\frac{2x}{1-x^2}.$$

Hence, $$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}.$$

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* Because the above result does not hold if $xy \geq 1$.
If $xy = 1$, then RHS of (1) is not defined.
If $xy > 1$ and $y < 0$, then $\frac{x+y}{1-xy} > 0$ and RHS of (1) is + ve while LHS of (1) is –ve. As such (1) does not hold.
Hence, (*iv*) does not hold for $xy \geq 1$.

KEY POINT

$\sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1}x$	if	$\lvert x\rvert \le 1$
$\cos^{-1}\frac{1-x^2}{1+x^2} = 2\tan^{-1}x$	if	$x \ge 0$
$\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}x$	if	$-1 < x < 1.$

(*e*) (*i*) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

(*ii*) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

(*iii*) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

(*iv*) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$.

Proof. (*i*) Let $\sin^{-1}x = \theta_1$ and $\sin^{-1}y = \theta_2$.

$\therefore$ $\sin\theta_1 = x$ and $\sin\theta_2 = y$

so that $\cos\theta_1 = \sqrt{1-x^2}$ and $\cos\theta_2 = \sqrt{1-y^2}$.

Now $\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = x.\sqrt{1-y^2} + \sqrt{1-x^2}\,.\,y$

$\Rightarrow$ $\theta_1+\theta_2 = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$.

Hence, $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$.

(*ii*) can be proved similarly.

(*iii*) Let $\cos^{-1}x = \theta_1$ and $\cos^{-1}y = \theta_2$.

$\therefore$ $\cos\theta_1 = x$ and $\cos\theta_2 = y$

so that $\sin\theta_1 = \sqrt{1-x^2}$ and $\sin\theta_2 = \sqrt{1-y^2}$.

Now $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = xy - \sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow$ $\theta_1+\theta_2 = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$.

Hence, $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$.

(*iv*) can be proved similarly.

Frequently Asked Questions

FAQs

Example 1. (i) If $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}x = \frac{\pi}{2}$, then find x.

(C.B.S.E. 2010 C)

(ii) If $\sec^{-1}(2) + \text{cosec}^{-1}(y) = \frac{\pi}{2}$, then find y.

(C.B.S.E. 2010 C)

Solution. (*i*) $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}x = \frac{\pi}{2}$

$\Rightarrow$ $x = \frac{1}{3}$. $\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

(*ii*) $\sec^{-1}(2) + \text{cosec}^{-1}(y) = \frac{\pi}{2}$

$\Rightarrow$ $y = 2$. $\left[\because \sec^{-1}(x) + \text{cosec}^{-1}(x) = \frac{\pi}{2}\right]$

Example 2. Prove the following :

$$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}.$$

(A.I.C.B.S.E. 2012)

Solution.

$$\text{LHS} = \cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$$

$$= \cos\left(\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{3}{\sqrt{13}}\right)$$

$$= \cos\left[\cos^{-1}\left(\frac{4}{5}.\frac{3}{\sqrt{13}}-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{9}{13}}\right)\right]$$

$$= \frac{12}{5\sqrt{13}}-\frac{3}{5}.\frac{2}{\sqrt{13}} = \frac{6}{5\sqrt{13}} = \text{RHS.}$$

Example 3. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1$, then write the value of $x + y + xy$. *(A.I.C.B.S.E. 2014)*

Solution. We have : $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\frac{x+y}{1-xy} = \frac{\pi}{4} \qquad \Rightarrow \frac{x+y}{1-xy} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy} = 1 \qquad \Rightarrow x + y = 1 - xy.$$

Hence, $x + y + xy = 1$.

Example 4. Prove that $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$; $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

(N.C.E.R.T.; C.B.S.E. 2018; Jammu B. 2013; H.B. 2012)

Solution. Put $\sin^{-1} x = \theta$ so that $x = \sin\theta$.

RHS. $= \sin^{-1}(3 \sin\theta - 4 \sin^3\theta)$
$= \sin^{-1}(\sin 3\theta)$
$= 3\theta$
$= 3 \sin^{-1} x =$ **LHS.**

Example 5. If $\sin^{-1} x = \tan^{-1} y$**, then show that :**

$$\frac{1}{x^2}-\frac{1}{y^2}=1.$$

(W. Bengal B. 2018)

Solution. We have : $\sin^{-1} x = \tan^{-1} y$

$$\Rightarrow \sin^{-1} x = \sin^{-1}\frac{y}{\sqrt{1+y^2}}$$

$$\Rightarrow x = \frac{y}{\sqrt{1+y^2}} \Rightarrow x\sqrt{1+y^2} = y.$$

Squaring, $x^2 (1 + y^2) = y^2 \Rightarrow y^2 - x^2 = x^2 y^2$.

Hence, $\frac{1}{x^2}-\frac{1}{y^2} = 1.$ *[Dividing by x^2y^2]*

Example 6. Prove that $\frac{1}{2} \le x \le 1$**, then :**

$$\cos^{-1} x+\cos^{-1}\left[\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right]=\frac{\pi}{3}.$$

(C.B.S.E. Sample Paper 2018)

Solution. LHS = $\cos^{-1} x+\cos^{-1}\left[\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right]$

$$= \cos^{-1} x+\cos^{-1}\left[\frac{1}{2}.x+\frac{\sqrt{3}}{2}.\sqrt{1-x^2}\right]$$

$$= \theta+\cos^{-1}\left[\cos\frac{\pi}{3}.\cos\theta+\sin\frac{\pi}{3}.\sin\theta\right]$$

[Putting $x = \cos\theta$ so that $\sqrt{1-x^2} = \sin\theta$]

$$= \theta+\cos^{-1}\left[\cos\left(\frac{\pi}{3}-\theta\right)\right]$$

$$= \theta+\left(\frac{\pi}{3}-\theta\right)$$

$$= \frac{\pi}{3} = \text{RHS.}$$

Example 7. Find the value of :

$$\sin^{-1}\left(2\tan^{-1}\frac{1}{4}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right).$$

(C.B.S.E. Sample Paper 2019)

Solution. To evaluate : $\sin^{-1}\left(2\tan^{-1}\frac{1}{4}\right)$.

Put $\tan^{-1}\frac{1}{4} = \theta$ so that $\tan\theta = \frac{1}{4}$.

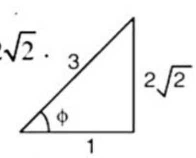
Now, $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\left(\frac{1}{4}\right)}{1+\left(\frac{1}{4}\right)^2} = \frac{1/2}{1+1/16}$

$$= \frac{1/2}{17/16} = \frac{8}{17} \qquad ...(1)$$

To evaluate : $\cos(\tan^{-1}2\sqrt{2})$.

Put $\tan^{-1}2\sqrt{2} = \phi$ so that $\tan\phi = 2\sqrt{2}$.

$\therefore \quad \cos\phi = \frac{1}{3} \qquad ...(2)$



Hence, $\sin^{-1}\left(2\tan^{-1}\frac{1}{4}\right)+\cos(\tan^{-1}2\sqrt{2})$

$$= \frac{8}{17}+\frac{1}{3} \qquad \text{[Using (1) \& (2)]}$$

$$= \frac{24+17}{51}=\frac{41}{51}.$$

Example 8. Prove that :

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$

(C.B.S.E. 2013; A.I.C.B.S.E. 2011; H.B. 2010)

Solution.

$$\text{LHS} = \left(\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\frac{5+2}{10-1}+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}} = \tan^{-1}\frac{56+9}{72-7}$$

$$= \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}(1) = \frac{\pi}{4} = \textbf{RHS.}$$

Example 9. Show that :

$$\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{84}{85}.$$

(N.C.E.R.T.; H.P.B. 2011)

Solution. Let $\sin^{-1}\frac{3}{5}=x$ and $\sin^{-1}\frac{8}{17}=y$.

$\therefore$ $\sin x=\frac{3}{5}$ and $\sin y=\frac{8}{17}$

so that $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$

and $\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\frac{64}{289}}=\sqrt{\frac{225}{289}}=\frac{15}{17}$.

Now $\cos(x-y)=\cos x\cos y+\sin x\sin y$

$$=\left(\frac{4}{5}\right)\left(\frac{15}{17}\right)+\left(\frac{3}{5}\right)\left(\frac{8}{17}\right)$$

$$=\frac{60+24}{85}=\frac{84}{85}$$

$\Rightarrow$ $x-y=\cos^{-1}\left(\frac{84}{85}\right)$.

Hence, $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}-\cos^{-1}\frac{84}{85}$.

Example 10. Prove that :

$$\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)=\frac{1}{2}\sin^{-1}\left(\frac{3696}{4225}\right).$$

(P.B. 2018)

Solution. $\text{LHS}=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)$

$$=\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$=\frac{1}{2}\left[2\sin^{-1}\left(\frac{5}{13}\right)+2\sin^{-1}\left(\frac{3}{5}\right)\right]$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{5}{13}\right)\right]+\left[\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{3}{5}\right)\right]$$

$$=\frac{1}{2}\left(\sin^{-1}\left(2.\frac{5}{13}\sqrt{1-\frac{25}{169}}\right)+\sin^{-1}\left(2.\frac{3}{5}\sqrt{1-\frac{9}{25}}\right)\right)$$

$$=\frac{1}{2}\left(\sin^{-1}\frac{10}{13}\times\frac{12}{13}\right)+\sin^{-1}\left(\frac{6}{5}\times\frac{4}{5}\right)$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{120}{169}\right)+\sin^{-1}\left(\frac{24}{25}\right)\right]$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{120}{169}\sqrt{1-\frac{576}{625}}+\frac{24}{25}\sqrt{1-\frac{14400}{28561}}\right)\right]$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{120}{169}\times\frac{7}{25}+\frac{24}{25}\times\frac{119}{169}\right)\right]$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{840+2856}{4225}\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{3696}{4225}\right)=\textbf{RHS.}$$

Example 11. Show that :

$$2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}.$$

(A.I.C.B.S.E. 2015)

Solution. $2\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(2.\frac{3}{5}\sqrt{1-\frac{9}{25}}\right)$

$\left[\because 2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})\right]$

$$=\sin^{-1}\left(2\times\frac{3}{5}\times\frac{4}{5}\right)$$

$$=\sin^{-1}\left(\frac{24}{25}\right)=\tan^{-1}\left(\frac{24}{7}\right) \quad ...(1)$$

Now $\text{LHS}=2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$$=\tan^{-1}\left(\frac{24}{7}\right)-\tan^{-1}\left(\frac{17}{31}\right) \quad \textit{[Using (1)]}$$

$$= \tan^{-1}\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}} = \tan^{-1}\frac{744-119}{217+408}$$

$$= \tan^{-1}\frac{625}{625} = \tan^{-1}(1)$$

$$= \frac{\pi}{4} = \textbf{RHS.}$$

Example 12. Prove that :

$$\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}=\frac{2b}{a}.$$

(C.B.S.E. 2017)

Solution. Put $\cos^{-1}\frac{a}{b} = \theta$ so that $\cos\theta = \frac{a}{b}$.

$$\text{LHS} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}+\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$= \frac{\left(1+\tan\frac{\theta}{2}\right)^2+\left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2(1+\tan^2\theta)}{1-\tan^2\frac{\theta}{2}} = \frac{2}{\cos 2 \cdot\frac{\theta}{2}} = \frac{2}{\cos\theta}$$

$$= \frac{2}{a/b} = \frac{2b}{a} = \textbf{RHS.}$$

Example 13. Prove the following :

$$\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

(Meghalaya B. 2016; H.P.B. 2015; C.B.S.E. (F) 2011)

Solution. Putting $\sin^{-1}\left(\frac{1}{3}\right) = x$ and $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y$,

we get :

$$\sin x = \frac{1}{3} \text{ and } \sin y = \frac{2\sqrt{2}}{3}.$$

$$\therefore \quad \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{1}{9}}$$

$$= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

and $\quad \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{8}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$

Now $\sin(x+y) = \sin x\cos y + \cos x \sin y$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)+\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right)$$

$$= \frac{1}{9}+\frac{8}{9} = 1 = \sin\frac{\pi}{2}$$

$$\Rightarrow \quad x+y = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{9}{4}\left(\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right) = \frac{9}{4}\left(\frac{\pi}{2}\right)$$

[*Multiplying by* $\frac{9}{4}$]

$$\Rightarrow \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}.$$

Hence, $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$

Example 14. Prove that :

$$\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right)-\tan^{-1}\left(\frac{4x}{1-4x^2}\right) = \tan^{-1} 2x;\ |2x| < \frac{1}{\sqrt{3}}.$$

(A.I.C.B.S.E. 2016)

Solution.

$$\text{LHS} = \tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right)-\tan^{-1}\left(\frac{4x}{1-4x^2}\right)$$

$$= \tan^{-1}\frac{\frac{6x-8x^3}{1-12x^2}-\frac{4x}{1-4x^2}}{1+\frac{6x-8x^3}{1-12x^2}\cdot\frac{4x}{1-4x^2}}$$

$$= \tan^{-1}\frac{(6x-8x^3)(1-4x^2)-4x(1-12x^2)}{(1-12x^2)(1-4x^2)+4x(6x-8x^3)}$$

$$= \tan^{-1}\frac{6x-8x^3-24x^3+32x^5-4x+48x^3}{1-12x^2-4x^2+48x^4+24x^2-32x^4}$$

$$= \tan^{-1}\frac{2x+16x^3+32x^5}{1+8x^2+16x^4} = \tan^{-1}\frac{2x(1+8x^2+16x^4)}{1+8x^2+16x^4}$$

$= \tan^{-1} 2x =$ **RHS.**

Example 15. Prove that :

$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}.$$

(C.B.S.E. 2016)

Solution.

$$\text{LHS} = \left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}} + \tan^{-1}\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}$$

$$= \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23} = \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}} = \tan^{-1}\frac{138+187}{391-66} = \tan^{-1}\frac{325}{325}$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{RHS.}$$

Example 16. Solve for x :

$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$. *(C.B.S.E. 2016)*

Solution. Here $2\tan^{-1}(\cos x) = \tan^{-1}(\cos x) + \tan^{-1}(\cos x)$

$$= \tan^{-1}\frac{\cos x+\cos x}{1\ \ \cos x \,.\, \cos x} = \tan^{-1}\frac{2\cos x}{1\ \ \cos^2 x}$$

$$= \tan^{-1}\frac{2\cos x}{\sin^2 x} = \tan^{-1}(2\cot x\operatorname{cosec} x) \qquad ...(1)$$

Now $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \quad \tan^{-1}(2\cot x\operatorname{cosec} x) = \tan^{-1}(2\operatorname{cosec} x)$

[*Using (1)*]

$\Rightarrow \quad 2\cot x\operatorname{cosec} x = 2\operatorname{cosec} x$

$\Rightarrow \quad \cot x\operatorname{cosec} x = \operatorname{cosec} x$

$\Rightarrow \quad \sin x = \tan x \sin x$

$\Rightarrow \quad$ either $\sin x = 0$ or $\tan x = 1$.

Hence, $x = n\pi \ \forall\ n \in \mathbf{Z}$ or $x = m\pi + \frac{\pi}{4} \ \forall\ m \in \mathbf{Z}$.

Example 17. If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, find the value of 'x'. *(A.I.C.B.S.E. 2014)*

Solution. We have :

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\frac{\left(\frac{x-2}{x-4}\right)+\left(\frac{x+2}{x+4}\right)}{1-\left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)} = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)-(x-2)(x+2)} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{(x^2-2x+4x-8)+(x^2+2x-4x-8)}{(x^2-16)-(x^2-4)} = 1$$

$\Rightarrow \quad \frac{2x^2-16}{-12} = 1 \qquad \Rightarrow \quad 2x^2 - 16 = -12$

$\Rightarrow \quad 2x^2 = 16 - 12 \qquad \Rightarrow \quad 2x^2 = 4$

$\Rightarrow \quad x^2 = 2.$

Hence, $\quad x = \pm\sqrt{2}$.

Example 18. If $\tan^{-1}\frac{x-3}{x-4}+\tan^{-1}\frac{x+3}{x+4}=\frac{\pi}{4}$, then find the value of x. *(A.I.C.B.S.E. 2017)*

Solution. We have : $\tan^{-1}\frac{x-3}{x-4}+\tan^{-1}\frac{x+3}{x+4} = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\frac{\frac{x-3}{x-4}+\frac{x+3}{x+4}}{1-\frac{x-3}{x-4}\cdot\frac{x+3}{x+4}} = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\frac{(x-3)(x+4)+(x-4)(x+3)}{(x^2-16)-(x^2-9)} = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{(x^2+x-12)+(x^2-x-12)}{-16+9} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x^2-24}{-7} = 1$$

$\Rightarrow \quad 2x^2 - 24 = -7$

$\Rightarrow \quad 2x^2 = 17 \qquad \Rightarrow \quad x^2 = \frac{17}{2}.$

Hence, $\quad x = \pm\sqrt{\frac{17}{2}}$.

Example 19. Solve : (i) $\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4}$; $\sqrt{6}>x>0$ *(P.B. 2015; C.B.S.E. 2010 C)*

(ii) $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$. *(N.C.E.R.T. ; H.B. 2017; Jammu B. 2013)*

Solution. (*i*) We have :

$$\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{\frac{x}{2}+\frac{x}{3}}{1-\left(\frac{x}{2}\right)\left(\frac{x}{3}\right)}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{3x+2x}{6-x^2}=\frac{\pi}{4} ; 6>x^2>0 \text{ i.e. } \sqrt{6}>x>0$$

$$\Rightarrow \frac{5x}{6-x^2}=\tan\frac{\pi}{4} \Rightarrow \frac{5x}{6-x^2}=1$$

$$\Rightarrow 5x=6-x^2 \Rightarrow x^2+5x-6=0$$

$$\Rightarrow (x+6)(x-1)=0 \quad \Rightarrow x=-6, 1.$$

Hence, $x=1$. $[\because x>0]$

(*ii*) We have : $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1=\sin\frac{\pi}{2}$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5}=\cos^{-1}\frac{1}{5}.$$

Hence, $x=\frac{1}{5}$.

Example 20. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, **then find 'x'.** *(C.B.S.E. 2015)*

Solution. We have : $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$ ...(1)

Put $\tan^{-1}x=t$ so that $\cot^{-1}x=\frac{\pi}{2}-t$.

$\therefore$ (1) becomes : $t^2+\left(\frac{\pi}{2}-t\right)^2=\frac{5\pi^2}{8}$

$$\Rightarrow t^2+\frac{\pi^2}{4}-\pi t+t^2=\frac{5\pi^2}{8}$$

$$\Rightarrow 2t^2-\pi t-\frac{3\pi^2}{8}=0$$

$$\Rightarrow 16t^2-8\pi t-3\pi^2=0.$$

Solving, $t=\frac{8\pi\pm\sqrt{64\pi^2+192\pi^2}}{32}$

$$=\frac{8\pi\pm16\pi}{32}=\frac{3\pi}{4},-\frac{\pi}{4}.$$

When $\tan^{-1}x=\frac{3\pi}{4}$, then $x=\tan\frac{3\pi}{4}=-1$.

When $\tan^{-1}x=-\frac{\pi}{4}$, then $x=\tan\left(-\frac{\pi}{4}\right)=-1$.

Hence, $x=-1$.

Example 21. Write : $\tan^{-1}\frac{1}{\sqrt{x^2-1}}$, $|x|>1$ **in the simplest form.** *(H.P.B. 2013 S, 10 S)*

Solution. Put $x=\sec\theta$.

Then $\sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=\tan\theta$.

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\left(\frac{1}{\tan\theta}\right)=\tan^{-1}(\cot\theta)$$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)$$

$$=\frac{\pi}{2}-\theta.$$

Hence, $\tan^{-1}\frac{1}{\sqrt{x^2-1}}=\frac{\pi}{2}-\sec^{-1}x$.

Example 22. Prove that :

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

(C.B.S.E. 2012)

Solution. Here

$$\frac{\cos x}{1+\sin x}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}$$

$$= \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}$$

$$= \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}$$

[*Dividing Num. & Denom. by* $\cos\frac{x}{2}$]

$$= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right).$$

Hence, $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}$.

Example 23. Prove that :

$$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^3}{1-3x^2},\ |x| < \frac{1}{\sqrt{3}}.$$

(N.C.E.R.T.; Jammu B. 2018; Kashmir B. 2012 ; A.I.C.B.S.E. 2010)

Solution. Put $x = \tan\theta$ so that $\theta = \tan^{-1}x$.

$$\text{RHS} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x$$

$$= \tan^{-1}x + 2\tan^{-1}x = \tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \text{LHS.}$$

[*Property (d) (vi) (III)*]

Example 24. Simplify : $\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$, **if** $\frac{a}{b}\tan x > -1$.

(N.C.E.R.T. ; H.P.B. 2013 S)

Solution. $\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$

$$= \tan^{-1}\left[\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x + a\sin x}{b\cos x}}\right]$$

[*Dividing Numerator and Denominator by* $b\cos x$]

$$= \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$$

$$= \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x) = \tan^{-1}\frac{a}{b} - x.$$

Example 25. Show that :

$$\tan^{-1}\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x; -\frac{1}{\sqrt{2}} \le x \le 1.$$

(N.C.E.R.T.; Assam B. 2018; Jammu B. 2015W, 13; H.P.B. 2015, 13 S; A.I.C.B.S.E. 2014, 11 ; H.B. 2013; P.B. 2013; Assam. B. 2013)

Solution. LHS $= \tan^{-1}\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$

$$= \tan^{-1}\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}$$

[*Putting* $x = \cos 2\theta$]

$$= \tan^{-1}\frac{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}$$

$$= \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

[*Dividing Numerator and Denominator by* $\cos\theta$]

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x = \text{RHS.}$$

$$\left[\because \cos 2\theta = x \Rightarrow \theta = \frac{1}{2}\cos^{-1}x\right]$$

Example 26. Prove the following :

$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}.$$

(H.B. 2015 ; A.I.C.B.S.E. 2010)

Solution. Put $\cot^{-1}x = \theta$ so that $x = \cot\theta$.

$$\therefore \quad \sin\theta = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \sin(\cot^{-1}x) = \frac{1}{\sqrt{1+x^2}} \quad ....(1)$$

$\therefore\ \tan^{-1}\{\sin(\cot^{-1}x)\} = \tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$ [*Using (1)*]

Put $\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \phi$ ...(2)

so that $\frac{1}{\sqrt{1+x^2}} = \tan\phi$.

$\therefore\ \cos\phi = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$

$\Rightarrow \cos\left(\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$. [*Using (2)*]

Hence, $\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}$. [*Using (1)*]

EXERCISE 2 (b)

Fast Track Answer Type Questions — FTATQ

1. (*i*) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$. **(True False)** *(Jammu B. 2018)*

(*ii*) Is $\sec^{-1}(-x) = \pi - \sec^{-1}x, |x| \geq 1$? **(True False)** *(Jammu B. 2014)*

(*iii*) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{4}$. **(True/False)** *(Kashmir B. 2016)*

2. Find the value of the following :

(*a*) (*i*) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ *(N.C.E.R.T.)*

(*ii*) $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ *(N.C.E.R.T.)*

(*iii*) $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$ *(A.I.C.B.S.E. 2010)*

(*iv*) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ *(N.C.E.R.T.; C.B.S.E. 2011)*

(*v*) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ *(N.C.E.R.T.; Kashmir B. 2018)*

(*vi*) $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ *(C.B.S.E. 2013)*

(*vii*) $\cos(\sec^{-1}x + \text{cosec}^{-1}x)$, $|x| > 1$ *(N.C.E.R.T.; H.B. 2012)*

(*viii*) $\cot(\tan^{-1}a + \cot^{-1}a)$. *(N.C.E.R.T.)*

(*b*) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right]$. *(Karnataka B. 2014; Bihar B. 2014; C.B.S.E. 2011)*

3. (*a*) Write down the value of $\text{cosec}^{-1}x + \sec^{-1}x$, where $|x| \geq 1$. *(Uttarakhand B. 2015)*

(*b*) If $4\sin^{-1}x + \cos^{-1}x = \pi$, then find the value of x. *(C.B.S.E. Sample Paper 2018)*

Very Short Answer Type Questions — VSATQ

4. Evaluate : (*i*) $\tan^{-1}1 + \cos^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{3}$ *(Jharkhand B. 2016)*

(*ii*) $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3)$. *(W. Bengal B. 2017)*

Prove that (5-9) :

5. (*a*) $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$. *(N.C.E.R.T. ; H.B. 2017, 12; Kashmir B. 2015)*

(*b*) $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ *(A.I.C.B.S.E. 2013)*

(*c*) $\tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$. *(Mizoram B. 2017)*

6. (*i*) $\cos^{-1}(\cos^2x - \sin^2x) = 2x$ *(H.B. 2012, 10)*

(*ii*) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$. *(N.C.E.R.T.; Jammu B. 2015 W; Karnataka B. 2014; H.P.B. 2012, 10)*

7. (*i*) $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$, $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$ *(N.C.E.R.T.; Kashmir B. 2011 ; H.B. 2010)*

(*ii*) $\sin^{-1}(2x\sqrt{1-x^2}) = 2\cos^{-1}x$, $\frac{1}{\sqrt{2}} \leq x \leq 1$ *(N.C.E.R.T.; Karnataka B. 2017; Uttarakhand B. 2015; H.B. 2012; H.P.B. 2012, 10)*

(*iii*) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$; $-1 \leq x \leq 1$ *(H.B. 2015)* (*iv*) $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right) = 2\sin^{-1}x$ when : $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$. *(Karnataka B. 2013)*

8. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0, 1]$.
(N.C.E.R.T.; H.B. 2017, 12; Jammu B. 2016; Nagaland B. 2016; C.B.S.E. 2012, 10)

9. $\tan^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) = \frac{\pi}{4} - \tan^{-1}\sqrt{x}$, where $x > 0$.
(H.B. 2011)

Short Answer Type Questions

SATQ

Prove that (10 – 12) :

10. (*a*) (*i*) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$
(H.B. 2018; C.B.S.E. 2009)

(*ii*) $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$.
(N.C.E.R.T.; Assam B. 2018; H.B. 2017)

(*b*) (*i*) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$
(C.B.S.E. 2010)

(*ii*) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$
(N.C.E.R.T.; Kerela B. 2018; H.B. 2017, 12; H.P.B. 2011)

(*iii*) $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$
(N.C.E.R.T.; Kashmir B. 2017; H.B. 2017; Jharkhand B. 2016; Kerala B. 2016, 15; P.B. 2016; H.P.B. 2013, 11, 10; P.B. 2012; A.I.C.B.S.E. 2011)

(*iv*) $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{32}{43}$
(Jharkhand B. 2013 ; P.B. 2012)

(*v*) $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} = \tan^{-1}\frac{42}{67}$ *(P.B. 2016)*

(*vi*) $2\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{9}{13}$ *(P.B. 2016)*

(*vii*) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} = \frac{1}{2}\cos^{-1}\frac{16}{65}$ *(P.B. 2017)*

(*viii*) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}$.
(Meghalaya B. 2017)

11. (*i*) $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$
(A.I.C.B.S.E. 2009 C)

(*ii*) $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} - \frac{\pi}{4}$
(N.C.E.R.T.; H.B. 2017; Kashmir B. 2013)

(*iii*) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.
(A.I.C.B.S.E. 2010, 09 C)

12. (*i*) $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$
(H.P.B. Model Paper 2018 ; H.P.B. 2017, 15; A.I.C.B.S.E. 2012 ; C.B.S.E. 2010 C; P.B. 2010)

(*ii*) $\sin^{-1}\frac{63}{65} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$
(C.B.S.E. (F) 2012)

(*iii*) $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$
(H.P.B. 2017, 15; A.I.C.B.S.E 2012; C.B.S.E. 2010; P.B. 2010)

(*iv*) $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$
(N.C.E.R.T.; H.P.B. 2016, 11)

(*v*) (I) $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$
(H.P.B. 2017, 15)

(II) $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$.
(Meghalaya B. 2018; H.B. 2017)

(*vi*) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{36}{85}$
(Meghalaya B. 2013 ; C.B.S.E. 2012, 10 C; P.B. 2012)

(*vii*) $2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$
(Mizoram B. 2018)

(*viii*) $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{56}{65}$
(C.B.S.E. 2010 C)

(*ix*) $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$
(H.P.B. 2016, 11; Kerala B. 2013; Kashmir B. 2013 ; Uttrakhand B. 2013; 11; P.B. 2010)

(*x*) $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$
(Meghalaya B. 2015)

(*xi*) $\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$.
(H.P.B. 2016; P.B. 2014)

13. (*a*) Find the value of :

(*i*) $4\left(2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}\right)$ *(W. Bengal B. 2016)*

(*ii*) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}$.

(H.B. 2015, 10 ; C.B.S.E. 2014)

(*b*) $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$. *(C.B.S.E. 2011)*

14. Prove that :

$$\cot^{-1}\frac{ab+1}{a-b}+\cot^{-1}\frac{bc+1}{b-c}+\cot^{-1}\frac{ca+1}{c-a}=0.$$

15. Find the value of :

$$\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right),$$

where $|x|<1$, $y>0$ and $xy<1$.

(N.C.E.R.T. ; Jammu B. 2015, 13 ; C.B.S.E. 2013)

16. Prove that $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$.

(W. Bengal B. 2018)

17. Solve the following equations :

(*i*) $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$, $x>0$

(N.C.E.R.T.; H.P.B. 2018; Kashmir B. 2016; Bihar B. 2014)

(*ii*) $\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$

(A.I.C.B.S.E. 2009 C)

(*iii*) $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}, |x|<1$

(N.C.E.R.T.; Kerala B. 2017; Kashmir B. 2016; Tripura B. 2016; H.B. 2015, 12 ; H.P.B. 2013, 10 ; Jammu B. 2013 ; A.I.C.B.S.E. 2010 ; C.B.S.E. 2010)

(*iv*) $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

(A.I.C.B.S.E. 2015)

(*v*) $2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x)$, $x\neq\frac{\pi}{2}$

(C.B.S.E. (F) 2012)

(*vi*) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

(N.C.E.R.T; H.P.B. 2018)

(*vii*) $\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right)$;

$x>0$ *(C.B.S.E. 2010 C)*

(*viii*) $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\left(\frac{8}{31}\right)$; $x>0$

(W. Bengal B. 2017, P.B. 2015, 14 S)

(*ix*) $\tan^{-1}(x+2)+\tan^{-1}(x-2)=\frac{\pi}{4}$; $x>0$

(P.B. 2015)

(*x*) $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$; $x>0$

(N.C.E.R.T.; Karnataka B. 2017; H.B. 2015, 12 ; C.B.S.E. (F) 2011)

(*xi*) $\tan^{-1}\frac{2x}{1-x^2}+\cot^{-1}\frac{1-x^2}{2x}=\frac{\pi}{3}$; $x>0$

(H.B. 2012)

(*xii*) $\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

(C.B.S.E. 2017; A.I.C.B.S.E. 2013)

(*xiii*) $\sin\left[\cot^{-1}(x+1)\right]=\cos(\tan^{-1}x)$

(C.B.S.E. 2015)

(*xiv*) $2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x)$; $0<x<\frac{\pi}{2}$

(C.B.S.E. 2010 C)

(*xv*) $\cos(\sin^{-1}x)=\frac{1}{2}$. *(Assam B. 2017)*

18. Solve for x : $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$. *(A.I.C.B.S.E. 2016)*

19. Solve for x :

$$3\sin^{-1}\left(\frac{2x}{1+x^2}\right)-4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{2}.$$

20. Write the following in the simplest form :

(*i*) $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2}<x<\frac{\pi}{2}$

(N.C.E.R.T.; H.P.B. 2017, 16, 14, 10 S; Meghalaya B. 2014 ; Jammu B. 2012)

(*ii*) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$, $x<\pi$

(N.C.E.R.T.; H.P.B. 2017, 16, 10; H.B. 2012, 11 ; Jammu B. 2012)

(*iii*) $\tan^{-1}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)$, $0<x<\pi$

(Kerala B. 2014)

(*iv*) $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$, $|x|<a$

(N.C.E.R.T. ; H.P.B. 2018, 17, 13S, 13, 10S, 10; H.B. 2014)

(*v*) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $x\neq 0$

(N.C.E.R.T.; H.B. 2018; H.P.B. 2018, 17, 13S; Karnataka B. 2017; Kashmir B. 2015; H.B. 2014 ; Jammu B. 2014, 13; P.B. 2010)

(vi) $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right), |x|>1.$ **(N.C.E.R.T.)**

21. Prove that :

(i) $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{1+x}\right) = \frac{1}{2}\cos^{-1}x$ **(P.B. 2010)**

(ii) $\frac{1}{2}\tan^{-1}x = \cos^{-1}\left\{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right\}^{\frac{1}{2}}.$

(Rajasthan B. 2013)

22. (a) Prove that :

(i) $\tan^{-1}\left[\frac{\sqrt{1+z}+\sqrt{1-z}}{\sqrt{1+z}-\sqrt{1-z}}\right] = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}z$ **(P.B. 2013)**

(ii) $\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$

(H.B. 2018, 15, 13; P.B. 2013)

(iii) $\cot^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x.$

(H.B. 2013)

(b) Prove that :

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x\in\left(0,\frac{\pi}{4}\right).$$

(H.B. 2018; Assam B. 2017; Jammu B. 2015; H.P.B. 2015, 13, S; C.B.S.E. 2014, 11)

23. If $\tan^{-1}x + \tan^{-1}y - \tan^{-1}z = 0$, then prove that : $x + y + xyz = z.$ **(H.B. 2011)**

24. (i) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, prove that : $x^2 + y^2 + z^2 + 2xyz = 1.$ **(H.B. 2010)**

(ii) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that :

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz.$$

(Assam B. 2016; W. Bengal B. 2016)

25. Show that $\sin\left[\cot^{-1}\left\{\cos(\tan^{-1}x)\right\}\right] = \sqrt{\frac{x^2+1}{x^2+2}}.$

Long Answer Type Questions

LATQ

26. If $\tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{zx}{yr}+\tan^{-1}\frac{xy}{zr}=\frac{\pi}{4}$, prove that : $x^2 + y^2 + z^2 = r^2.$ **(H.B. 2010)**

27. (i) If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, prove that :

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha.$$

(ii) If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$, prove that $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta.$

Answers

1. (i) False (ii) True (iii) False.

2. (a) (i) $\frac{\pi}{3}$ (ii) $\frac{2\pi}{5}$ (iii) $\frac{\pi}{5}$ (iv) $-\frac{\pi}{4}$ (v) $\frac{\pi}{6}$ (vi) $\frac{5}{12}$ (vii) – (viii) 0 (b) 1.

3. (a) $\frac{\pi}{2}$ (b) $\frac{1}{2}$. **4.** (i) $\frac{3\pi}{4}$ (ii) 15.

13. (a) (i) π (ii) $\frac{\pi}{4}$ (b) $\frac{\pi}{4}$. **15.** $\frac{x+y}{1-xy}$.

17. (i) $\frac{1}{6}$ (ii) 2 (iii) $\pm\frac{1}{\sqrt{2}}$ (iv) $\frac{1}{2}$ (v) $\frac{\pi}{4}$ (vi) $n\pi$ or $m\pi+\frac{\pi}{4}$; $m, n \in$ **I**

(vii) – (viii) $\frac{1}{4}$ (ix) $\sqrt{6}-1$ (x) $\frac{1}{\sqrt{3}}$ (xi) $2-\sqrt{3}$ (xii) $\frac{3}{4}$ (xiii) $-\frac{1}{2}$ (xiv) $\frac{\pi}{4}$ (xv) $\frac{\sqrt{3}}{2}$.

18. $x=\pm\frac{1}{\sqrt{2}}$. **19.** $x=1$.

20. (i) $\frac{\pi}{4}+\frac{x}{2}$ (ii) $\frac{\pi}{4}-x$ (iii) $\frac{\pi}{4}+x$ (iv) $\sin^{-1}\frac{x}{a}$ (v) $\frac{1}{2}\tan^{-1}x$ (vi) $\frac{x}{2}-\sec^{-1}x$.

Hints to Selected Questions

5. (*a*) $2\sin^{-1}\frac{3}{5} = \sin^{-1}\left(2.\frac{3}{5}\sqrt{1-\frac{9}{25}}\right)$

$\left[\because \sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x\right]$

$= \sin^{-1}\left(\frac{6}{5}\times\frac{4}{5}\right) = \sin^{-1}\left(\frac{24}{25}\right) = \tan^{-1}\left(\frac{24}{7}\right).$

(*b*) Put $\sin^{-1}\frac{3}{4} = x$ so that $\sin x = \frac{3}{4}$.

$\therefore \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \tan\frac{x}{2} = \frac{\sin x/2}{\cos x/2}$

$= \frac{2\sin\frac{x}{2}\sin\frac{x}{2}}{2\cos\frac{x}{2}\sin\frac{x}{2}} = \frac{2\sin^2\frac{x}{2}}{\sin x} = \frac{1-\cos x}{\sin x}$

$= \frac{1-\sqrt{7}/4}{3/4} = \frac{4-\sqrt{7}}{3}.$

6. (*i*) $\cos^{-1}(\cos^2 x - \sin^2 x) = \cos^{-1}(\cos 2x) = 2x$

(*ii*) Put $\cos^{-1}x = \theta$.

7. (*iii*) Put $\tan^{-1}x = \theta$ so that $x = \tan\theta$.

8. Put $\tan^{-1}\sqrt{x} = \theta$.

10. (*a*) (*i*) Put $\sin^{-1}\frac{4}{5} = \theta$ and $\sin^{-1}\frac{5}{13} = \phi$

so that $\sin\theta = \frac{4}{5}$ and $\sin\phi = \frac{5}{13}$.

$\therefore \cos\theta = \frac{3}{5}$ and $\cos\phi = \frac{12}{13}$.

Now $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$= \frac{3}{5}\times\frac{12}{13} - \frac{4}{5}\times\frac{5}{13} = \frac{16}{65}.$

$\therefore \theta+\phi = \cos^{-1}\frac{16}{65}$

$\Rightarrow \sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13} = \frac{\pi}{2}-\sin^{-1}\frac{16}{65}.$

11. (*i*) LHS $= \left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)$

$= \tan^{-1}\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}} - \tan^{-1}\left(\frac{8}{19}\right)$

$= \tan^{-1}\left(\frac{27}{11}\right)-\tan^{-1}\left(\frac{8}{19}\right).$

14. LHS. $= \tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}\frac{c-a}{1+ca}$

$= \left(\tan^{-1}a-\tan^{-1}b\right)+\left(\tan^{-1}b-\tan^{-1}c\right)$

$+\left(\tan^{-1}c-\tan^{-1}a\right)=0.$

17. (*ii*) $\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$

$\Rightarrow \tan^{-1}\left[\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\times\frac{x-1}{x}}\right] = \tan^{-1}(-7)$

$\Rightarrow \frac{x(x+1)+(x-1)^2}{x(x-1)-(x^2-1)} = -7 \Rightarrow 2x^2-8x+8 = 0$; etc.

(*vi*) $2\tan^{-1}(\cos x) = \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)$

$= \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right)$

$= \tan^{-1}(2\cot x \operatorname{cosec} x).$

(*x*) $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

$\Rightarrow \tan^{-1}x = 2\tan^{-1}\left(\frac{1-x}{1+x}\right)$

$= \tan^{-1}\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}$; etc.

18. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1) = \tan^{-1}3x$

$\Rightarrow \tan^{-1}(x-1)+\tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$

$\Rightarrow \tan^{-1}\frac{(x-1)+(x+1)}{1-(x-1)(x+1)} = \tan^{-1}\frac{3x-x}{1+3x(x)}$

$\Rightarrow \tan^{-1}\frac{2x}{2-x^2} = \tan^{-1}\frac{2x}{1+3x^2}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2}$; etc.

20. (*i*) $\dfrac{\cos x}{1-\sin x} = \dfrac{\sin\left(\frac{\pi}{2}+x\right)}{1+\cos\left(\frac{\pi}{2}+x\right)}$

$= \dfrac{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}$

$= \dfrac{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$

(*iv*) Put $x = a\sin\theta$.

(*v*) Put $x = \tan\theta$.

(*vi*) Put $x = \sec\theta$.

22. (*b*) $\sqrt{1+\sin x} = \cos\frac{x}{2}+\sin\frac{x}{2}$ and

$\sqrt{1-\sin x} = \cos\frac{x}{2}-\sin\frac{x}{2}$.

26. Put $\tan^{-1}\frac{yz}{xr} = A, \tan^{-1}\frac{zx}{yr} = B, \tan^{-1}\frac{xy}{zr} = C$.

NCERT-FILE

Questions from NCERT Book

(For each unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 2.1

Find the principal values of the following :

1. $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution : Let $\sin^{-1}(-1) = y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$\therefore \quad \sin y = -1 \quad \Rightarrow \quad y = -\frac{\pi}{2}$.

Hence, the reqd. principal value $= -\frac{\pi}{2}$.

2. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Solution : Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y,\ 0 \le y \le \pi$.

$\therefore \quad \cos y = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \cos y = \cos\frac{\pi}{6}$

$\Rightarrow \quad y = \frac{\pi}{6}$.

Hence, the reqd. principal value $= \frac{\pi}{6}$.

3. $\mathrm{cosec}^{-1}(2)$.

[Solution : Refer Q. 7(*i*) ; Ex. 2(*a*)]

4. $\tan^{-1}(-\sqrt{3})$.

[Solution : Refer Q. 4(*i*) ; Ex. 2(*a*)]

5. $\cos^{-1}\left(-\frac{1}{2}\right)$.

[Solution : Refer Q. 3(*i*) ; Ex. 2(*a*)]

6. $\tan^{-1}(-1)$.

[Solution : Refer Q. 4(*ii*) ; Ex. 2(*a*)]

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

[Solution : Refer Q. 6(*i*) ; Ex. 2(*a*)]

8. $\cot^{-1}(\sqrt{3})$.

Solution : Let $\cot^{-1}(\sqrt{3}) = y,\ 0 < y < \pi$.

$\therefore \quad \cot y = \sqrt{3}$

$\Rightarrow \quad y = \frac{\pi}{6}$.

Hence, the reqd. principal value $= \frac{\pi}{6}$.

9. $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

[Solution : Refer Q. 3(*iii*) ; Ex. 2(*a*)]

10. $\mathrm{cosec}^{-1}(-\sqrt{2})$

[Solution : Refer Q. 7(*ii*) ; Ex. 2(*a*)]

Find the values of the following :

11. $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution : (I) Let $\tan^{-1}(1) = x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore \quad \tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4}$ ...(1)

(II) Let $\cos^{-1}\left(-\frac{1}{2}\right) = y, 0 \le y \le \pi$

$\therefore \quad \cos y = -\frac{1}{2} = -\cos\frac{\pi}{3}$

$= \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\Rightarrow \quad y = \frac{2\pi}{3}$ ...(2)

(III) Let $\sin^{-1}\left(-\frac{1}{2}\right) = z, -\frac{\pi}{2} \le z \le \frac{\pi}{2}$

$\therefore \quad \sin z = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad z = -\frac{\pi}{6}$ ...(3)

$\therefore \quad \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$ [*Using (1), (2) & (3)*]

$= \frac{1}{12}(3\pi + 8\pi - 2\pi) = \frac{9\pi}{12} = \frac{3\pi}{4}$.

12. $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

[Solution : Refer Q. 11 ; Ex. 2(*a*)].

13. If $\sin^{-1} x = y$, then :

(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

[Ans. (B)]

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to :

(A) π (B) $-\frac{\pi}{3}$

(C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$. **[Ans.** (B)]

Exercise 2.2

Prove the following :

1. $3\sin^{-1}x = \sin^{-1}(3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

[Solution : Refer Ex. 4 ; Page 2/12]

2. $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$

[Solution : Refer Q. 6(*ii*) ; Ex. 2(*b*)]

3. $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$. ***(H.P.B. 2016, 14)***

Solution : LHS = $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$

$= \tan^{-1}\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \times \frac{7}{24}}$

$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$

$= \tan^{-1}\frac{48+77}{264-14}$

$= \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2}$ = **RHS.**

4. $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$.

[Solution : Refer Q. 10 (*b*)(*iii*) ; Ex. 2(*b*)]

Write the following functions in the simplest form :

5. $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \ne 0$.

[Solution : Refer Q. 20 (*v*) ; Ex. 2(*b*)]

6. $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$.

[Solution : Refer Q. 20 (*vi*) ; Ex. 2(*b*)]

7. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x < \pi$.

Solution : LHS = $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$

$\left[\because 1 - \cos 2\theta = 2\sin^2\theta; 1 + \cos 2\theta = 2\cos^2\theta\right]$

$= \tan^{-1}\left(\sqrt{\tan^2\frac{x}{2}}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$.

8. $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), 0 < x < \pi$.

[Solution : Refer Q. 20 (*ii*) ; Ex. 2(*b*)]

9. $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}, |x| < a$.

[Solution : Refer Q. 20 (*iv*) ; Ex. 2(*b*)]

10. $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right), a>0; \dfrac{-a}{\sqrt{3}} \le x \le \dfrac{a}{\sqrt{3}}$.

(H.P.B. 2016, 13)

Solution : $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left(\dfrac{3\frac{x}{a}-\frac{x^3}{a^3}}{1-\frac{3x^2}{a^2}}\right)$.

Put $\dfrac{x}{a} = \tan\theta$ so that $\theta = \tan^{-1}\dfrac{x}{a}$.

$\therefore$ Given expression

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}\frac{x}{a}.$$

Find the values of each of the following :

11. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right]$.

Solution : $\tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right]$

$$= \tan^{-1}\left[2\cos\left(2.\frac{\pi}{6}\right)\right] = \tan^{-1}\left(2\cos\frac{\pi}{3}\right)$$

$$= \tan^{-1}\left(2.\frac{1}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

12. $\cot(\tan^{-1} a + \cot^{-1} a)$.

[Solution : Refer Q. 2 (*viii*) ; Ex. 2(*b*)]

13. $\tan\dfrac{1}{2}\left[\sin^{-1}\dfrac{2x}{1+x^2}+\cos^{-1}\dfrac{1-y^2}{1+y^2}\right], |x| < 1, y > 0$ and $xy < 1$.

[Solution. Refer Q. 15; Ex. 2(*b*)]

14. If $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right) = 1$, then find the value of 'x'.

Solution : $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right) = 1 = \sin\dfrac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2}-\sin^{-1}\frac{1}{5}$$

$$\Rightarrow \quad \cos^{-1}x = \cos^{-1}\frac{1}{5}$$

Hence, $\quad x = \dfrac{1}{5}$.

15. If $\tan^{-1}\dfrac{x-1}{x-2}+\tan\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$, then find the value of 'x'.

[Solution : Refer Q. 17(*iii*); Ex. 2(*b*)]

Find the values of each of the expressions in Exercise 16 to 18 :

16. $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$.

[Solution : Refer Q. 2(*i*); Ex. 2(*b*)]

17. $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$.

[Solution : Refer Q. 2(*iv*); Ex. 2(*b*)]

18. $\tan\left(\sin^{-1}\dfrac{3}{5}+\cot^{-1}\dfrac{3}{2}\right)$.

Solution : $\tan\left(\sin^{-1}\dfrac{3}{5}+\cot^{-1}\dfrac{3}{2}\right)$

$$= \tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$$

$$= \tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right) = \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}.$$

19. $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$ is equal to :

(A) $\dfrac{7\pi}{6}$ (B) $\dfrac{5\pi}{6}$

(C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{6}$. **[Ans. (B)]**

20. $\sin\left(\dfrac{\pi}{3}-\sin^{-1}\left(-\dfrac{1}{2}\right)\right)$ is equal to :

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$

(C) $\dfrac{1}{4}$ (D) 1. **[Ans. (D)]**

21. $\tan^{-1}(\sqrt{3})-\cot^{-1}(-\sqrt{3})$ is equal to :

(A) π (B) $-\dfrac{\pi}{2}$

(C) 0 (D) $2\sqrt{3}$. **[Ans. (B)]**

Miscellaneous Exercise on Chapter 2

Find the value of the following :

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Solution : $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{6}\right)\right)$

$= \cos^{-1}\left(\cos\frac{\pi}{6}\right) = \frac{\pi}{6}$.

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$.

[Solution : Refer Q. 2(*v*); Ex. 2(*b*)]

Prove that :

3. $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$.

[Solution : Refer Q. 5(*a*); Ex. 2(*b*)]

4. $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$.

Solution : Let **LHS** = θ.

Then $\sin\theta = \sin\left[\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)\right]$

$= \sin\left(\sin^{-1}\left(\frac{8}{17}\right)\right)\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

$+\cos\left(\sin^{-1}\left(\frac{8}{17}\right)\right)\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

$= \frac{8}{17}\sqrt{1-\frac{9}{25}}+\sqrt{1-\frac{64}{289}}\left(\frac{3}{5}\right)$

$= \frac{8}{17}\times\frac{4}{5}+\frac{15}{17}\times\frac{3}{5} = \frac{77}{85}$

$\Rightarrow \quad \theta = \sin^{-1}\frac{77}{85}$.

Hence, $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{85}$.

5. $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$.

[Solution : Refer Q. 12(*i*); Ex. 2(*b*)]

6. $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$.

[Solution : Refer Q. 12(*iii*); Ex. 2(*b*)]

7. $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$.

[Solution : Refer Q. 12(*iv*); Ex. 2(*b*)]

8. $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

[Solution : Refer Q. 11(*ii*); Ex. 2(*b*)]

Prove that :

9. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in[0,1]$.

[Solution : Refer Q. 8; Ex. 2(*b*)]

10. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x\in\left(0,\frac{\pi}{4}\right)$.

[Solution : Refer Q. 22 (*b*); Ex. 2(*b*)]

11. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, -\frac{1}{\sqrt{2}}\le x\le 1$.

[Hint : Put $x = \cos 2\theta$]

[Solution : Refer Ex. 25; Page 2/17]

12. $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$.

Solution. Putting

$\sin^{-1}\left(\frac{1}{3}\right) = x$ and $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y$, we get :

$\sin x = \frac{1}{3}$ and $\sin y = \frac{2\sqrt{2}}{3}$.

$\therefore \quad \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{1}{9}}$

$= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

and $\quad \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{8}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

Now $\sin(x+y) = \sin x\cos y+\cos x\sin y$

$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)+\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right)$

$= \frac{1}{9}+\frac{8}{9} = 1 = \sin\frac{\pi}{2}$

$\Rightarrow \quad x+y = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{9}{4}\left(\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right) = \frac{9}{4}\left(\frac{\pi}{2}\right)$

[*Multiplying by* $\frac{9}{4}$]

$\Rightarrow \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}$.

Hence, $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.

Solve the following equations :

13. $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)$.

[Solution : Refer Q. 17 (*vi*) ; Ex. 2(*b*)]

14. $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{1}{2}\tan^{-1}x, (x > 0)$.

[Solution : Refer Q. Ex. 17(*x*); Ex. 2(*b*)]

15. $\sin(\tan^{-1}x), |x| < 1$ is equal to :

(A) $\dfrac{x}{\sqrt{1-x^2}}$ (B) $\dfrac{1}{\sqrt{1-x^2}}$

(C) $\dfrac{1}{\sqrt{1+x^2}}$ (D) $\dfrac{x}{\sqrt{1+x^2}}$. **[Ans. (D)]**

16. $\sin(1-x) - 2\sin^{-1}x = \dfrac{\pi}{2}$, then x is equal to :

(A) $0, \dfrac{1}{2}$ (B) $1, \dfrac{1}{2}$

(C) 0 (D) $\dfrac{1}{2}$. **[Ans. (C)]**

17. $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan\dfrac{x-y}{x+y}$ is equal to :

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{3}$

(C) $\dfrac{\pi}{4}$ (D) $-\dfrac{3\pi}{4}$. **[Ans. (C)]**

Questions From NCERT Exemplar

Example 1. Prove that $\tan(\cot^{-1}x) = \cot(\tan^{-1}x)$.

State with reason whether the equality is valid for all values of x.

Solution. Let $\cot^{-1}x = \theta$. Then $\cot\theta = x$

$$\Rightarrow \tan\left(\frac{\pi}{2}-\theta\right) = x \Rightarrow \tan^{-1}x = \frac{\pi}{2}-\theta.$$

$\therefore \tan(\cot^{-1}x) = \tan\theta$

$$= \cot\left(\frac{\pi}{2}-\theta\right) = \cot\left(\frac{\pi}{2}-\cot^{-1}x\right)$$

$= \tan(\cot^{-1}x) = \cot(\tan^{-1}x)$.

The equality is valid for all values of x.

[$\because$ *$\tan^{-1}x$ and $\cot^{-1}x$ are true for $x \in \mathbf{R}$*]

Example 2. Find the value of $\tan(\cos^{-1}x)$ and hence evaluate $\tan\left(\cos^{-1}\dfrac{8}{17}\right)$.

Solution. Let $\cos^{-1}x = \theta$ so that $\cos\theta = x$, where $\theta \in [0, \pi]$.

$$\therefore \tan(\cos^{-1}x) = \tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}.$$

$$\text{Hence, } \tan\left(\cos^{-1}\frac{8}{17}\right) = \frac{\sqrt{1-\left(\frac{8}{17}\right)^2}}{\frac{8}{17}} = \frac{\frac{15}{17}}{\frac{8}{17}} = \frac{15}{8}.$$

Example 3. Prove that : $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$.

Solution. **LHS** $= \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$

$$= \left(\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}\right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right) + \tan^{-1}\left(\frac{1}{18}\right)$$

$$= \tan^{-1}\frac{3}{11} + \tan^{-1}\left(\frac{1}{18}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right) \qquad [\because xy < 1]$$

$$= \tan^{-1}\frac{65}{195} = \tan^{-1}\left(\frac{1}{3}\right) = \cot^{-1}3 = \textbf{RHS}.$$

Example 4. Find the value of :

$$\sin\left(2\tan^{-1}\frac{2}{3}\right) + \cos\left(\tan^{-1}\sqrt{3}\right).$$

Solution. Let $\tan^{-1}\dfrac{2}{3} = x$ and $\tan^{-1}\sqrt{3} = y$

so that $\tan x = \dfrac{2}{3}$ and $\tan y = \sqrt{3}$.

Now $\sin\left(2\tan^{-1}\dfrac{2}{3}\right) + \cos\left(\tan^{-1}\sqrt{3}\right)$

$$= \sin 2x + \cos y = \frac{2\tan x}{1+\tan^2 x} + \frac{1}{\sqrt{1+\tan^2 y}}$$

$$= \frac{2.\frac{2}{3}}{1+\frac{4}{9}} + \frac{1}{\sqrt{1+(\sqrt{3})^2}} = \frac{12}{13} + \frac{1}{2} = \frac{37}{26}.$$

Exercise

1. Find the value of $\cos^{-1}\left(\cos\dfrac{3\pi}{2}\right)$.

2. Evaluate : $\tan^{-1}\left\{\sin\left(\dfrac{-\pi}{2}\right)\right\}$.

3. Evaluate : $\sin^{-1}\left[\cos\left(\sin^{-1}\dfrac{\sqrt{3}}{2}\right)\right]$.

4. Find the value of $\sec\left(\tan^{-1}\dfrac{y}{2}\right)$ in terms of y.

5. Find the value of $\sin\left[2\cot^{-1}\left(-\dfrac{5}{12}\right)\right]$.

6. Which is greater $\tan 1$ or $\tan^{-1} 1$?

7. Evaluate : $\cos\left[\sin^{-1}\dfrac{1}{4}+\sec^{-1}\dfrac{4}{3}\right]$.

8. Find the value of the expression :
$\sin\left(2\tan^{-1}\dfrac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$.

9. Solve for x :
(*i*) $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$
(*ii*) $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\dfrac{3}{4}\right)$.

10. Find the solution of the equation :
$\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$.

Answers

1. $\dfrac{\pi}{2}$. 2. $-\dfrac{\pi}{4}$. 3. $\dfrac{\pi}{6}$. 4. $\dfrac{1}{2}\sqrt{4+y^2}$. 5. $-\dfrac{120}{169}$. 6. tan 1.

7. $\dfrac{3\sqrt{15}-\sqrt{7}}{16}$. 8. $\dfrac{14}{15}$. 9. (*i*) 0, $\dfrac{1}{2}$ (*ii*) $\dfrac{3}{4}$. 10. $\sqrt{3}$.

Revision Exercise

1. Show that :

(*i*) $\sec(\text{cosec}^{-1}x)=\dfrac{|x|}{\sqrt{x^2-1}}$, for $|x|>1$

(*ii*) $\cos(2\tan^{-1}x)=\dfrac{1-x^2}{1+x^2}$.

2. Prove that :

(*i*) $\sin(\tan^{-1}1)=\dfrac{1}{\sqrt{2}}$

(*ii*) $\text{cosec}\left[\tan^{-1}(-\sqrt{3})\right]=-\dfrac{2}{\sqrt{3}}$.

3. Solve : $3\tan^{-1}\left(\dfrac{1}{2+\sqrt{3}}\right)-\tan^{-1}\left(\dfrac{1}{x}\right)=\tan^{-1}\left(\dfrac{1}{3}\right)$.

4. Solve : $\sin^{-1}\left(\dfrac{5}{x}\right)+\sin^{-1}\left(\dfrac{12}{x}\right)=\dfrac{\pi}{2}$.

Solution. The given equation is :

$$\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2} \quad ...(1)$$

$$\Rightarrow \sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}-\sin^{-1}\left(\frac{5}{x}\right)$$

$$\Rightarrow \frac{12}{x}=\sin\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{5}{x}\right)\right]$$

$$\Rightarrow \frac{12}{x}=\cos\left(\sin^{-1}\left(\frac{5}{x}\right)\right)=\sqrt{1-\left(\frac{5}{x}\right)^2}$$

$$\left[\because \cos(\sin^{-1}x)=\sqrt{1-x^2}, |x|<1\right]$$

$$\Rightarrow \left(\frac{12}{x}\right)^2=1-\left(\frac{5}{x}\right)^2 \Rightarrow 144=x^2-25$$

$$\Rightarrow x^2=169.$$

Hence, $x=13$. [$\because x=-13$ *does not satisfy (1)*]

5. Show that : (*i*) $\sin^{-1}\left[\sin\dfrac{3\pi}{4}\right]\neq\dfrac{3\pi}{4}$.

What is its value ?

(*ii*) $\tan^{-1}\left[\tan\dfrac{5\pi}{6}\right]\neq\dfrac{5\pi}{6}$.

What is its value ?

6. Prove that :

(*i*) $\tan^{-1} x + \cot^{-1}(x+1) = \tan^{-1}(x^2 + x + 1)$

(*ii*) $\cot^{-1} 3 + \text{cosec}^{-1} \sqrt{5} = \dfrac{\pi}{4}$.

7. Prove that :

$\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3) = 15$.

8. Prove that

$$2\tan^{-1}\left[\sqrt{\left(\frac{a-b}{a+b}\right)}\frac{\tan\theta}{2}\right] = \cos^{-1}\left(\frac{a\cos\theta+b}{a+b\sin\theta}\right)$$

(Assam B. 2017)

Answers

3. $x = 2$.

5. (*i*) $\dfrac{\pi}{4}$ (*ii*) $-\dfrac{\pi}{6}$.

Hints to Selected Questions

1. (*i*) Put $\text{cosec}^{-1} x = \theta$ (*ii*) $\tan^{-1} x = \theta$.

3. $\tan^{-1}\dfrac{1}{2+\sqrt{3}} = \tan^{-1}\dfrac{2-\sqrt{3}}{4-3} = \tan^{-1}(2-\sqrt{3}) = 15°$.

7. Put $\tan^{-1} 2 = \theta$ and $\cot^{-1} 3 = \phi$

so that $\tan\theta = 2$ and $\cot\phi = 3$.

LHS $= \sec^2\theta + \text{cosec}^2\theta = (1+\tan^2\theta) + (1+\cot^2\phi) = (1+4)+(1+9) = 15$.

CHECK YOUR UNDERSTANDING

1. Write the domain of $f(x) = \tan^{-1} x$. *(Jammu B. 2015 W)*

Ans. **R**.

2. Write the principal value of $\sin^{-1}\left(\dfrac{1}{2}\right)$. *(Jammu B. 2016)*

Ans. $\dfrac{\pi}{6}$.

3. Write the principal value of $\tan^{-1}(\sqrt{3}) + \text{cosec}^{-1}(-2)$.

Ans. $\dfrac{\pi}{6}$.

4. If $\sec^{-1}(x) + \text{cosec}^{-1}\left(\dfrac{1}{3}\right) = \dfrac{\pi}{2}$, then find x.

Ans. $x = \dfrac{1}{3}$.

5. Is $\cos^{-1}(-x) = \pi - \cos^{-1} x,\ x \in [-1, 1]$? (State True/False)

Ans. True.

6. Find the value of $\sin(\sec^{-1} x + \text{cosec}^{-1} x), |x| > 1$.

Ans. 1.

7. Write down the value of $\tan^{-1} x + \cot^{-1} x, x \in \mathbf{R}$.

Ans. $\dfrac{\pi}{2}$.

8. Write down the value of $2\sin^{-1}\dfrac{3}{5}$.

Ans. $\sin^{-1}\dfrac{24}{25}$.

9. Find the solution of $\tan^{-1} 2x + \tan^{-1} 3x = \dfrac{\pi}{4},\ x > 0$.

Ans. $\dfrac{1}{6}$.

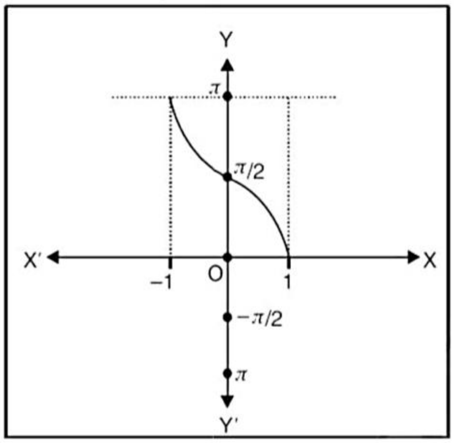
10. Is $\tan^{-1}\left(\sqrt{\dfrac{1-x^2}{1+x^2}}\right) = \dfrac{1}{2}\cos^{-1} x$ true ?

Ans. Yes.

SUMMARY INVERSE-TRIGONOMETRIC FUNCTIONS

Inverse Trigonometric Functions

Inverse Trigonometric Functions



Properties of inverse trigonometric functions

1. TABLE

Function	Principal Value Branch	
	Domain	Range
(i) $y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii) $y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii) $y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv) $y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
(v) $y = \sec^{-1} x$	$1 \le x < \infty$	$0 \le y < \frac{\pi}{2}$
	$-\infty < x \le -1$	$\frac{\pi}{2} < y \le \pi$
(vi) $y = \text{cosec}^{-1} x$	$1 \le x < \infty$	$0 < y \le \frac{\pi}{2}$
	$-\infty < x \le -1$	$-\frac{\pi}{2} \le y < 0$

2. Properties :

(a) $x = \sin^{-1}(\sin x) = \cos^{-1}(\cos x) = \tan^{-1}(\tan x)$; etc.

(b) (i) $\text{cosec}^{-1} x = \sin^{-1}\frac{1}{x}, x \ge 1$ or $x \le -1$

(ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}, x \ge 1$ or $x \le -1$

(iii) $\cot^{-1} x = \tan^{-1}\frac{1}{x}, x > 0$.

(c) (i) $\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in \mathbf{R}$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in \mathbf{R}$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \ge 1$

(vi) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x, |x| \ge 1$.

(d) (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \; x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \; x \in \mathbf{R}$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}, \; |x| \ge 1$

(iv) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}, xy < 1$

(v) $\tan^{-1} x - \tan^{-1} y = \frac{x-y}{1+xy}, xy > -1$

(vi) $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x}$

(e) (i) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

(ii) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$.

MULTIPLE CHOICE QUESTIONS

For Board Examinations

1. If $\cos^{-1} x = y$, then :

(A) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ (B) $-\pi \le y \le \pi$

(C) $0 \le y \le \frac{\pi}{2}$ (D) $0 \le y \le \pi$. *(P.B. 2018)*

2. The principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is :

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$ *(H.P.B. 2018)*

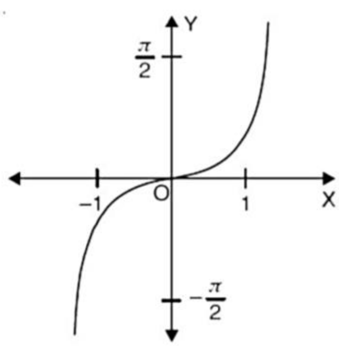
3. The value of $\sin\left(\cos^{-1}\frac{3}{5}\right)$ is :

(A) $\frac{4}{5}$ (B) $\frac{3}{5}$

(C) $\frac{2}{5}$ (D) None of these.

(H.B. 2018)

4. Identify the function from the following figure :



(A) $\tan^{-1} x$ (B) $\sin^{-1} x$

(C) $\cos^{-1} x$ (D) $\text{cosec}^{-1} x$.

(Kerela B. 2018)

5. $\sin^{-1}\left(\frac{1}{2}\right)$ is equal to :

(A) 0 (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{3}$ *(P.B. 2017)*

6. (*i*) Principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is :

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$

(C) $-\frac{\pi}{3}$ (D) $\frac{\pi}{6}$ *(H.P.B. 2017)*

7. (*i*) $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ is equal to :

(A) $-\frac{\pi}{2}$ (B) $2\sqrt{3}$

(C) 0 (D) π *(H.B. 2017)*

8. Principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is :

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{2\pi}{3}$ (D) π *(H.B. 2017)*

9. The value of $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ is :

(A) $\frac{4\pi}{3}$ (B) $\frac{2\pi}{3}$

(C) 0 (D) π. *(Nagaland B. 2017)*

10. The value of $\sin^{-1}\left\{\sin\left(\frac{2\pi}{3}\right)\right\}$ is :

(A) $\frac{2\pi}{3}$ (B) $\frac{\pi}{3}$

(C) $-\frac{\pi}{3}$ (D) $-\frac{2\pi}{3}$.

(Mizoram B. 2017)

11. If $\sin^{-1} x = y$, then :

(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

(H.P.B. 2016)

12. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ equals :

(A) $\frac{3\pi}{4}$ (B) $-\frac{\pi}{4}$

(C) $\frac{\pi}{4}$ (D) None of these.

(H.P.B. 2016)

13. Principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is :

(A) $\frac{5\pi}{6}$ (B) $\frac{\pi}{6}$

(C) $-\frac{\pi}{6}$ (D) $-\frac{5\pi}{6}$. *(P.B. 2016)*

14. If $\sec^{-1} x = \text{cosec}^{-1} y$, then the value of

$\cos^{-1}\dfrac{1}{x} + \cos^{-1}\dfrac{1}{y}$ will be :

(A) π (B) $\dfrac{2\pi}{3}$

(C) $\dfrac{5\pi}{6}$ (D) $\dfrac{\pi}{2}$. *(W. Bengal B. 2016)*

15. The value of $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right\}$ is :

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{3}$

(C) $\dfrac{\pi}{4}$ (D) $\dfrac{\pi}{6}$. *(Mizoram B. 2016)*

RCQ Pocket

(Single Correct Answer Type)

(JEE-Main and Advanced)

16. If $\cos^{-1} x - \cos^{-1}\dfrac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to :

(A) 4 (B) $2\sin^2\alpha$

(C) $-4\sin^2\alpha$ (D) $4\sin^2\alpha$.

(A.I.E.E.E. 2005)

17. If $\sin^{-1}\left(\dfrac{x}{5}\right) + \text{cosec}^{-1}\left(\dfrac{5}{4}\right) = \dfrac{\pi}{2}$, then a value of x is :

(A) 3 (B) 4

(C) 5 (D) 1. *(A.I.E.E.E. 2007)*

18. The value of $\cot\left(\text{cosec}^{-1}\dfrac{5}{3} + \tan^{-1}\dfrac{2}{3}\right)$ is :

(A) $\dfrac{5}{17}$ (B) $\dfrac{6}{17}$

(C) $\dfrac{3}{17}$ (D) $\dfrac{4}{17}$.

(A.I.E.E.E. 2008)

19. If $0 < x < 1$, then :

$$\sqrt{1+x^2}\left[\left\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x\right\}^2 - 1\right]^{1/2} =$$

(A) $\dfrac{x}{\sqrt{1+x^2}}$ (B) x

(C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$.

(I.I.T. 2008)

20. The value of $\cot\left\{\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right\}$ is :

(A) $\dfrac{23}{25}$ (B) $\dfrac{25}{23}$

(C) $\dfrac{23}{24}$ (D) $\dfrac{24}{23}$.

(I.I.T. (Advanced) 2013)

21. If $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, when $|x| < \dfrac{1}{\sqrt{3}}$, then a value of y is :

(A) $\dfrac{3x - x^3}{1 - 3x^2}$ (B) $\dfrac{3x + x^3}{1 - 3x^2}$

(C) $\dfrac{3x - x^3}{1 + 3x^2}$ (D) $\dfrac{3x + x^3}{1 + 3x^2}$.

(J.E.E. (Main) 2015)

Answers

1. (D) **2.** (D) **3.** (A) **4.** (B) **5.** (B) **6.** (B) **7.** (A) **8.** (C) **9.** (B) **10.** (B)
11. (B) **12.** (B) **13.** (C) **14.** (D) **15.** (C) **16.** (D) **17.** (A) **18.** (B) **19.** (C) **20.** (B)
21. (A).

Hints/Solutions

RCQ Pocket

16. (D) We have : $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$

$$\Rightarrow x = \cos\left(\cos^{-1}\frac{y}{2} + \alpha\right)$$

$$= \cos\left(\cos^{-1}\frac{y}{2}\right)\cos\alpha - \sin\left(\cos^{-1}\frac{y}{2}\right)\sin\alpha$$

$$= \frac{y}{2}\cos\alpha - \sqrt{1-\frac{y^2}{4}}\sin\alpha$$

$$\Rightarrow 2x = y\cos\alpha - \sin\alpha\sqrt{4-y^2}$$

$$\Rightarrow 2x - y\cos\alpha = -\sin\alpha\sqrt{4-y^2}.$$

Squaring, $4x^2 + y^2\cos^2\alpha - 4xy\cos\alpha$

$$= 4\sin^2\alpha - y^2\sin^2\alpha$$

$$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha.$$

17. (A) $\sin^{-1}\frac{x}{5} + \operatorname{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1}\frac{x}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\frac{x}{5} = \frac{\pi}{2} - \sin^{-1}\frac{4}{5} = \cos^{-1}\frac{4}{5} = \sin^{-1}\frac{3}{5}$$

$$\Rightarrow x = 3.$$

18. (B) $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$

$$= \cot\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$$

$$= \cot\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right) = \cot\left(\tan^{-1}\frac{17}{6}\right)$$

$$= \cot\left(\cot^{-1}\frac{6}{17}\right) = \frac{6}{17}.$$

19. (C)
Here

$$\sqrt{1+x^2}\left[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\frac{x}{\sqrt{1+x^2}}\right) + \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right)^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\,(x^2+1-1)^{1/2} = x\sqrt{1+x^2}.$$

20. (B) Let $\cot\sum_{n=1}^{23}\left\{\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right\}$

$$= \cot\sum_{n=1}^{23}\cot^{-1}(1+2(1+2+......+n))$$

$$= \cot\sum_{n=1}^{23}\cot^{-1}\left(1+2\frac{n(n+1)}{2}\right)$$

$$= \cot\sum_{n=1}^{23}\cot^{-1}(1+n(n+1))$$

$$= \cot\sum_{n=1}^{23}\tan^{-1}\left(\frac{1}{1+n(n+1)}\right)$$

$$= \cot\sum_{n=1}^{23}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)$$

$$= \cot\sum_{n=1}^{23}\left(\tan^{-1}(n+1) - \tan^{-1}n\right)$$

$$= \cot[\{(\tan^{-1}(2) - \tan^{-1}(1)\} + \{(\tan^{-1}(3) - \tan^{-1}(2))\} + ...+ \{\tan^{-1}(24) - \tan^{-1}(23)\}$$

$$= \cot[\tan^{-1}(24) - \tan^{-1}(1)] = \cot\left[\tan^{-1}\frac{24-1}{1+(24)(1)}\right]$$

$$= \cot\left[\tan^{-1}\frac{23}{25}\right] = \cot\left[\cot^{-1}\frac{25}{23}\right] = \frac{25}{23}.$$

21. (A) We have : $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$= \tan^{-1}\frac{x+\frac{2x}{1-x^2}}{1-x.\frac{2x}{1-x^2}} = \tan^{-1}\frac{3x-x^3}{1-3x^2}.$$

Hence, $y = \dfrac{3x-x^3}{1-3x^2}.$