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CBSE

COMPLEX NUMBER AND QUADRATIC EQUATION

Imaginary Numbers

The square root of a negative real number is called imaginary number, e.g. $\sqrt{-2}, \sqrt{-5}$ etc.

The quantity $\sqrt{-1}$ is an imaginary unit and it is denoted by 'i' called **iota**.

Integral Power of IOTA (i)

$$i=\sqrt{-1}, i^2=-1, i^3=-i, i^4=1$$

So, $i^{4n+1}=i, i^{4n+2}=-1, i^{4n+3}=-i, i^{4n}=1$

- For any two real numbers a and b, the result $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ is true only, when atleast one of the given numbers is either zero or positive.
  $\sqrt{-a}\times\sqrt{-b}\neq\sqrt{ab}$ So, $i^2=\sqrt{-1}\times\sqrt{-1}\neq 1$
- 'i' is neither positive, zero nor negative.
- $i^n+i^{n+1}+i^{n+2}+i^{n+3}=0$

Complex Numbers

A number of the form $x+iy$, where x and y are real numbers, is called a complex number. Here, x is called real part and y is called imaginary part of the complex number, i.e. $\text{Re}(z)=x$ and $\text{Im}(z)=y$.

Purely Real and Purely Imaginary Complex Numbers

A complex number $z=x+iy$ is a purely real if its imaginary part is 0, i.e. $\text{Im}(z)=0$ and purely imaginary if its real part is 0 i.e. $\text{Re}(z)=0$.

Equality of Complex Numbers

Two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ are said to be equal, iff $x_1=x_2$ and $y_1=y_2$
i.e. $\text{Re}(z_1)=\text{Re}(z_2)$ and $\text{Im}(z_1)=\text{Im}(z_2)$
Other relation 'greater than' and 'less than' are not defined for complex number.

Algebra of Complex Numbers

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ be any two complex numbers.

(i) **Addition of Complex Numbers**

$$z_1+z_2=(x_1+iy_1)+(x_2+iy_2)$$
$$=(x_1+x_2)+i(y_1+y_2)$$

Properties of addition

- **Closure** z_1+z_2 is also a complex number.
- **Commutative** $z_1+z_2=z_2+z_1$
- **Associative** $z_1+(z_2+z_3)=(z_1+z_2)+z_3$
- **Existence of additive identity**
  $$z+0=z=0+z$$
  Here, 0 is additive identity.
- **Existence of Additive inverse**
  $$z+(-z)=0=(-z)+z$$
  Here, $-z$ is additive inverse.

(ii) **Subtraction of Complex Numbers**

$$z_1-z_2=(x_1+iy_1)-(x_2+iy_2)$$
$$=(x_1-x_2)+i(y_1-y_2)$$

(iii) **Multiplication of Complex Numbers**

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Properties of multiplication

- **Closure** $z_1 z_2$ is also a complex number.
- **Commutative** $z_1 z_2 = z_2 z_1$
- **Associative** $z_1 (z_2 z_3) = (z_1 z_2) z_3$
- **Existence of multiplicative identity**

$$z \cdot 1 = z = 1 \cdot z$$

Here, 1 is multiplicative identity.

- **Existence of multiplicative inverse** For every non-zero complex number z, there exists a complex number z_1 such that $z \cdot z_1 = 1 = z_1 \cdot z$
- **Distributive law** $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

(iv) **Division of Complex Numbers**

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

where, $z_2 \neq 0$

Conjugate of a Complex Number

Let $z = x + iy$, if 'i' is replaced by $(-i)$, then it is said to be conjugate of the complex number z and denoted by $\bar{z}$, i.e. $\bar{z} = x - iy$.

Properties of Conjugate

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}(z)$, $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z = \bar{z}$, if z is purely real

(iv) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(v) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vi) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(vii) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

(ix) $z \cdot \bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2$

(x) $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2\,\text{Re}(\bar{z}_1 z_2) = 2\,\text{Re}(z_1 \bar{z}_2)$

(xi) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

(xii) $(\bar{z})^n = \overline{(z^n)}$

Modulus (Absolute Value) of a Complex Number

Let $z = x + iy$ be a complex number. Then, the positive square root of the sum of square of real part and square of imaginary part is called modulus (absolute values) of z and it is denoted by $|z|$ i.e. $|z| = \sqrt{x^2 + y^2}$.

Properties of Modulus

(i) $|z| \geq 0$

(ii) If $|z| = 0$, then $z = 0$ i.e. $\text{Re}(z) = 0 = \text{Im}(z)$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im} z) \leq |z|$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(v) $z \cdot \bar{z} = |z|^2$

(vi) $|z_1 z_2| = |z_1||z_2|$

(vii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \bar{z}_2)$

(ix) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1 \bar{z}_2)$

(x) $|z_1 + z_2| \leq |z_1| + |z_2|$

(xi) $|z_1 - z_2| \geq |z_1| - |z_2|$

Argand Plane

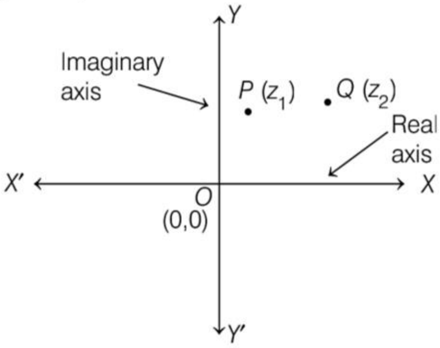
A complex number $z = a + ib$ can be represented by a unique point $P(a, b)$ in the cartesian plane referred to a pair of rectangular axes.

A purely real number a, i.e. $(a + 0i)$ is represented by the point $(a, 0)$ on X-axis. Therefore, X-axis is called **real axis**.

A purely imaginary number ib i.e. $(0 + ib)$ is represented by the point $(0, b)$ on Y-axis. Therefore, Y-axis is called **imaginary axis.** The intersection (common) of two axes is called zero complex number i.e. $z = 0 + 0i$.

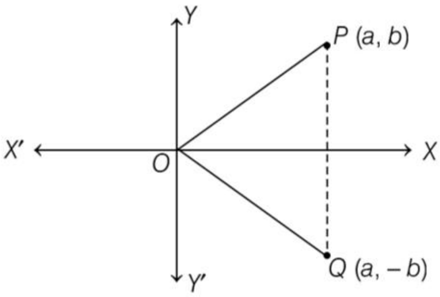
Similarly, the representation of complex numbers as points in the plane is known as **argand diagram**. The plane representing complex numbers as points, is called **complex plane** or **argand plane** or **gaussian plane.**

If two complex numbers z_1 and z_2 are represented by the points P and Q in the complex plane, then $|z_1 - z_2| = PQ$ = Distance between P and Q



Representation of Conjugate of z on Argand Plane

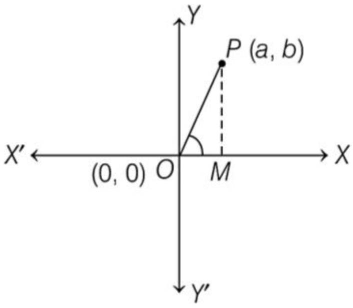
Geometrically, the mirror image of the complex number $z = a + ib$ (represented by the ordered pair (a, b) about the X-axis is called **conjugate of z** which is represented by the ordered pair $(a, -b)$.
If $z = a + ib$, then $\bar{z} = a - ib$.



Representation of Modulus of z on Argand Plane

Geometrically, the distance of the complex number $z = a + ib$ [represented by the ordered pair (a, b)] from origin, is called the modulus of z.

$$\therefore \quad OP = \sqrt{(a-0)^2 + (b-0)^2}$$
$$= \sqrt{a^2 + b^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2} = |a + ib|$$



Quadratic Equation

An equation of the form $ax^2 + bx + c, a \neq 0$ is called quadratic equation in variable x, where a, b and c are numbers (real or complex).

Nature of Roots of Quadratic Equation

The roots of quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Now, if we look at these roots of quadratic equation $ax^2 + bx + c = 0$; $a \neq 0$, we observe that the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is known as the **discriminant** of the quadratic equation and denoted by D.

There are following four cases arise :

***Case* I** If $b^2 - 4ac = 0$ i.e. $D = 0$,
then $\alpha = \beta = -\dfrac{b}{2a}$.

Thus, if $b^2 - 4ac = 0$, then the quadratic equation has real and equal roots and each equal to $-b / 2a$.

***Case* II** If a, b and c are rational numbers and $b^2 - 4ac > 0$ and it is a perfect square, then $D = \sqrt{b^2 - 4ac}$ is a rational number and hence α and β are rational and unequal.

***Case* III** If $b^2 - 4ac > 0$ and it is not a perfect square, then roots are irrational and unequal.

***Case* IV** If $b^2 - 4ac < 0$, then the roots are complex conjugate of each other.

Quadratic Equations with Real Coefficients

Let us consider the following quadratic equation $ax^2 + bx + c = 0$ with real coefficients a, b, c and $a \neq 0$. Also, let us assume that $b^2 - 4ac < 0$. Now, we can find the square root of negative real numbers in the set of complex numbers.
Therefore, the solutions of the above equation are available in the set of complex numbers which are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{-(4ac - b^2)}}{2a}$$
$$= \frac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$$

SECTION A NCERT EXERCISES

EXERCISE 5.1

Express each of the complex number given in exercise 1 to 7 in the form $a + ib$

1. $(5i)\left(-\frac{3}{5}i\right)$

Sol. $(5i)\left(-\frac{3}{5}i\right) = -3i^2 = 3(-1) = 3$

2. $i^9 + i^{19}$

Sol. $i^9 + i^{19} = (i^2)^4.i + (i^2)^9.i$

$= (-1)^4 i + (-1)^9 i$

$= i - i = 0$

3. i^{-39}

Sol. $(i)^{-39} = (i^2)^{-19} i^{-1} = (-1)^{-19}. i^{-1}$

$$= \frac{1}{(-1)^{19}} \times \frac{1}{i} = -\frac{1}{i} \times \frac{i}{i} = \frac{-i}{(i^2)} = \frac{-i}{-1} = i$$

4. $3(7 + i7) + i(7 + i7)$

Sol. $3(7 + i7) + i(7 + i7) = 21 + 21i + 7i + 7i^2$

$= 21 + 28i + 7(-1)$

$= 14 + 28i.$

5. $(1 - i) - (-1 + i6)$

Sol. $(1 - i) - (-1 + i.6) = (1 - i) + (1 - 6i)$

$= 1 + 1 - i - 6i$

$= 2 - 7i = (a + ib),$

where $a = 2, b = -7$

6. $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$

Sol. $\left(\frac{1}{5}+i.\frac{2}{5}\right)-\left(4+i.\frac{5}{2}\right)=\left(\frac{1}{5}+\frac{2}{5}i\right)+\left(-4-\frac{5}{2}i\right)$

$$=\frac{1}{5}-4+\frac{2}{5}i-\frac{5}{2}i=-\frac{19}{5}-\left(-\frac{2}{5}+\frac{5}{2}\right)i$$

$$=-\frac{21}{5}-\frac{21}{10}i$$

7. $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i.\frac{1}{3}\right)-\left(-\frac{4}{3}+i\right)\right]$

Sol. $\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)-\left(-\frac{4}{3}+i\right)$

$$=\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)+\left(\frac{4}{3}-i\right)$$

$$=\left(\frac{1}{3}+4+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-1\right) = \frac{17}{3}+i.\frac{5}{3}$$

$$= (a + ib) \text{ where } a = \frac{17}{3}, b = \frac{5}{3}$$

8. **Express each of the complex number $(1 - i)^4$ in the form $a + ib$.**

Sol. $(1 - i)^4 = [(1 - i)^2]^2$

$= (1 + i^2 - 2i)^2 = (1 - 1 - 2i)^2$

$= (-2i)^2 = 4i^2 = 4(-1) = -4$

9. **Express the complex number $\left(\frac{1}{3}+3i\right)^3$ in the form $a + ib$.**

Sol. $\left(\frac{1}{3}+3i\right)^3=\left(\frac{1}{3}\right)^3+3\left(\frac{1}{3}\right)^2(3i)+3.\left(\frac{1}{3}\right)(3i)^2+(3i)^3$

$$=\frac{1}{27}+i+9(-1)+27i^3$$

$$=\frac{1}{27}+i+9(-1)+27i.(i^2)$$

$$=\frac{1}{27}+i+9(-1)+27i.(-1)$$

$$=\frac{1}{27}+i-9-27i \quad =-\frac{242}{27}-26i$$

10. Express $\left(-2-\frac{1}{3}i\right)^3$ of the complex number in the form $a+ib$

Sol. $\left(-2-\frac{1}{3}i\right)^3$

$$=(-2)^3-3(-2)^2.\left(\frac{1}{3}i\right)+3(-2)\left(-\frac{1}{3}i\right)^2-\left(\frac{1}{3}i\right)^3$$

$$=-8-4i-6\times\frac{1}{9}(i^2)-\frac{1}{27}i^3$$

$$=-8-4i-\frac{2}{3}(-1)-\frac{1}{27}i.i^2$$

$$=-8-4i+\frac{2}{3}-\frac{1}{27}i.(-1)$$

$$=-8-4i+\frac{2}{3}+\frac{1}{27}i$$

$$=-\frac{22}{3}-\frac{107}{27}i$$

11. Find the multiplicative inverse of $4-3i$

Sol. We have multiplicative inverse of $4-3i$

$$=\frac{1}{4-3i}\times\frac{4+3i}{4+3i}$$

$$=\frac{4+3i}{4^2-9i^2}=\frac{4+3i}{16+9}=\frac{4+3i}{25}=\frac{4}{25}+i\frac{3}{25}$$

12. Find the multiplicative inverse of $(\sqrt{5}+3i)$.

Sol. We have multiplicative inverse of $\sqrt{5}+3i$

$$=\frac{1}{\sqrt{5}+3i}\times\frac{\sqrt{5}-3i}{\sqrt{5}-3i} \text{ (multiply by conjugate)}$$

$$=\frac{\sqrt{5}-3i}{5-9i^2}=\frac{\sqrt{5}-3i}{5+9}=\frac{\sqrt{5}-3i}{14}=\frac{\sqrt{5}}{14}-\frac{3}{14}i$$

$[\because (a+ib)(a-ib)=a^2+b^2]$

13. Find the multiplicative inverse of $-i$

Sol. We have multiplicative inverse of $-i = \frac{1}{-i}$.

Multiply by conjugate

$$=\frac{1}{-i}\times\frac{i}{i}=\frac{-i}{i^2}=\frac{-i}{-1}=i=0+i.1$$

14. Express the following expression in the form of $a+ib$: $\dfrac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+i\sqrt{2})-(\sqrt{3}-i\sqrt{2})}$

Sol. We have $\dfrac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+i\sqrt{2})-(\sqrt{3}-i\sqrt{2})}$

$$=\frac{(3)^2-(i\sqrt{5})^2}{\sqrt{3}+i\sqrt{2}-\sqrt{3}+i\sqrt{2}}$$

$$=\frac{9-5i^2}{2\sqrt{2}i}=\frac{9-5(-1)}{2\sqrt{2}i}=\frac{14}{2\sqrt{2}i}$$

$$=\frac{7}{\sqrt{2}i}\times\frac{\sqrt{2}i}{\sqrt{2}i}=\frac{7\sqrt{2}i}{2i^2}=-\frac{7\sqrt{2}i}{2}=0-\frac{7\sqrt{2}}{2}i$$

EXERCISE 5.2

1. Find the modulus and the argument of the complex number $z=-1-i\sqrt{3}$

Sol. Let $z=-1-i\sqrt{3}$

Here $x=-1$, $y=-\sqrt{3}$

$$\therefore \alpha=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right|$$

$$=\tan^{-1}\tan\frac{\pi}{3}=\frac{\pi}{3}$$

Since $x<0$, and $y<0$

$$\therefore \arg|i|=-(\pi-\alpha)=-(\pi-\pi/3)=-2\pi/3$$

$$|z|=\sqrt{(-1)^2+(-\sqrt{3})^2}=\sqrt{1+3}=2$$

2. Find the modulus and the argument of $z=-\sqrt{3}+i$

Sol. We know that the polar form of

$z=r(\cos\theta+i\sin\theta)$

$\therefore$ Let $-\sqrt{3}+i=r(\cos\theta+i\sin\theta)$

$\Rightarrow r\cos\theta = -\sqrt{3}$ and $r\sin\theta - 1$

By squaring and adding, we get

$r^2(\cos^2\theta + \sin^2\theta) = (\sqrt{3})^2 + 1$

$r^2 . 1 = 4 \quad \therefore\ r = 2$

By dividing, $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{-1}{\sqrt{3}}$

i.e θ lies in second quadrant

$\Rightarrow \quad \tan\theta = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = 180° - 30° = 150° \Rightarrow \theta = \dfrac{5\pi}{6}$

$\therefore \quad |z| = 2$ and $\arg z = \dfrac{5\pi}{6}$

3. Convert (1 – *i*) in the polar form.

Sol. We have $1 - i = r(\cos\theta + i\sin\theta)$

$\Rightarrow r\cos\theta = 1,\ r\sin\theta = -1$

By squaring and adding, we get

$r^2(\cos^2\theta + \sin^2\theta) = 1^2 + (-1)^2$

$\Rightarrow r^2 . 1 = 1 + 1 \Rightarrow r^2 = 2$

$\therefore \quad r = \sqrt{2}$, By dividing $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{-1}{1} = -1$

$\Rightarrow \quad \tan\theta = -1$ i.e., θ lies in fourth quadrant.

$\Rightarrow \quad \theta = -45 \Rightarrow \theta = -\dfrac{\pi}{4}$

$\therefore$ Polar form of $1 - i$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

4. Convert (–1 + *i*) in the polar form.

Sol. We have $-1 + i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad r\cos\theta = -1$ and $r\sin\theta = 1$

By squaring and adding, we get

$r^2(\cos^2\theta + \sin^2\theta) = (-1)^2 + 1^2 \Rightarrow r^2 . 1 = 1 + 1$

$\therefore \quad r^2 = 2 \quad \therefore \quad r = \sqrt{2}$

By dividing, $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{1}{-1} = -1 \Rightarrow \tan\theta = -1$

$\therefore \quad \theta$ lies in second quadrant ;

$\theta = 180° - 45° = 135°$ i.e $\theta = \dfrac{3\pi}{4}$

$\therefore$ Polar form of $-1 + i$

$$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

5. Convert (–1 – *i*) in the polar form.

Sol. We have $-1 - i = r(\cos\theta + i\sin\theta)$

$\Rightarrow r\cos\theta = -1$ and $r\sin\theta = -1$

By squaring and adding, we get

$r^2(\cos^2\theta + \sin^2\theta) = (-1)^2 + (-1)^2$

$\Rightarrow \quad r^2 . 1 = 1 + 1 \Rightarrow r^2 = 2$

$\therefore \quad r = \sqrt{2}$

By dividing $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{-1}{-1} = 1 \quad \Rightarrow \quad \tan\theta = 1$

$\therefore$ θ lies in IIIrd quadrant.

$\theta = -180° + 45° = -135°$ or $\theta = -\dfrac{3\pi}{4}$

$\therefore$ Polar form of $-1 - i$

$$= \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

Convert each of the complex numbers given in Exercises 6 to 8 in the polar form.

6. – 3

Sol. $z = -3 = r(\cos\theta + i\sin\theta)$

$\therefore \quad r\cos\theta = -3,\ r\sin\theta = 0$

Squaring and adding $r^2 = (-3)^2 \therefore r = 3$

$\tan\theta = 0 \therefore \theta = \pi \because \cos\pi = 0$

$\therefore \quad -3 = 3(\cos\pi + i\sin\pi)$

7. $\sqrt{3} + i$

Sol. $r = \sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$\therefore \quad r\cos\theta = \sqrt{3},\ r\sin\theta = 1$

Squaring and adding $r^2 = 3 + 1 = 4,\ r = 2$

Also $\tan\theta = \dfrac{1}{\sqrt{3}}$, $\sin\theta$ and $\cos\theta$ both are positive.

$\therefore \theta$ lies in the I quadrant

$\therefore \quad \theta = 30° = \dfrac{\pi}{6}$

$\therefore$ Polar form of z is $2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

8. i

Sol. $z = i = r(\cos\theta + i\sin\theta)$

$\therefore \quad r\cos\theta = 0,\ r\sin\theta = 1$

Squaring and adding $r^2 = 1,\ \therefore r = 1$

Now $\sin\theta = 1, \cos\theta = 0$ at $\theta = \frac{\pi}{2}$

$\therefore$ Polar form of z is $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

EXERCISE 5.3

Solve each of the following equations :

1. $x^2 + 3 = 0$

Sol. $x^2 + 3 = 0 \therefore x^2 = -3$

$\therefore x = \pm\sqrt{-3} = \pm\sqrt{3}i$

2. $2x^2 + x + 1 = 0$

Sol. $2x^2 + x + 1 = 0$ Comparing with $ax^2 + bx + c = 0$

$a = 2, b = 1, c = 1$

$b^2 - 4ac = 1^2 - 4.2.1 = 1 - 8 = -7$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{-7}}{2.2}$$

$$= \frac{-1 \pm \sqrt{7}i}{4}$$

3. $x^2 + 3x + 9 = 0$

Sol. $x^2 + 3x + 9 = 0 \therefore a = 1, b = 3, c = 9$

$b^2 - 4ac = 3^2 - 4.1.9 = 9 - 36 = -27$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{-27}}{2\times 1}$$

$$= \frac{-3 \pm (3\sqrt{3})i}{2}$$

4. $-x^2 + x - 2 = 0$

Sol. $-x^2 + x - 2 = 0$ or $x^2 - x + 2 = 0$

$a = 1, b = -1, c = 2$

Hence, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times 1\times 2}}{2\times 1} = \frac{1 \pm \sqrt{1-8}}{2}$$

$$= \frac{1 \pm \sqrt{-7}}{2} = \frac{1 \pm \sqrt{7}i}{2} \text{ (Solutions)}$$

5. $x^2 + 3x + 5 = 0$

Sol. $x^2 + 3x + 5 = 0$

$a = 1, b = 3, c = 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4\times 1\times 5}}{2\times 1} \quad x = \frac{-3 \pm \sqrt{9-20}}{2}$$

$$x = \frac{-3 \pm \sqrt{-11}}{2} \quad x = \frac{-3 \pm \sqrt{11}i}{2}$$

6. $x^2 - x + 2 = 0$

Sol. $x^2 - x + 2 = 0$

$a = 1, b = -1, c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times 1\times 2}}{2\times 1}$$

$$x = \frac{1 \pm \sqrt{1-8}}{2} \quad x = \frac{1 \pm \sqrt{-7}}{2} = \frac{1 \pm \sqrt{7}i}{2}$$

7. $\sqrt{2}x^2 + x + \sqrt{2} = 0$

Sol. $\sqrt{2}x^2 + x + \sqrt{2} = 0$

$a = \sqrt{2}, b = 1, c = \sqrt{2}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\times\sqrt{2}\times\sqrt{2}}}{2\sqrt{2}}$$

$$x = \frac{-1 \pm \sqrt{1-8}}{2\sqrt{2}} \quad x = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$$

8. $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Sol. We have $a = \sqrt{3}$, $b = -\sqrt{2}$, $c = 3\sqrt{3}$

$\therefore$ Discriminant of the equation is $D = b^2 - 4$

$ac = (-\sqrt{2})^2 - 4\times\sqrt{3}\times 3\sqrt{3}$

$\Rightarrow D = 2 - 36 = -34$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{\sqrt{2} \pm \sqrt{-34}}{2\times\sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}}$$

$$\therefore x = \frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}}$$

9. $x^2+x+\frac{1}{\sqrt{2}}=0$

Sol. $x^2+x+\frac{1}{\sqrt{2}}=0$ or $\sqrt{2}x^2+\sqrt{2}x+1=0$

we have, $a=\sqrt{2}$, $b=\sqrt{2}$, $c=1$

$\therefore$ Discriminant of the equation is D

$=b^2-4ac$

$=(\sqrt{2})^2-4\times\sqrt{2}\times 1$ or $D=2-4\sqrt{2}$

$$\therefore\ x=\frac{-b\pm\sqrt{D}}{2a}=\frac{-\sqrt{2}\pm\sqrt{2-4\sqrt{2}}}{2\sqrt{2}}$$

$$=\frac{-\sqrt{2}\pm\sqrt{2}\sqrt{1-2\sqrt{2}}}{2\sqrt{2}}$$

$$=\frac{\sqrt{2}(-1\pm\sqrt{1-2\sqrt{2}})}{2\sqrt{2}}$$

$$\therefore\ x=\frac{-1\pm\sqrt{1-2\sqrt{2}})}{2}$$

10. $x^2+\frac{x}{\sqrt{2}}+1=0$

Sol. $x^2+\frac{x}{\sqrt{2}}+1=0$ or $\sqrt{2}x^2+x+\sqrt{2}=0$,

We have $a=\sqrt{2}$, $b=1$, $c=\sqrt{2}$

$\therefore$ Discriminant of the equation is

$D=b^2-4ac=1^2-4\times\sqrt{2}\times\sqrt{2}$

$\Rightarrow$ $D=1-8=-7$

$$\therefore\ x=\frac{-b\pm\sqrt{D}}{2a}=\frac{-1\pm\sqrt{-7}}{2\times\sqrt{2}}=\frac{-1\pm\sqrt{7}i}{2\sqrt{2}}$$

$$\therefore\ x=\frac{-1\pm\sqrt{7}i}{2\sqrt{2}}$$

MISCELLANEOUS EXERCISE

1. **Evaluate :** $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

Sol. $\frac{1}{i}=\frac{1}{i}\times\frac{i}{i}=\frac{i}{i^2}=\frac{i}{-1}=-i$

$$\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3=\left[i^{18}+(-i)^{25}\right]^3$$

$$=\left[(i^2)^9+(-i)(i^2)^{12}\right]^3$$

$$=\left[(-1)^9-i(-1)^{12}\right]^3=[-1-i]^3$$

$=(-1)^3-(i)^3-3(-1)^2(i)+3(-1)(i)^2$

$=-1-i^3-3i+3=-1-i(i^2)-3i+3$

$=-1-i(-1)-3i+3=-1+i-3i+3$

$$\therefore\ \left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3=2-2i$$

2. **For any two complex numbers z_1 and z_2, prove that**

$\mathrm{Re}(z_1z_2)=\mathrm{Re}(z_1)\mathrm{Re}(z_2)-\mathrm{Im}(z_1)\mathrm{Im}(z_2)$

Sol. Let z_1-a+ib

Now, $a=\mathrm{Re}(z_1)$, $b=\mathrm{Im}(z_1)$ and

$z_2=c+id\ \Rightarrow c=\mathrm{Re}(z_2)$, $d=\mathrm{Im}(z_2)$

$z_1z_2=(a+ib)(c+id)=ac-bd+i(ad+bc)$

$\therefore\ \mathrm{Re}(z_1z_2)=ac-bd$

$=\mathrm{Re}\,z_1\,\mathrm{Re}\,z_2-\mathrm{Im}\,z_1\,\mathrm{Im}\,z_2$

3. **Reduce $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.**

Sol. $$\frac{1}{1-4i}=\frac{1}{1-4i}\times\frac{1+4i}{1+4i}=\frac{1+4i}{1^2-(4i)^2}$$

$$=\frac{1+4i}{1+16}=\frac{1}{17}+\frac{4}{17}i$$

$$\frac{2}{1+i}=\frac{2}{1+i}\times\frac{1-i}{1-i}=\frac{2(1-i)}{1^2-(i)^2}=\frac{2(1-i)}{1+1}$$

$$=\frac{2(1-i)}{2}=1-i$$

$$\frac{3-4i}{5+i}=\frac{3-4i}{5+i}\times\frac{5-i}{5-i}=\frac{15-3i-20i+4(i^2)}{(5)^2-(i)^2}$$

$$=\frac{11-23i}{26}=\frac{11}{26}-\frac{23}{26}i$$

Now, $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$

$$=\left[\left(\frac{1}{17}+\frac{4}{17}i\right)-(1-i)\right]\left[\frac{11}{26}-\frac{23}{26}i\right]$$

$$=\left[\frac{1}{17}+\frac{4}{17}i-1+i\right]\left[\frac{11}{26}-\frac{23}{26}i\right]$$

$$=\left(\frac{21}{17}i-\frac{16}{17}\right)\left(\frac{11}{26}-\frac{23}{26}i\right)$$

$$=\left(\frac{-16}{17}+\frac{21}{17}i\right)\left(\frac{11}{26}-\frac{23}{26}i\right)$$

$$=\frac{-16}{17}\times\frac{11}{26}+\frac{16}{17}\times\frac{23}{26}i+\frac{21}{17}i\times\frac{11}{26}\ -\frac{21}{17}\times\frac{23}{26}i^2$$

$$=\frac{-176}{442}+\frac{368}{442}i+\frac{231}{442}i+\frac{483}{442}$$

$$=\frac{307}{442}+\frac{599}{442}i$$

4. If $x-iy=\sqrt{\frac{a-ib}{c-id}}$ then prove that $(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$

Sol. We have, $x-iy=\sqrt{\frac{a-ib}{c-id}}$ ... (i)

Replacing i by $-i$ both side, we have

$$x+iy=\sqrt{\frac{a+ib}{c+id}} \quad \ldots \text{(ii)}$$

Multiply eq. (i) and (ii)

$$(x-iy)(x+iy)=\sqrt{\frac{a-ib}{c-id}}\times\sqrt{\frac{a+ib}{c+id}}$$

$$x^2-i^2y^2=\sqrt{\frac{a^2-i^2b^2}{c^2-i^2d^2}}$$

$$\Rightarrow x^2+y^2=\sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

$$\Rightarrow (x^2+y^2)^2=\left(\sqrt{\frac{a^2+b^2}{c^2+d^2}}\right)^2$$

(squaring on both side)

$\therefore\ (x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$, Hence proved

5. Convert the following in the polar form :

(i) $\frac{1+7i}{(2-i)^2}$ **(ii)** $\frac{1+3i}{1-2i}$

Sol. (i) $\frac{1+7i}{(2-i)^2}=\frac{1+7i}{4+i^2-4i}=\frac{1+7i}{3-4i}$

$$=\frac{1+7i}{3-4i}\times\frac{3+4i}{3+4i}=\frac{3+4i+21i+28i^2}{9-16i^2}$$

$$=\frac{-25+25i}{9+16}=\frac{-25+25i}{25}=-1+i$$

Put in the form of $z=r(\cos\theta+i\sin\theta)$

$\Rightarrow r\cos\theta=-1$ and $r\sin\theta=1$

Squaring and adding, we get

$$r^2(\cos^2\theta+\sin^2\theta)=(-1)^2+1$$

$$\Rightarrow r^2.\,1=1+1\Rightarrow r^2=2\Rightarrow r=\sqrt{2}$$

By dividing $\frac{r\sin\theta}{r\cos\theta}=\frac{1}{-1}=-1$,

$$\tan\theta=-1\Rightarrow\tan\theta=-\tan\left(\frac{\pi}{4}\right)=\tan\left(\pi-\frac{\pi}{4}\right)$$

$\theta=\frac{3\pi}{4}$, θ lies in second quadrant,

Hence, $\frac{1+7i}{(2-i)^2}=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

(ii) $\frac{1+3i}{1-2i}=\frac{1+3i}{1-2i}\times\frac{1+2i}{1+2i}=\frac{1+2i+3i+6i^2}{1^2-4i^2}$

$$=\frac{-5+5i}{1+4}=\frac{-5+5i}{5}=-1+i$$

solution (ii) same as above part (i). Hence, required polar form is $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

6. Solve the equation : $3x^2 - 4x + \frac{20}{3} = 0$

Sol. $3x^2 - 4x + \frac{20}{3} = 0$ or $9x^2 - 12x + 20 = 0$

We have $a = 9, b = -12, c = 20$

$\therefore$ Discriminant of the equation is

$D = b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20$

$= 144 - 720 = -576$

$\therefore \ x = \frac{-b \pm \sqrt{D}}{2a} = \frac{12 \pm \sqrt{-576}}{2\times 9} = \frac{12 \pm \sqrt{576}\, i}{18}$

$= \frac{12 \pm 24i}{18} = \frac{6(2 \pm 4i)}{18} \Rightarrow x = \frac{2 \pm 4i}{3}$

7. Solve the equation : $x^2 - 2x + \frac{3}{2} = 0$

Sol. $x^2 - 2x + \frac{3}{2} = 0$ or $2x^2 - 4x + 3 = 0$

we have, $a = 2, b = -4, c = 3$

$\therefore$ Discriminant of the equation is

$D = b^2 - 4ac = (-4)^2 - 4 \times 2 \times 3 = 16 - 24 = -8$

$\therefore \ x = \frac{-b \pm \sqrt{D}}{2a} = \frac{4 \pm \sqrt{-8}}{2 \times 2}$

$= \frac{4 \pm i\sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}i}{4}$

$= \frac{2(2 \pm \sqrt{2}i)}{4} = \frac{(2 \pm \sqrt{2}i)}{2}$

8. Solve the equation : $27x^2 - 10x + 1 = 0$

Sol. $27x^2 - 10x + 1 = 0$

We have, $a = 27, b = -10, c = 1$

$\therefore$ Discriminant of the equation is

$D = b^2 - 4ac$

$= (-10)^2 - 4 \times 27 \times 1 = 100 - 108 = -8$

$\therefore \ x = \frac{-b \pm \sqrt{D}}{2a} = \frac{10 \pm \sqrt{-8}}{2 \times 27} = \frac{2(5 \pm \sqrt{2}i)}{2 \times 27}$

$\Rightarrow x = \frac{5 \pm \sqrt{2}i}{27}$

9. Solve the equation : $21x^2 - 28x + 10 = 0$

Sol. $21x^2 - 28x + 10 = 0$ ·

We have, $a = 21, b = -28, c = 10$

$\therefore$ Discriminant of the equation is

$D = b^2 - 4ac$

$= (-28)^2 - 4 \times 21 \times 10 = 784 - 840 = -56$

$\therefore \ x = \frac{-b \pm \sqrt{D}}{2a} = \frac{28 \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm 2\sqrt{14}i}{2 \times 21}$

$= \frac{2(14 \pm \sqrt{14}i)}{2 \times 21} \Rightarrow x = \frac{14 \pm \sqrt{14}i}{21}$

10. If $z_1 = 2 - i, z_2 = 1 + i$ find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

Sol. $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right| = \left|\frac{(2-i)+(1+i)+1}{(2-i)-(1+i)+i}\right|$

$\Rightarrow \left|\frac{2 - i + 1 + i + 1}{2 - i - 1 - i + i}\right| = \left|\frac{4}{1-i}\right|$

$= \left|\frac{4}{1-i} \times \frac{1+i}{1+i}\right| = \left|\frac{4(1+i)}{1-i^2}\right|$

$\Rightarrow 4\left|\frac{(1+i)}{1+1}\right| = 4\left|\frac{(1+i)}{2}\right| = 2|(1+i)|$

$\therefore \ \left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

$= 2|(1+i)| = 2.\sqrt{(1)^2 + (1)^2} = 2\sqrt{2}$

11. If $a + ib = \frac{(x+i)^2}{2x^2+1}$, Prove that

$$a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

Sol. We have $a + ib = \frac{(x+i)^2}{2x^2+1}$ ... (i)

Replacing i by $-i$ both side we have

$a - ib = \frac{(x-i)^2}{2x^2+1}$ ... (ii)

Multiplying eqs. (i) and (ii)

$(a+ib)(a-ib) = \frac{(x+i)^2}{2x^2+1} \times \frac{(x-i)^2}{2x^2+1}$

$\Rightarrow a^2 - i^2b^2 = \frac{[(x+i)(x-i)]^2}{(2x^2+1)^2}$

$= \frac{(x^2-i^2)^2}{(2x^2+1)^2} = \frac{(x^2+1)^2}{(2x^2+1)^2}$

$\therefore \ a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$ Hence proved

12. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

(i) $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $\text{Im}\left(\frac{1}{z_1 \bar{z}_1}\right)$

Sol. (i) $\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \frac{(2-i)(-2+i)}{\overline{2-i}} = \frac{-(2-i)^2}{2+i}$

$= \frac{-(4+i^2-4i)}{2+i} = \frac{-(3-4i)}{2+i} = \frac{-3+4i}{2+i}$

$= \frac{-3+4i}{2+i} \times \frac{2-i}{2-i} = \frac{-6+3i+8i-4i^2}{4-i^2}$

$= \frac{-2+11i}{5} = \frac{-2}{5} + \frac{11}{5}i$

$\therefore$ $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \frac{-2}{5}$

(ii) $\frac{1}{z_1\bar{z}_1} = \frac{1}{(2-i)(2+i)} = \frac{1}{4-i^2} = \frac{1}{5} = \frac{1}{5} + i.0$

$\therefore$ $\text{Im}\left(\frac{1}{z_1\bar{z}_2}\right) = 0$

13. Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$

Sol. We have,

$z = \frac{1+2i}{1-3i} = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1-9i^2}$

$= \frac{1-6+5i}{1+9} = \frac{-5+5i}{10}$

$= \frac{-5}{10} + \frac{5}{10}i = -\frac{1}{2} + \frac{1}{2}i$

Put in the polar form $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow r\cos\theta = -\frac{1}{2}$ and $r\sin\theta = \frac{1}{2}$

By squaring and adding, we get

$r^2(\cos^2\theta + \sin^2\theta) = \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$\Rightarrow$ $r^2.1 = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$ $\Rightarrow$ $r^2 = \frac{1}{2}$

$\therefore$ $r = \frac{1}{\sqrt{2}}$

By dividing $\frac{r\sin\theta}{r\cos\theta} = \frac{1/2}{-1/2} = -1$

$\Rightarrow$ $\tan\theta = -1$

$= -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$

$\Rightarrow$ $\theta = \frac{3\pi}{4}$ $\therefore$ θ lies in second quadrant.

$\therefore$ $\left|\frac{1+2i}{1-3i}\right| = \frac{1}{\sqrt{2}}$ and $\arg\left|\frac{1+2i}{1-3i}\right| = \frac{3\pi}{4}$

14. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Sol. Congugate of $-6 - 24i$ is $-6 + 24i$ ...(i)

we have, $(x - iy)(3 + 5i)$

$= 3x + 5xi - 3yi - 5i^2y$

$= (3x + 5y) + (5x - 3y)i$ ...(ii)

From equations (i) & (ii)

$(3x + 5y) + (5x - 3y)i = -6 + 24i$

$\Rightarrow$ $3x + 5y = -6$ or $3x + 5y + 6 = 0$

and $5x - 3y = 24$ or $5x - 3y - 24 = 0$

By cross multiplication method, we get,

$\frac{x}{-120+18} = \frac{y}{30+72} = \frac{1}{-9-25}$

$\frac{x}{-102} = \frac{y}{102} = \frac{1}{-34}$

$\therefore x = \frac{-102}{-34} = 3$ and $y = \frac{102}{-34} = -3$

hence, $x = 3$ and $y = -3$

15. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$

Sol. $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2}$

$= \frac{1+i^2+2i}{1+1} = \frac{2i}{2} = i$

$\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1-i^2}$

$= \frac{1+i^2-2i}{1+1} = \frac{-2i}{2} = -i$

We have, $\frac{1+i}{1-i} - \frac{1-i}{1+i} = i - (-i) = 2i$

$\therefore \left|\frac{1+i}{1-i} - \frac{1-i}{1+i}\right| = 2$

16. If $(x+iy)^3 = u+iv$, then show that $\frac{u}{x}+\frac{v}{y}=4(x^2-y^2)$

Sol. $(x+iy)^3 = x^3+3x^2iy+3x.(iy)^2+(iy)^3$

$= x^3+3x^2iy+3x\,i^2y^2+i^3y^3$

$= x^3+i3x^2y-3xy^2-iy^3$

$= (x^3-3xy^2)+(3x^2y-y^3)i = u+iv$

Equating real and imaginary parts

$u = x^3-3xy^2$ or $\frac{u}{x}=x^2-3y^2$ ... (i)

and $v=3x^2y-y^3$ or $\frac{v}{y}=3x^2-y^2$ ... (ii)

Adding equations (i) & (ii)

$$\frac{u}{x}+\frac{v}{y}=x^2-3y^2+3x^2-y^2$$

$\Rightarrow \frac{u}{x}+\frac{v}{y}=4(x^2-y^2)$ Hence proved

17. If α and β are different complex numbers with $|\beta|=1$, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$

Sol. Consider $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2=\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\overline{\frac{\beta-\alpha}{1-\bar{\alpha}\beta}}\right)$

$=\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}}\right)$ $(\because |z|^2=z.\bar{z})$

$=\frac{\beta\bar{\beta}-\bar{\alpha}\beta-\alpha\bar{\beta}+\alpha\bar{\alpha}}{1-\bar{\alpha}\beta-\alpha\bar{\beta}+(\alpha\bar{\alpha})(\beta\bar{\beta})}$

$=\frac{|\beta|^2-\bar{\alpha}\beta-\alpha\bar{\beta}+|\alpha|^2}{1-\bar{\alpha}\beta-\alpha\bar{\beta}+|\alpha|^2|\beta|^2}$

But $|\beta|^2=1$

$\therefore \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2=\frac{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}=1.$

Hence $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=1$

18. Find the number of non-zero integral solution of the equation $|(1-i)|^x=2^x$

Sol. $|1-i|=\sqrt{1^2+(-1)^2}=\sqrt{1+1}=\sqrt{2}=2^{1/2}$

$\therefore |1-i|^x=(2^{1/2})^x=2^{x/2}$ $\because |1-i|^x=2^x$ (given)

$\Rightarrow 2^{x/2}=2^x$

$\Rightarrow \frac{x}{2}=x$ is admissible except $x=0$

$\Rightarrow$ There is no non-zero integral solution. Its only solution is $x=0$

19. If $(a+ib)(c+id)(e+if)(g+ih)=A+iB$, then show that

$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

Sol. We have $(a+ib)(c+id)(e+if)(g+ih)=A+iB$ ... (i)

Replacing i by $-i$ both side, we get

$(a-ib)(c-id)(e-if)(g-ih)=A-iB$ ... (ii)

Multiplying eqn. (i) & (ii), we get

$[(a+ib)(a-ib)][(c+id)(c-id)]$
$[(e+if)(e-if)][(g+ih)(g-ih)]$
$=(A+iB)(A-iB)$

or $(a^2-i^2b^2)(c^2-i^2d^2)(e^2-i^2f^2)(g^2-i^2h^2)$
$=(A^2-i^2B^2)$

$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)$
$=(A^2+B^2)$ $(\because i^2=-1)$. Hence proved

20. If $\left(\frac{1+i}{1-i}\right)^m=1$, then find the least integral value of *m*.

Sol. $\frac{1+i}{1-i}=\frac{1+i}{1-i}\times\frac{1+i}{1+i}=\frac{(1+i)^2}{1^2-i^2}$

$=\frac{1+i^2+2i}{1+1}=\frac{2i}{2}=i$

$\left(\frac{1+i}{1-i}\right)^m - i^m - 1$ or $(i^4)^{m/4}=1$

$\Rightarrow$ m is a multiple of 4

$\Rightarrow$ least value of $m=4$

SECTION B PRACTICE QUESTIONS

Short Answer Questions

1. Express $\left(\frac{3+i}{3-i}\right)$ in the form $P+iQ$

2. Express $\left(\frac{1-i}{1+i}\right)^2$ in the form $A+iB$.

3. Find the value of p if the equation $3x^2-2x+p=0$ and $6x^2-17x+12=0$ have a common root.

4. Find the equation with rational coefficient for which one root is $i-1$.

5. If $3+4i$ is a root of the equation $x^2+px+q=0$.
Then find the value of p and q.

6. Express $\frac{2-\sqrt{-25}}{1-\sqrt{-16}}$ in the form $x+iy$

7. If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2-bx+1=0$ has imaginary roots then what condition on $a+b+1$ should be ?

8. If the roots of the equation $x^2-2ax+a^2+a-3=0$ are less than 3 then what is the value of a ?

9. Find the locus of a complex variable z satisfying the following condition and show that it is circle: $|z+4+i|=5$
Note : Equation of circle is $x^2+y^2+2gx+2fy+c=0$

10. Express each of the following in the form $(a+ib)$:
$\sqrt{-4}(\sqrt{-9}+3)+\sqrt{-49}(i^3+\sqrt{3})-\sqrt{-36}$
$(2-\sqrt{-121})+\sqrt{5}i^5$

11. Solve each of the following for x and y
$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$$

12. What is the locus of z, if amplitude of $z-2-3i$ is $\frac{\pi}{4}$?

13. Locate the points for which $3<|z|<4$

14. Find the root of the equation $2(1+i)x^2-4(2-i)x-5-3i=0$ which has greater modulus.

Long Answer Questions

1. Let a and b be two roots of the equation $x^3+px^2+qx+r=0$ satisfy the relation $ab+1=0$. Prove that $r^2+pr+q+1=0$ $(r \neq 0)$

2. Find the amplitude of the complex number $\sin\frac{6\pi}{5}+i\left(1-\cos\frac{6\pi}{5}\right)$

3. If $i=\sqrt{-1}$ prove the following :
$(x+1+i)(x+1-i)(x-1+i)(x-1-i)=x^4+4$

4. Given $z_1=1-i$, $z_2=-2+4i$. Calculate the values of a and b if $a+bi=\frac{z_1z_2}{z_1}$

5. Find the modulus and amplitude of $\frac{1+7i}{(2-i)^2}$

6. Evaluate the argument of $(1-i\sqrt{3})/(1+i\sqrt{3})$

7. If z_1, z_2, z_3, z_4 are the roots of the equation $z^4=1$, then find the value of $\sum_{i=1}^{4} z_i^3$.

8. If $z=x+iy$ and $\frac{|z-1-i|+4}{3|z-1-i|-2}=1$. Show that $x^2+y^2-2x-2y-7=0$

9. Find the locus of a complex number $z=x+iy$ satisfying the relation $\left|\frac{z-2i}{z+2i}\right|=\sqrt{2}$, Illustrate the locus of z in the argand plane.

10. Find the greatest and least values of the modulus of complex number z satisfying the equation $\left|z-\frac{4}{z}\right|=2$

11. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, $ac \neq 0$, then prove that $P(x)Q(x) = 0$ has at least two real roots.

12. Find the square roots of the following :
(i) $1 - i$ (ii) i

13. Solve the equation $2z = |z| + 2i$.

14. If $|a + ib| = 1$. Convert into simplified form of $\frac{1+b+ai}{1+b-ai}$.

15. If $\frac{(\cos x + i\sin x)(\cos y + i\sin y)}{(\cot u + i)(1 + i\tan v)} = A + iB$, then find the value of A and B.

16. Find the value of k if for the complex numbers z_1 and z_2,

$$|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1 - |z_1|^2)(1 - |z_2|^2)$$

17. Simplify:
$(x - 1 - i)(x - 1 + i)(x + 1 + i)(x + 1 - i)$

18. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and if $x + y + z = 0$, then prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

PRACTICE QUESTION'S SOLUTIONS

Short Answer Questions

1-3. **Do it yourself.**

4. **Hint** - One root is $-1 + i$
$\therefore$ the other root will be $-1 - i$
$\therefore$ the equation is
$x^2 - (-1 + i - 1 - i)x + (-1 + i)(-1 - i) = 0$
or, $x^2 + 2x + 2 = 0$

5. **Hint** - Roots are $3 + 4i$ and $3 - 4i$
$\therefore$ sum $= 3 + 4i + 3 - 4i = -p$ and
product $= (3 + 4i)(3 - 4i) = q$
$\therefore p = -6, q = 25$

6. **Do it yourself.**

7. Let $f(x) = ax^2 - bx + 1$.
Now, $f(0) = 1$ and roots are imaginary.
$\therefore f(x) > 0 \ \forall x \in R \Rightarrow f(-1) = a + b + 1 > 0$

8. Discriminant $\geq 0 \Rightarrow 4a^2 - 4(a^2 + a - 3) \geq 0$
$\Rightarrow a \leq 3$ ... (i)
Let $f(x) = x^2 - 2ax + a^2 + a - 3$
$\because$ 3 lie outside the range of roots
$\therefore f(3) > 0 \Rightarrow a < 2$ or $a > 3$ ... (ii)
from (i), (ii), we have $a < 2$.

9. Consider $|z + 4 + i| = 5$
$\Rightarrow |(x + iy) + 4 + i| = 5 \Rightarrow |(x + 4) + i(y + 1)| = 5$
$\Rightarrow \sqrt{(x+4)^2 + (y+1)^2} = 5$
Squaring both sides,
$\Rightarrow (x + 4)^2 + (y + 1)^2 = 25$
$\Rightarrow x^2 + y^2 + 8x + 2y - 8 = 0$
This equation represents a circle.

10. $2i(3i + 3) + 7i(-i + \sqrt{3}) - 6i(2 - 11i) + \sqrt{5}i$
$= -6 + 6i + 7 + 7\sqrt{3}i - 12i - 66 + \sqrt{5}i$
$= -65 + (7\sqrt{3} + \sqrt{5} - 6)i$

11. $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$

$\Rightarrow \frac{[(1+i)x - 2i](3-i) + [(2-3i)y + i](3+i)}{(3+i)(3-i)} = i$

$\Rightarrow (1 + i)(3 - i)x - 2i(3 - i) + (2 - 3i)(3 + i)y + i(3 + i) = 10i$
$\Rightarrow (4 + 2i)x - 6i - 2 + (9 - 7i)y + 3i - 1 = 10i$
$\Rightarrow (4x + 9y) + (2x - 7y)i = 3 + 13i$
$\Rightarrow 4x + 9y = 3; 2x - 7y = 13$
$\Rightarrow x = 3, y = -1$

12. Let $z = x + iy$. Then $z - 2 - 3i = (x - 2) + i(y - 3)$
Let θ be the amplitude of $z - 2 - 3i$.

Then $\tan\theta = \frac{y-3}{x-2}$

$\Rightarrow \tan\frac{\pi}{4} = \frac{y-3}{x-2}\left(\text{since } \theta = \frac{\pi}{4}\right)$

$\Rightarrow 1 = \frac{y-3}{x-2}$ i.e. $x - y + 1 = 0$

Hence, the locus of z is a straight line.

13. $|z| < 4 \Rightarrow x^2 + y^2 < 16$ which is the interior of circle with centre at origin and radius 4 units, and $|z| > 3 \Rightarrow x^2 + y^2 > 9$ which is exterior of circle with centre at origin and radius 3 units. Hence $3 < |z| < 4$ is the portion between two circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.

14. Roots

$$= \frac{4(2-i) \pm \sqrt{16(2-i)^2 + 8(1+i)(5+3i)}}{4(1+i)}$$

$= \frac{4-i}{1+i}$ or $\frac{-i}{1+i} = \frac{3-5i}{2}$ or $\frac{-1-i}{2}$

$= \left|\frac{3-5i}{2}\right| = \sqrt{\frac{9+25}{4}} = \sqrt{\frac{17}{2}}$

and $\left|\frac{-1-i}{2}\right| = \sqrt{\frac{1+1}{4}} = \sqrt{\frac{1}{2}}$

Long Answer Questions

1–5. **Do it yourself.**

6. **Hint:**

$$\frac{(1-i\sqrt{3})}{(1+i\sqrt{3})} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \arg\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -(\pi - \tan^{-1}\sqrt{3}) = -\frac{2\pi}{3}$$

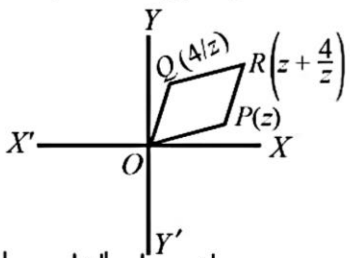
$-\pi < \arg \le \pi$, so $\arg\left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right) \ne \frac{4\pi}{3}$

7. **Hint** - i.e., $(z^2 - 1)(z^2 + 1) = 0$ are $1, -1, i, -i$

$$\therefore \sum_{i=1}^{4} z_i^3 = 1 - 1 + i^3 + (-i)^3 = 1 - 1 - i + i = 0$$

8–9. **Do it yourself.**

10. We have $|OP - OQ| \le QP$



$$\left||z| - \left|\frac{4}{z}\right|\right| \le \left|z - \frac{4}{z}\right| = 2$$

$\Rightarrow -2 \le |z| - \frac{4}{|z|} \le 2 \Rightarrow |z^2| + 2|z| - 4 \ge 0$

or $|z^2| - 2|z| - 4 \le 0$

$\Rightarrow (|z|+1)^2 - 5 \ge 0$ *or* $(|z|-1)^2 \le 5$

$\Rightarrow (|z|+1+\sqrt{5})(|z|+1-\sqrt{5}) \ge 0$

$\Rightarrow (|z|-1+\sqrt{5})(|z|-1-\sqrt{5}) \le 0$

$\Rightarrow \sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$

$\Rightarrow$ Greatest value of $|z| = \sqrt{5} + 1$ and least value of

$|z| = \sqrt{5} - 1$

11. We have $P(x)Q(x) = 0$. If $P(x) = 0$ has real roots, then $b^2 - 4ac \ge 0$

If $Q(x) = 0$ has real roots, then $d^2 + 4ac \ge 0$

Now, $ac \ne 0$. If $ac < 0$, $b^2 - 4ac \ge 0$. Hence $P(x) = 0$ has real roots.

If $ac > 0$, then $d^2 + 4ac \ge 0$. Hence $Q(x) = 0$ has real roots.

Hence, at least two roots of $P(x)Q(x) = 0$ are real.

12. (i) $\left(\text{Ans. } \pm\sqrt{\frac{\sqrt{2}-1}{2}} \pm \sqrt{\frac{\sqrt{2}-1}{2}}\,i\right)$

(ii) $\left(\text{Ans. } \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i\right)$

13. Let $z = x + iy$, where $x, y \in R$ so that

$|z| = \sqrt{x^2 + y^2}$

Given $2z = |z| +$

$2i \Rightarrow 2(x+iy) = \sqrt{x^2+y^2} + 2i$

$\Rightarrow 2x = \sqrt{x^2+y^2}$ and $2y = 2$

$\Rightarrow 4x^2 = x^2 + y^2$ and $y = 1$

$\Rightarrow 3x^2 = 1$ and $y = 1$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$ and $y = 1$

$\Rightarrow z = \frac{1}{\sqrt{3}} + i, -\frac{1}{\sqrt{3}} + i$

But $z = \frac{-1}{\sqrt{3}} + i$ does not satisfy the given equation

$2z = |z| + 2i$

$\therefore z = \frac{1}{\sqrt{3}} + i$

14. As $|a+ib| = 1 \Rightarrow a^2 + b^2 = 1$

$$= \frac{a+b+ai}{1+b-ai} = \frac{(1+b+ai)^2}{(1+b-ai)(1+b+ai)}$$

$$= \frac{(1+b)^2 - a^2 + 2(1+b)ai}{(1+b)^2 + a^2}$$

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$$= \frac{(1-a^2)+b^2+2b+2ai+2abi}{1+(a^2+b^2)+2b}$$

$$= \frac{b^2+b^2+2b+2ai+2abi}{1+1+2b}$$

$$= \frac{b^2+b+ai+abi}{1+b} = \frac{b(1+b)+a(1+b)i}{1+b} = b+ai$$

15. We have, $\dfrac{(\cos x+i\sin x)(\cos y+i\sin y)}{(\cot u+i)(1+i\tan v)}$

$$= \frac{(\cos x+i\sin x)(\cos y+i\sin y)}{\left(\frac{\cos u}{\sin u}+i\right)\left(1+i\frac{\sin v}{\cos v}\right)}$$

$$= \frac{\sin u\cos v(\cos x+i\sin x)(\cos y+i\sin y)}{(\cos u+i\sin u)(\cos v+i\sin v)}$$

$$= \frac{\sin u\cos v[\cos(x+y)+i\sin(x+y)]}{[\cos(u+v)+i\sin(u+v)]}$$

$$\times \frac{[\cos(u+v)-i\sin(u+v)]}{[\cos(u+v)-i\sin(u+v)]}$$

$$= \frac{\sin u\cos v[\cos(x+y-u-v)+i\sin(x+y-u-v)]}{\cos^2(u+v)+\sin^2(u+v)}$$

$$= \sin u\cos v\cos(x+y-u-v)$$
$$+i\sin u\cos v\sin(x+y-u-v)$$

$\therefore A = \sin u\cos v\cos(x+y-u-v)$

and $B = \sin u\cos v\sin(x+y-u-v)$

16. L.H.S. $= |1-\bar{z}_1 z_2|^2 - |z_1 - z_2|^2$

$= (1-\bar{z}_1 z_2)(\overline{1-\bar{z}_1 z_2}) - (z_1 - z_2)(\overline{z_1 - z_2})$

$= (1-\bar{z}_1 z_2)(1-z_1\bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= 1 - z_1\,\bar{z}_1\,z_2\,\bar{z}_2 - z_1\bar{z}_1 - z_2\bar{z}_2$

$= 1+|z_1|^2 . |z_2|^2 - |z_1|^2 - |z_2|^2$

$= (1-|z_1|^2)(1-|z_2|^2)$

R.H.S. $= k(1-|z_1|^2)(1-|z_1|^2)$

$\Rightarrow k = 1$

Hence, equating LHS and RHS, we get $k = 1$.

17. Given expression :

$= [((x-1)-i)((x-1)+i)]\ [((x+1+i)((x+1)-i)]$

$= [((x-1)^2 - i^2)((x+1)^2 - i^2)]$

$= [((x^2-2x+1)-(-1))((x^2+2x+1-(-1))]$

$= [(x^2-2x+2)(x^2+2x+2)]$

$= (x^2+2)^2 - 4x^2 = x^4+4x^2+4-4x^2 = x^4+4$

18. Do it yourself.

SECTION C NCERT EXEMPLAR QUESTIONS

Fill in the Blanks

1. For any two complex numbers z_1, z_2 and any real numbers a, b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$ = __________.

2. The value of $\sqrt{-25}\times\sqrt{-9}$ __________.

3. The number $\dfrac{(1-i)^3}{1-i^3}$ is equal to __________.

4. The sum of the series $i + i^2 + i^3 + \ldots$ upto 1000 terms is __________.

5. Multiplicative inverse of $1+i$ is __________.

6. If z_1 and z_2 are comlex numbers such that $z_1 + z_2$ is a real number, then $z_2 =$ __________.

7. $\arg(z) + \arg\bar{z}\ (\bar{z} \neq 0)$ is __________.

8. If $|z+4| \le 3$, then the greatest and least values of $|z+1|$ are __________ and __________.

9. If $\left|\dfrac{z-2}{z+2}\right| = \dfrac{5\pi}{6}$, then the locus of z is __________.

10. If $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$, then $z =$ __________.

TRUE OR FALSE

1. The order relation is defined on the set of complex numbers.
2. Multiplication of a non zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.
3. For any complex number z, the minimum value of $|z| + |z-1|$ is 1.
4. The locus represented by $|z-1| = |z-i|$ is a line perpendicular to the join of the points (1, 0) and (0, 1).
5. If z is a complex number such that $z \neq 0$ and Re $(z) = 0$, then Im $(z^2) = 0$.
6. The inequality $|z-4| < |z-2|$ represents the region given by $x > 3$.
7. Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg $(z_1 - z_2) = 0$.
8. 2 is not a complex number.

SHORT ANSWER QUESTIONS

1. For a positive integer n, find the value of $(1-i)^n \left(1 - \frac{1}{i}\right)^n$.
2. Evaluate $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $n \in N$.
3. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y).
4. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.
5. If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).
6. If $a = \cos\theta + i\sin\theta$, then find the value of $\frac{1+a}{1-a}$.
7. If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.
8. If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in R$, represents a circle.
9. If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.
10. Show that the complex number z, satisfying the condition arg $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ lies on a circle.
11. Solve the equation $|z| = z + 1 + 2i$.

LONG ANSWER QUESTIONS

1. If $|z+1| = z + 2(1+i)$, then find the value of z.
2. If arg $(z-1)$ = arg $(z+3i)$, then find $x - 1 : y$, where $z = x + iy$.
3. Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.
4. If $\frac{z-1}{z+1}$ is purely imaginary number $(z \neq -1)$, then find the value of $|z|$.
5. z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and arg (z_1) + arg $(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.
6. If $|z_1| = 1$ $(z_1 \neq -1)$ and $z_2 = \frac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.
7. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find arg $\left(\frac{z_1}{z_4}\right)$ + arg $\left(\frac{z_2}{z_3}\right)$.
8. If for the complex numbers z_1 and z_2, arg (z_1) − arg $(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$.
9. Solve the system of equations

   $$\text{Re}(z^2) = 0, \ |z| = 2.$$
10. Find the complex number satisfying the equation $z + \sqrt{2}\,|(z+1)| + i = 0$.
11. Write the complex number $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in polar form.
12. If z and w are two complex numbers such that $|zw| = 1$ and arg (z) − arg $(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.

NCERT EXEMPLAR SOLUTIONS

Fill in the Blanks

1. $(a^2+b^2)(|z_1|^2+|z_2|^2)$
2. -15 3. -2 4. 0
5. $\frac{1}{2}-\frac{i}{2}$ 6. $\bar{z}_1$ 7. 0
8. 6 and 0 9. a circle
10. $-2\sqrt{3}+2i$

True or False

1. F 2. F 3. T 4. T
5. IF 6. T 7. F 8. T

Short Answer Questions

1. We have $(1-i)^n\left(1-\frac{1}{i}\right)^n$

$$=(1-i)^n(i-1)^n\cdot i^{-n}=(1-i)^n(1-i)^n(-1)^n\cdot i^{-n}$$
$$=[(1-i)^2]^n(-1)^n\cdot i^{-n}=(1+i^2-2i)^n(-1)^n i^{-n}$$
$$[\because i^2=-1]$$
$$=(1-1-2i)^n(-1)^n i^{-n}$$
$$=(-2)^n\cdot i^n(-1)^n i^{-n}$$
$$=(-1)^{2n}\cdot 2^n=2^n$$

2. Given that, $\sum_{n=1}^{13}(i^n+i^{n+1}), n\in N$

$$=(i+i^2+i^3+i^4+i^5+i^6+i^7+i^8+i^9+i^{10}+i^{11}+i^{12}+i^{13})+(i^2+i^3+i^4+i^5+i^6+i^7+i^8+i^9+i^{10}+i^{11}+i^{12}+i^{13}+i^{14})$$
$$=(i+2i^2+2i^3+2i^4+2i^5+2i^6+2i^7+2i^8+2i^9+2i^{10}+2i^{11}+2i^{12}+2i^{13}+i^{14})$$
$$=i-2-2i+2+2i+2(i^4)i^2+2(i)^4 i^3+2(i^2)^4+2(i^2)^4 i+2(i^2)^5+2(i^2)^5\cdot i+2(i^2)^6+2(i^2)^6\cdot i+(i^2)^7$$
$$=i-2-2i+2+2i-2-2i+2+2i-2-2i+2+2i-1=i-1$$

3. Now, $\left(\frac{1+i}{1-i}\right)^3=\frac{1+i^3+3i(1+i)}{1-i^3-3i(1-i)}$

$$=\frac{1-i+3i+3i^2}{1+i-3i+3i^2}$$
$$=\frac{2i-2}{-2i-2}=\frac{i-1}{-i-1}=\frac{1-i}{1+i}$$
$$=\frac{(1-i)(1-i)}{(1+i)(1-i)}=\frac{1+i^2-2i}{1+1}$$
$$=\frac{1-1-2i}{2}$$
$$\Rightarrow \left(\frac{1+i}{1-i}\right)^3=-i$$

Similarly, $\left(\frac{1-i}{1+i}\right)^3=\frac{-1}{i}=\frac{i^2}{i}=i$

Since $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=x+iy$

$$\Rightarrow -i-i=x+iy$$
$$\Rightarrow -2i=x+iy$$

On comparing real and imaginary parts we get

$$x=0 \text{ and } y=-2$$

Therefore, $(x,y)=(0,-2)$

4. We have, $x+iy=\frac{(1+i)^2}{2-i}$

$$=\frac{(1+i^2+2i)}{2-i}=\frac{2i}{2-i}$$
$$=\frac{2i(2+i)}{(2-i)(2+i)}=\frac{4i-2}{4+1}$$
$$\Rightarrow x+iy=\frac{-2}{5}+\frac{4i}{5} \qquad [\because i^2=-1]$$

On comparing both sides, we get

$$x=-2/5 \text{ and } y=4/5$$
$$\Rightarrow x+y=\frac{-2}{5}+\frac{4}{5}=2/5$$

5. We have $\left(\frac{1-i}{1+i}\right)^{100}=a+ib$

$\Rightarrow \left[\frac{(1-i)}{(1+i)}\cdot\frac{(1-i)}{(1-i)}\right]^{100} = a+ib$

$\Rightarrow \left(\frac{1+i^2-2i}{1-i^2}\right)^{100} = a+ib$

$\Rightarrow \left(\frac{-2i}{2}\right)^{100} = a+ib \quad [\because i^2 = -1]$

$\Rightarrow (i^4)^{25} = a+ib$

$\Rightarrow 1 = a+ib$

On comparing $a = 1$ and $b = 0$ $[\because i^4 = 1]$

$\Rightarrow (a, b) = (1, 0)$

6. We have $a = \cos\theta + i\sin\theta$

$\therefore \frac{1+a}{1-a} = \frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$

$= \frac{1+2\cos^2\theta/2-1+2i\sin\theta/2\cdot\cos\theta/2}{1-1+2\sin^2\theta/2-2i\sin\theta/2\cdot\cos\theta/2}$

$= \frac{2\cos\theta/2\,(\cos\theta/2+i\sin\theta/2)}{2\sin\theta/2\,(\sin\theta/2-i\cos\theta/2)}$

$= -\frac{2\cos\theta/2(\cos\theta/2+i\sin\theta/2)}{2i\sin\theta/2(\cos\theta/2+i\sin\theta/2)} = -\frac{1}{i}\cot\theta/2$

$= \frac{+i^2}{i}\cot\theta/2 = i\cot\theta/2 \quad \left[\because \frac{-1}{i} = \frac{i^2}{i}\right]$

7. Given $(1+i)z = (1-i)\bar{z}$

$\Rightarrow \frac{z}{\bar{z}} = \frac{(1-i)}{(1+i)}$

$\Rightarrow \frac{z}{\bar{z}} = \frac{(1-i)(1-i)}{(1+i)(1-i)} \Rightarrow \frac{z}{\bar{z}} = \frac{1+i^2-2i}{1-i^2}$

$[\because i^2 = -1]$

$\Rightarrow \frac{z}{\bar{z}} = \frac{1-1-2i}{2}$

$\Rightarrow \frac{z}{\bar{z}} = -i$

$\therefore z = -i\bar{z}$

8. We have $z = x+iy$

Then $\bar{z} = x-iy$

Now, $z\bar{z}+2(z+\bar{z})+b = 0$

$\Rightarrow (x+iy)(x-iy)+2(x+iy+x-iy)+b = 0$

$\Rightarrow x^2+y^2+4x+b = 0$, which is the equation of a circle.

9. Let $z = x+iy$

Now, $\frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1}$

$= \frac{[(x+2)-iy][(x-1)+iy]}{[(x-1)-iy][(x-1)+iy]}$

[Rationalizing]

$= \frac{(x-1)(x+2)-iy(x-1)+iy(x+2)+y^2}{(x-1)^2+y^2}$

$= \frac{(x-1)(x+2)+y^2+i[(x+2)y-(x-1)y]}{(x-1)^2+y^2}$

$[\because -i^2 = 1]$

Taking real part, $\frac{(x-1)(x+2)+y^2}{(x-1)^2+y^2} = 4$

$\Rightarrow x^2-x+2x-2+y^2 = 4(x^2-2x+1+y^2)$

$\Rightarrow 3x^2+3y^2-9x+6 = 0$, which represents a circle.

Hence, z lies on the circle.

10. Let $z = x+iy$

Since $\arg\left(\frac{z-1}{z+1}\right) = \pi/4$

$\left[\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2)\right]$

$\Rightarrow \arg(z-1)-\arg(z+1) = \pi/4$

$\Rightarrow \arg(x+iy-1)-\arg(x+iy+1) = \pi/4$

$\Rightarrow \arg(x-1+iy)-\arg(x+1+iy) = \pi/4$

$\Rightarrow \tan^{-1}\frac{y}{x-1}-\tan^{-1}\frac{y}{x+1} = \pi/4$

$\left[\arg(z) = \theta = \tan^{-1}\frac{y}{x}\right]$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\left(\frac{y}{x-1}\right)\left(\frac{y}{x+1}\right)}\right]=\pi/4$$

$$\Rightarrow \frac{y\left[\frac{x+1-x+1}{x^2-1}\right]}{\frac{x^2-1+y^2}{x^2-1}}=\tan \pi/4$$

$$\Rightarrow \frac{2y}{x^2+y^2-1}=1$$

$\Rightarrow x^2+y^2-2y-1=0$ which represents a circle.

11. Since, $|z| = z + 1 + 2i$ ...(1)

Let $z = x + iy$

From Eq. (1) $|x + iy| = x + iy + 1 + 2i$

$$\Rightarrow \sqrt{x^2+y^2} = x + iy + 1 + 2i \quad \left[\because |z| = \sqrt{x^2+y^2}\right]$$

$$\Rightarrow \sqrt{x^2+y^2} = (x+1) + i(y+2)$$

On squaring both sides, we get

$$x^2 + y^2 = (x+1)^2 + i^2(y+2)^2 + 2i(x+1)(y+2)$$

$$\Rightarrow x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4 + 2i(x+1)(y+2)$$

On comparing real and imaginary parts,

$$x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4$$

$\Rightarrow \quad 2y^2 = 2x - 4y - 3$ ...(2)

and $\quad 2(x+1)(y+2) = 0$

$(x+1) = 0$ or $(y+2) = 0$

$\Rightarrow \quad x = -1$ or $y = -2$

For $\quad x = -1$,

We get, $\quad 2y^2 = -2 - 4y - 3$

$2y^2 + 4y + 5 = 0$ [using Eq. (2)]

$$\Rightarrow y = \frac{-4 \pm \sqrt{16-2\times4\times5}}{4}$$

$$\Rightarrow y = \frac{-4 \pm \sqrt{-24}}{4} \notin R$$

Now, for $\quad y = -2$,

Then, $\quad 2(-2)^2 = 2x - 4(-2) - 3$ [using Eq. (2)]

$8 = 2x + 8 - 3$

$2x = 3$

$\Rightarrow \quad x = 3/2$

So, $\quad z = x + iy = 3/2 - 2i$

Long Answer Questions

1. We have $|z + 1| = z + 2(1 + i)$ ...(1)

Let $\quad z = x + iy$

Then, $|x + iy + 1| = x + iy + 2(1 + i)$

$\Rightarrow \quad |x + 1 + iy| = (x + 2) + i(y + 2)$

$$\Rightarrow \sqrt{(x+1)^2+y^2} = (x+2) + i(y+2) \quad \left[\because |z| = \sqrt{x^2+y^2}\right]$$

On squaring both sides, we get

$$(x+1)^2 + y^2 = (x+2)^2 + i^2(y+2)^2 + 2i(x+2)(y+2)$$

$$\Rightarrow x^2 + 2x + 1 + y^2 = x^2 + 4x + 4 - y^2 - 4y - 4 + 2i(x+2)(y+2)$$

$$\Rightarrow x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y + 2i(x+2)(y+2)$$

On comparing real and imaginary parts,we get

$$x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y$$

$\Rightarrow \quad 2y^2 - 2x + 4y + 1 = 0$ ...(2)

and $\quad 2(x+2)(y+2) = 0$

$\Rightarrow \quad x + 2 = 0$ or $y + 2 = 0$

$x = -2$ or $y = -2$ ...(3)

For $x = -2$, $2y^2 + 4 + 4y + 1 = 0$ [using Eq. (2)]

$\Rightarrow \quad 2y^2 + 4y + 5 = 0$

So $\quad D = b^2 - 4ac = 16 - 4 \times 2 \times 5 = -24 < 0$

$\Rightarrow \quad 2y^2 + 4y + 5$ has no real roots.

For $y = -2$, $2(-2)^2 - 2x + 4(-2) + 1 = 0$ [using Eq. (2)]

$\Rightarrow \quad 8 - 2x - 8 + 1 = 0$

$\Rightarrow \quad x = 1/2$

$$\therefore \quad z = x + iy = \frac{1}{2} - 2i$$

2. We have $\arg(z - 1) = \arg(z + 3i)$

and $\quad z = x + iy$

$\Rightarrow$ arg $(x + iy - 1)$ = arg $(x + iy + 3i)$

$\Rightarrow$ arg $(x - 1 + iy)$ = arg $[x + i(y + 3)]$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x-1} = \tan^{-1}\frac{y+3}{x}$$

$$\left[\because \arg(z) = \theta = \tan^{-1}\frac{y}{x}\right]$$

$$\Rightarrow \quad \frac{y}{x-1} = \frac{y+3}{x}$$

$\Rightarrow \quad xy = (x-1)(y+3)$

$\Rightarrow \quad xy = xy - y + 3x - 3$

$\Rightarrow \quad 3x - 3 = y$

$$\Rightarrow \quad \frac{3(x-1)}{y} = 1$$

$$\Rightarrow \quad \frac{x-1}{y} = \frac{1}{3}$$

Therefore, $(x - 1) : y = 1 : 3$

3. Let $z = x + iy$

Since $\left|\frac{z-2}{z-3}\right| = 2$

$$\Rightarrow \quad \left|\frac{x+iy-2}{x+iy-3}\right| = 2$$

$\Rightarrow \quad |x - 2 + iy| = 2\,|x - 3 + iy|$

$$\Rightarrow \quad \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2} \quad ...(1)$$

$$\left[\because |x+iy| = \sqrt{x^2+y^2}\right]$$

On squaring eqn. (1), we get

$x^2 - 4x + 4 + y^2 = 4(x^2 - 6x + 9 + y^2)$

$\Rightarrow 3x^2 + 3y^2 - 20x + 32 = 0$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0 \quad ...(2)$$

On comparing the above equation with

$x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$\Rightarrow \quad 2g = \frac{-20}{3} \Rightarrow g = \frac{-10}{3}$$

and $\quad 2f = 0 \Rightarrow f = 0$ and $c = \dfrac{32}{3}$

$\therefore \quad$ Centre $= (-g, -f) = (10/3, 0)$

Also radius $(r) = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(10/3)^2 + 0 - 32/3}$$

$$= \frac{1}{3}\sqrt{(100-96)} = 2/3$$

4. $z = x + iy$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1},\ z \neq -1$$

$$= \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$$

$$= \frac{(x^2-1) + iy(x+1) - iy(x-1) - i^2y^2}{(x+1)^2 - (iy)^2}$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{(x^2-1) + y^2 + i[y(x+1) - y(x-1)]}{(x+1)^2 + y^2}$$

As $\dfrac{z-1}{z+1}$ is a purely imaginary number

So $\quad \dfrac{(x^2-1) + y^2}{(x+1)^2 + y^2} = 0$

$\Rightarrow \quad x^2 - 1 + y^2 = 0$

$\Rightarrow \quad x^2 + y^2 = 1$

$\Rightarrow \quad \sqrt{x^2 + y^2} = \sqrt{1}$

$\Rightarrow \quad |z| = 1 \quad \left[\because |z| \sqrt{x^2 + y^2}\right]$

5. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers.

We have $\quad |z_1| = |z_2|$

and $\quad \arg(z_1) + \arg(z_2) = \pi$

Since $\quad |z_1| = |z_2|$

$\Rightarrow \quad r_1 = r_2 \quad ...(i)$

and if $\arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \quad \theta_1 + \theta_2 = \pi$

$\Rightarrow \quad \theta_1 = \pi - \theta_2$

Now, $\quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow \quad z_1 = r_2 [\cos(\pi - \theta_2) + i \sin(\pi - \theta_2)]$

$[\because r_1 = r_2 \text{ and } \theta_1 = (\pi - \theta_2)]$

$\Rightarrow \quad z_1 = r_2 (-\cos\theta_2 + i \sin\theta_2)$

$\Rightarrow \quad z_1 = -r_2 (\cos\theta_2 - i \sin\theta_2)$

$\Rightarrow \quad z_1 = -[r_2 (\cos\theta_2 - i \sin\theta_2)]$

$\Rightarrow \quad z_1 = -\overline{z}_2$

$[\because \overline{z}_2 = r_2 (\cos\theta_2 - i\sin\theta_2)]$

6. Let $z_1 = x + iy$

$\Rightarrow \quad |z_1| = \sqrt{x^2 + y^2} = 1 \qquad [\because |z_1| = 1]$

Now, $z_2 = \dfrac{z_1 - 1}{z_1 + 1} = \dfrac{x + iy - 1}{x + iy + 1}$

$= \dfrac{x - 1 + iy}{x + 1 + iy} = \dfrac{(x - 1 + iy)(x + 1 - iy)}{(x + 1 + iy)(x + 1 - iy)}$

$= \dfrac{x^2 - 1 - ixy + iy + ixy + iy + y^2}{(x+1)^2 + y^2}$

$= \dfrac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2} = \dfrac{1 - 1 + 2iy}{(x+1)^2 + y^2}$

$[\because x^2 + y^2 = 1]$

$= 0 + \dfrac{2yi}{(x+1)^2 + y^2}$

Hence, the real part of z_2 is zero.

7. Let $z_1 = r_1 (\cos\theta_1 + i \sin\theta_1)$,

Therefore, $z_2 = \overline{z}_1 = r_1 (\cos\theta_1 - i \sin\theta_1)$

$= r_1 [\cos(-\theta_1) + \sin(-\theta_1)]$

Also, let $z_3 = r_2 (\cos\theta_2 + i \sin\theta_2)$,

Therefore, $z_4 = \overline{z}_3 = r_2 (\cos\theta_2 - i \sin\theta_2)$

Then, $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right)$

$= \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3)$

$= \theta_1 - (-\theta_2) + (-\theta_1) - \theta_2 \qquad [\because \arg(z) = \theta]$

$= \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0$

8. Let $z_1 = r_1 (\cos\theta_1 + i \sin\theta_1)$

and $z_2 = r_2 (\cos\theta_2 + i \sin\theta_2)$

$\Rightarrow \quad \arg(z_1) = \theta_1 \text{ and } \arg(z_2) = \theta_2$

Since $\arg(z_1) - \arg(z_2) = 0$

$\theta_1 - \theta_2 = 0 \quad \Rightarrow \quad \theta_1 = \theta_2$

$z_2 = r_2 (\cos\theta_1 + i \sin\theta_1)$

$[\because \theta_1 = \theta_2]$

$z_1 - z_2 = (r_1 \cos\theta_1 - r_2 \cos\theta_1) + i (r_1 \sin\theta_1 - r_2 \sin\theta_1)$

$|z_1 - z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_1)^2 + (r_1\sin\theta_1 - r_2\sin\theta_1)^2}$

$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2(\sin^2\theta_1 + \cos^2\theta_1)}$

$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2} = \sqrt{(r_1 - r_2)^2}$

$\Rightarrow \quad |z_1 - z_2| = r_1 - r_2 \qquad [\because r = |z|]$

$= |z_1| - |z_2|$

Hence proved.

9. We have $\text{Re}(z^2) = 0, |z| = 2$

Let $z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

Now $\sqrt{x^2 + y^2} = 2$

$\Rightarrow \quad x^2 + y^2 = 4 \qquad \ldots(1)$

Since, $z = x + iy$

Squaring, we get

$\Rightarrow \quad z^2 = x^2 + 2ixy - y^2$

$\Rightarrow \quad z^2 = (x^2 - y^2) + 2ixy$

$\Rightarrow \quad \text{Re}(z^2) = x^2 - y^2$

$\Rightarrow \quad x^2 - y^2 = 0 \quad [\because \text{Re}(z^2) = 0] \qquad \ldots(2)$

From Eqs. (1) and (2),

$x^2 + x^2 = 4$

$\Rightarrow \quad 2x^2 = 4$

$\Rightarrow \quad x^2 = 2$

$\Rightarrow \quad x = \pm\sqrt{2}$

$\therefore \quad y = \pm\sqrt{2}$

Since $z = x + iy$

$\Rightarrow \quad z = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2}$

10. We have $z + \sqrt{2}\,|(z + 1)| + i = 0$

Let $z = x + iy \qquad \ldots(1)$

$\Rightarrow \quad x + iy + \sqrt{2}\ |x + iy + 1| + i = 0$

$\Rightarrow \quad x + i(1 + y) + \sqrt{2}\left[\sqrt{(x+1)^2 + y^2}\right] = 0$

$\Rightarrow \quad x + i(1 + y) + \sqrt{2}\sqrt{(x^2 + 2x + 1 + y^2)} = 0$

$\Rightarrow \quad x + \sqrt{2}\sqrt{x^2 + 2x + 1 + y^2} = 0$

On squaring both sides

$\Rightarrow \quad x^2 = 2(x^2 + 2x + 1 + y^2)$

$\Rightarrow \quad x^2 + 4x + 2y^2 + 2 = 0 \qquad ...(2)$

Also $\quad 1 + y = 0$

$\Rightarrow \quad y = -1$

For $y = -1,\ x^2 + 4x + 2 + 2 = 0$ [using Eq. (2)]

$\Rightarrow \quad x^2 + 4x + 4 = 0$

$\Rightarrow \quad (x + 2)^2 = 0$

$\Rightarrow \quad x + 2 = 0 \Rightarrow x = -2$

Therefore, $\quad z = x + iy = -2 - i$

11. We have

$$z = \frac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \frac{-\sqrt{2}\left[\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \frac{-\sqrt{2}\,[\cos(\pi - \pi/4) + i\sin(\pi - \pi/4)]}{\cos\pi/3 + i\sin\pi/3}$$

$$= \frac{-\sqrt{2}\,[\cos 3\pi/4 + i\sin 3\pi/4]}{\cos\pi/3 + i\sin\pi/3}$$

$$= -\sqrt{2}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right]$$

$$= -\sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$$

12. Let $\quad z = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$\quad w = r_2(\cos\theta_2 + i\sin\theta_2)$

So, $\quad |zw| = |z||w| = r_1 r_2 = 1$

Further, $\quad \arg(z) = \theta_1$ and $\arg(w) = \theta_2$

But $\quad \arg(z) - \arg(w) = \frac{\pi}{2}$

$\Rightarrow \quad \theta_1 - \theta_2 = \frac{\pi}{2}$

Now,

$\bar{z}w = r_1(\cos\theta_1 - i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2[\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)]$

$= r_1 r_2[\cos(-\pi/2) + i\sin(-\pi/2)]$

$= 1\,[0 - i]$

$\Rightarrow \bar{z}w = -i$