AEP STUDY CIRCLE
SINCE 2001...
IIT-NEET-CBSE
ACCENTS EDUCATIONAL PROMOTERS

ONLINE-OFFLINE LEARNING ACADEMY

24 YEARS OF LEGACY

XI IIT-JEE

LINEAR INEQUALITIES

MATHEMATICS

YOUR GATEWAY TO EXCELLENCE IN

IIT-JEE, NEET AND CBSE EXAMS

IIT-JEE
NEET
CBSE

LINEAR INEQUALITIES

THEORY AND ILLUSTRATIVE EXAMPLES

CONTACT US:
+91-9939586130
+91-9955930311

www.aepstudycircle.com

aepstudycircle@gmail.com

2ND FLOOR, SATKOUDI COMPLEX, THANA CHOWK, RAMGARH - 829122-JH

LINEAR INEQUALITIES

- Inequalities Involving Simple A.M., G.M., H.M.
- Inequalities Involving Arithmetic Mean of m^{th} Power
- Inequalities Involving Weighted Means

CONSTANT AND VARIABLES

In *mathematics*, a **variable** is a *value* that may change within the scope of a given problem or set of operations.

In contrast, a **constant** is a value that remains unchanged, though often unknown or undetermined.

Dependent and Independent Variables

Variables are further distinguished as being either a **dependent variable** or an **independent variable**. Independent variables are regarded as inputs to a system and may take on different values freely.

Dependent variables are those values that change as a consequence to changes in other values in the system.

When one value is completely determined by another, or of several others, then it is called a function of the other value or values. In this case the value of the function is a dependent variable and the other values are independent variables. The notation $f(x)$ is used for the value of the function f with x representing the independent variable.

For example, $y = f(x) = 3x^2$, here we can take x as any real value, hence x is independent variable. But value of y depends on value of x, hence y is dependent variable.

WHAT IS FUNCTION

To provide the classical understanding of functions, think of a *function* as a kind of machine. You feed the machine raw materials, and the machine changes the raw materials into a finished product based on a specific set of instructions. The kinds of functions we consider here, for the most part, take in a real number, change it in a formulaic way, and give out a real number (possibly the same as the one it took in). Think of this as an *input–output machine*; you give the function an input, and it gives you an output.

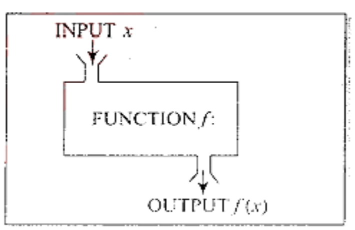


Fig. 1.1

For example, the squaring function takes the input 4 and gives the output value 16. The same squaring function takes the input 1 and gives the output value 1.

A function is always defined as "of a variable" which tells us what to replace in the formula for the function.

For example, $f(x) = 3x + 2$ tells us:

- The function f is a function of x.
- To evaluate the function at a certain number, replace the x with that number.
- Replacing x with that number in the right side of the function will produce the function's output for that certain input.
- In English, the above definition of f is interpreted, "Given *a* number, f will return *two more than the triple of that number*."

Thus, if we want to know the value (or output) of the function at 3:

$$f(x) = 3x + 2$$
$$f(3) = 3(3) + 2 = 11$$

Thus, the value of f at 3 is 11.

Note that $f(3)$ means the value of the dependent variable when "x" takes on the value of 3. So we see that the number 11 is the output of the function when we give the number 3 as the input. We refer to the input as the **argument** of the function (or the **independent variable**), and to the output as the **value** of the function at the given argument (or the **dependent variable**). A good way to think of it is the dependent variable $f(x)$ depends on the value of the independent variable x.

The formal definition of a function states that a function is actually a *rule* that associates elements of one set called the *domain* of the function with the elements of another set called the *range* of the function. For each value, we select from the domain of the function, there exists exactly one corresponding element in the range of the function. The definition of the function tells us which element in the range corresponds to the element we picked from the domain. Classically, the element picked from the domain is pictured as something that is fed into the function and the corresponding element in the range is pictured as the output. Since we "pick" the element in the domain whose corresponding element in the range we want to find, we have control over what element we pick and hence this element is also known as the "independent variable". The element mapped in the range is beyond our control and is "mapped to" by the function. This element is hence also known as the "dependent variable", for it depends on which independent variable we pick. Since the elementary idea of functions is better understood from the classical viewpoint, we shall use it hereafter. However, it is still important to remember the correct definition of functions at all times.

To make it simple, for the function $f(x)$, all of the possible x values constitute the domain, and all of the values $f(x)$ (y on the x-y plane) constitute the range.

Example 1.1 **A function is defined as $f(x) = x^2 - 3x$.**

(i) Find the value of $f(2)$.

(ii) Find the value of x for which $f(x) = 4$.

Sol.

(i) $f(2) = (2)^2 - 3(2) = -2$

(ii) $f(x) = 4$

$\Rightarrow x^2 - 3x = 4 \quad \Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x - 4)(x + 1) = 0 \quad \Rightarrow x = 4$ or -1

This means $f(4) = 4$ and $f(-1) = 4$.

Example 1.2 **If f is linear function and $f(2) = 4, f(-1) = 3$, then find $f(x)$.**

Sol. Let linear function is $f(x) = ax + b$

Given $f(2) = 4 \Rightarrow 2a + b = 4$ (1)

Also $f(-1) = 3 \Rightarrow -a + b = 3$ (2)

Solving (1) and (2) we get $a = \frac{1}{3}$ and $b = \frac{10}{3}$

Hence, $f(x) = \frac{x+10}{3}$

Example 1.3 **A function is defined as $f(x) = \frac{x^2+1}{3x-2}$. Can $f(x)$ take a value 1 for any real x?**

Also find the value/values of x for which $f(x)$ takes the value 2.

Sol. Here $f(x) = \frac{x^2+1}{3x-2} = 1$

$\Rightarrow x^2 + 1 = 3x - 2$

$\Rightarrow x^2 - 3x + 3 = 0.$

Now this equation has no real roots as $D < 0$.

Hence, value of $f(x)$ cannot be 1 for any real x.

For $f(x) = 2$ we have $\frac{x^2+1}{3x-2} = 2$

or $x^2 + 1 = 6x - 4$ or $x^2 - 6x + 5 = 0$

or $(x-1)(x-5) = 0$

or $x = 1, 5$

Example 1.4 **Find the values of x for which the following functions are defined. Also find all possible values which functions take.**

(i) $f(x) = \frac{1}{x+1}$ **(ii) $f(x) = \frac{x-2}{x-3}$** **(iii) $f(x) = \frac{2x}{x-1}$**

Sol.

(i) $f(x) = \frac{1}{x+1}$ is defined for all real values of x except when $x + 1 = 0$

Hence, $f(x)$ is defined for $x \in R - \{-1\}$.

Let $y = \frac{1}{x+1}$

Here we cannot find any real x for which $y = \frac{1}{x+1} = 0$

For y other than '0', there exists a real number x.

Hence, $\frac{1}{x+1} \in R - \{0\}$.

(ii) $f(x) = \frac{x-2}{x-3}$ is defined for all real values of x except when $x - 3 = 0$.

Hence, $f(x)$ is defined for $x \in R - \{3\}$

Let $y = \frac{x-2}{x-3}$

Here we cannot find any real x for which $y = \frac{x-2}{x-3} = 1$

Note: *When $\frac{x-2}{x-3} = 1$, we have $x - 2 = x - 3$ or $-2 = -3$ which is absurd.*

For y other than '1' there exists a real number x.

Hence, $\frac{1}{x+1} \in R - \{1\}$.

(iii) $f(x) = \frac{2x}{x-1}$ is defined for all real values of x except when $x - 1 = 0$

Hence, $f(x)$ is defined for $x \in R - \{1\}$

Let $y = \frac{2x}{x-1}$

Here we cannot find any real x for which $y = \frac{2x}{x-1} = 2$

Note: *When $\frac{2x}{x-1} = 2$, we have $2x = 2x - 2$ or $0 = -2$ which is absurd.*

For y other than '2' there exists a real number x.

Hence, $\frac{2x}{x-1} \in R - \{2\}$.

Example 1.5 **If $f(x) = \begin{cases} x^3, & x < 0 \\ 3x-2, & 0 \le x \le 2 \\ x^2+1, & x > 2 \end{cases}$, then find the value of $f(-1) + f(1) + f(3)$.**

Also find the value/values of x for which $f(x) = 2$.

Sol. Here function is differently defined for three different intervals mentioned.

For $x = -1$, consider $f(x) = x^3$

$\Rightarrow f(-1) = -1$

For $x = 1$, consider $f(x) = 3x - 2$

$\Rightarrow f(1) = 1$

For $x = 3$, consider $f(x) = x^2 + 1$

$\Rightarrow f(3) = 10$

$\Rightarrow f(-1) + f(1) + f(3) = -1 + 1 + 10 = 10$

Also when $f(x) = 2$,

for $x^3 = 2, x = 2^{1/3}$, which is not possible as $x < 0$.

for $3x - 2 = 2, x = 4/3$, which is possible as $0 \le x \le 2$.

For $x^2 + 1 = 2, x = \pm 1$, which is not possible as $x > 2$.

Hence, for $f(x) = 2$, we have $x = 4/3$.

INTERVALS

The set of numbers between any two real numbers is called interval. The following are the types of interval.

Closed Interval

$x \in [a, b] \equiv \{x : a \le x \le b\}$

Fig. 1.2

Open Interval

$x \in (a, b)$ or $]a, b[\equiv \{x : a < x < b\}$

Fig. 1.3

Semi-Open or Semi Closed Interval

$x \in [a, b]$ or $[a, b[= \{x: a \le x < b\}$

Fig. 1.4

$x \in]a, b]$ or $(a, b] = \{x: a < x \le b\}$

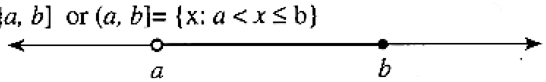


Fig. 1.5

Note:

- *A set of all real numbers can be expressed as* $(-\infty, \infty)$
- $x \in (-\infty, a) \cup (b, \infty) \Rightarrow x \in R - [a, b]$
- $x \in (-\infty, a] \cup [b, \infty) \Rightarrow x \in R - (a, b)$

INEQUALITIES

Some Important Facts about Inequalities

The following are some very useful points to remember:

(i) $a \le b$ either $a < b$ or $a = b$

(ii) $a < b$ and $b < c \Rightarrow a < c$ (transition property)

(iii) $a < b \Rightarrow -a > -b$, i.e., inequality sign reverses if both sides are multiplied by a negative number

(iv) $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$.

(v) If both sides of inequality are multiplied (or divided) by a positive number, inequality does not change. When both of its sides are multiplied (or divided) by a negative number, inequality gets reversed.

i.e., $a < b \Rightarrow ka < kb$ if $k > 0$ and $ka > kb$ if $k < 0$

(vi) $0 < a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$

(vii) $a + \frac{1}{a} \ge 2$ for $a > 0$ and equality holds for $a = 1$

(viii) $a + \frac{1}{a} \le -2$ for $a < 0$ and equality holds for $a = -1$

(ix) **Squaring an inequality:**
If $a < b$, then $a^2 < b^2$ does not follows always:
Consider the following illustrations:

$2 < 3 \Rightarrow 4 < 9$, but $-4 < 3 \Rightarrow 16 > 9$

Also if $x > 2 \Rightarrow x^2 > 4$, but for $x < 2 \Rightarrow x^2 \ge 0$

If $2 < x < 4 \Rightarrow 4 < x^2 < 16$

If $-2 < x < 4 \Rightarrow 0 \le x^2 < 16$

If $-5 < x < 4 \Rightarrow 0 \le x^2 < 25$

In fact $a < b \Rightarrow a^2 < b^2$ follows only when absolute value of a is less than the absolute value of b or distance of a from zero is less than the distance of b from zero on real number line.

(x) **Law of reciprocal:**
If both sides of inequality have same sign, while taking its reciprocal the sign of inequality gets reversed. i.e., $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$ and $a < b < 0 \Rightarrow \frac{1}{a} > \frac{1}{b}$

But if both sides of inequality have opposite sign, then while taking reciprocal sign of inequality does not change, i.e.

$$a < 0 < b \Rightarrow \frac{1}{a} < \frac{1}{b}$$

$$\text{If } x \in [a, b] \Rightarrow \begin{cases} \frac{1}{x} \in \left[\frac{1}{b}, \frac{1}{a}\right], \text{ if } a \text{ and } b \text{ have same sign} \\ \frac{1}{x} \in \left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right), \text{ if } a \text{ and } b \text{ have opposite signs} \end{cases}$$

Example 1.6 **Find the values of x^2 for the given values of x.**

(i) $x < 2$ **(ii) $x > -1$** **(iii) $x \ge 2$** **(iv) $x < -1$**

Sol.

(i) When $x < 2$ we have $x \in (-\infty, 0) \cup [0, 2)$
for $x \in [0, 2)$, $x^2 \in [0, 4)$
for $x \in (-\infty, 0)$, $x^2 \in (0, \infty)$
$\Rightarrow$ for $x < 2$, $x^2 \in [0, 4) \cup (0, \infty)$
$\Rightarrow$ $x \in [0, \infty)$

(ii) When $x > -1$ we have $x \in (-1, 0) \cup [0, \infty)$
for $x \in (-1, 0)$, $x^2 \in (0, 1)$
for $x \in [0, \infty)$, $x^2 \in [0, \infty)$
$\Rightarrow$ for $x > -1$, $x^2 \in (0, 1) \cup [0, \infty)$
$\Rightarrow$ $x \in [0, \infty)$

(iii) Here $x \in [2, \infty)$
$\Rightarrow$ $x^2 \in [4, \infty)$

(iv) Here $x \in (-\infty, -1)$
$\Rightarrow$ $x^2 \in (1, \infty)$

Example 1.7 **Find the values of $1/x$ for the given values of x.**

(i) $x > 3$ **(ii) $x < -2$** **(iii) $x \in (-1, 3) - \{0\}$**

Sol.

(i) We have $3 < x < \infty$

$\Rightarrow \frac{1}{3} > \frac{1}{x} > \frac{1}{\to \infty}$ ($\to \infty$ means tends to infinity)

$\Rightarrow 0 < \frac{1}{x} < \frac{1}{3}$

(ii) We have $-\infty < x < -2$

$$\Rightarrow \frac{1}{\to -\infty} > \frac{1}{x} > \frac{1}{-2}$$

$$\Rightarrow \frac{1}{\to -\infty} > \frac{1}{x} > \frac{1}{-2}$$

$$\Rightarrow 0 > \frac{1}{x} > -\frac{1}{2}$$

(iii) $x \in (-1, 3) - \{0\}$

$\Rightarrow \quad x \in (-1, 0) \cup (0, 3)$

For $x \in (-1, 0)$

$$\frac{1}{-1} > \frac{1}{x} > \frac{1}{\to 0^-}$$

(here $\to 0^-$ means value of x approaches to 0 from its left hand side or negative side)

$$\Rightarrow -1 > \frac{1}{x} > -\infty$$

$$\Rightarrow -\infty < \frac{1}{x} < -1 \qquad (1)$$

For $x \in (0, 3)$

$$\frac{1}{\to 0^+} > \frac{1}{x} > \frac{1}{3}$$

(here $\to 0^+$ means value of x approaches to 0 from its right hand side or positive side)

$$\Rightarrow \infty > \frac{1}{x} > \frac{1}{3}$$

$$\Rightarrow \frac{1}{3} < \frac{1}{x} < \infty \qquad (2)$$

From (1) and (2), $\frac{1}{x} \in (-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$

Note: For $x \in R - \{0\}$, $\frac{1}{x} \in R - \{0\}$

Example 1.8 **Find all possible values of the following expressions:**

(i) $\frac{1}{x^2+2}$ (ii) $\frac{1}{x^2-2x+3}$ (iii) $\frac{1}{x^2-x-1}$

Sol.

(i) We know that $x^2 \geq 0\ x \in R$.

$\Rightarrow \quad x^2 + 2 \geq 2, \forall x \in R$.

or $\quad 2 \leq (x^2+2) < \infty$

$$\Rightarrow \frac{1}{2} \geq \frac{1}{x^2+2} > 0$$

$$\Rightarrow 0 < \frac{1}{x^2+2} \leq \frac{1}{2}$$

(ii) $$\frac{1}{x^2-2x+3} = \frac{1}{(x-1)^2+2}$$

Now we know that $(x-1)^2 \geq 0\ \forall x \in R$.

$\Rightarrow \quad (x-1)^2 + 2 \geq 2\ \forall x \in R$.

or $\quad 2 \leq (x-1)^2 + 2 < \infty$

$$\Rightarrow \frac{1}{2} \geq \frac{1}{(x-1)^2+2} > 0$$

$$\Rightarrow \frac{1}{x^2-2x+3} \in \left(0, \frac{1}{2}\right]$$

(iii) $$\frac{1}{x^2-x-1} = \frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}}$$

$$\left(x-\frac{1}{2}\right)^2 \geq 0, \forall x \in R$$

$$\Rightarrow \left(x-\frac{1}{2}\right)^2 - \frac{5}{4} \geq -\frac{5}{4}, \forall x \in R$$

For $\frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}}$, we must have

$$\left(x-\frac{1}{2}\right)^2 - \frac{5}{4} \in \left[-\frac{5}{4}, 0\right) \cup (0, \infty)$$

$$\Rightarrow \frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}} \in \left(-\infty, -\frac{4}{5}\right] \cup (0, \infty)$$

Example 1.9 **Find all possible values of the following expressions:**

(i) $\sqrt{x^2-4}$ (ii) $\sqrt{9-x^2}$ (iii) $\sqrt{x^2-2x+10}$

Sol.

(i) $\sqrt{x^2-4}$

Least value of square root is 0 when $x^2 = 4$ or $x = \pm 2$. Also $x^2 - 4 \geq 0$

Hence, $\sqrt{x^2-4} \in [0, \infty)$.

(ii) $\sqrt{9-x^2}$

Least value of square root is 0 when $9 - x^2 = 0$ or $x = \pm 3$. Also, the greatest value of $9 - x^2$ is 9 when $x = 0$.

Hence, we have $0 \leq 9 - x^2 \leq 9 \Rightarrow \sqrt{9-x^2} \in [0, 3]$.

(iii) $\sqrt{x^2-2x+10} = \sqrt{(x-1)^2+9}$

Here, the least value of $\sqrt{(x-1)^2+9}$ is 3 when $x - 1 = 0$.

Also $(x-1)^2 + 9 \geq 9 \Rightarrow \sqrt{(x-1)^2+9} \geq 3$

Hence, $\sqrt{x^2-2x+10} \in [3, \infty)$.

GENERALIZED METHOD OF INTERVALS FOR SOLVING INEQUALITIES

Let $F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2}...(x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n}$

where $k_1, k_2, \ldots, k_n \in Z$ and $a_1, a_2, \ldots, a_n$ are fixed real numbers satisfying the condition

$$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n$$

For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm:

- We mark the numbers $a_1, a_2, \ldots, a_n$ on the number axis and put the plus sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n.
- Then we put the plus sign in the interval on the left of a_n if k_n is an even number and the minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule:
  - When passing through the point a_{n-1} the polynomial $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we have put the plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we have put the minus sign.

Frequently used Inequalities

(i) $(x-a)(x-b) < 0 \Rightarrow x \in (a, b)$, where $a < b$

(ii) $(x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$

(iii) $x^2 \le a^2 \Rightarrow x \in [-a, a]$

(iv) $x^2 \ge a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

(v) If $ax^2 + bx + c < 0$, $(a > 0) \Rightarrow x \in (\alpha, \beta)$, where α, β $(\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

(vi) If $ax^2 + bx + c > 0$, $(a > 0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where α, β $(\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

Example 1.10 **Solve $x^2 - x - 2 > 0$.**

Sol. $x^2 - x - 2 > 0$

$\Rightarrow$ $(x-2)(x+1) > 0$

Now $x^2 - x - 2 = 0 \Rightarrow x = -1, 2$.

Now on number line (x-axis) mark $x = -1$ and $x = 2$.

Now when $x > 2$, $x + 1 > 0$ and $x - 2 > 0$

$\Rightarrow$ $(x+1)(x-2) > 0$

when $-1 < x < 2$, $x + 1 > 0$ but $x - 2 < 0$

$\Rightarrow$ $(x+1)(x-2) < 0$

when $x < -1$, $x + 1 < 0$ and $x - 2 < 0$

$\Rightarrow$ $(x+1)(x-2) > 0$

Hence, sign scheme of $x^2 - x - 2$ is

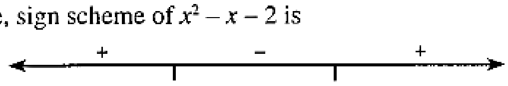


Fig. 1.6

From the figure, $x^2 - x - 2 > 0$, $x \in (-\infty, -1) \cup (2, \infty)$.

Example 1.11 **Solve $x^2 - x - 1 < 0$.**

Sol. Let's first factorize $x^2 - x - 1$.

For that let $x^2 - x - 1 = 0$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Now on number line (x-axis) mark $x = \dfrac{1 \pm \sqrt{5}}{2}$

Fig. 1.7

From the sign scheme of $x^2 - x - 1$ which shown in the given figure,

$$x^2 - x - 1 < 0 \Rightarrow x \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

Example 1.12 **Solve $(x-1)(x-2)(1-2x) > 0$.**

Sol. We have $(x-1)(x-2)(1-2x) > 0$

or $\quad -(x-1)(x-2)(2x-1) > 0$

or $\quad (x-1)(x-2)(2x-1) < 0$

On number line mark $x = 1/2, 1, 2$

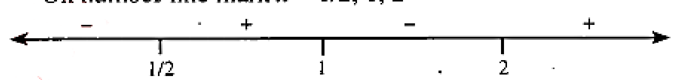


Fig. 1.8

When $x > 2$, all factors $(x-1)$, $(2x-1)$ and $(x-2)$ are positive.

Hence, $(x-1)(x-2)(2x-1) > 0$ for $x > 2$.

Now put positive and negative sign alternatively as shown in figure.

Hence, solution set of $(x-1)(x-2)(1-2x) > 0$ or $(x-1)(x-2)(2x-1) < 0$ is $(-\infty, 1/2) \cup (1, 2)$.

Example 1.13 **Solve $(2x + 1)(x - 3)(x + 7) < 0$.**

Sol. $(2x + 1)(x - 3)(x + 7) < 0$

Sign scheme of $(2x + 1)(x - 3)(x + 7)$ is as follows:

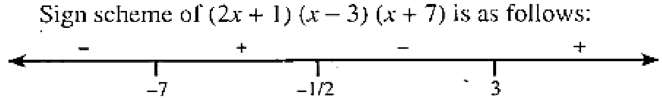


Fig. 1.9

Hence, solution is $(-\infty, -7) \cup (-1/2, 3)$.

Example 1.14 **Solve $\dfrac{2}{x} < 3$.**

Sol. $\dfrac{2}{x} < 3$

$\Rightarrow \quad \dfrac{2}{x} - 3 < 0$ (We cannot crossmultiply with x as x can be negative or positive)

$$\Rightarrow \quad \frac{2-3x}{x}<0$$

$$\Rightarrow \quad \frac{3x-2}{x}>0$$

$$\Rightarrow \quad \frac{(x-2/3)}{x}>0$$

Sign scheme of $\frac{(x-2/3)}{x}$ is as follows:

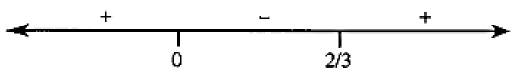


Fig. 1.10

$\Rightarrow \quad x \in (-\infty, 0) \cup (2/3, \infty)$

Example 1.15 Solve $\frac{2x-3}{3x-5} \geq 3$.

Sol. $\frac{2x-3}{3x-5} \geq 3$

$$\Rightarrow \quad \frac{2x-3}{3x-5}-3 \geq 0$$

$$\Rightarrow \quad \frac{2x-3-9x+15}{3x-5} \geq 0$$

$$\Rightarrow \quad \frac{-7x+12}{3x-5} \geq 0$$

$$\Rightarrow \quad \frac{7x-12}{3x-5} \leq 0$$

Sign scheme of $\frac{7x-12}{3x-5}$ is as follows:

Fig. 1.11

$\Rightarrow \quad x \in (5/3, 12/7]$

$x = 5/3$ is not included in the solutions as at $x = 5/3$ denominator becomes zero.

Example 1.16 Solve $x > \sqrt{(1-x)}$.

Sol. Given inequality can be solved by squaring both sides.

But sometimes squaring gives extraneous solutions which do not satisfy the original inequality. Before squaring we must restrict x for which terms in the given inequality are well defined.

$x > \sqrt{(1-x)}$. Here x must be positive.

Here $\sqrt{1-x}$ is defined only when $1-x \geq 0$ or $x \leq 1$ (1)

Squaring given inequality but sides $x^2 > 1-x$

$$\Rightarrow x^2+x-1>0 \Rightarrow \left(x-\frac{-1-\sqrt{5}}{2}\right)\left(x-\frac{-1+\sqrt{5}}{2}\right)>0$$

$$\Rightarrow x<\frac{-1-\sqrt{5}}{2} \text{ or } x>\frac{-1+\sqrt{5}}{2} \quad (2)$$

From (1) and (2) $x \in \left(\frac{\sqrt{5}-1}{2}, 1\right]$ (as x is +ve)

Example 1.17 Solve $\frac{2}{x^2-x+1}-\frac{1}{x+1}-\frac{2x-1}{x^3+1} \leq 0$.

Sol. $\frac{2}{x^2-x+1}-\frac{1}{x+1}-\frac{2x-1}{x^3+1} \geq 0$

$$\Rightarrow \quad \frac{2(x+1)-(x^2-x+1)-(2x-1)}{(x+1)(x^2-x+1)} \geq 0$$

$$\Rightarrow \quad \frac{-(x^2-x-2)}{(x+1)(x^2-x+1)} \geq 0$$

$$\Rightarrow \quad \frac{-(x-2)(x+1)}{(x+1)(x^2-x+1)} \geq 0$$

$$\Rightarrow \quad \frac{2-x}{x^2-x+1} \geq 0, \text{ where } x \neq 1$$

$\Rightarrow \quad 2-x \geq 0, x \neq -1$, (as $x^2-x+1>0$ for $\forall x \in R$)

$\Rightarrow \quad x \leq 2, x \neq -1$

Example 1.18 **Solve $x(x+2)^2(x-1)^5(2x-3)(x-3)^4 \geq 0$.**

Sol. Let $E = x(x+2)^2(x-1)^5(2x-3)(x-3)^4$.

Here for x, $(x-1)$, $(2x-3)$ exponents are odd, hence sign of E changes while crossing $x = 0, 1, 3/2$. Also for $(x+2)$, $(x-3)$ exponents are even, hence sign of E does not change while crossing $x = -2$ and $x = 3$.

Further for $x > 3$, all factors are positive, hence sign of the expression starts with positive sign from the right hand side.

The sign scheme of the expression is as shown in the following figure.

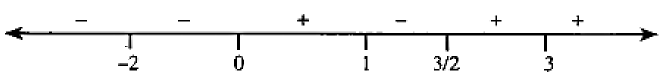


Fig. 1.12

Hence, for $E \geq 0$, we have $x \in [0, 1] \cup [3/2, \infty)$

Example 1.19 **Solve $x(2^x-1)(3^x-9)(x-3)<0$.**

Sol. Let $E = x(2^x-1)(3^x-9)(x-3)$

Here $2^x-1=0 \Rightarrow x=0$ and when $3^x-9=0 \Rightarrow x=2$

Now mark $x = 0, 2$ and 3 on real number line.

Sign of E starts with positive sign from right hand side.

Also at $x = 0$, two factors are 0, x and 2^x-1, hence sign of E does not change while crossing $x = 0$.

Sign scheme of E is as shown in the following figure.

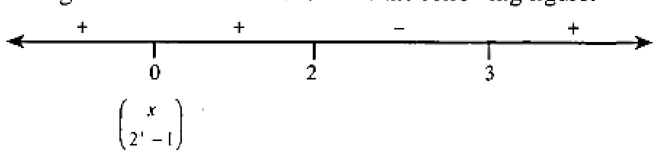


Fig. 1.13

From the figure, we have $E < 0$ for $x \in (2, 3)$.

Example 1.20 Find all possible values of $\frac{x^2+1}{x^2-2}$.

Sol. Let $y = \frac{x^2+1}{x^2-2}$

$\Rightarrow \quad yx^2 - 2y = x^2 + 1$

$\Rightarrow \quad x^2 = \frac{2y+1}{y-1}$

Now $x^2 \geq 0 \Rightarrow \frac{2y+1}{y-1} \geq 0$

Now $x^2 \geq 0 \Rightarrow \frac{2y+1}{y-1} \geq 0$

$\Rightarrow \quad y \leq -1/2$ or $y > 1$

Solving Irrational Inequalities

Example 1.21 Solve $\sqrt{x-2} \geq -1$.

Sol. We must have $x - 2 \geq 0$ for $\sqrt{x-2}$ to get defined, thus $x \geq 2$.

Now $\sqrt{x-2} \geq -1$, as square roots are always non-negative.

Hence, $x \geq 2$.

Note: *Some students solve it by squaring it both sides for which $x - 2 \geq 1$ or $x \geq 3$ which cause loss of interval [2, 3).*

Example 1.22 Solve $\sqrt{x-1} > \sqrt{3-x}$.

Sol. $\sqrt{x-1} > \sqrt{3-x}$ is meaningful if $x - 1 \geq 0$ and $3 - x \geq 0$

or $\quad 1 \leq x \leq 3$ (1)

Also $\quad \sqrt{x-1} > \sqrt{3-x}$

Squaring, we have $x - 1 > 3 - x$

$\Rightarrow \quad x > 2$ (2)

From (1) and (2), we have $2 < x \leq 3$.

Example 1.23 Solve $x + \sqrt{x} \geq \sqrt{x} - 3$.

Sol. $x + \sqrt{x} \geq \sqrt{x} - 3$ is meaningful only when $x \geq 0$ (1)

Now $x + \sqrt{x} \geq \sqrt{x} - 3$

$\Rightarrow \quad x \geq -3$ (2)

From (1) and (2), we have $x \geq 0$.

Example 1.24 Solve $(x^2 - 4)\sqrt{x^2-1} < 0$.

Sol. $(x^2 - 4)\sqrt{x^2-1} < 0$

We must have $x^2 - 1 \geq 0$

or $\quad (x - 1)(x + 1) \geq 0$

or $\quad x \leq -1$ or $x \geq 1$ (1)

Also $(x^2 - 4)\sqrt{x^2-1} < 0$

$\Rightarrow \quad x^2 - 4 < 0$

$\Rightarrow \quad -2 < x < 2$ (2)

From (1) and (2), we have $x \in (-2, -1] \cup [1, 2)$

ABSOLUTE VALUE OF x

Absolute value of any real number x is denoted by $|x|$ (read as modulus of x).

The absolute value is closely related to the notions of *magnitude*, *distance*, and *norm* in various mathematical and physical contexts.

From an *analytic geometry* point of view, the absolute value of a real number is that number's *distance* from zero along the *real number line*, and more generally the absolute value of the difference of two real numbers is the distance between them.

Let's look at the number line:

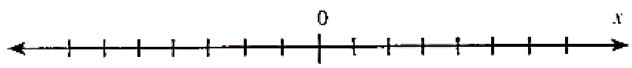


Fig. 1.14

The absolute value of x, denoted "$|x|$" (and which is read as "the absolute value of x"), is the distance of x from zero. This is why absolute value is never negative; absolute value only asks "how far?", not "in which direction?". This means not only that $|3| = 3$, because 3 is three units to the right of zero, but also that $|-3| = 3$, because -3 is three units to the left of zero.

When the number inside the absolute value (the "argument" of the absolute value) was positive anyway, we did not change the sign when we took the absolute value. But when the argument was negative, we did change the sign.

If $x > 0$ (that is, if x is positive), then the value would not change when you take the absolute value. For instance, if $x = 2$, then you have $|x| = |2| = 2 = x$. In fact, for *any* positive value of x (or if x equals zero), the sign would be unchanged, so:

For $x \geq 0$, $|x| = x$

On the other hand, if $x < 0$ (that is, if x is negative), then it will change its sign when you take the absolute value. For instance, if $x = -4$, then $|x| = |-4| = +4 = -(-4) = -x$. In fact, for *any* negative value of x, the sign would have to be changed, so:

For $x < 0$, $|x| = -x$

Thus $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Also $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

i.e., $2 = \sqrt{2^2} = \sqrt{(-2)^2} = \left[(-2)^2\right]^{1/2} = -2$ is absurd as $\sqrt{x^2} = |x|$

$\Rightarrow \sqrt{(-2)^2} = |-2| = 2$

Thus square root exists only for non-negative numbers and its value is also non-negative.

Some students consider $\sqrt{4} = \pm 2$, which is wrong.

In fact $\sqrt{(-4)^2} = |-4| = 4$

$\sqrt{(1-\sqrt{2})^2} = |1-\sqrt{2}| = \sqrt{2}-1$ etc.

Also some students write $\sqrt{x^2} = \pm x$ which is wrong, infact,

$$\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Also $a^2 < b^2 \Rightarrow \sqrt{a^2} < \sqrt{b^2} \Rightarrow |a| < |b|$

Graph of function $f(x) = y = |x|$

x	0	±1	±2	±3
y	0	1	4	9

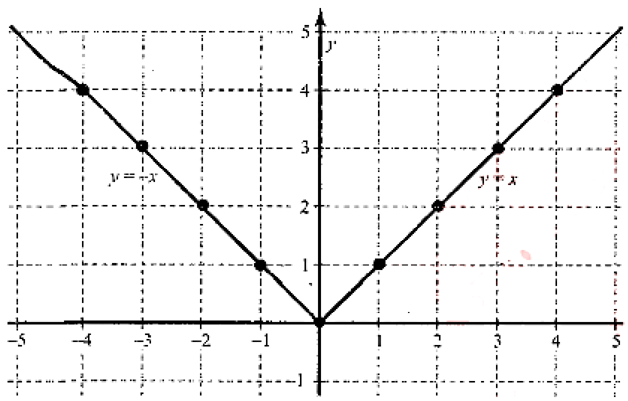


Fig. 1.15

We can see that graph of $y = |x|$ is in 1st and 2nd quadrant only where $y \geq 0$, hence $|x| \geq 0$.

Example 1.25 **Solve the following:**

(i) $|x| = 5$ **(ii) $x^2 - |x| - 2 = 0$**

Sol.

(i) $|x| = 5$, i.e., those points on real number line which are at distance 5 units from "0", which are –5 and 5.

Thus, $|x| = 5 \Rightarrow x = \pm 5$

(ii)

$x^2 - |x| - 2 = 0$

$\Rightarrow |x|^2 - |x| - 2 = 0$

$\Rightarrow (|x| - 2)(|x| + 1) = 0$

$\Rightarrow |x| = 2$ ($\because |x| + 1 \neq 0$)

$\Rightarrow x = \pm 2$

Example 1.26 **Find the value of x for which following expressions are defined:**

(i) $\dfrac{1}{\sqrt{x-|x|}}$ **(ii)** $\dfrac{1}{\sqrt{x+|x|}}$

Sol.

(i) $x - |x| = \begin{cases} x - x = 0, \text{ if } x \geq 0 \\ x + x = 2x, \text{ if } x < 0 \end{cases}$

$\Rightarrow x - |x| \leq 0$ for all x

$\Rightarrow \dfrac{1}{\sqrt{x-|x|}}$ does not take real values for any $x \in R$

$\Rightarrow \dfrac{1}{\sqrt{x-|x|}}$ is not defined for any $x \in R$.

(ii)

$x + |x| = \begin{cases} x + x = 2x, \text{ if } x \geq 0 \\ x - x = 0, \text{ if } x < 0 \end{cases}$

$\Rightarrow \dfrac{1}{\sqrt{x+|x|}}$ is defined only when $x > 0$

What is the geometric meaning of $|x - y|$?

$|x - y|$ is the distance between x and y on the real number line.

Example 1.27 **Solve the following:**

(i) $|x - 2| = 1$ **(ii)** $2|x + 1|^2 - |x + 1| = 3$

Sol.

(i) $|x - 2| = 1$, i.e., those points on real number line which are distance 1 units from 2.

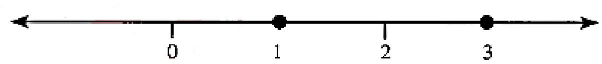


Fig. 1.16

As shown in the figure $x = 1$ and $x = 3$ are at distance 1 units from 2,

Hence, $x = 1$ or $x = 3$.

Thus $|x - 2| = 1$

$\Rightarrow x - 2 = \pm 1$

$\Rightarrow x = 1$ or $x = 3$

(ii)

$2|x + 1|^2 - |x + 1| = 3$

$\Rightarrow 2|x + 1|^2 - |x + 1| - 3 = 0$

$\Rightarrow 2|x + 1|^2 - 3|x + 1| + 2|x + 1| - 3 = 0$

$\Rightarrow (2|x + 1| - 3)(|x + 1| + 1) = 0$

$\Rightarrow 2|x + 1| - 3 = 0$

$\Rightarrow |x + 1| = 3/2$

$\Rightarrow x + 1 = \pm 3/2$

$\Rightarrow x = 1/2$ or $x = -5/2$

$$|x-a| = \begin{cases} x-a, & x \geq a \\ a-x, & x < a \end{cases}, \text{ where } a > 0$$

In general, $|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$, where $y = f(x)$ is any real-valued function.

Example 1.28 **Solve the following:**

(i) $|x - 2| = (x - 2)$

(ii) $|x + 3| = -x - 3$

(iii) $|x^2 - x| = x^2 - x$

(iv) $|x^2 - x - 2| = 2 + x - x^2$

Sol.

(i) $|x - 2| = (x - 2)$, if $x - 2 \geq 0$ or $x \geq 2$

(ii) $|x + 3| = -x - 3$, if $x + 3 \leq 0$ or $x \leq -3$

(iii) $|x^2 - x| = x^2 - x$, if $x^2 - x \geq 0$

$\Rightarrow \quad x(x-1) \geq 0$

$\Rightarrow \quad x \in (-\infty, 0] \cup [1, \infty)$

(iv) $|x^2 - x - 2| = 2 + x - x^2$

$\Rightarrow \quad x^2 - x - 2 \leq 0$

$\Rightarrow \quad (x-2)(x+1) \leq 0$

$\Rightarrow \quad -1 \leq x \leq 2$

Example 1.29 Solve $1 - x = \sqrt{x^2 - 2x + 1}$.

Sol. $1 - x = \sqrt{x^2 - 2x + 1}$

$\Rightarrow \quad 1 - x = \sqrt{(x-1)^2}$

$\Rightarrow \quad 1 - x = |x - 1|$

$\Rightarrow \quad 1 - x \geq 0$

$\Rightarrow \quad x \leq 1$

Example 1.30 Solve $|3x - 2| = x$.

Sol. $|3x - 2| = x$

Case (i)

When $3x - 2 \geq 0$ or $x \geq 2/3$

For which we have $3x - 2 = x$ or $x = 1$.

Case (ii)

When $3x - 2 < 0$ or $x < 2/3$

For which we have $2 - 3x = x$ or $x = 1/2$.

Hence, solution set is $\{1/2, 1\}$.

Example 1.31 Solve $|x| = x^2 - 1$.

Sol. $x^2 - 1 = |x|$

$\Rightarrow \quad x^2 - 1 = x$ when $x \geq 0$

or $\quad x^2 - 1 = -x$ when $x < 0$

$$x^2 - x - 1 = 0 \Rightarrow x = \frac{1+\sqrt{5}}{2} \quad (\text{as } x \geq 0)$$

$$x^2 + x - 1 = 0 \Rightarrow x = \frac{-1-\sqrt{5}}{2} \quad (\text{as } x < 0)$$

Example 1.32 Solve

$$\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$$

Sol. $\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$

$\Rightarrow \quad \sqrt{x - 1 - 4\sqrt{x-1} + 4} + \sqrt{x - 1 - 6\sqrt{x-1} + 9} = 1$

$\Rightarrow \quad \sqrt{|\sqrt{x-1} - 2|^2} + \sqrt{|\sqrt{x-1} - 3|^2} = 1$

$\Rightarrow \quad |\sqrt{x-1} - 2| + |\sqrt{x-1} - 3| = 1$

$\Rightarrow \quad |\sqrt{x-1} - 2| + |\sqrt{x-1} - 3| = (\sqrt{x-1} - 2) - (\sqrt{x-1} - 3)$

$\Rightarrow \quad \sqrt{x-1} - 2 \geq 0$ and $\sqrt{x-1} - 3 \leq 0$

$\Rightarrow \quad 2 \leq \sqrt{x-1} \leq 3$

$\Rightarrow \quad 4 \leq x - 1 \leq 9$

$\Rightarrow \quad 5 \leq x \leq 10$

Example 1.33 Prove that

$$\sqrt{x^2 + 2x + 1} - \sqrt{x^2 - 2x + 1} = \begin{cases} -2, & x < -1 \\ 2x, & -1 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$$

Sol. $\sqrt{x^2 + 2x + 1} - \sqrt{x^2 - 2x + 1}$

$$= \sqrt{(x+1)^2} - \sqrt{(x-1)^2}$$

$$= |x+1| - |x-1|$$

$$= \begin{cases} -x - 1 - (1 - x), & x < -1 \\ x + 1 - (1 - x), & -1 \leq x \leq 1 \\ x + 1 - (x - 1), & x > 1 \end{cases}$$

$$= \begin{cases} -2, & x < -1 \\ 2x, & -1 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$$

Example 1.34

(i) **For $2 < x < 4$, find the values of $|x|$.**

(ii) **For $-3 \leq x \leq -1$, find the values of $|x|$.**

(iii) **For $-3 \leq x < 1$, find the values of $|x|$.**

(iv) **For $-5 < x < 7$, find the values of $|x - 2|$.**

(v) **For $1 \leq x \leq 5$, find the values of $|2x - 7|$.**

Sol.

(i) $2 < x < 4$

Here values on real number line whose distance lies between 2 and 4.

Here values of x are positive $\Rightarrow |x| \in (2, 4)$

(ii) $-3 \leq x \leq -1$

Here values on real number line whose distance lies between 1 and 3 or at distance 1 or 3.

$\Rightarrow 1 \leq |x| \leq 3$

(iii) $-3 \leq x < 1$

For $-3 \leq x < 0$, $|x| \in (0, 3]$

For $0 \leq x < 1$, $|x| \in [0, 1)$

So for $-3 \leq x < 1$, $|x| \in [0, 1) \cup (0, 3]$ or $x \in [0, 3]$

(iv) $-5 < x < 7$

$\Rightarrow -7 < x - 2 < 5$

$\Rightarrow 0 \leq |x - 2| < 7$

(v) $1 \le x \le 5$

$\Rightarrow 2 \le 2x \le 10$

$\Rightarrow -5 \le 2x - 7 \le 3$

$\Rightarrow |2x-7| \in [0, 5]$

Example 1.35 For $x \in R$, find all possible values of

(i) $|x-3|-2$ **(ii)** $4-|2x+3|$

Sol.

(i) We know that $|x-3| \ge 0 \ \forall x \in R$

$\Rightarrow |x-3|-2 \ge -2$

$\Rightarrow |x-3|-2 \in [-2, \infty)$

(ii) We know that $|2x+3| \ge 0 \ \forall x \in R$

$\Rightarrow -|2x+3| \le 0$

$\Rightarrow 4-|2x+3| \le 4$

or $4-|2x+3| \in (-\infty, 4]$

Example 1.36 Find all possible values of

(i) $\sqrt{|x|-2}$ (ii) $\sqrt{3-|x-1|}$ (iii) $\sqrt{4-\sqrt{x^2}}$

Sol.

(i) $\sqrt{|x|-2}$

We know that square roots are defined for non-negative values only.

It implies that we must have $|x|-2 \ge 0$.

$\Rightarrow \sqrt{|x|-2} \ge 0$

(ii) $\sqrt{3-|x-1|}$ is defined when $3-|x-1| \ge 0$

But the maximum value of $3-|x-1|$ is 3 when $|x-1|$ is 0.

Hence, for $\sqrt{3-|x-1|}$ to get defined, $0 \le 3-|x-1| \le 3$.

$\Rightarrow \sqrt{3-|x-1|} \in \left[0, \sqrt{3}\right]$

Alternatively, $|x-1| \ge 0$

$\Rightarrow -|x-1| \le 0$

$\Rightarrow 3-|x-1| \le 3$

But for $\sqrt{3-|x-1|}$ to get defined, we must have

$0 \le 3-|x-1| \le 3 \Rightarrow 0 \le \sqrt{3-|x-1|} \le \sqrt{3}$

(iii) $\sqrt{4-\sqrt{x^2}} = \sqrt{4-|x|}$

$|x| \ge 0$

$\Rightarrow -|x| \le 0$

$\Rightarrow 4-|x| \le 4$

But for $\sqrt{4-|x|}$ to get defined $0 \le 4-|x| \le 4$

$\Rightarrow 0 \le \sqrt{4-|x|} \le 2$

Example 1.37 Solve $|x-3|+|x-2|=1$.

Sol. $|x-3|+|x-2|=1$

$\Rightarrow |x-3|+|x-2| = (3-x)+(x-2)$

$\Rightarrow x-3 \le 0$ and $x-2 \ge 0$

$\Rightarrow x \le 3$ and $x \ge 2$

$\Rightarrow 2 \le x \le 3$

Inequalities Involving Absolute Value

(i) $|x| \le a$ (where $a > 0$)

It implies those values of x on real number line which are at distance a or less than a.

Fig. 1.17

$\Rightarrow -a \le x \le a$

e.g. $|x| \le 2 \Rightarrow -2 \le x \le 2$

$|x| < 3 \Rightarrow -3 < x < 3$

In general, $|f(x)| \le a$ (where $a > 0$) $\Rightarrow -a \le f(x) \le a$.

(ii) $|x| \ge a$ (where $a > 0$)

It implies those values of x on real number line which are at distance a or more than a

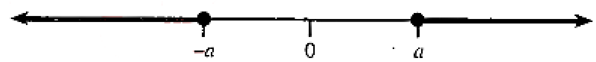


Fig. 1.18

$\Rightarrow x \le -a$ or $x \ge a$

e.g. $|x| \ge 3 \Rightarrow x \le -3$ or $x \ge 3$.

$|x| > 2 \Rightarrow x < -2$ or $x > 2$

In general, $|f(x)| \ge a \Rightarrow f(x) \le -a$ or $f(x) \ge a$.

(iii) $a \le |x| \le b$ (where $a, b > 0$)

It implies those value of x on real number line which are at distance equal a or b or between a and b.

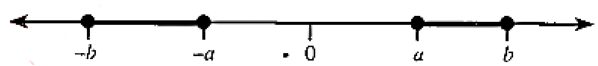


Fig. 1.19

$\Rightarrow [-b, -a] \cup [a, b]$

e.g. $2 \le |x| \le 4 \Rightarrow x \in [-4, -2]$ [2, 4]

(iv) $|x+y| < |x|+|y|$ if x and y have opposite signs.

$|x-y| < |x|+|y|$ if x and y have same sign.

$|x+y| = |x|+|y|$ if x and y have same sign or at least one of x and y is zero.

$|x-y| = |x|+|y|$ if x and y have opposite signs or at least one of x and y is zero.

Example 1.38 Solve $x^2 - 4|x| + 3 < 0$.

Sol. $x^2 - 4|x| + 3 < 0$

$\Rightarrow (|x|-1)(|x|-3) < 0$

$\Rightarrow 1 < |x| < 3$

$\Rightarrow -3 < x < -1$ or $1 < x < 3$

$\Rightarrow x \in (-3, -1) \cup (1, 3)$

Example 1.39 Solve $0 < |x| < 2$.

Sol. We know that $|x| \ge 0, \ \forall x \in R$

But given $|x| > 0 \Rightarrow x \ne 0$

Now $0 < |x| < 2$

$\Rightarrow x \in (-2, 2), x \ne 0$

$\Rightarrow$ $x \in (-2, 2) - \{0\}$

Example 1.40 **Solve $|3x - 2| < 4$.**

Sol. $|3x - 2| < 4$

$\Rightarrow$ $-4 < 3x - 2 < 4$

$\Rightarrow$ $-2 < 3x < 6$

$\Rightarrow$ $-2/3 < x < 2$

Example 1.41 **Solve $1 \le |x - 2| \le 3$.**

Sol. $1 \le |x - 2| \le 3$

$\Rightarrow$ $-3 \le x - 2 \le -1$ or $1 \le x - 2 \le 3$

$\Rightarrow$ $-1 \le x \le 1$ or $3 \le x \le 5$

$\Rightarrow$ $x \in [-1, 1] \cup [3, 5]$

Example 1.42 **Solve $0 < |x - 3| \le 5$.**

Sol. $0 < |x - 3| \le 5$

$\Rightarrow$ $-5 \le x - 3 < 0$ or $0 < x - 3 \le 5$

$\Rightarrow$ $-2 \le x < 3$ or $3 < x \le 8$

$\Rightarrow$ $x \in [-2, 3) \cup (3, 8]$

Example 1.43 **Solve $||x - 1| - 2| < 5$.**

Sol. $||x - 1| - 2| < 5$

$\Rightarrow$ $-5 < |x - 1| - 2 < 5$

$\Rightarrow$ $-3 < |x - 1| < 7$

$\Rightarrow$ $|x - 1| < 7$

$\Rightarrow$ $-7 < x - 1 < 7$

$\Rightarrow$ $-6 < x < 8$

Example 1.44 **Solve $|x - 3| \ge 2$.**

Sol. $|x - 3| \ge 2$

$\Rightarrow$ $x - 3 \le -2$ or $x - 3 \ge 2$

$\Rightarrow$ $x \le 1$ or $x \ge 5$

Example 1.45 **Solve $||x| - 3| > 1$.**

Sol. $||x| - 3| > 1$

$\Rightarrow$ $|x| - 3 < -1$ or $|x| - 3 > 1$

$\Rightarrow$ $|x| < 2$ or $|x| > 4$

$\Rightarrow$ $-2 < x < 2$ or $x < -4$ or $x > 4$

Example 1.46 **Solve $|x - 1| + |x - 2| \ge 4$.**

Sol. Let $f(x) = |x - 1| + |x - 2|$

A.	B. $f(x)$	C. $f(x) \ge 4$	D. $A \cap C$
$x < 1$	$1 - x + 2 - x = 3 - 2x$	$3 - 2x \ge 4 \Rightarrow x \le -2/3$	$x \le -1/2$
$1 \le x \le 2$	$x - 1 + 2 - x = 1$	$1 \ge 4$, not possible	
$x > 2$	$x - 1 + x - 2 = 2x - 3$	$2x - 3 \ge 4 \Rightarrow x \ge 7/2$	$x \ge 7/2$

Hence, solutions is $x \in (-\infty, -1/2] \cup [7/2, \infty)$.

Example 1.47 **Solve $|x + 1| + |2x - 3| = 4$.**

Sol. Let $f(x) = |x + 1| + |2x - 3|$

A.	B. $f(x)$	C. $f(x) \ge 4$	D. $A \cap C$
$x < -1$	$-1 - x + 3 - 2x$	$2 - 3x = 4 \Rightarrow x = -2/3$	No such x exists
$-1 \le x \le 3/2$	$x + 1 + 3 - 2x$	$4 - x = 4 \Rightarrow x = 0$	$x = 0$
$x > 3/2$	$x + 1 + 2x - 3$	$3x - 2 = 4 \Rightarrow x = 2$	$x = 2$

Hence, solutions set is $\{0, 2\}$

Example 1.48 **Solve $|x| + |x - 2| = 2$.**

Sol. We have $|x| + |x - 2| = 2$

$\Rightarrow$ $|x| + |x - 2| = x - (x - 2)$

$\Rightarrow$ $x(x - 2) \le 0$

$\Rightarrow$ $0 \le x \le 2$

Example 1.49 **Solve $|2x - 3| + |x - 1| = |x - 2|$.**

Sol. $|2x - 3| + |x - 1| = |(2x - 3) - (x - 1)|$

$\Rightarrow$ $(2x - 3)(x - 1) \le 0$

$\Rightarrow$ $1 \le x \le 3/2$

Example 1.50 **Solve $|x^2 + x - 4| = |x^2 - 4| + |x|$.**

Sol. $|x^2 + x - 4| = |x^2 - 4| + |x|$

$\Rightarrow$ $x(x^2 - 4) \ge 0$

$\Rightarrow$ $x(x - 2)(x + 2) \ge 0$

$\Rightarrow$ $x \in [-2, 0] \cup [2, \infty)$

Example 1.51 **If $|\sin x + \cos x| = |\sin x| + |\cos x|$ ($\sin x, \cos x \ne 0$), then in which quadrant does x lie?**

Sol. Here we have $|\sin x + \cos x| = |\sin x| + |\cos x|$.
It implies that $\sin x$ and $\cos x$ must have the same sign.
Therefore, x lies in the first or third quadrant.

Example 1.52 **Is $|\tan x + \cot x| < |\tan x| + |\cot x|$ true for any x ? If it is true, then find the values of x.**

Sol. Since $\tan x$ and $\cot x$ have always the same sign, $|\tan x + \cot x| < |\tan x| + |\cot x|$ does not hold true for any value of x.

Example 1.53 **Solve $\left|\frac{x-3}{x+1}\right| \le 1$.**

Sol. $\left|\frac{x-3}{x+1}\right| \le 1$

$\Rightarrow$ $-1 \le \frac{x-3}{x+1} \le 1$

$\Rightarrow$ $\frac{x-3}{x+1} - 1 \le 0$ and $0 \le \frac{x-3}{x+1} + 1$

$\Rightarrow$ $\frac{-4}{x+1} \le 0$ and $0 \le \frac{2x-2}{x+1}$

$\Rightarrow\ x > -1$ and $\{x < -1$ or $x \geq 1\}$
$\Rightarrow\ x \geq 1$

Example 1.54 Solve $|x^2 - 2x| + |x - 4| > |x^2 - 3x + 4|$.

Sol. We have $|x^2 - 2x| + |4 - x| > |x^2 - 2x + 4 - x|$
$\Rightarrow\ (x^2 - 2x)(4 - x) < 0$
$\Rightarrow\ x\ (x - 2)(x - 4) > 0$
$\Rightarrow\ x \in (0, 2) \cup (4, \infty)$

Concept Application Exercise 1.1

1. If $f(x) = \begin{cases} x+3, & x<1 \\ x^2, & 1 \leq x \leq 3 \\ 2-3x, & x>3 \end{cases}$, then which of the following is greatest ?
$f(0), f(3), f(4), f(2)$
2. If $f(x)$ is quadratic function such that $f(0) = -4$, $f(1) = -5$ and $f(-1) = -1$, then find the value of $f(3)$.
3. Find the value of x^2 for the following values of x:
(i) $[-5, -1]$ (ii) $(3, 6)$
(iii) $(-2, 3]$ (iv) $(-3, \infty)$ (v) $(-\infty, 4)$
4. Find the values of $1/x$ for the following values of x:
(i) $(2, 5)$ (ii) $[-5, -1)$
(iii) $(3, \infty)$ (iv) $(-\infty, -2]$
(v) $[-3, 4]$
5. Which of the following is always true?
(a) If $a < b$, then $a^2 < b^2$
(b) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$
(c) If $a < b$, then $|a| < |b|$
6. Find the values of x which satisfy the inequalities simultaneously:
(i) $-3 < 2x - 1 < 19$ (ii) $-1 \leq \frac{2x+3}{5} \leq 3$
7. Find all the possible values which the following expressions take.
(i) $\frac{2-5x}{3x-4}$
(ii) $\sqrt{x^2 - 7x + 6}$
(iii) $\frac{x^2 - x - 6}{x - 3}$
8. Solve $\frac{x(3-4x)(x+1)}{(2x-5)} < 0$.
9. Solve $\frac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2 x^5} \leq 0$.
10. Solve $\frac{(x-3)(x+5)(x-7)}{|x-4|(x+6)} \leq 0$.
11. Find all possible values of $f(x) = \frac{1-x^2}{x^2+3}$.
12. Solve $\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$.
13. Solve (i) $\frac{\sqrt{x-1}}{x-2} < 0$ (ii) $\sqrt{x-2} \leq 3$
14. Which of the following equations has maximum number of real roots?
(i) $x^2 - |x| - 2 = 0$
(ii) $x^2 - 2|x| + 3 = 0$
(iii) $x^2 - 3|x| + 2 = 0$
(iv) $x^2 + 3|x| + 2 = 0$
15. Find the number of solutions of the system of equation $x + 2y = 6$ and $|x - 3| = y$.
16. Find the values of x for which $f(x) = \sqrt{\frac{1}{|x-2|-(x-2)}}$ is defined.
17. Find all values of x for which $f(x) = x + \sqrt{x^2}$.
18. Solve $\left|\frac{x+2}{x-1}\right| = 2$.
19. If $|x^2 - 7| \leq 9$, then find the values of x.
20. Find the values of x for which $\sqrt{5-|2x-3|}$ is defined.
21. Solve $||x - 2| - 3| < 5$.
22. Which of the following is/are true?
(a) If $|x + y| = |x| + |y|$, then points (x, y) lie in 1st or 3rd quadrant or any of the x-axis or y-axis.
(b) If $|x + y| < |x| + |y|$, then points (x, y) lie in 2nd or 4th quadrant.
(c) If $|x - y| = |x| + |y|$, then points (x, y) lie in 2nd or 4th quadrant.
23. Solve $|x^2 - x - 2| + |x + 6| = |x^2 - 2x - 8|$.
24. Solve $|x| = 2x - 1$.
25. Solve $|2^x - 1| + |2^x + 1| = 2$.
26. Solve $|x^2 - 4x + 3| = x + 1$.
27. Solve $|x^2 - 1| + |x^2 - 4| > 3$.
28. Solve $|x - 1| - |2x - 5| = 2x$.

SOME DEFINITIONS

Real Polynomial

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is called a real polynomial of real variable x with real coefficients.

Complex Polynomial

If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is called a complex polynomial or a polynomial of complex coefficients.

Rational Expression or Rational Function

An expression of the form

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials in x is called a rational expression.

In the particular case when $Q(x)$ is a non-zero constant,

$$\frac{P(x)}{Q(x)}$$

reduces to a polynomial. Thus every polynomial is a rational expression but the converse is not true. Some of the examples are as follows:

(1) $\frac{x^2-5x+4}{x-2}$ (2) x^2-5x+4

(3) $\frac{1}{x-2}$ (4) $x+\frac{1}{x}$, i.e., $\frac{x^2+1}{x}$.

Degree of a Polynomial

A polynomial $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$.

The polynomials $2x^3-7x^2+x+5$ and $(3-2i)x^2-ix+5$ are polynomials of degree 3 and 2, respectively

A polynomial of second degree is generally called a quadratic polynomial, and polynomials of degree 3 and 4 are known as cubic and bi-quadratic polynomials, respectively.

Polynomial Equation

If $f(x)$ is a polynomial, then $f(x)=0$ is called a polynomial equation.

If $f(x)$ is a quadratic polynomial, then $f(x)=0$ is called a quadratic equation. The general form of a quadratic equation is $ax^2+bx+c=0$, $a \neq 0$. Here, x is the variable and a, b and c are called coefficients, real or imaginary.

Roots of an Equation

The values of the variable satisfying a given equation are called its roots.

Thus, $x=\alpha$ is a root of the equation $f(x)=0$, if $f(\alpha)=0$.

For example, $x=1$ is a root of the equation $x^3-6x^2+11x-6=0$, because $1^3-6\times1^2+11\times1-6=0$.

Similarly, $x=\omega$ and $x=\omega^2$ are roots of the equation $x^2+x+1=0$ as they satisfy it (where ω is the complex cube root of unity).

Solution Set

The set of all roots of an equation, in a given domain, is called the solution set of the equation.

For example, the set {1, 2, 3} is the solution set of the equation $x^3-6x^2+11x-6=0$.

Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining all its roots.

Example 1.55 **If $x=1$ and $x=2$ are solutions of the equation $x^3+ax^2+bx+c=0$ and $a+b=1$, then find the value of b.**

Sol. Since $x=1$ is a root of the given equation it satisfies the equation.

Hence, putting $x=1$ in the given equation, we get

$$a+b+c=-1 \quad (1)$$

but given that

$$a+b=1 \quad (2)$$

$\Rightarrow$ $c=-2$

Now put $x=2$ in the given equation, we have

$8+4a+2b-2=0$

$\Rightarrow$ $6+2a+2(a+b)=0$

$\Rightarrow$ $6+2a+2=0$

$\Rightarrow$ $a=-4$

$\Rightarrow$ $b=5$

Example 1.56 **Let $f(x)=ax^2+bx+c$, where $a, b, c \in R$ and $a \neq 0$. It is known that $f(5)=-3f(2)$ and that 3 is a root of $f(x)=0$, then find the other root of $f(x)=0$.**

Sol. $f(x)=ax^2+bx+c$

Given that $f(5)=-3f(2)$

$25a+5b+c=-3(4a+2b+c)$

or $37a+11b+4c=0$ (1)

Also $x=3$ satisfies $f(x)=0$

$\therefore$ $9a+3b+c=0$ (2)

or $36a+12b+4c=0$ (3)

[Multiplying Eq. (2) by 4]

Subtracting (3) from (1), we have

$a-b=0$

$\Rightarrow$ $a=b$ $\Rightarrow$ In (2) put $b=a$,

$\Rightarrow$ $12a+c=0$ or $c=-12a$

Hence, equation $f(x)=0$ becomes

$ax^2+ax-12a=0$

or $x^2+x-12=0$

or $(x-3)(x+4)=0$ or $x=-4, 3$

Example 1.57 **A polynomial in x of degree three vanishes when $x=1$ and $x=-2$, and has the values 4 and 28 when $x=-1$ and $x=2$, respectively. Then find the value of polynomial when $x=0$.**

Sol. From the given data $f(x)=(x-1)(x+2)(ax+b)$

Now $f(-1)=4$ and $f(2)=28$

$\Rightarrow$ $(-1-1)(-1+2)(-a+b)=4$

and $(2-1)(2+2)(2a+b)=28$

$\Rightarrow$ $a-b=2$ and $2a+b=7$

Solving, $a=3$ and $b=1$

$\Rightarrow$ $f(x)=(x-1)(x+2)(3x+1)$

$\Rightarrow$ $f(0)=-2$

Example 1.58 **If $(1-p)$ is a root of quadratic equation $x^2+px+(1-p)=0$, then find its roots.**

Sol. Since $(1-p)$ is the root of quadratic equation

$$x^2+px+(1-p)=0 \quad (1)$$

So $(1-p)$ satisfies the above equation

$\therefore$ $(1-p)^2+p(1-p)+(1-p)=0$

$\therefore$ $(1-p)[1-p+p+1]=0$

$\therefore$ $(1-p)(2)=0$

$\Rightarrow$ $p=1$

On putting this value of p in Eq. (1), we get

$x^2+x=0$

$\Rightarrow$ $x(x+1)=0$

$\Rightarrow$ $x=0, -1$

Example 1.59 The quadratic polynomial $p(x)$ has the following properties:

$p(x)$ can be positive or zero for all real numbers

$p(1) = 0$ and $p(2) = 2$.

Then find the quadratic polynomial.

Sol. $p(x)$ is positive or zero for all real numbers

also $\quad p(1) = 0$

then we have $p(x) = k(x-1)^2$, where $k > 0$

Now $\quad p(2) = 2$

$\Rightarrow \quad k = 2$

$\therefore \quad p(x) = 2(x-1)^2$

GEOMETRICAL MEANING OF ROOTS (ZEROS) OF AN EQUATION

We know that that a real number k is a zero of the polynomial $f(x)$ if $f(k) = 0$. But why are the zeroes of a polynomial so important? To answer this, first we will see the *geometrical* representations of polynomials and the geometrical meaning of their zeroes.

We know that graph of the linear function $y = f(x) = ax + b$ is a straight line.

Consider the function $f(x) = x + 3$.

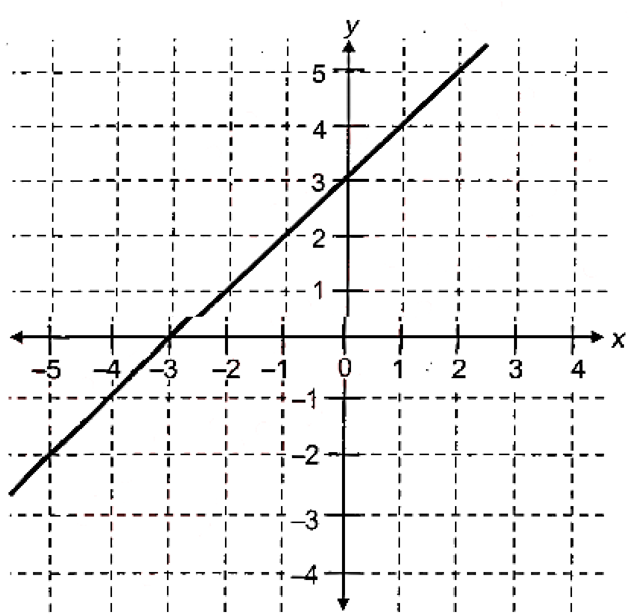


Fig. 1.20

Now we can see that this graph cuts the x-axis at $x = -3$, where value of $y = 0$ or we can say $x + 3 = 0$ (or $y = 0$) when value of $x = -3$. Thus, $x = -3$ which is a root (zero) of equation $x + 3 = 0$ is actually the value of x where graph of $y = f(x) = x + 3$ intersects the x-axis.

Consider the function $f(x) = x^2 - x - 2$, now for $f(x) = 0$ or $x^2 - x - 2 = 0$, we have $(x-2)(x+1) = 0$ or $x = -1$ or $x = 2$. Then graph of $f(x) = x^2 - x - 2$ cuts the x-axis at two values of x, $x = -1$ and $x = 2$.

Following is the graph of $y = f(x)$.

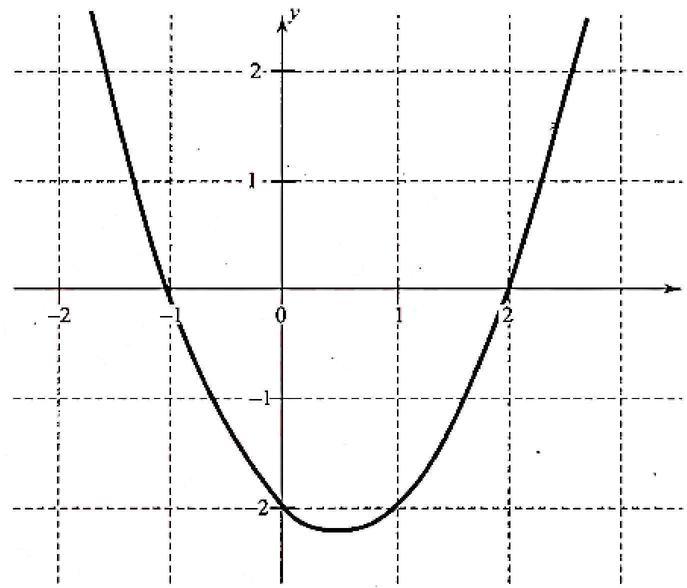


Fig. 1.21

Consider the function $f(x) = x^3 - 6x^2 + 11x - 6$, now for $f(x) = 0$ we have $(x-1)(x-2)(x-3) = 0$ or $x = 1, 2, 3$. Then graph of $y = f(x)$ cuts x-axis at three values of x, $x = 1, 2, 3$.

Following is the graph of $y = f(x)$.

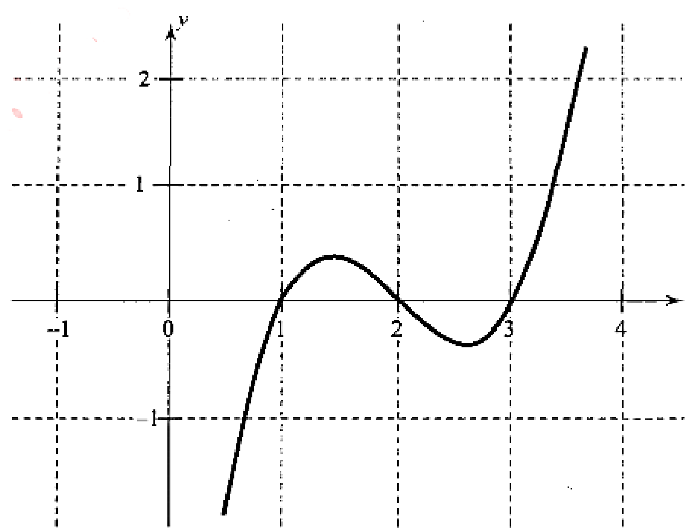


Fig. 1.22

Consider the function $f(x) = (x^2 - 3x + 2)(x^2 - x + 1)$, now for $f(x) = 0$ we have $x = 1$ or $x = 2$, as $x^2 - x + 1 = 0$ is not possible for any real value of x. Hence, $f(x) = 0$ has only two real roots and cuts x-axis for only two values of x, $x = 1$ and $x = 2$.

Following is the graph of $y = f(x)$.

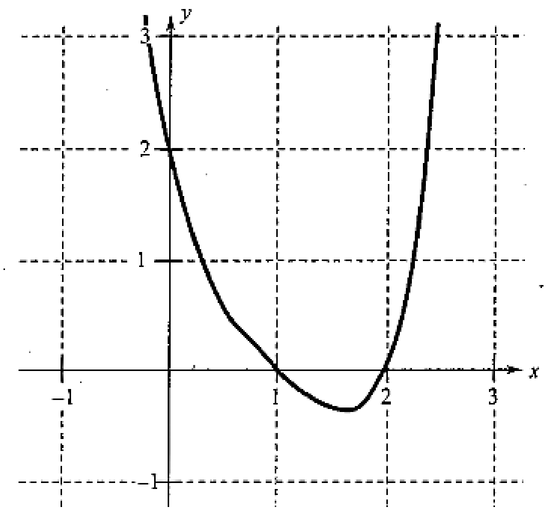


Fig. 1.23

Thus, roots of equation $f(x) = 0$ are actually those values of x where graph $y = f(x)$ meets x-axis.

Roots (Zeros) of the Equation $f(x) = g(x)$

Now we know that zeros of the equation $f(x) = 0$ are the x-coordinates of the points where graph of $y = f(x)$ intersect the x-axis, where $y = 0$ or zeros are x-coordinate of the point of intersection of $y = f(x)$ and $y = 0$ (x-axis)

Consider the equation $x + 5 = 2$.

Let's draw the graph of $y = x + 5$ and $y = 2$, which are as shown in the following figure.

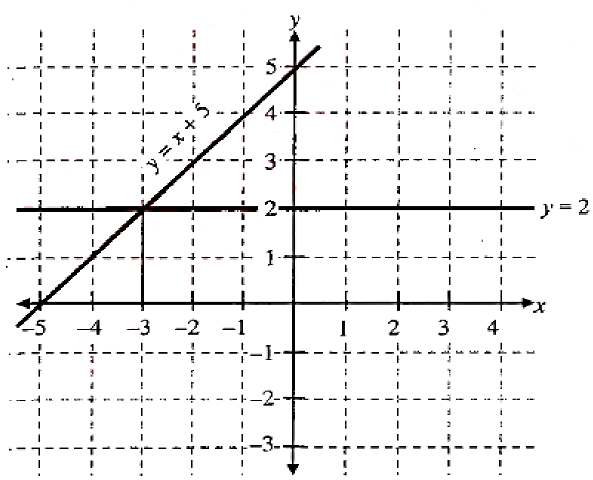


Fig. 1.24

Graph of $y = 2$ is a line parallel to x-axis at height 2 unit above x-axis. Now in the figure, we can see that graphs of $y = x + 5$ and $y = 2$ intersect at point $(-3, 2)$ where value of $x = -3$.

Also from $x + 5 = 2$, we have $x = 2 - 5$ or $x = -3$, which is a root of the equation $x + 2 = 5$. Thus root of the equation $x + 5 = 2$ occurs at point of intersection of graphs $y = x + 5$ and $y = 2$.

Consider the another example $x^2 - 2x = 2 - x$. Let's draw the graph of $y = x^2 - 2x$ and $y = 2 - x$ as shown in the following figure.

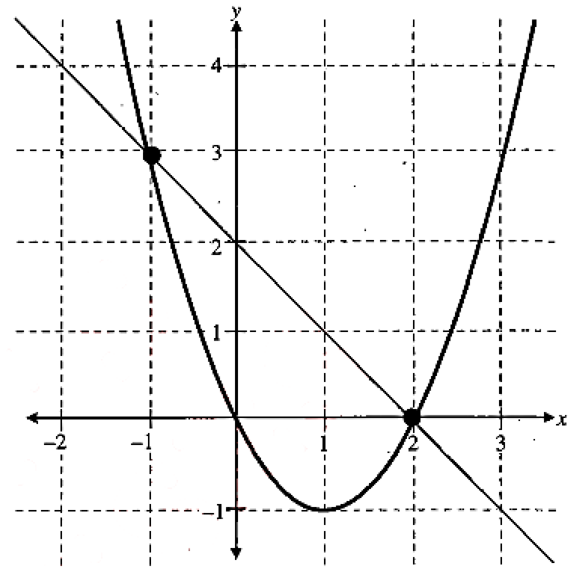


Fig. 1.25

Now in the figure, we can see that graphs of $y = x^2 - 2x$ and $y = 2 - x$ intersect at points $(-1, 3)$ and $(2, 0)$ or where values of x are $x = -1$ and $x = 2$, which are in fact zeros or roots of the equation $x^2 - 2x = 2 - x$ or $x^2 - x - 2 = 0$.

The given equation simplifies to $x^2 - x - 2 = 0$. So one can also locate the roots of the same equation by plotting the graph of $y = x^2 - x - 2$, then the roots of equation are x-coordinates of points where graph of $y = x^2 - x - 2$ intersects with the x-axis (where $y = 0$), as shown in the following figure.

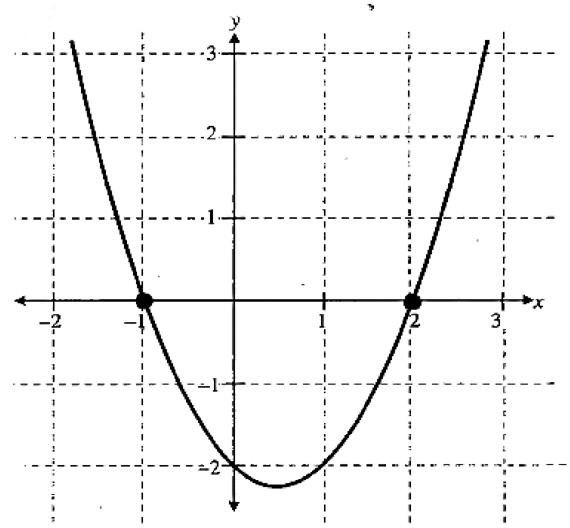


Fig. 1.26

From the above discussion we understand that roots of the equation $f(x) = g(x)$ are the x-coordinate of the points of intersection of graphs $y = f(x)$ and $y = g(x)$.

Example 1.60 **In how many points graph of $y = x^3 - 3x^2 + 5x - 3$ intersect x-axis?**

Sol. Number of point in which $y = x^3 - 3x^2 + 5x - 3$ intersect the x-axis is same as number of real roots of the equation $x^3 - 3x^2 + 5x - 3 = 0$.

Now we can see that $x = 1$ satisfies the equation, hence one root of the equation is $x = 1$.

Now dividing $x^3 - 3x^2 + 5x - 3$ by $x - 1$, we have quotient $x^2 - 2x + 3$.

Hence equation reduces to $(x - 1)(x^2 - 2x + 3) = 0$.

Now $x^2 - 2x + 3 = 0$ or $(x - 1)^2 + 2 = 0$ is not true for any real value of x.

Hence, the only root of the equation is $x = 1$.

Therefore, the graph of $y = x^3 - 3x^2 + 5x - 3$ cuts the x-axis in one point only.

Example 1.61 **In the following diagram, the graph of $y = f(x)$ is given.**

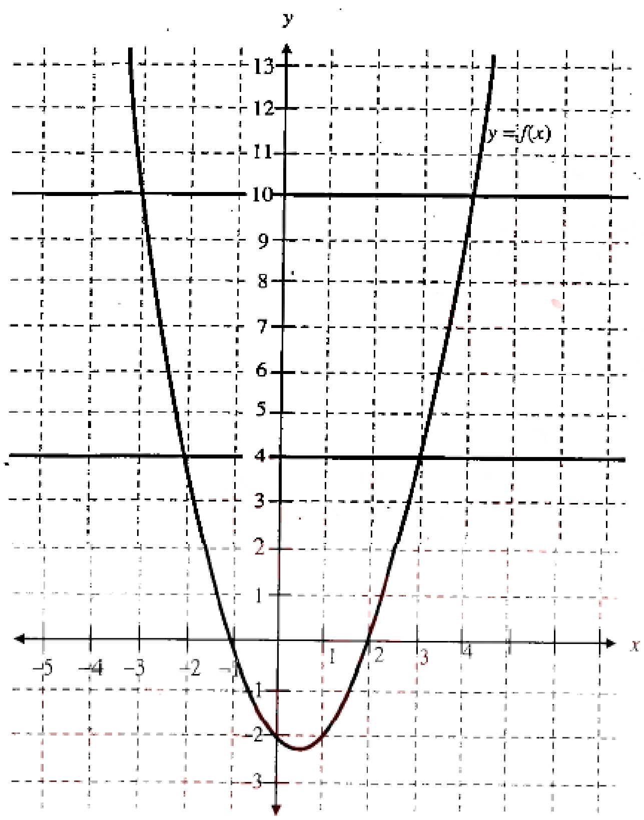


Fig. 1.27

Answer the following questions:

(a) what are the roots of the $f(x) = 0$?

(b) what are the roots of the $f(x) = 4$?

(c) what are the roots of the $f(x) = 10$?

Sol.

(a) The root of the equation $f(x) = 0$ occurs for the values of x where the graphs of $y = f(x)$ and $y = 0$ intersect.

From the diagram, for these point of intersection $x = -1$ and $x = 2$. Hence, roots of the equation $f(x) = 0$ are $x = -1$ and $x = 2$.

(b) The root of the equation $f(x) = 4$ occurs for the values of x where the graphs of $y = f(x)$ and $y = 4$ intersect.

From the diagram, for these point of intersection $x = -2$ and $x = 3$. Hence, roots of the equation $f(x) = 0$ are $x = -2$ and $x = 3$.

(c) Also roots of the equation $f(x) = 10$ are -3 and 4.

Example 1.62 **Which of the following pair of graphs intersect?**

(i) $y = x^2 - x$ and $y = 1$

(ii) $y = x^2 - 2x + 3$ and $y = \sin x$

(iii) $y = x^2 - x + 1$ and $y = x - 4$

Sol. $y = x^2 - x$ and $y = 1$ intersect if $x^2 - x = 1 \Rightarrow x^2 - x - 1 = 0$, which has real roots.

$y = x^2 - 2x + 3$ and $y = \sin x$ intersect if $x^2 - 2x + 3 = \sin x$ or $(x - 1)^2 + 2 = \sin x$, which is not possible as L.H.S. has the least value 2, while R.H.S. has the maximum value 1.

$y = x^2 - x + 1$ and $y = x - 4$ intersect if $x^2 - x + 1 = x - 4$ or $x^2 - 2x + 5 = 0$, which has non-real roots. Hence, graphs do not intersect.

Example 1.63 **Prove that graphs $y = 2x - 3$ and $y = x^2 - x$ never intersect.**

Sol. $y = 2x - 3$ and $y = x^2 - x$ intersect only when $x^2 - x = 2x - 3$ or $x^2 - 3x + 3 = 0$

Now discriminant $D = (-3)^2 - 4(3) = -3 < 0$

Hence, roots of the equation are not real, or we can say that there is no real number for which $2x - 3$ and $x^2 - x$ are equal (or $y = 2x - 3$ and $y = x^2 - x$ intersect).

Hence, proved.

KEY POINTS IN SOLVING AN EQUATION

Domain of Equation

It is a set of the values of independent variables x for which each function used in the equation is defined, i.e., it takes up finite real values. In other words, the final solution obtained while solving any equation must satisfy the domain of the expression of the parent equation.

Example 1.64 **Solve $\dfrac{x^2 - 2x - 3}{x + 1} = 0$.**

Sol. Equation $\dfrac{x^2 - 2x - 3}{x + 1} = 0$ is solvable over $R - \{-1\}$

Now $\dfrac{x^2 - 2x - 3}{x + 1} = 0$

$\Rightarrow$ $x^2 - 2x - 3 = 0$ or $(x - 3)(x + 1) = 0$

$\Rightarrow$ $x = 3$ (as $x \in R - \{-1\}$)

Example 1.65 **Solve $(x^3 - 4x)\sqrt{x^2 - 1} = 0$.**

Sol. Given equation is solvable for $x^2 - 1 \geq 0$

or $x \in (-\infty, -1] [1, \infty)$

$(x^3 - 4x)\sqrt{x^2 - 1} = 0$

$\Rightarrow$ $x(x - 2)(x + 2)\sqrt{x^2 - 1} = 0$

$\Rightarrow$ $x = 0, -2, 2, -1, 1$

But $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow$ $x = \pm 1, \pm 2$

Example 1.66 Solve $\dfrac{2x-3}{x-1} + 1 = \dfrac{6x - x^2 - 6}{x-1}$.

Sol. $\dfrac{2x-3}{x-1} + 1 = \dfrac{6x - x^2 - 6}{x-1}, x \neq 1$

$\Rightarrow$ $\dfrac{3x-4}{x-1} = \dfrac{6x - x^2 - 6}{x-1}, x \neq 1$

$\Rightarrow$ $3x - 4 = 6x - x^2 - 6, x \neq 1$

$\Rightarrow$ $x^2 - 3x + 2 = 0, x \neq 1$

$\Rightarrow$ $x = 2$

Extraneous Roots

While simplifying the equation, the domain of the equation may expand and give the extraneous roots.
For example, consider the equation $\sqrt{x} = x - 2$.
For solving, we first square it
so $\sqrt{x} = x - 2$

$\Rightarrow$ $x = (x - 2)^2$ [on squaring both sides]

$\Rightarrow$ $x^2 - 5x + 4 = 0$

$\Rightarrow$ $(x - 1)(x - 4) = 0$

$\Rightarrow$ $x = 1, 4$

We observe that $x = 4$ satisfies the given equation but $x = 1$ does not satisfy it.
Hence, $x = 4$ is the only solution of the given equation.
The domain of actual equation is $[2, \infty)$.
While squaring the equation, domain expands to R, which gives extra root $x = 1$.

Loss of Root

Cancellation of common factors from both sides of equation leads to loss of root.

For example, consider an equation $x^2 - 2x = x - 2$

$\Rightarrow$ $x(x - 2) = x - 2$

$\Rightarrow$ $x = 1$

Here we have cancelled factor $x - 2$ which causes the loss of root, $x = 2$
The correct way of solving is

$x^2 - 2x = x - 2$

$\Rightarrow$ $x^2 - 3x + 2 = 0$

$\Rightarrow$ $(x - 1)(x - 2) = 0$

$\Rightarrow$ $x = 1$ and $x = 2$.

GRAPHS OF POLYNOMIAL FUNCTIONS

When the polynomial function is written in standard form, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $(a_n \neq 0)$, the leading term is $a_n x^n$. In other words, the leading term is the term that the variable has its highest exponent. The degree of a term of a polynomial function is the exponent on the variable. The degree of the polynomial is the largest degree of all of its terms.

For drawing the graph of the polynomial function, we consider the following tests.

Test 1: Leading Co-efficient

If n is odd and the leading coefficient a_n is positive, then the graph falls to the left and rises to the right:

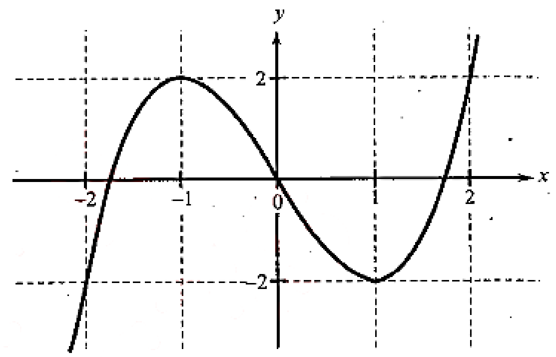


Fig. 1.28

If n is odd and the leading coefficient a_n is negative, the graph rises to the left and falls to the right.

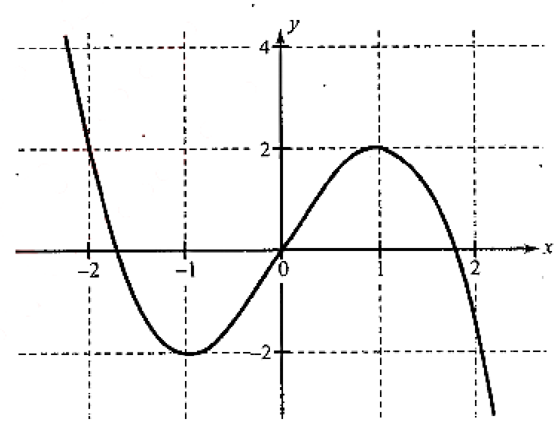


Fig. 1.29

If n is even and the leading coefficient a_n is positive, the graph rises to the left and to the right.

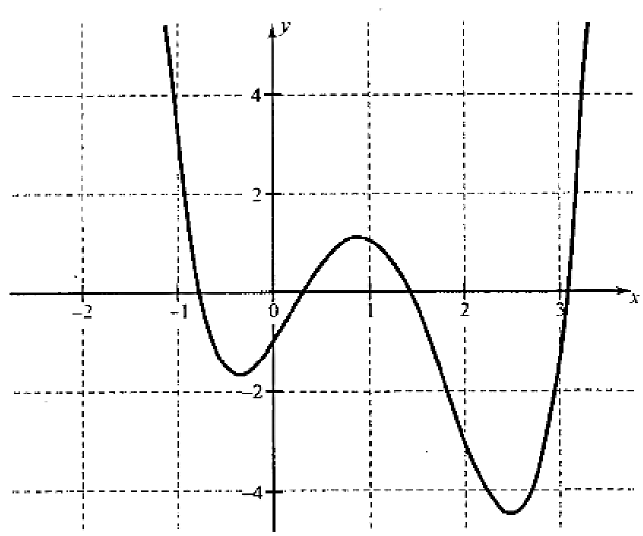


Fig. 1.30

If n is even and the leading coefficient a_n is negative, the graph falls to the left and to the right

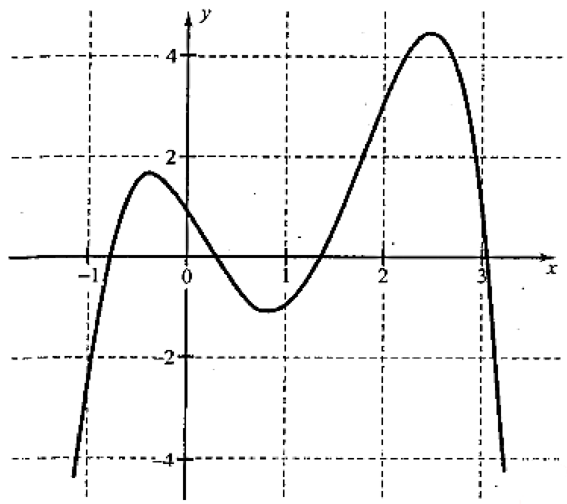


Fig. 1.31

Test 2: Roots (Zeros) of Polynomial

In other words, when a polynomial function is set equal to zero and has been completely factored and each different factor is written with the highest appropriate exponent, depending on the number of times that factor occurs in the product, the exponent on the factor that the zero is a solution for it gives the multiplicity of that zero.

The exponent indicates how many times that factor would be written out in the product, this gives us a multiplicity.

Multiplicity of Zeros and the x-Intercept

If r is a zero of even multiplicity:

This means the graph touches the x-axis at r and turns around.

This happens because the sign of $f(x)$ does not change from one side to the other side of r.

See the graph of $f(x) = (x-2)^2(x-1)(x+1)$.

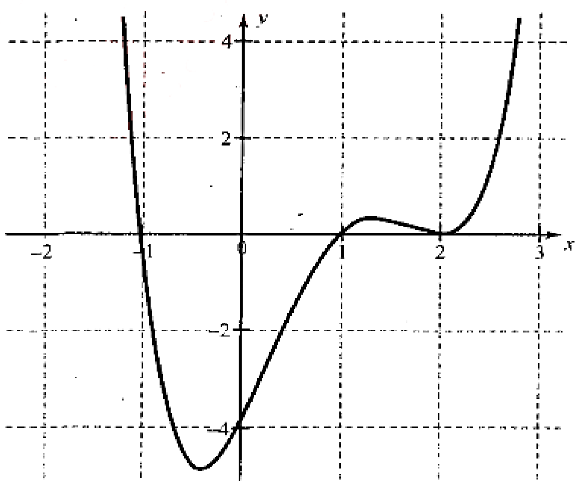


Fig. 1.32

If r is a zero of odd multiplicity:

This means the graph crosses (also touches if exponent is more than 1) the x-axis at r. This happens because the sign of $f(x)$ changes from one side to the other side or r.

See the graph of $f(x) = (x-1)(3x-2)(x-3)^3$

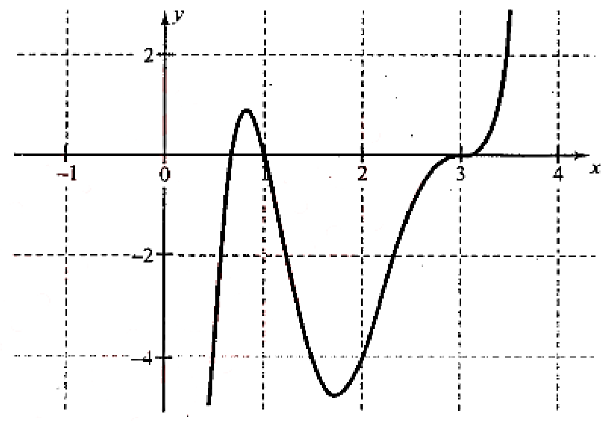


Fig. 1.33

Thus, in general, polynomial function graphs consist of a smooth line with a series of hills and valleys. The hills and valleys are called **turning points**. The maximum possible number of turning points is one less than the degree of the polynomial. The point where graph has turning point, derivative of function $f(x)$ becomes zero, which provides point of local minima or local maxima. Knowledge of derivative provides great help in drawing the graph of the function, hence finding its point of intersection with x-axis or roots of the equation $f(x) = 0$. Also we know that geometrically the derivative of function at any point of the graph of the function is equal to the slope of tangent at that point to the curve.

Consider the following graph of the function $y = f(x)$ as shown in the following figure.

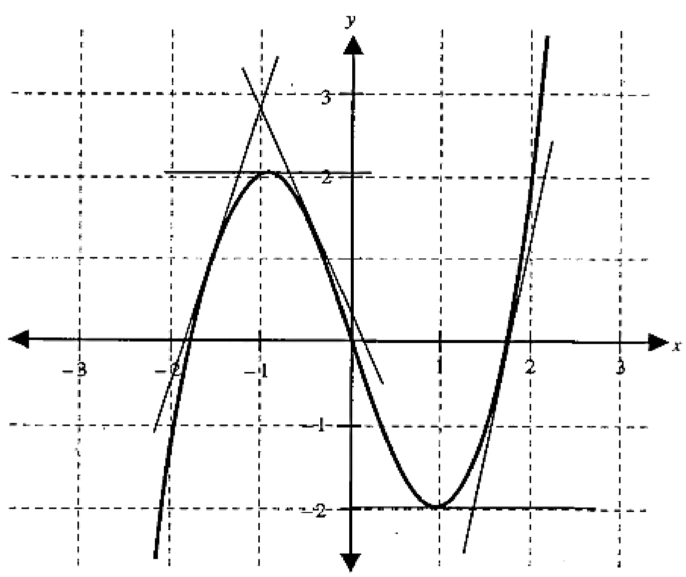


Fig. 1.34

In the figure, we can see that tangent to the curve at point for which $x < -1$ and $x > 1$ makes acute angle with the positive direction of x-axis, hence derivative is positive for these points. For $-1 < x < 1$, tangent to the curve makes obtuse angle with the positive direction of x-axis, hence derivative is negative at these points. At $x = -1$ and $x = 1$, tangent is parallel to x-axis, where derivative is zero.

Here $x = -1$ is called point of maxima, where derivative changes sign from positive to negative (from left to right), and $x = 1$ is called point of minima, where derivative changes sign from negative to positive (from left to right).

At point of maxima and minima, derivative of the function is zero.

Example 1.67 **Using differentiation method check how many roots of the equation $x^3 - x^2 + x - 2 = 0$ are real?**

Sol. Let $y = f(x) = x^3 - x^2 + x - 2$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 2x + 1$$

Let $3x^2 - 2x + 1 = 0$, now this equation has non-real roots, i.e., derivative never becomes zero or graph of $y = f(x)$ has no turning point.

Also when $x \to \infty, f(x) \to \infty$ and when $x \to -\infty, f(x) \to -\infty$

Further $3x^2 - 2x + 1 > 0 \; \forall \; x \in R$

Thus graph of the function is as shown in the following figure.

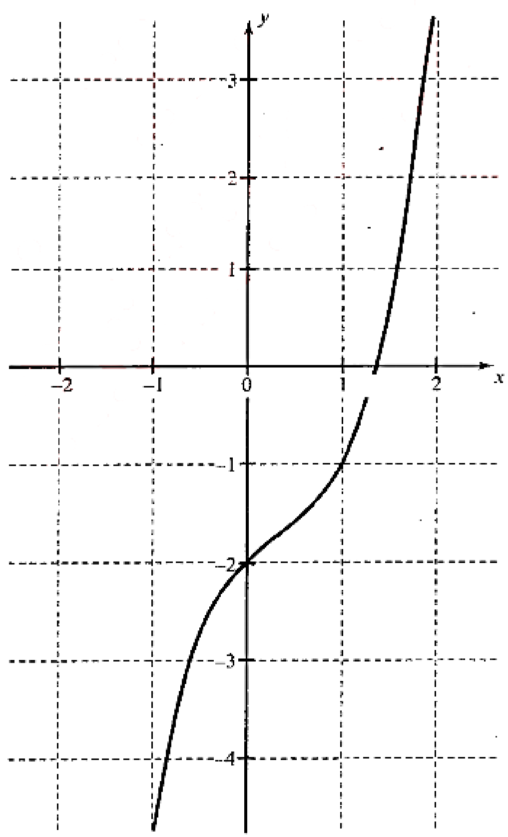


Fig. 1.35

Also $f(0) = -2$, hence graph cuts the x-axis for some positive value of x.

Hence, the only root of the equation is positive.

Thus we can see that differentiation and then graph of the function is much important in analyzing the equation.

Example 1.68 **Analyze the roots of the following equations:**

(i) $2x^3 - 9x^2 + 12x - (9/2) = 0$

(ii) $2x^3 - 9x^2 + 12x - 3 = 0$

Sol.

(i) Let $f(x) = 2x^3 - 9x^2 + 12x - (9/2)$

Then $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$

Now $f'(x) = 0 \Rightarrow x = 1$ and $x = 2$.

Hence, graph has turn at $x = 1$ and at $x = 2$.

Also $f(1) = 2 - 9 + 12 - (9/2) > 0$

and $f(2) = 16 - 36 + 24 - (9/2) < 0$

Hence, graph of the function $y = f(x)$ is as shown in the following figure.

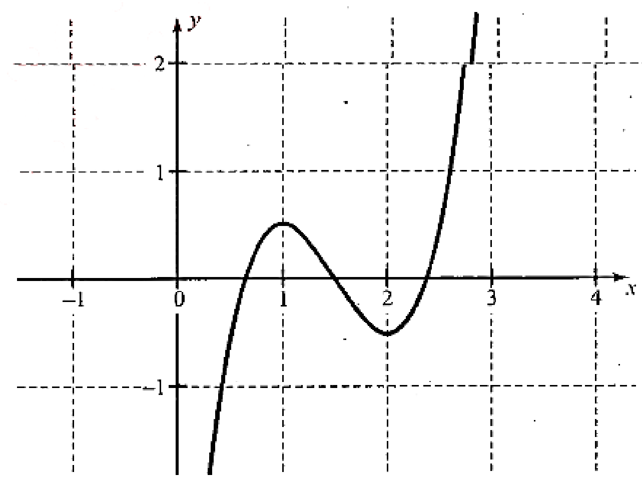


Fig. 1.36(a)

As shown in the figure, graph cuts x-axis at three distinct point.

Hence, equation $f(x) = 0$ has three distinct roots.

(ii) For $2x^3 - 9x^2 + 12x - 3 = 0, f(x) = 2x^3 - 9x^2 + 12x - 3$

$f'(x) = 0 \Rightarrow x = 1$ and $x = 2$

Also $f(1) = 2 - 9 + 12 - 3 = 2$ and $f(2) = 16 - 36 + 24 - 3 = 1$

Hence, graph of $y = f(x)$ is as shown in the following figure.

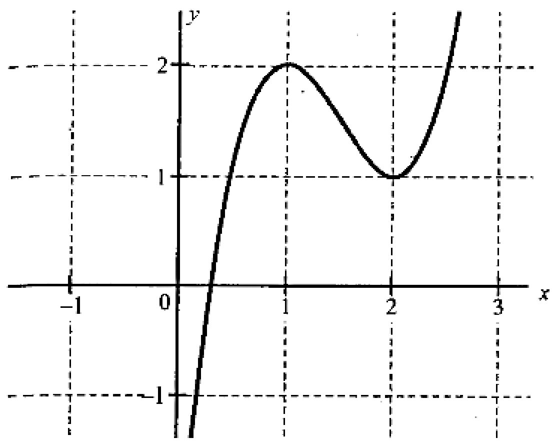


Fig. 1.36(b)

Thus from the graph, we can see that $f(x) = 0$ has only one real root, though $y = f(x)$ has two turning points.

Example 1.69 **Find how many roots of the equation $x^4 + 2x^2 - 8x + 3 = 0$ are real.**

Sol. Let $f(x) = x^4 + 2x^2 - 8x + 3$

$\Rightarrow f'(x) = 4x^3 + 4x - 8 = 4(x-1)(x^2 + x + 2)$

Now $f'(x) = 0 \Rightarrow x = 1$

Hence graph of $y = f(x)$ has only one turn (maxima/minima).

Now $f(1) = 1 + 2 - 8 + 3 < 0$

Also when $x \to \pm\infty$, $f(x) \to \infty$

Then graph of the function is as shown in the following figure.

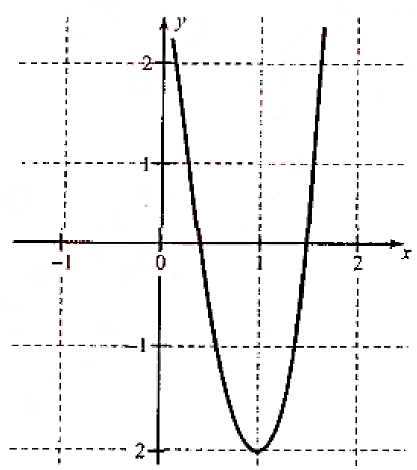


Fig. 1.37

Hence, equation $f(x) = 0$ has only two real roots.

EQUATIONS REDUCIBLE TO QUADRATIC

Example 1.70 Solve $\sqrt{5x^2-6x+8} - \sqrt{5x^2-6x-7} = 1$.

Sol. Let $5x^2 - 6x = y$. Then,

$$\sqrt{5x^2-6x+8} - \sqrt{5x^2-6x-7} = 1$$

$\Rightarrow \sqrt{y+8} - \sqrt{y-7} = 1$

$\Rightarrow (\sqrt{y+8} - \sqrt{y-7})^2 = 1$

$\Rightarrow y = \sqrt{y^2 + y - 56}$

$\Rightarrow y^2 = y^2 + y - 56$

$\Rightarrow y = 56$

$\Rightarrow 5x^2 - 6x = 56$ $\quad [\because y = 5x^2 - 6x]$

$\Rightarrow 5x^2 - 6x - 56 = 0$

$\Rightarrow (5x + 14)(x - 4) = 0$

$\Rightarrow x = 4, \frac{-14}{5}$

Clearly, both the values satisfy the given equation. Hence, the roots of the given equation are 4 and –14/5.

Example 1.71 **Solve $(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1$.**

Sol. We have,

$(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1$

$\Rightarrow (x^2 - 5x + 7)^2 - (x^2 - 5x + 7) = 0$

$\Rightarrow y^2 - y = 0$, where $-y = x^2 - 5x + 7$

$\Rightarrow y(y - 1) = 0$

$\Rightarrow y = 0, 1$

Now,

$y = 0$

$\Rightarrow x^2 - 5x + 7 = 0$

$$\Rightarrow x = \frac{5 \pm \sqrt{25-28}}{2} = \frac{5 \pm \sqrt{-3}}{2} = \frac{5 \pm i\sqrt{3}}{2}$$

where $i = \sqrt{-1}$

and

$y = 1$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x - 3)(x - 2) = 0$

$\Rightarrow 3, 2$

Hence, the roots of the equation are 2, 3, $(5 + i\sqrt{3})/2$ and $(5 - i\sqrt{3})/2$.

Example 1.72 **Solve the equation $4^x - 5 \times 2^x + 4 = 0$.**

Sol. We have,

$4^x - 5 \times 2^x + 4 = 0$

$\Rightarrow (2^x)^2 - 5(2^x) + 4 = 0$

$\Rightarrow y^2 - 5y + 4 = 0$, where $y = 2^x$

$\Rightarrow (y - 4)(y - 1) = 0$

$\Rightarrow y = 1, 4$

$\Rightarrow 2^x = 1, 2^x = 4$

$\Rightarrow 2^x = 2^0, 2^x = 2^2$

$\Rightarrow x = 0, 2$

Hence, the roots of the given equation are 0 and 2.

Example 1.73 **Solve the equation $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.**

Sol. The given equation is

$12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$

Dividing by x^2, we get

$$12x^2-56x+89-\frac{56}{x}+\frac{12}{x^2}=0$$

$$\Rightarrow\ 12\left(x^2+\frac{1}{x^2}\right)-56\left(x+\frac{1}{x}\right)+89=0$$

$$\Rightarrow\ 12\left[\left(x+\frac{1}{x}\right)^2-2\right]-56\left(x+\frac{1}{x}\right)+89=0$$

$$\Rightarrow\ 12\left(x+\frac{1}{x}\right)^2-56\left(x+\frac{1}{x}\right)+65=0$$

$$\Rightarrow\ 12y^2-56y+65=0,\text{ where } y=x+\frac{1}{x}$$

$$\Rightarrow\ 12y^2-26y-30y+65=0$$

$$\Rightarrow\ (6y-13)(2y-5)=0$$

$$\Rightarrow\ y=\frac{13}{6}\text{ or } y=\frac{5}{2}$$

If $y = 13/6$, then

$$x+\frac{1}{x}=\frac{13}{6}$$

$$\Rightarrow\ 6x^2-13x+6=0$$

$$\Rightarrow\ (3x-2)(2x-3)=0$$

$$\Rightarrow\ x=\frac{2}{3},\frac{3}{2}$$

If $y = 5/2$, then

$$x+\frac{1}{x}=\frac{5}{2}$$

$$\Rightarrow\ 2x^2-5x+2=0$$

$$\Rightarrow\ (x-2)(2x-1)=0$$

$$\Rightarrow\ x=2,\ \frac{1}{2}$$

Hence, the roots of the given equation are 2, 1/2, 2/3, 3/2.

Example 1.74 **Solve the equation $3^{x^2-x}+4^{x^2-x}=25$.**

Sol. We have,

$$3^{x^2-x}+4^{x^2-x}=25$$

$$\Rightarrow\ 3^{x^2-x}+4^{x^2-x}=3^2+4^2$$

$$\Rightarrow\ x^2-x=2$$

$$\Rightarrow\ x^2-x-2=0$$

$$\Rightarrow\ (x-2)(x+1)=0$$

$$\Rightarrow\ x=-1,2$$

Hence, the roots of the given equation are –1 and 2.

Example 1.75 **Solve the equation $(x-1)^4+(x-5)^4=82$.**

Sol. Let

$$y=\frac{(x-1)+(x-5)}{2}=x-3$$

$$\Rightarrow\ x=y+3$$

Putting $x = y + 3$ in the given equation, we obtain

$$(y+2)^4+(y-2)^4=82$$

$$\Rightarrow\ (y^2+4y+4)^2+(y^2-4y+4)^2=82$$

$$\Rightarrow\ \{(y^2+4)^2+4y\}^2+\{(y^2+4)-4y\}^2=82$$

$$\Rightarrow\ 2\{(y^2+4)^2+16y^2\}=82$$

$$[\because (a+b)^2+(a-b)^2=2(a^2+b^2)]$$

$$\Rightarrow\ y^4+8y^2+16+16y^2=41$$

$$\Rightarrow\ y^4+24y^2-25=0$$

$$\Rightarrow\ (y^2+25)(y^2-1)=0$$

$$\Rightarrow\ y^2+25=0,\ y^2-1=0$$

$$\Rightarrow\ y=\pm 5i,\ y=\pm 1 \quad (\text{where } i=\sqrt{-1})$$

$$\Rightarrow\ x-3=\pm 5i,\ x-3=\pm 1$$

$$\Rightarrow\ x=3\pm 5i,\ x=4,2 \quad [\because y=x-3]$$

Hence, the roots of the given equation are $3\pm5i$, 2 and 4.

Example 1.76 **Solve the equation $(x+2)(x+3)(x+8)\times(x+12)=4x^2$.**

Sol. $(x+2)(x+3)(x+8)(x+12)=4x^2$

$$\Rightarrow\ \{(x+2)(x+12)\}\{(x+3)(x+8)\}=4x^2$$

$$\Rightarrow\ (x^2+14x+24)(x^2+11x+24)=4x^2$$

Dividing throughout by x^2, we get

$$\left(x+14+\frac{24}{x}\right)\left(x+11+\frac{24}{x}\right)=4$$

$$\Rightarrow\ (y+14)(y+11)=4,\text{ where } x+\frac{24}{x}=y$$

$$\Rightarrow\ y^2+25y+154=4$$

$$\Rightarrow\ y^2+25y+150=0$$

$$\Rightarrow\ (y+15)(y+10)=0$$

$$\Rightarrow\ y=-15,-10$$

If $y=-15$, then

$$x+\frac{24}{x}=-15$$

$$\Rightarrow\ x^2+15x+24=0$$

$$\Rightarrow\ x=\frac{-15-\sqrt{129}}{2}$$

If $y=-10$, then

$$x+\frac{24}{x}=-10$$

$$\Rightarrow\ x^2+10x+24=0$$

$$\Rightarrow\ (x+4)(x+6)=0$$

$$\Rightarrow\ x=-4,-6$$

Hence, the roots of the given equation are –4, –6, $(-15-\sqrt{129})/2$.

Example 1.77 **Evaluate $\sqrt{6+\sqrt{6+\sqrt{6+\cdots\infty}}}$.**

Sol. Let $x=\sqrt{6+\sqrt{6+\sqrt{6+\cdots\infty}}}$. Then,

$$x=\sqrt{6+x}$$

$$\Rightarrow\ x^2=6+x$$

$$\Rightarrow\ x^2-x-6=0$$

$$\Rightarrow\ (x-3)(x+2)=0$$

$$\Rightarrow\ x=3\text{ or } x=-2$$

But, the given expression is positive. So, $x = 3$. Hence, the value of the given expression is 3.

Example 1.78 Solve $\sqrt{x+5}+\sqrt{x+21}=\sqrt{6x+40}$.

Sol. $\sqrt{x+5}+\sqrt{x+21}=\sqrt{6x+40}$

$\Rightarrow (\sqrt{x+5}+\sqrt{x+21})^2=6x+40$

$\Rightarrow (x+5)+(x+21)+2\sqrt{(x+5)(x+21)}=6x+40$

$\Rightarrow \sqrt{(x+5)(x+21)}=2x+7$

$\Rightarrow (x+5)(x+21)=(2x+7)^2$

$\Rightarrow 3x^2+2x-56=0$

$\Rightarrow (3x+14)(x-4)=0$

$\Rightarrow x=4$ or $x=-14/3$

Clearly, $x=-14/3$ does not satisfy the given equation. Hence, $x=4$ is the only root of the given equation.

Concept Application Exercise 1.2

1. Prove that graph of $y=x^2+2$ and $y=3x-4$ never intersect.
2. In how many points the line $y+14=0$ cuts the curve whose equation is $x(x^2+x+1)+y=0$?
3. Consider the following graphs:

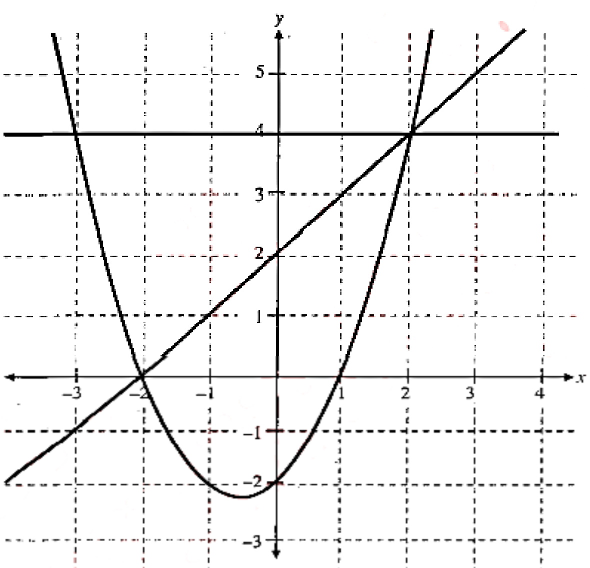


Fig. 1.38

Answer the following questions:
   (i) sum of roots of the equation $f(x)=0$
   (ii) product of roots of the equation $f(x)=4$
   (iii) the absolute value of the difference of the roots of equation $f(x)=x+2$
4. Solve $\dfrac{x^2+3x+2}{x^2-6x-7}=0$.
5. Solve $\sqrt{x-2}+\sqrt{4-x}=2$.
6. Solve $\sqrt{x-2}(x^2-4x-5)=0$.
7. Solve the equation $x(x+2)(x^2-1)=-1$.
8. Find the value of $2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots\infty}}}$.
9. Solve $4^x+6^x=9^x$.
10. Solve $3^{2x^2-7x+7}=9$.
11. Find the number of real roots of the equation $(x-1)^2+(x-2)^2+(x-3)^2=0$.
12. Solve $\sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x+5$.
13. If $x=\sqrt{7+4\sqrt{3}}$, prove that $x+1/x=4$
14. Solve $\sqrt{5x^2-6x+8}-\sqrt{5x^2-6x-7}=1$.
15. Solve $\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$.
16. How many roots of the equation $3x^4+6x^3+x^2+6x+3=0$ are real ?
17. Find the value of k if $x^3-3x+a=0$ has three real distinct roots.
18. Analyze the roots of the equation $(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3+(x-5)^3=0$ by differentiation method.
19. In how many points the graph of $f(x)=x^3+2x^2+3x+4$ meets x-axis.

REMAINDER AND FACTOR THEOREMS

Remainder Theorem

The remainder theorem states that if a polynomial $f(x)$ is divided by a linear function $x-k$, then the remainder is $f(k)$.

Proof:

In any division,

Dividend = Divisor × Quotient + Remainder

Let $Q(x)$ be the quotient and R be the remainder. Then,

$f(x)=(x-k)\,Q(x)+R$

$\Rightarrow f(k)=(k-k)\,Q(x)+R=0+R=R$

Note: *If a n-degree polynomial is divided by a m-degree polynomial, then the maximum degree of the remainder polynomial is m – 1.*

Example 1.79 **Find the remainder when x^3+4x^2-7x+6 is divided by $x-1$.**

Sol. Let $f(x)=x^3+4x^2-7x+6$. The remainder when $f(x)$ is divided by $x-1$ is

$f(1)=1^3+4\times(1)^2-7+6=4$

Example 1.80 **If the expression $ax^4+bx^3-x^2+2x+3$ has remainder $4x+3$ when divided by x^2+x-2, find the value of a and b.**

Sol. Let $f(x)=ax^4+bx^3-x^2+2x+3$.

Now, $x^2+x-2=(x+2)\times(x-1)$.

Given, $f(-2)=a(-2)^4+b(-2)^3-(-2)^2+2(-2)+3$

$= 4(-2) + 3$

$\Rightarrow \quad 16a - 8b - 4 - 4 + 3 = -5$

$\Rightarrow \quad 2a - b = 0 \qquad (1)$

Also,

$f(1) = a + b - 1 + 2 + 3 = 4(1) + 3$

$\Rightarrow \quad a + b = 3 \qquad (2)$

From (1) and (2), $a = 1, b = 2$.

Factor Theorem

Factor Theorem Is a Special Case of Remainder Theorem

Let,

$f(x) = (x - k)\,Q(x) + R$

$\Rightarrow \quad f(x) = (x - k)\,Q(x) + f(k)$

When $f(k) = 0$, $f(x) = (x - k)\,Q(x)$. Therefore, $f(x)$ is exactly divisible by $x - k$.

Example 1.81 **Given that $x^2 + x - 6$ is a factor of $2x^4 + x^3 - ax^2 + bx + a + b - 1$, find the values of a and b.**

Sol. We have,

$x^2 + x - 6 = (x + 3)(x - 2)$

Let,

$f(x) = 2x^4 + x^3 - ax^2 + bx + a + b - 1$

Now,

$f(-3) = 2(-3)^4 + (-3)^3 - a(-3)^2 - 3b + a + b - 1 = 0$

$\Rightarrow \quad 134 - 8a - 2b = 0$

$\Rightarrow \quad 4a + b = 67 \qquad (1)$

$\Rightarrow \quad f(2) = 2(2)^4 + 2^3 - a(2)^2 + 2b + a + b - 1 = 0$

$\Rightarrow \quad 39 - 3a + 3b = 0$

$\Rightarrow \quad a - b = 13 \qquad (2)$

From (1) and (2), $a = 16, b = 3$.

Example 1.82 **Use the factor theorem to find the value of k for which $(a + 2b)$, where $a, b \neq 0$ is a factor of $a^4 + 32b^4 + a^3b(k + 3)$.**

Sol. Let $f(a) = a^4 + 32b^4 + a^3b(k + 3)$. Now,

$f(-2b) = (-2b)^4 + 32b^4 + (-2b)^3b(k + 3) = 0$

$\Rightarrow \quad 48b^4 - 8b^4(k + 3) = 0$

$\Rightarrow \quad 8b^4[6 - (k + 3)] = 0$

$\Rightarrow \quad 8b^4(3 - k) = 0$

Since $b \neq 0$, so, $3 - k = 0$ or $k = 3$.

Example 1.83 **If c, d are the roots of the equation $(x - a)(x - b) - k = 0$, prove that a, b are the roots of the equation $(x - c)(x - d) + k = 0$.**

Sol. Since c and d are the roots of the equation $(x - a)(x - b) - k = 0$, therefore,

$(x - a)(x - b) - k = (x - c)(x - d)$

$\Rightarrow \quad (x - a)(x - b) = (x - c)(x - d) + k$

$\Rightarrow \quad (x - c)(x - d) + k = (x - a)(x - b)$

Clearly, a and b are roots of the equation $(x - a)(x - b) = 0$. Hence, a, b are roots of $(x - c)(x - d) + k = 0$.

Concept Application Exercise 1.3

1. Given that the expression $2x^3 + 3px^2 - 4x + p$ has a remainder of 5 when divided by $x + 2$, find the value of p.
2. Determine the value of k for which $x + 2$ is a factor of $(x + 1)^7 + (2x + k)^3$.
3. Find the value of p for which $x + 1$ is a factor of $x^4 + (p - 3)x^3 - (3p - 5)x^2 + (2p - 9)x + 6$.
   Find the remaining factors for this value of p.
4. If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$, then find the conditions.
5. If $f(x) = x^3 - 3x^2 + 2x + a$ is divisible by $x - 1$, then find the remainder when $f(x)$ is divided by $x - 2$.
6. If $f(x) = x^3 - x^2 + ax + b$ is divisible by $x^2 - x$, then find the value of $f(2)$.

Identity

A relation which is true for every value of the variable is called an identity.

Example 1.84 **If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then find the value of a.**

Sol. The given relation is satisfied for all real values of x, so all the coefficients must be zero. Then,

$$\left.\begin{aligned} a^2 - 1 = 0 &\Rightarrow a = \pm 1 \\ a - 1 = 0 &\Rightarrow a = 1 \\ a^2 - 4a + 3 = 0 &\Rightarrow a = 1, 3 \end{aligned}\right\} \text{common value of } a \text{ is } 1$$

Example 1.85 **Show that $\dfrac{(x+b)(x+c)}{(b-a)(c-a)} + \dfrac{(x+c)(x+a)}{(c-b)(a-b)} + \dfrac{(x+a)(x+b)}{(a-c)(b-c)} = 1$ is an identity.**

Sol. Given relation is

$$\frac{(x+b)(x+c)}{(b-a)(c-a)} + \frac{(x+c)(x+a)}{(c-b)(a-b)} + \frac{(x+a)(x+b)}{(a-c)(b-c)} = 1 \qquad (1)$$

When $x = -a$,

$$\text{L.H.S.} = \frac{(b-a)(c-a)}{(b-a)(c-a)} = 1 = \text{R.H.S.}$$

Similarly, when $x = -b$,

$$\text{L.H.S.} = \frac{(c-b)(a-b)}{(c-b)(a-b)} = 1 = \text{R.H.S.}$$

When $x = -c$,

$$\text{L.H.S.} = \frac{(a-c)(b-c)}{(a-c)(b-c)} = 1 = \text{R.H.S.}$$

Thus, the highest power of x occurring in relation (1) is 2 and this relation is satisfied by three distinct values a, b and c of x; therefore, it is an equation but an identity.

Example 1.86 **A certain polynomial $P(x)$, $x \in R$ when divided by $x - a$, $x - b$ and $x - c$ leaves remainders a, b and c, respectively. Then find the remainder when $P(x)$ is divided by $(x - a)$ $(x - b)$ $(x - c)$ where a, b, c are distinct.**

Sol. By remainder theorem, $P(a) = a$, $P(b) = b$ and $P(c) = c$.

Let the required remainder be $R(x)$. Then,

$$P(x) = (x - a)(x - b)(x - c)\,Q(x) + R(x)$$

where $R(x)$ is a polynomial of degree at most 2. We get $R(a) = a$, $R(b) = b$ and $R(c) = c$. So, the equation $R(x) - x = 0$ has three roots a, b and c. But its degree is at most 2. So, $R(x) - x$ must be zero polynomial (or identity). Hence $R(x) = x$.

QUADRATIC EQUATION

Quadratic Equation with Real Coefficients

Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad (1)$$

where $a, b, c \in R$ and $a \neq 0$.

Roots of the equation are given by

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Now, we observe that the nature of the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the discriminant of the quadratic equation [Eq.(1)].

We also observe the following results:

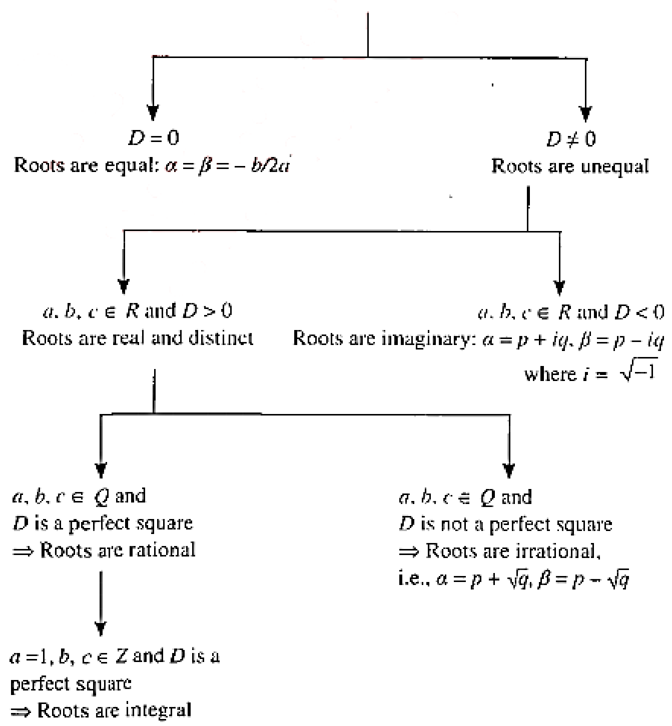


Fig. 1.39

Note:

- *If a, b, $c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots are irrational and they always occur in conjugate pair like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, if a, b, c are irrational numbers and $b^2 - 4ac$ is positive but not a perfect square, then the roots may not occur in conjugate pairs. For example, the roots of the equation $x^2 - (5 + \sqrt{2})x + 5\sqrt{2} = 0$ are 5 and $\sqrt{2}$, which do not form a conjugate pair.*
- *If $b^2 - 4ac < 0$, then roots of equations are complex. If a, b and c are real then complex roots occur in conjugate pair such as of the form $p + iq$ and $p - iq$. If all the coefficients are not real then complex roots may not conjugate.*

Example 1.87 **If $a, b, c \in R^+$ and $2b = a + c$, then check the nature of roots of equation $ax^2 + 2bx + c = 0$.**

Sol. Given equation is $ax^2 + 2bx + c = 0$. Hence,

$$\begin{aligned} D &= 4b^2 - 4ac \\ &= (a + c)^2 - 4ac \\ &= (a - c)^2 > 0 \end{aligned}$$

Thus, the roots are real and distinct.

Example 1.88 **If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, show that $2/b = 1/a + 1/c$.**

Sol. Since the roots of the given equations are equal, therefore its discriminant is zero, i.e.,

$$b^2(c - a)^2 - 4a(b - c)\,c(a - b) = 0$$

$$\Rightarrow \quad b^2(c^2 + a^2 - 2ac) - 4ac(ba - ca - b^2 + bc) = 0$$

$$\Rightarrow \quad a^2b^2 + b^2c^2 + 4a^2c^2 + 2b^2ac - 4a^2bc - 4abc^2 = 0$$

$$\Rightarrow \quad (ab + bc - 2ac)^2 = 0$$

$$\Rightarrow \quad ab + bc - 2ac = 0$$

$$\Rightarrow \quad ab + bc = 2ac$$

$$\Rightarrow \quad \frac{1}{c} + \frac{1}{a} = \frac{2}{b} \quad \text{[Dividing both sides by } abc]$$

$$\Rightarrow \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

Example 1.89 **Prove that the roots of the equation $(a^4 + b^4)x^2 + 4abcdx + (c^4 + d^4) = 0$ cannot be different, if real.**

Sol. The discriminant of the given equation is

$$\begin{aligned} D &= 16a^2b^2c^2d^2 - 4(a^4 + b^4)(c^4 + d^4) \\ &= -4[(a^4 + b^4)(c^4 + d^4) - 4a^2b^2c^2d^2] \\ &= -4[a^4c^4 + a^4d^4 + b^4c^4 + b^4d^4 - 4a^2b^2c^2d^2] \\ &= -4[(a^4c^4 + b^4d^4 - 2a^2b^2c^2d^2) + (a^4d^4 + b^4c^4 - 2a^2b^2c^2d^2)] \\ &= -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \qquad (1) \end{aligned}$$

Since roots of the given equation are real, therefore

$$D \geq 0$$

$$\Rightarrow \quad -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \geq 0$$

$\Rightarrow \quad (a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 \le 0$

$\Rightarrow \quad (a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 = 0 \quad (2)$

(since sum of two positive quantities cannot be negative)

From (1) and (2), we get $D = 0$. Hence, the roots of the given quadratic equation are not different, if real.

Example 1.90 **If the roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real distinct, then find all possible values of a.**

Sol. Since the roots of the given equation are real and distinct, we must have

$D > 0$

$\Rightarrow \quad 64 - 4(a^2 - 6a) > 0$

$\Rightarrow \quad 4[16 - a^2 + 6a] > 0$

$\Rightarrow \quad -4(a^2 - 6a - 16) > 0$

$\Rightarrow \quad a^2 - 6a - 16 < 0$

$\Rightarrow \quad (a - 8)(a + 2) < 0$

$\Rightarrow \quad -2 < a < 8$

Hence, the roots of the given equation are real if a lies between -2 and 8.

Example 1.91 **Find the quadratic equation with rational coefficients whose one root is $1/(2 + \sqrt{5})$.**

Sol. If the coefficients are rational, then irrational roots occur in conjugate pair. Given that if one root is $\alpha = 1/(2 + \sqrt{5}) = \sqrt{5} - 2$, then the other root is $\beta = 1/(2 - \sqrt{5}) = -(2 + \sqrt{5})$.

Sum of roots $\alpha + \beta = -4$ and product of roots $\alpha\beta = -1$. Thus, required equation is $x^2 + 4x - 1 = 0$.

Example 1.92 **If $f(x) = ax^2 + bx + c$, $g(x) = -ax^2 + bx + c$, where $ac \neq 0$, then prove that $f(x)\,g(x) = 0$ has at least two real roots.**

Sol. Let D_1 and D_2 be discriminants of $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$, respectively. Then,

$D_1 = b^2 - 4ac,\ D_2 = b^2 + 4ac$

Now,

$ac \neq 0 \Rightarrow$ either $ac > 0$ or $ac < 0$

If $ac > 0$, then $D_2 > 0$. Therefore, roots of $-ax^2 + bx + c = 0$ are real.

If $ac < 0$, then $D_1 > 0$. Therefore, roots of $ax^2 + bx + c = 0$ are real.

Thus, $f(x)\,g(x)$ has at least two real roots.

Example 1.93 **If $a, b, c \in R$ such that $a + b + c = 0$ and $a \neq c$, then prove that the roots of $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are real and distinct.**

Sol. Given equation is

$(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$

or

$(-2a)x^2 + (-2b)x + (-2c) = 0$

or

$ax^2 + bx + c = 0$

$\Rightarrow \quad D = b^2 - 4ac$

$= (-c - a)^2 - 4ac$

$= (c - a)^2$

> 0

Hence, roots are real and distinct.

Example 1.94 **If $\cos\theta$, $\sin\phi$, $\sin\theta$ are in G.P., then check the nature of roots of $x^2 + 2\cot\phi \cdot x + 1 = 0$.**

Sol. We have,

$\sin^2\phi = \cos\theta\sin\theta$

The discriminant of the given equation is

$$D = 4\cot^2\phi - 4$$

$$= 4\left[\frac{\cos^2\varphi - \sin^2\varphi}{\sin^2\varphi}\right]$$

$$= \frac{4(1 - 2\sin^2\varphi)}{\sin^2\varphi}$$

$$= \frac{4(1 - 2\sin\theta\cos\theta)}{\sin^2\varphi}$$

$$= \left[\frac{2(\sin\theta - \cos\theta)}{\sin\varphi}\right]^2 \ge 0$$

Example 1.95 **If a, b and c are odd integers, then prove that roots of $ax^2 + bx + c = 0$ cannot be rational.**

Sol. Discriminant $D = b^2 - 4ac$. Suppose the roots are rational. Then, D will be a perfect square.

Let $b^2 - 4ac = d^2$. Since a, b and c are odd integers, d will be odd. Now,

$b^2 - d^2 = 4ac$

Let $b = 2k + 1$ and $d = 2m + 1$. Then

$b^2 - d^2 = (b - d)(b + d)$

$= 2(k - m)\,2(k + m + 1)$

Now, either $(k - m)$ or $(k + m + 1)$ is always even. Hence $b^2 - d^2$ is always a multiple of 8. But, $4ac$ is only a multiple of 4 (not of 8), which is a contradiction. Hence, the roots of $ax^2 + bx + c = 0$ cannot be rational.

Quadratic Equations with Complex Coefficients

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are complex numbers and $a \neq 0$. Roots of equation are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Here nature of roots should not be analyzed by sign of $b^2 - 4ac$.

Note: *In case of quadratic equations with real coefficients, imaginary (complex) roots always occur in conjugate pairs. However, it is not true for quadratic equations with complex coefficients. For example, the equation $4x^2 - 4ix - 1 = 0$ has both roots equal to $1/(2i)$.*

Concept Application Exercise 1.4

1. Find the values of a for which the roots of the equation $x^2 + a^2 = 8x + 6a$ are real.
2. Find the condition if the roots of $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{ac}x + b = 0$ are simultaneously real.
3. If $a < c < b$, then check the nature of roots of the equation $(a - b)^2 x^2 + 2(a + b - 2c)x + 1 = 0$.
4. If $a + b + c = 0$ then check the nature of roots of the equation $4ax^2 + 3bx + 2c = 0$ where $a, b, c \in R$.
5. Find the greatest value of a non-negative real number λ for which both the equations $2x^2 + (\lambda - 1)x + 8 = 0$ and $x^2 - 8x + \lambda + 4 = 0$ have real roots.

Relations Between Roots and Coefficients

Let α and β be the roots of quadratic equation $ax^2 + bx + c = 0$. Then by factor theorem,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta) = a(x^2 - (\alpha + \beta)x + \alpha\beta)$$

Comparing coefficients, we have $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

Thus, we find that

$$\alpha + \beta = \frac{-b}{a} = -\frac{\text{coeff of } x}{\text{coeff of } x^2} \text{ and } \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coeff of } x^2}$$

Also, if sum of roots is S and product is P, then quadratic equation is given by $x^2 - Sx + P = 0$.

Example 1.96 **Form a quadratic equation whose roots are –4 and 6.**

Sol. We have sum of the roots, $S = -4 + 6 = 2$ and, product of the roots, $P = -4 \times 6 = -24$. Hence, the required equation is

$$x^2 - Sx + P = 0$$
$$\Rightarrow \quad x^2 - 2x - 24 = 0$$

Example 1.97 **Form a quadratic equation with real coefficients whose one root is $3 - 2i$.**

Sol. Since the complex roots always occur in pairs, so the other root is $3 + 2i$. The sum of the roots is $(3 + 2i) + (3 - 2i) = 6$. The product of the roots is $(3 + 2i)(3 - 2i) = 9 - 4i^2 = 9 + 4 = 13$. Hence, the equation is

$$x^2 - Sx + P = 0$$
$$\Rightarrow \quad x^2 - 6x + 13 = 0$$

Example 1.98 **If roots of the equation $ax^2 + bx + c = 0$ are α and β, find the equation whose roots are**

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$

(ii) $-\alpha, -\beta$

(iii) $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$

Sol. Here in all cases functions of α and β are symmetric.

(i) Let $\frac{1}{\alpha} = y \Rightarrow \alpha = \frac{1}{y}$

Now α is a root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$
$$\frac{a}{y^2} + \frac{b}{y} + c = 0$$
$$\Rightarrow \quad cy^2 + by + a = 0$$

Hence, the required equation is $cx^2 + bx + a = 0$.

We get same equation if we start with $1/\beta$.

(ii) Let $-\alpha = y \Rightarrow \alpha = -y$

Now α is root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$
$$\Rightarrow \quad a(-y)^2 + b(-y) + c = 0$$

Hence, the required equation is $ax^2 - bx + c = 0$.

(iii) Let $\frac{1-\alpha}{1+\alpha} = y \Rightarrow \alpha = \frac{1-y}{1+y}$

Now α is root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$
$$\Rightarrow \quad a\left(\frac{1-y}{1+y}\right)^2 + b\left(\frac{1-y}{1+y}\right) + c = 0$$

Hence required equation is $a(1 - x)^2 + b(1 - x^2) + c(1 + x)^2 = 0$.

Example 1.99 **If a, b and c are in A.P. and one root of the equation $ax^2 + bx + c = 0$ is 2, then find the other root.**

Sol. Let α be the other root. Then,

$$4a + 2b + c = 0 \text{ and } 2b = a + c$$
$$\Rightarrow \quad 5a + 2c = 0$$
$$\Rightarrow \quad \frac{c}{a} = \frac{-5}{2}$$

Now,

$$2 \times \alpha = \frac{c}{a} = -\frac{5}{2}$$
$$\therefore \quad \alpha = -\frac{5}{4}$$

Example 1.100 **If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, respectively, then find the value of $2 + q - p$.**

Sol. The equation $x^2 + px + q = 0$ has roots tan 30° and tan 15°. Therefore,

$$\tan 30° + \tan 15° = -p \quad (1)$$
$$\tan 30° \tan 15° = q \quad (2)$$

Now,

$$\tan 45° = \tan(30° + 15°)$$
$$\Rightarrow \quad 1 = \frac{\tan 30 + \tan 15}{1 - \tan 30 \tan 15}$$
$$\Rightarrow \quad 1 = \frac{-p}{1-q} \quad \text{[Using (1) and (2)]}$$
$$\Rightarrow \quad 1 - q = -p \Rightarrow q - p = 1$$

$\Rightarrow \quad 2 + q - p = 3$

Example 1.101 **If the sum of the roots of the equation $1/(x+a) + 1/(x+b) = 1/c$ is zero, then prove that the product of the roots is $(-1/2)(a^2 + b^2)$.**

Sol. We have,

$$\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$$

$\Rightarrow \quad x^2 + (a+b-2c)x + (ab - bc - ca) = 0$

Let α, β be the roots of this equation. Then,

$\alpha + \beta = -(a+b-2c)$ and $\alpha\beta = ab - bc - ca$

It is given that

$\alpha + \beta = 0$

$\Rightarrow \quad -(a+b-2c) = 0$

$\Rightarrow \quad c = \frac{a+b}{2}$ (1)

$\therefore \quad \alpha\beta = ab - bc - ca$

$= ab - c(a+b)$

$= ab - \left(\frac{a+b}{2}\right)(a+b)$ [Using (1)]

$= \frac{2ab - (a+b)^2}{2} = -\frac{1}{2}(a^2 + b^2)$

Example 1.102 **Solve the equation $x^2 + px + 45 = 0$. It is given that the squared difference of its roots is equal to 144.**

Sol. Let α, β be the roots of the equation $x^2 + px + 45 = 0$. Then,

$\alpha + \beta = -p$ (1)

$\alpha\beta = 45$ (2)

It is given that

$(\alpha - \beta)^2 = 144$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = 144$

$\Rightarrow \quad p^2 - 4 \times 45 = 144$

$\Rightarrow \quad p^2 = 324$

$\Rightarrow \quad p = \pm 18$

Substituting $p = 18$ in the given equation, we obtain

$x^2 + 18x + 45 = 0$

$\Rightarrow \quad (x+3)(x+15) = 0$

$\Rightarrow \quad x = -3, -15$

Substituting $p = -18$ in the given equation, we obtain

$x^2 + 18x + 45 = 0$

$\Rightarrow \quad (x-3)(x-15) = 0$

$\Rightarrow \quad x = 3, 15$

Hence, the roots of the given equation are $-3, -15$ or $3, 15$.

Example 1.103 **If the ratio of the roots of the equation $x^2 + px + q = 0$ are equal to the ratio of the roots of the equation $x^2 + bx + c = 0$, then prove that $p^2c = b^2q$.**

Sol. Let α, β be the roots of $x^2 + px + q = 0$ and γ, δ be the roots of the equation $x^2 + bx + c = 0$. Then,

$\alpha + \beta = -p, \alpha\beta = q$ (1)

$\gamma + \delta = -b, \gamma\delta = c$ (2)

We have,

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{\alpha+\beta}{\alpha-\beta} = \frac{\gamma+\delta}{\gamma-\delta}$$

[Using componendo and dividendo]

$$\Rightarrow \quad \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} = \frac{(\gamma-\delta)^2}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{(\alpha+\beta)^2 - 4\alpha\beta}{(\alpha+\beta)^2} = \frac{(\gamma+\delta)^2 - 4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad 1 - \frac{4\alpha\beta}{(\alpha+\beta)^2} = 1 - \frac{4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{\alpha\beta}{(\alpha+\beta)^2} = \frac{\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{q}{p^2} = \frac{c}{b^2}$$

$\Rightarrow \quad p^2c = b^2q$

Example 1.104 **If $\sin\theta$, $\cos\theta$ be the roots of $ax^2 + bx + c = 0$, then prove that $b^2 = a^2 + 2ac$.**

Sol. We have,

$$\sin\theta + \cos\theta = -\frac{b}{a}, \sin\theta\cos\theta = \frac{c}{a}$$

Now, we know that

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \quad (\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta = 1$

$$\Rightarrow \quad \frac{b^2}{a^2} = 1 + 2\frac{c}{a} \Rightarrow b^2 = a^2 + 2ac$$

Example 1.105 **If a and b ($\neq 0$) are the roots of the equation $x^2 + ax + b = 0$, then find the least value of $x^2 + ax + b$ ($x \in R$).**

Sol. Since a and b are the roots of the equation $x^2 + ax + b = 0$, so

$a + b = -a, ab = b$

Now,

$ab = b \Rightarrow (a-1)b = 0 \Rightarrow a = 1 \quad (\because b \neq 0)$

Putting $a = 1$ in $a + b = -a$, we get $b = -2$. Hence,

$x^2 + ax + b = x^2 + x - 2 = (x + 1/2)^2 - 1/4 - 2$

$= (x + 1/2)^2 - 9/4$

which has a minimum value $-9/4$.

Example 1.106 **If the sum of the roots of the equation $(a+1)x^2 + (2a+3)x + (3a+4) = 0$ is -1, then find the product of the roots.**

Sol. Let α, β be roots of the equation $(a+1)x^2 + (2a+3)x + (3a+4) = 0$. Then,

$$\alpha + \beta = -1 \Rightarrow -\left(\frac{2a+3}{a+1}\right) = -1 \Rightarrow a = -2$$

Now, product of the roots is $(3a+4)/(a+1) = (-6+4)/(-2+1) = 2$.

Example 1.107 Find the value of 'a' for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other.

Sol. Let the roots be α and 2α. Then,

$$\alpha + 2\alpha = \frac{1-3a}{a^2-5a+3}, \alpha \times 2\alpha = \frac{2}{a^2-5a+3}$$

$$\Rightarrow 2\left[\frac{1}{9}\frac{(1-3a)^2}{(a^2-5a+3)^2}\right] = \frac{2}{a^2-5a+3}$$

$$\Rightarrow \frac{(1-3a)^2}{a^2-5a+3} = 9 \Rightarrow 9a^2 - 6a + 1 = 9a^2 - 45a + 27$$

$$\Rightarrow 39a = 26 \Rightarrow a = \frac{2}{3}$$

Example 1.108 If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then find the set of possible values of a.

Sol. If α, β are roots of $x^2 + ax + 1 = 0$, then

$$|\alpha - \beta| < \sqrt{5}$$

$$\Rightarrow \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} < \sqrt{5}$$

$$\Rightarrow \sqrt{a^2 - 4} < \sqrt{5}$$

$$\Rightarrow \sqrt{a^2 - 4} < \sqrt{5}$$

$$\Rightarrow a^2 - 4 < 5$$

$$\Rightarrow a^2 < 9$$

$$\Rightarrow -3 < a < 3$$

$$\therefore a \in (-3, 3)$$

Example 1.109 Find the values of the parameter a such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\alpha/\beta + \beta/\alpha < 2$.

Sol. We have $\alpha + \beta = -3$ and $\alpha\beta = a/2$. Now,

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$$

$$\Rightarrow \frac{\alpha^2 + \beta^2}{\alpha\beta} < 2$$

$$\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} < 2$$

$$\Rightarrow \frac{9-a}{a/2} < 2$$

$$\Rightarrow \frac{9-a}{a} < 1$$

$$\Rightarrow \frac{9-a}{a} - 1 < 0$$

$$\Rightarrow \frac{9-2a}{a} < 0$$

$$\Rightarrow \frac{2a-9}{a} > 0$$

$$\Rightarrow a < 0 \text{ or } a > 9/2$$

Example 1.110 If the harmonic mean between roots of $(5 + \sqrt{2})x^2 - bx + 8 + 2\sqrt{5} = 0$ is 4, then find the value of b.

Sol. Let α, β be the roots of the given equation whose H.M. is 4. Then,

$$4 = \frac{2\alpha\beta}{\alpha+\beta}$$

$$\Rightarrow 4 = 2 \times \frac{\frac{8+2\sqrt{5}}{5+\sqrt{2}}}{\frac{b}{5+\sqrt{2}}}$$

$$\Rightarrow 2 = \frac{8+2\sqrt{5}}{b} \Rightarrow b = 4 + \sqrt{5}$$

Example 1.111 If α, β are the roots of the equation $2x^2 - 3x - 6 = 0$, find the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$.

Sol. Since α, β are roots of the equation $2x^2 - 3x - 6 = 0$, so

$$\alpha + \beta = 3/2 \text{ and } \alpha\beta = -3$$

$$\Rightarrow \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{9}{4} + 6 = \frac{33}{4}$$

Now,

$$(\alpha^2 + 2) + (\beta^2 + 2) = (\alpha^2 + \beta^2) + 4 = \frac{33}{4} + 4 = \frac{49}{4}$$

and

$$(\alpha^2 + 2)(\beta^2 + 2) = \alpha^2\beta^2 + 2(\alpha^2 + \beta^2) + 4$$

$$= (3)^2 + 2\left(\frac{33}{4}\right) + 4$$

$$= \frac{59}{2}$$

So, the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$ is

$$x^2 - x[(\alpha^2 + 2) + (\beta^2 + 2)] + (\alpha^2 + 2)(\beta^2 + 2) = 0$$

$$\Rightarrow x^2 - \frac{49}{4}x + \frac{59}{2} = 0$$

$$\Rightarrow 4x^2 - 49x + 118 = 0$$

Example 1.112 If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, find the equation whose roots are α/β and β/α.

Sol. We have $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$. Hence, α, β are roots of $x^2 = 5x - 3$, i.e., $x^2 - 5x + 3 = 0$. Therefore,

$$\alpha + \beta = 5 \text{ and } \alpha\beta = 3$$

Now,

$$S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{25-6}{3} = \frac{19}{3}$$

and

$$P = \frac{\alpha}{\beta}\frac{\beta}{\alpha} = 1$$

So, the required equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow \quad x^2 - \frac{19}{3}x + 1 = 0$$

$$\Rightarrow \quad 3x^2 - 19x + 3 = 0$$

Example 1.113 **If α, β are the roots of the equation $ax^2 + bx + c = 0$, then find the roots of the equation $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$ in terms of α and β.**

Sol. $ax^2 - bx(x-1) + c(x-1)^2 = 0$

$$\Rightarrow \quad \frac{ax^2}{(1-x)^2} + \frac{bx}{1-x} + c = 0 \qquad (1)$$

Now, α is a root of $ax^2 + bx + c = 0$. Then let

$$\alpha = \frac{x}{1-x}$$

$$\Rightarrow \quad x = \frac{\alpha}{\alpha+1}$$

Hence, the roots of (1) are $\alpha/(1+\alpha)$, $\beta/(1+\beta)$.

Concept Application Exercise 1.5

1. If the product of the roots of the equation $(a+1)x^2 + (2a+3)x + (3a+4) = 0$ is 2, then find the sum of roots.
2. Find the value of a for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - a - 1 = 0$ assumes the least value.
3. If x_1 and x_2 are the roots of $x^2 + (\sin\theta - 1)x - 1/2\cos^2\theta = 0$, then find the maximum value of $x_1^2 + x_2^2$.
4. If $\tan\theta$ and $\sec\theta$ are the roots of $ax^2 + bx + c = 0$, then prove that $a^4 = b^2(4ac - b^2)$.
5. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then find the value of $b^2 - 4c$.
6. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2:3, then find the value of m.
7. If α, β are the roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$, then prove that $q^2 - p^2 = (\alpha-\gamma)(\beta-\gamma)(\alpha+\delta) \times (\beta+\delta)$.
8. If the equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, find the condition.
9. If α, β be the roots of $x^2 - a(x-1) + b = 0$, then find the value of $1/(\alpha^2 - a\alpha) + 1/(\beta^2 - b\beta) + 2/a + b$.
10. If α, β are roots of $375x^2 - 25x - 2 = 0$ and $s_n = \alpha^n + \beta^n$, then find the value of $\lim_{n\to\infty}\sum_{r=1}^{n} s_r$.
11. If α and β are the roots of the equation $2x^2 + 2(a+b)x + a^2 + b^2 = 0$, then find the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$.
12. If the sum of the roots of an equation is 2 and sum of their cubes is 98, then find the equation.
13. Let α, β be the roots of $x^2 + bx + 1 = 0$. Then find the equation whose roots are $-(\alpha + 1/\beta)$ and $-(\beta + 1/\alpha)$.

COMMON ROOT(S)

Condition for One Common Root

Let us find the condition that the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ may have a common root. Let α be the common root of the given equations. Then,

$$a_1\alpha^2 + b_1\alpha + c_1 = 0$$

and

$$a_2\alpha^2 + b_2\alpha + c_2 = 0$$

Solving these two equations by cross-multiplication, we have

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

$$\Rightarrow \quad \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{(from first and third)}$$

and

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \quad \text{(from second and third)}$$

$$\Rightarrow \quad \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)^2$$

$$\Rightarrow \quad (c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$$

This condition can easily be remembered by cross-multiplication method as shown in the following figure.

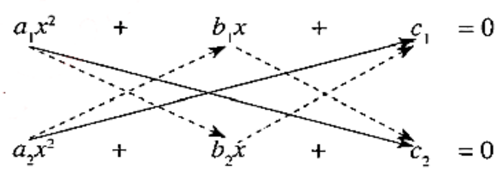


Fig. 1.40

(Bigger cross product)^2
= Product of the two smaller crosses

This is the condition required for a root to be common to two quadratic equations. The common root is given by

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

or

$$\alpha = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

Note: *The common root can also be obtained by making the coefficient of x^2 common to the two given equations and then subtracting the two equations. The other roots of the given equations can be determined by using the relations between their roots and coefficients.*

Condition for Both the Common Roots

Let α, β be the common roots of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$. Then, both the equations are identical, hence,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Note:

- *If two quadratic equations with real coefficients have a non-real complex common root then both the roots will be common, i.e. both the equations will be the same. So the coefficients of the corresponding powers of x will have proportional values.*
- *If two quadratic equations with rational coefficients have a common irrational root $p + \sqrt{q}$ then both the roots will be common, i.e. no two different quadratic equations with rational coefficients can have a common irrational root $p + \sqrt{q}$.*

Example 1.114 **Determine the values of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.**

Sol. Let α be the common root of the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$. Then, α must satisfy both the equations. Therefore,

$$3\alpha^2 + 4m\alpha + 2 = 0$$
$$2\alpha^2 + 3\alpha - 2 = 0$$

Using cross-multiplication method, we have

$$(-6 - 4)^2 = (9 - 8m)(-8m - 6)$$
$$\Rightarrow 50 = (8m - 9)(4m + 3)$$
$$\Rightarrow 32m^2 - 12m - 77 = 0$$
$$\Rightarrow 32m^2 - 56m + 44m - 77 = 0$$
$$\Rightarrow 8m(4m - 7) + 11(4m - 7) = 0$$
$$\Rightarrow (8m + 11)(4m - 7) = 0$$
$$\Rightarrow m = -\frac{11}{8}, \frac{7}{4}$$

Example 1.115 **If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have common root/roots and $a, b, c \in N$, then find the minimum value of $a + b + c$.**

Sol. The roots of $x^2 + 3x + 5 = 0$ are non-real. Thus given equations will have two common roots. We have,

$$\frac{a}{1} = \frac{b}{3} = \frac{c}{5} = \lambda$$
$$\Rightarrow a + b + c = 9\lambda$$

Thus minimum value of $a + b + c$ is 9.

Example 1.116 **If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and a, b and c are non-zero real numbers then find the value of $(a^3 + b^3 + c^3)/abc$.**

Sol. Given that $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root. Hence,

$$(bc - a^2)^2 = (ab - c^2)(ac - b^2)$$
$$\Rightarrow b^2c^2 + a^4 - 2a^2bc = a^2bc - ab^3 - ac^3 + b^2c^2$$
$$\Rightarrow a^4 + ab^3 + ac^3 = 3a^2bc$$
$$\Rightarrow \frac{a^3 + b^3 + c^3}{abc} = 3$$

Example 1.117 **a, b, c are positive real numbers forming a G.P. If $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then prove that $d/a, e/b, f/c$ are in A.P.**

Sol. For first equation $D = 4b^2 - 4ac = 0$ (as given a, b, c are in G.P.). The equation has equal roots which are equal to $-b/a$ each. Thus, it should also be the root of the second equation. Hence,

$$d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0$$
$$\Rightarrow d\frac{b^2}{a^2} - 2\frac{be}{a} + f = 0$$
$$\Rightarrow d\frac{ac}{a^2} - 2\frac{be}{a} + f = 0 \quad (\because b^2 = ac)$$
$$\Rightarrow \frac{d}{a} + \frac{f}{c} = 2\frac{eb}{ac} = 2\frac{e}{b}$$

Example 1.118 **If the equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common positive root, then find the values of a and b.**

Sol. We have,

$$x^2 + ax + 12 = 0 \quad (1)$$
$$x^2 + bx + 15 = 0 \quad (2)$$

Adding (1) and (2), we get

$$2x^2 + (a + b)x + 27 = 0$$

Now subtracting it from the third given equation, we get

$$x^2 - 9 = 0 \Rightarrow x = 3, -3$$

Thus, common positive root is 3. Hence,

$$9 + 12 + 3a = 0$$
$$\Rightarrow a = -7 \text{ and } 9 + 3b + 15 = 0$$
$$\Rightarrow b = -8$$

Example 1.119 **The equations $ax^2 + bx + a = 0$ and $x^3 - 2x^2 + 2x - 1 = 0$ have two roots common. Then find the value of $a + b$.**

Sol. By observation, $x = 1$ is a root of equation $x^3 - 2x^2 + 2x - 1 = 0$. Thus we have

$$(x - 1)(x^2 - x + 1) = 0$$

Now roots of $x^2 - x + 1 = 0$ are non-real.

Then equation $ax^2 + bx + a = 0$ has both roots common with $x^2 - x + 1 = 0$. Hence, we have

$$\frac{a}{1} = \frac{b}{-1} = \frac{a}{1}$$

or $a + b = 0$

Concept Application Exercise 1.6

1. If $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ $(a \neq b)$ have a common root, then prove that their other roots satisfy the equation $x^2 + cx + ab = 0$.

2. Find the condition that the expressions $ax^2 + bxy + cy^2$ and $a_1x^2 + b_1xy + c_1y^2$ may have factors $y - mx$ and $my - x$, respectively.
3. If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, then find $a:b:c$.
4. Find the condition on a, b, c, d such that equations $2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.
5. Let $f(x)$, $g(x)$ and $h(x)$ be the quadratic polynomials having positive leading coefficients and real and distinct roots. If each pair of them has a common root, then find the roots of $f(x) + g(x) + h(x) = 0$.

RELATION BETWEEN COEFFICIENT AND ROOTS OF *n*-DEGREE EQUATIONS

- Let α and β be roots of quadratic equation $ax^2 + bx + c = 0$. Then by factor theorem
$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$
$$= a(x^2 - (\alpha + \beta)x + \alpha\beta)$$
Comparing coefficients, we have
$$\alpha + \beta = -b/a \text{ and } \alpha\beta = c/a$$
- Let α, β, γ are roots of cubic equation $ax^3 + bx^2 + cx + d = 0$. Then,
$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$
$$= a(x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma)$$
Comparing coefficients, we have
$$\alpha + \beta + \gamma = -b/a$$
$$\alpha\beta + \beta\gamma + \alpha\gamma = c/a$$
$$\alpha\beta\gamma = -d/a$$
- If $\alpha, \beta, \gamma, \delta$ are roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then
$$\alpha + \beta + \gamma + \delta = -b/a$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c/a$$
(sum of product taking two at a time)
$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -d/a$$
(sum of product taking three at a time)
$$\alpha\beta\gamma\delta = e/a$$

In general, if $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$, then sum of the roots is
$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = -\frac{a_1}{a_0}$$
Sum of the product taken two at a time is
$$\left.\begin{matrix}\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_1\alpha_n \\ \cdots + \alpha_2\alpha_3 + \cdots + \alpha_2\alpha_n \\ \cdots + \alpha_{n-1}\alpha_n\end{matrix}\right\} = \frac{a_2}{a_0}$$
Sum of the product taken three at a time is $-a_3/a_0$ and so on.
Product of all the root is
$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note:
- *A polynomial equation of degree n has n roots (real or imaginary).*
- *If all the coefficients are real then the imaginary roots occur in conjugate pairs, i.e., number of imaginary roots is always even.*
- *If the degree of a polynomial equation is odd, then the number of real roots will also be odd. It follows that at least one of the roots will be real.*

SOLVING CUBIC EQUATION

By using factor theorem together with some intelligent guessing, we can factorise polynomials of higher degree.

In summary, to solve a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$,
1. obtain one factor $(x - \alpha)$ by trial and error
2. factorize $ax^3 + bx^2 + cx + d = 0$ as $(x - \alpha)(hx^2 + kx + s) = 0$
3. solve the quadratic expression for other roots

Example 1.120 **If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, then find the value of $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$.**

Sol. For the given equation $\alpha + \beta + \gamma = 0$,
$$\alpha\beta + \beta\gamma + \alpha\gamma = 4, \quad \alpha\beta\gamma = -1$$
Now,
$$(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1} = (-\gamma)^{-1} + (-\alpha)^{-1} + (-\beta)^{-1}$$
$$= -\frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$$
$$= -\frac{4}{(-1)}$$
$$= 4$$

Example 1.121 **Let $\alpha + i\beta$ ($\alpha, \beta \in R$) be a root of the equation $x^3 + qx + r = 0$, $q, r \in R$. Find a real cubic equation, independent of α and β, whose one root is 2α.**

Sol. If $\alpha + i\beta$ is a root then, $\alpha - i\beta$ will also be a root. If the third root is γ, then
$$(\alpha + i\beta) + (\alpha - i\beta) + \gamma = 0$$
$$\Rightarrow \quad \gamma = -2\alpha$$
But γ is a root of the given equation $x^3 + qx + r = 0$. Hence,
$$(-2\alpha)^3 + q(-2\alpha) + r = 0$$
$$\Rightarrow \quad (2\alpha)^3 + q(2\alpha) - r = 0$$
Therefore, 2α is a root of $t^3 + qt - r = 0$, which is independent of α and β.

Example 1.122 **In equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ if two of its roots are equal in magnitude but opposite in sign, find the roots.**

Sol. Given that $\alpha + \beta = 0$ but $\alpha + \beta + \gamma + \delta = 2$. Hence,
$$\gamma + \delta = 2$$
Let $\alpha\beta = p$ and $\gamma\delta = q$. Therefore, given equation is equivalent to $(x^2 + p)(x^2 - 2x + q) = 0$. Comparing the coefficients, we get

$p+q=4$, $-2p=6$, $pq=-21$. Therefore, $p=-3$, $q=7$ and they satisfy $pq=-21$. Hence,

$$(x^2-3)(x^2-2x+7)=0$$

Therefore, the roots are $\pm\sqrt{3}$ and $1\pm i\sqrt{6}$. (where $i=\sqrt{-1}$)

Example 1.123 **Solve the equation $x^3-13x^2+15x+189=0$ if one root exceeds the other by 2.**

Sol. Let the roots be $\alpha, \alpha+2, \beta$. Sum of roots is $2\alpha+\beta+2=13$.

$$\therefore\ \beta=11-2\alpha \quad (1)$$

Sum of the product of roots taken two at a time is

$$\alpha(\alpha+2)+(\alpha+2)\beta+\beta\alpha=15$$

or

$$\alpha^2+2\alpha+2(\alpha+1)\beta=15 \quad (2)$$

Product of the roots is

$$\alpha\beta(\alpha+2)=-189 \quad (3)$$

Eliminating β from (1) and (2), we get

$$\alpha^2+2\alpha+2(\alpha+1)(11-2\alpha)=15$$

or

$$3\alpha^2-20\alpha-7=0$$

$$\therefore\ (\alpha-7)(3\alpha+1)=0$$

$$\therefore\ \alpha=7 \text{ or } -\frac{1}{3}$$

$$\therefore\ \beta=-3,\ \frac{35}{3}$$

Out of these values, $\alpha=7$, $\beta=-3$ satisfy the third relation $\alpha\beta(\alpha+2)=-189$, i.e., $(-21)(9)=-189$. Hence, the roots are 7, 7 + 2, –3 or 7, 9, –3.

REPEATED ROOTS

In equation $f(x)=0$, where $f(x)$ is a polynomial function, and if it has roots $\alpha, \alpha, \beta, \ldots$ or α is a repeated root, then $f(x)=0$ is equivalent to $(x-\alpha)^2(x-\beta)\cdots=0$, from which we can conclude that $f'(x)=0$ or $2(x-\alpha)[(x-\beta)\cdots]+(x-\alpha)^2[(x-\beta)\cdots]'=0$ or $(x-\alpha)[2\{(x-\beta)\cdots\}+(x-\alpha)\{(x-\beta)\cdots\}']=0$ has root α.

Thus if α root occurs twice in equation then it is common in equations $f(x)=0$ and $f'(x)=0$.

Similarly, if root α occurs thrice in equation, then it is common in the equations $f(x)=0$, $f'(x)=0$ and $f''(x)=0$.

Example 1.124 **If $x-c$ is a factor of order m of the polynomial $f(x)$ of degree n $(1<m<n)$, then find the polynomials for which $x=c$ is a root.**

Sol. From the given information we have $f(x)=(x-c)^m g(x)$, where $g(x)$ is polynomial of degree $n-m$. Then $x=c$ is common root for the equations $f(x)=0$, $f^1(x)=0$, $f^2(x)=0$, …, $f^{m-1}(x)=0$, where $f^r(x)$ represents r^{th} derivative of $f(x)$ w.r.t. x.

Example 1.125 **If $a_1x^3+b_1x^2+c_1x+d_1=0$ and $a_2x^3+b_2x^2+c_2x+d_2=0$ have a pair of repeated roots common, then prove that**

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1-a_1b_2 & c_1a_2-c_2a_1 & d_1a_2-d_2a_1 \end{vmatrix}=0$$

Sol. If $f(x)=a_1x^3+b_1x^2+c_1x+d_1=0$ has roots α, α, β, then $g(x)=a_2x^3+b_2x^2+c_2x+d_2=0$ must have roots α, α, γ. Hence,

$$a_1\alpha^3+b_1\alpha^2+c_1\alpha+d_1=0 \quad (1)$$

$$a_2\alpha^3+b_2\alpha^2+c_2\alpha+d_2=0 \quad (2)$$

Now, α is also a root of equations $f'(x)=3a_1x^2+2b_1x+c_1=0$ and $g'(x)=3a_2x^2+2b_2x+c_2=0$. Therefore,

$$3a_1\alpha^2+2b_1\alpha+c_1=0 \quad (3)$$

$$3a_2\alpha^2+2b_2\alpha+c_2=0 \quad (4)$$

Also, from $a_2\times(1)-a_1\times(2)$, we have

$$(a_2b_1-a_1b_2)\alpha^2+(c_1a_2-c_2a_1)\alpha+d_1a_2-d_2a_1=0 \quad (5)$$

Eliminating α from (3), (4) and (5), we have

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1-a_1b_2 & c_1a_2-c_2a_1 & d_1a_2-d_2a_1 \end{vmatrix}=0$$

Concept Application Exercise 1.7

1. If $b^2<2ac$, then prove that $ax^3+bx^2+cx+d=0$ has exactly one real root.
2. If two roots of $x^3-ax^2+bx-c=0$ are equal in magnitude but opposite in signs, then prove that $ab=c$.
3. If α, β and γ are the roots of $x^3+8=0$, then find the equation whose roots are α^2, β^2 and γ^2.
4. If α, β, γ are the roots of the equation $x^3-px+q=0$, then find the cubic equation whose roots are $\alpha/(1+\alpha)$, $\beta/(1+\beta)$, $\gamma/(1+\gamma)$.
5. If the roots of equation $x^3+ax^2+b=0$ are α_1, α_2 and α_3 $(a, b\neq 0)$, then find the equation whose roots are
$$\frac{\alpha_1\alpha_2+\alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3},\ \frac{\alpha_2\alpha_3+\alpha_3\alpha_1}{\alpha_1\alpha_2\alpha_3},\ \frac{\alpha_1\alpha_3+\alpha_1\alpha_2}{\alpha_1\alpha_2\alpha_3}$$

QUADRATIC EXPRESSION IN TWO VARIABLES

The general quadratic expression $ax^2+2hxy+by^2+2gx+2fy+c$ can be factorized into two linear factors. Given quadratic expression is

$$ax^2+2hxy+by^2+2gx+2fy+c \quad (1)$$

Corresponding equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

or

$$ax^2+2(hy+g)x+by^2+2fy+c=0 \quad (2)$$

$$\therefore\ x=\frac{-2(hy+g)\pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}$$

$$\Rightarrow\ x=\frac{-(hy+g)\pm\sqrt{h^2y^2+g^2+2ghy-aby^2-2afy-ac}}{a}$$

$$\Rightarrow ax+hy+g=\pm\sqrt{h^2y^2+g^2+2ghy-aby^2-2afy-ac} \quad (3)$$

Now, expression (1) can be resolved into two linear factors if $(h^2-ab)y^2+2(gh-af)y+g^2-ac$ is a perfect square and $h^2-ab>0$. But $(h^2-ab)y^2+2(gh-af)y+g^2-ac$ will be a perfect square if

$\Rightarrow \quad 4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$ and $h^2 - ab > 0$

$\Rightarrow \quad g^2h^2 + a^2f^2 - 2afgh - h^2g^2 + abg^2 + ach^2 - a^2bc = 0$

and

$h^2 - ab > 0$

$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

and

$h^2 - ab > 0$

This is the required condition.

Example 1.126 **Find the values of m for which the expression $2x^2 + mxy + 3y^2 - 5y - 2$ can be resolved into two rational linear factors.**

Sol. We know that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be resolved into two linear factors if and only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Given expression is

$$2x^2 + mxy + 3y^2 - 5y - 2 \quad (1)$$

Here, $a = 2$, $h = m/2$, $b = 3$, $g = 0$, $f = -5/2$, $c = -2$. Therefore, expression $2x^2 + mxy + 3y^2 - 5y - 2$ will have two linear factors if and only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow \quad 2 \times 3(-2) + 2\left(\frac{-5}{2}\right)(0)\left(\frac{m}{2}\right) - 2\left(\frac{-5}{2}\right)^2 - 3 \times 0^2 - (-2)\left(\frac{m}{2}\right)^2 = 0$$

$$\Rightarrow \quad -12 - \frac{25}{2} + \frac{m^2}{2} = 0$$

$$\Rightarrow \quad m^2 = 49 \Rightarrow m = \pm 7$$

Example 1.127 **Find the linear factors of $2x^2 - y^2 - x + xy + 2y - 1$.**

Sol. Given expression is

$$2x^2 - y^2 - x + xy + 2y - 1 \quad (1)$$

Its corresponding equation is

$$2x^2 - y^2 - x + xy + 2y - 1 = 0$$

or

$$2x^2 - (1 - y)x - (y^2 - 2y + 1) = 0$$

$$\therefore \quad x = \frac{1 - y \pm \sqrt{(1-y)^2 + 4.2(y^2 - 2y + 1)}}{4}$$

$$= \frac{1 - y \pm \sqrt{(1-y)^2 + 8(y-1)^2}}{4}$$

$$= \frac{1 - y \pm \sqrt{9(1-y)^2}}{4}$$

$$= \frac{1 - y \pm 3(1-y)}{4}$$

$$= 1 - y, -\frac{1-y}{2}$$

Hence, the required linear factors are $(x + y - 1)$ and $(2x - y + 1)$.

FINDING THE RANGE OF A FUNCTION INVOLVING QUADRATIC EXPRESSION

In this section, some examples are given to illustrate the range of a function involving quadratic expression.

Example 1.128 **Find the range of the function $f(x) = x^2 - 2x - 4$.**

Sol. Let

$$x^2 - 2x - 4 = y$$

$$\Rightarrow \quad x^2 - 2x - 4 - y = 0$$

Now if x is real, then

$$D \geq 0$$

$$\Rightarrow \quad (-2)^2 - 4(1)(-4 - y) \geq 0$$

$$\Rightarrow \quad 4 + 16 + 4y \geq 0$$

$$\Rightarrow \quad y \geq -5$$

Hence range of $f(x)$ is $[-5, \infty)$.

Alternative method:

$$f(x) = x^2 - 2x - 4$$
$$= (x - 1)^2 - 5$$
$$\geq -5$$

Hence, range is

$$[-5, \infty)$$

Example 1.129 **Find the least value of $\dfrac{(6x^2 - 22x + 21)}{(5x^2 - 18x + 17)}$ for real x.**

Sol. Let,

$$\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17} = y$$

$$\Rightarrow \quad (6 - 5y)x^2 - 2x(11 - 9y) + 21 - 17y = 0$$

Since x is real

$$4(11 - 9y)^2 - 4(6 - 5y)(21 - 17y) \geq 0$$

$$\Rightarrow \quad -4y^2 + 9y - 5 \geq 0$$

$$\Rightarrow \quad 4y^2 - 9y + 5 \leq 0$$

$$\Rightarrow \quad 4(y - 1)(y - 5/4) \leq 0$$

$$\Rightarrow \quad 1 \leq y \leq 5/4$$

Hence, the least value of the given expression is 1.

Example 1.130 **Prove that if the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied for real value of x and y, then x must lie between 1 and 3 and y must lie between $-1/3$ and $1/3$.**

Sol. Given equation is

$$x^2 + 9y^2 - 4x + 3 = 0 \quad (1)$$

$$\Rightarrow \quad x^2 - 4x + 9y^2 + 3 = 0$$

Since x is real,

$$(-4)^2 - 4(9y^2 + 3) \geq 0$$

$$\Rightarrow \quad 16 - 4(9y^2 + 3) \geq 0$$

$$\Rightarrow \quad 4 - 9y^2 - 3 \geq 0$$

$$\Rightarrow \quad 9y^2 - 1 \leq 0$$

$$\Rightarrow \quad 9y^2 \leq 1$$

$$\Rightarrow \quad y^2 \leq \frac{1}{9}$$

$$\Rightarrow \quad -\frac{1}{3} \leq y \leq \frac{1}{3} \quad (2)$$

Equation (1) can also be written as

$$9y^2 + 0y + x^2 - 4x + 3 = 0 \quad (3)$$

Since y is real, so

$$0^2 - 4.9(x^2 - 4x + 3) \ge 0$$

or

$$x^2 - 4x + 3 \le 0 \qquad (4)$$

or

$$(x - 3)(x - 1) \le 0$$

or

$$1 \le x \le 3$$

Example 1.131 **Find the domain and the range of**

$$f(x) = \sqrt{3 - 2x - x^2}.$$

Sol. $f(x) = \sqrt{3 - 2x - x^2}$ is defined if

$$3 - 2x - x^2 \ge 0$$

$$\Rightarrow \quad x^2 + 2x - 3 \le 0$$

$$\Rightarrow \quad (x - 1)(x + 3) \le 0$$

$$\Rightarrow \quad x \in [-3, 1]$$

Also, $f(x) = \sqrt{4 - (x + 1)^2}$ has maximum value when $x + 1 = 0$. Hence range is $[0, 2]$.

Example 1.132 **Find the domain and range of**

$$f(x) = \sqrt{x^2 - 3x + 2}.$$

Sol. $x^2 - 3x + 2 \ge 0$

$$\Rightarrow \quad (x - 1)(x - 2) \ge 0$$

$$\Rightarrow \quad x \in (-\infty, 1] \cup [2, \infty)$$

Now,

$$f(x) = \sqrt{x^2 - 3x + 2}$$

$$= \sqrt{\left(x - \frac{3}{2}\right)^2 + 2 - \frac{9}{4}}$$

$$= \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}}$$

Now, the least permissible value of $(x - 3/2)^2 - 1/4$ is 0 when $(x - 3/2) = \pm 1/2$. Hence, the range is $[0, \infty)$.

Concept Application Exercise 1.8

1. Find the range of $f(x) = x^2 - x - 3$.
2. Find the range of

   (i) $f(x) = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$

   (ii) $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$
3. Find the range of $f(x) = \sqrt{x - 1} + \sqrt{5 - x}$.
4. Find the range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.
5. Find the domain and range of $f(x) = \sqrt{x^2 - 4x + 6}$.

QUADRATIC FUNCTION

Let $f(x) = ax^2 + bx + c$, where $a, b, c, \in R$ and $a \neq 0$. We have,

$$f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

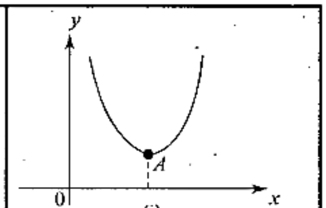
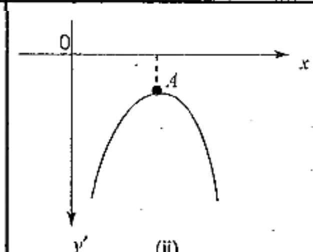
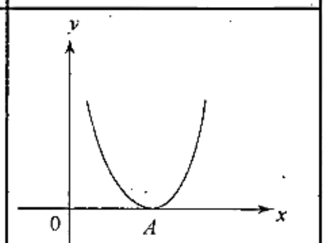
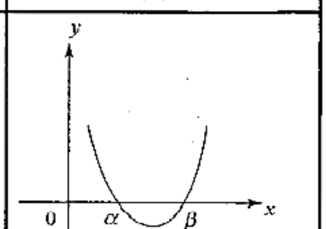
$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

$$\Rightarrow \quad \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

Thus $y = f(x)$ represents a parabola whose axis is parallel to y-axis and vertex is $A(-b/2a, -D/4a)$. For some values of x, $f(x)$ may be positive, negative or zero and for $a > 0$, the parabola opens upwards and for $a < 0$, the parabola opens downwards. This gives the following cases:

1. $a > 0$ and $D < 0$, so $f(x) > 0$, $\forall x \in R$, i.e., $f(x)$ is positive for all values of x. Range of function is $[-D/(4a), \infty)$. $x = -b/(2a)$ is a point of minima.	 (i)
2. $a < 0$ and $D < 0$ so $f(x) < 0$, $\forall x \in R$, i.e., $f(x)$ is negative for all values of x. Range of function is $(-\infty, -D/(4a)]$. $x = -b/(2a)$ is a point of maxima.	 (ii)
3. $a > 0$ and $D = 0$, so $f(x) \ge 0$, $\forall x \in R$, i.e., $f(x)$ is positive for all values of x except at vertex where $f(x) = 0$.	 (iii)
4. $a > 0$ and $D > 0$. Let $f(x) = 0$ have two real roots α and β. If $\alpha < \beta$, then $f(x) > 0$, $\forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0$, $\forall x \in (\alpha, \beta)$.	 (iv)

Example 1.139 **If c is positive and $2ax^2 + 3bx + 5c = 0$ does not have any real roots, then prove that $2a - 3b + 5c > 0$.**

Sol. Given $c > 0$ and $2ax^2 + 3bx + 5c = 0$ does not have real roots. Let

$f(x) = 2ax^2 + 3bx + 5c$

$\Rightarrow$ $f(x) > 0, \forall x \in R$, if $a > 0$ or $f(x) < 0\ \forall x \in R$, if $a < 0$

But

$5c = f(0) > 0$

$\Rightarrow$ $f(x) > 0, \forall x \in R$

$\Rightarrow$ $2ax^2 + 3bx + 5c > 0, \forall x \in R$

$\Rightarrow$ $2a - 3b + 5c > 0$ (for $x = -1$)

Example 1.140 **If $ax^2 + bx + 6 = 0$ does not have distinct real roots, then find the least value of $3a + b$.**

Sol. Given equation $ax^2 + bx + 6 = 0$ does not have distinct real roots. Hence,

$\Rightarrow$ $f(x) = ax^2 + bx + 6 \le 0, \forall x \in R$, if $a < 0$

or

$f(x) = ax^2 + bx + 6 \ge 0, \forall x \in R$, if $a > 0$

But

$f(0) = 6 > 0$

$\Rightarrow$ $f(x) = ax^2 + bx + 6 \ge 0, \forall x \in R$

$\Rightarrow$ $f(3) = 9a + 3b + 6 \ge 0$

$\Rightarrow$ $3a + b \ge -2$

Therefore, the least value of $3a + b$ is -2.

Example 1.141 **A quadratic trinomial $P(x) = ax^2 + bx + c$ is such that the equation $P(x) = x$ has no real roots. Prove that in this case the equation $P(P(x)) = x$ has no real roots either.**

Sol. Since the equation $ax^2 + bx + c = x$ has no real roots, the expression $P(x) - x = ax^2 + (b - 1)x + c$ assumes values of one sign $\forall x \in R$, say $P(x) - x > 0$. Then

$P(P(x_0)) - P(x_0) > 0$

for any $x = x_0$, i.e., $P(x_0) > x_0$ and hence $P(P(x_0)) > x_0$. Therefore, x_0 cannot be a root of the 4th degree equation $P(P(x)) = x$.

Example 1.142 **Prove that for real values of x the expression $(ax^2 + 3x - 4)/(3x - 4x^2 + a)$ may have any value provided a lies between 1 and 7.**

Sol. Let,

$$y = \frac{ax^2 + 3x - 4}{3x - 4x^2 + a}$$

$\Rightarrow$ $(a + 4y)x^2 + (3 - 3y)x - 4 - ay = 0$

Now, x is real. So,

$D \ge 0$

$\Rightarrow$ $9(1 - y)^2 + 4(a + 4y)(4 + ay) \ge 0$

$\Rightarrow$ $(9 + 16a)y^2 + (-18 + 4a^2 + 64)y + (9 + 16a) \ge 0$, $\forall y \in R$ ($\because$ y takes any real value)

$\Rightarrow$ $9 + 16a > 0$ and $(4a^2 + 46)^2 - 4(9 + 16a)^2 \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $(4a^2 + 46 - 18 - 32a)(4a^2 + 46 + 18 + 32a) \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $(a^2 - 8a + 7)(a^2 + 8a + 16) \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $1 \le a \le 7$ or $a = -4$

$\Rightarrow$ $1 \le a \le 7$

Example 1.143 **Let a, b and c be real numbers such that $a + 2b + c = 4$. Find the maximum value of $(ab + bc + ca)$.**

Sol. Given,

$a + 2b + c = 4 \Rightarrow a = 4 - 2b - c$

Let,

$ab + bc + ca = x \Rightarrow a(b + c) + bc = x$

$\Rightarrow$ $(4 - 2b - c)(b + c) + bc = x$

$\Rightarrow$ $4b + 4c - 2b^2 - 2bc - bc - c^2 + bc = x$

$\Rightarrow$ $2b^2 - 4b + 2bc - 4c + c^2 + x = 0$

$\Rightarrow$ $2b^2 + 2(c - 2)b - 4c + c^2 + x = 0$

Since $b \in R$, so

$4(c - 2)^2 - 4 \times 2(-4c + c^2 + x) \ge 0$

$\Rightarrow$ $c^2 - 4c + 4 + 8c - 2c^2 - 2x \ge 0$

$\Rightarrow$ $c^2 - 4c + 2x - 4 \le 0$

Since $c \in R$, so

$16 - 4(2x - 4) \ge 0 \Rightarrow x \le 4$

$\therefore$ $\max(ab + bc + ac) = 4$

Example 1.144 **Prove that for all real values of x and y, $x^2 + 2xy + 3y^2 - 6x - 2y \ge -11$.**

Sol. Let,

$x^2 + 2xy + 3y^2 - 6x - 2y + 11 \ge 0, \forall x, y \in R$

$\Rightarrow$ $x^2 + (2y - 6)x + 3y^2 - 2y + 11 \ge 0, \forall x \in R$

$\Rightarrow$ $(2y - 6)^2 - 4(3y^2 - 2y + 11) \le 0, \forall y \in R$

$\Rightarrow$ $(y - 3)^2 - (3y^2 - 2y + 11) \le 0, \forall y \in R$

$\Rightarrow$ $2y^2 + 4y + 2 \ge 0, \forall y \in R$

$\Rightarrow$ $(y + 1)^2 \ge 0, \forall y \in R$, which is always true

Concept Application Exercise 1.9

1. If $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x, then find the values of a.
2. If $ax^2 + bx + c = 0$, $a, b, c \in R$ has no real zeros, and if $c < 0$, then which of the following is true?
   (i) $a < 0$
   (ii) $a + b + c > 0$
   (iii) $a > 0$
3. If $ax^2 + bx + c = 0$ has imaginary roots and $a + c < b$, then prove that $4a + c < 2b$.

4. Let $x, y, z \in R$ such that $x + y + z = 6$ and $xy + yz + zx = 7$. Then find the range of values of x, y and z.
5. If x is real and $(x^2 + 2x + c)/(x^2 + 4x + 3c)$ can take all real values, then show that $0 \le c \le 1$.
6. If $x \in R$, and a, b, c are in ascending or descending order of magnitude, show that $(x - a)(x - c)/(x - b)$ (where $x \ne b$) can assume any real value.
7. Find the complete set of values of a such that $(x^2 - x)/(1 - ax)$ attains all real values.
8. If the quadratic equation $ax^2 + bx + 6 = 0$ does not have real roots and $b \in R^+$, then prove that
$$a > \max\left\{\frac{b^2}{24}, b - 6\right\}$$
9. If x be real and the roots of the equation $ax^2 + bx + c = 0$ are imaginary, then prove that $a^2x^2 + abx + ac$ is always positive.

LOCATION OF ROOTS

In some problems, we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c.

1. $\alpha, \beta > 0$

Conditions:

(a) sum of roots, $\alpha + \beta > 0$

(b) product of roots, $\alpha\beta > 0$

(c) $D \ge 0$

Graphically:

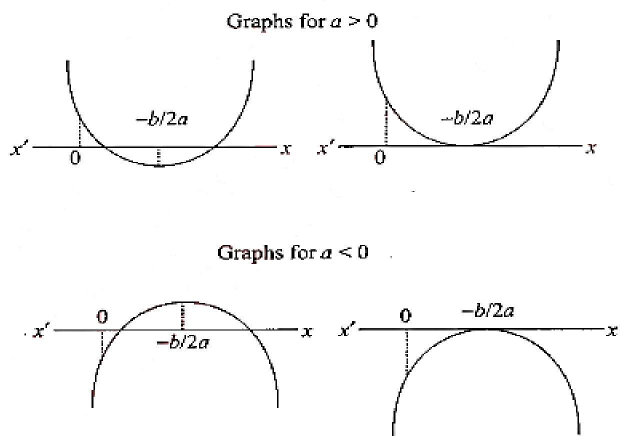


Fig. 1.41

Conditions:

(a) $af(0) > 0$ ($\because$ when $a > 0, f(0) > 0$ and when $a < 0, f(0) < 0$)

(b) $-b/2a > 0$

(c) $D \ge 0$

2. $\alpha, \beta < 0$

Conditions:

(a) sum of roots, $\alpha + \beta < 0$

(b) product of roots, $\alpha\beta > 0$

(c) $D \ge 0$

Graphically:

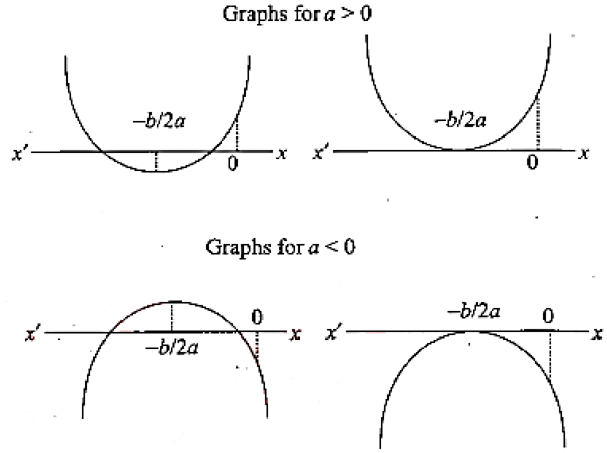


Fig. 1.42

Conditions:

(a) $af(0) > 0$

(b) $-b/2a < 0$

(c) $D \ge 0$

3. $\alpha < 0 < \beta$ (roots of opposite sign)

Product of roots, $\alpha\beta < 0$

Note That when $\alpha\beta = \frac{C}{a} 0$, $ac < 0$

$\Rightarrow$ $D = b^2 - 4ac > 0$.

Graphically:

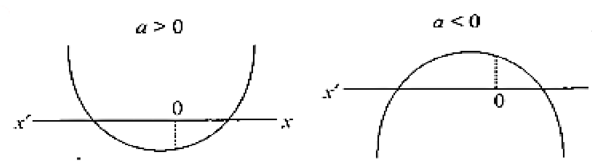


Fig. 1.43

When $a > 0, f(0) < 0$ and when $a < 0$, then $f(0) > 0$

$\Rightarrow$ $af(0) < 0$.

4. $\alpha, \beta > k$

Graphically:

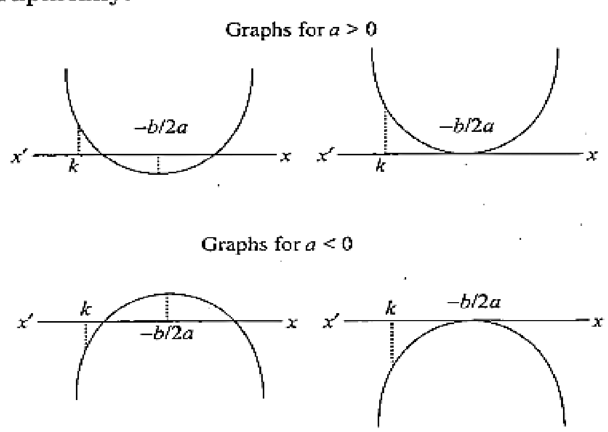


Fig. 1.44

Conditions:
(a) $af(k) > 0$ ($\because$ when $a > 0, f(k) > 0$ and when $a < 0, f(k) < 0$)
(b) $-b/2a > k$
(c) $D \geq 0$

5. $\alpha, \beta < k$
Graphically:

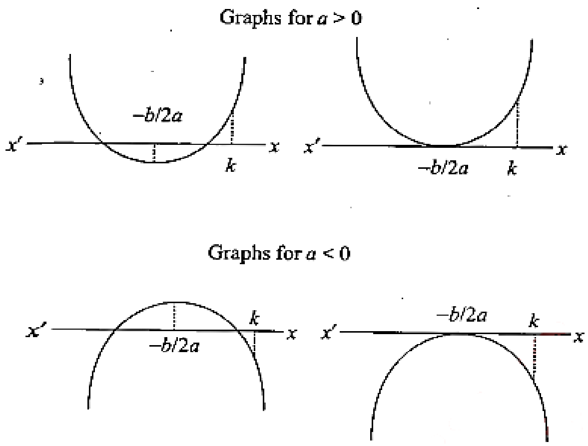


Fig. 1.45

Conditions:
(a) $af(k) > 0$
($\because$ when $a > 0, f(k) > 0$ and when $a < 0, f(k) < 0$)
(b) $-b/(2a) < k$
(c) $D \geq 0$

6. $\alpha < k < \beta$ (one root is smaller than k and other root is greater than k)

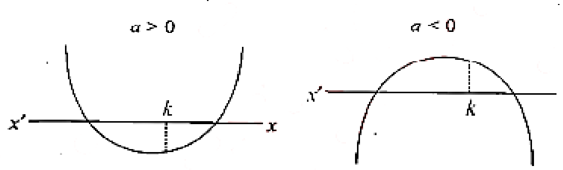


Fig. 1.46

When $a > 0, f(k) < 0$ and when $a < 0$, then $f(k) > 0$
$\Rightarrow \quad af(k) < 0.$

7. Exactly one root lying in (k_1, k_2)

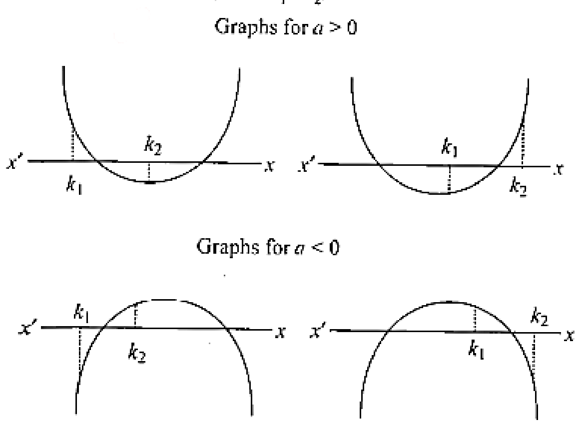


Fig. 1.47

From the graphs, we can see that $f(k_1)$ and $f(k_2)$ have opposite sign. Hence, $f(k_1)f(k_2) < 0$.

8. Both the roots lying in the interval (k_1, k_2).

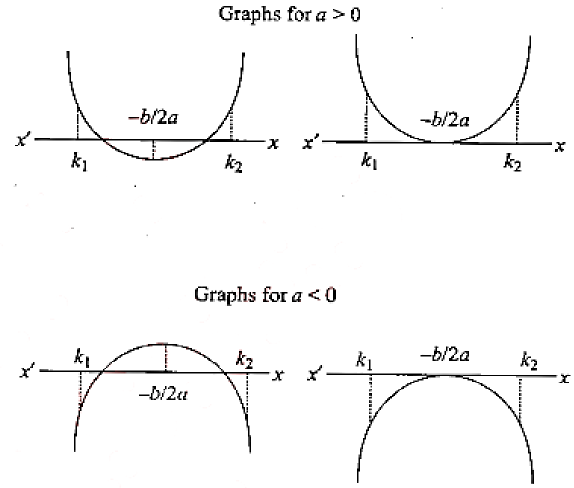


Fig. 1.48

From the graphs,
(a) $af(k_1) > 0$ and $af(k_2) > 0$
(b) $k_1 < -b/(2a) < k_2$
(c) $D \geq 0$

9. One root is smaller than k_1 and other root is greater than k_2
In this case k_1 and k_2 lie between the roots.

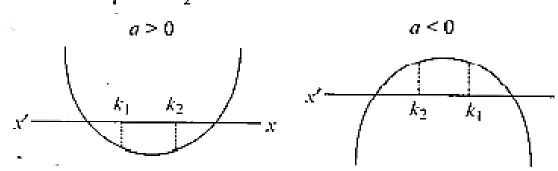


Fig. 1.49

From the graphs, $af(k_1) < 0$ and $af(k_2) < 0$.

Example 1.145 **Let $x^2 - (m - 3)x + m = 0$ ($m \in R$) be a quadratic equation. Find the values of m for which the roots are**

(i) real and distinct
(ii) equal
(iii) not real
(iv) opposite in sign
(v) equal in magnitude but opposite in sign
(vi) positive
(vii) negative
(viii) such that at least one is positive
(ix) one root is smaller than 2 and the other root is greater than 2
(x) both the roots are greater than 2

(xi) **both the roots are smaller than 2**

(xii) **exactly one root lies in the interval (1, 2)**

(xiii) **both the roots lie in the interval (1, 2)**

(xiv) **at least one root lies in the interval (1, 2)**

(xv) **one root is greater than 2 and the other root is smaller than 1**

Sol. Let $f(x) = x^2 - (m-3)x + m = 0$

(i)

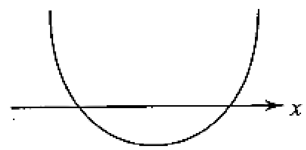


Fig. 1.50

Both the roots are real and distinct. So,

$D > 0$

$\Rightarrow \quad (m-3)^2 - 4m > 0$

$\Rightarrow \quad m^2 - 10m + 9 > 0$

$\Rightarrow \quad (m-1)(m-9) > 0$

$\Rightarrow \quad m \in (-\infty, 1) \cup (9, \infty)$

(ii)

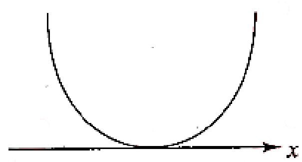


Fig. 1.51

Both the roots are equal. So,

$D = 0 \Rightarrow m = 9$ or $m = 1$

(iii)

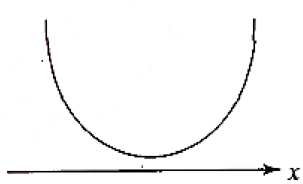


Fig. 1.52

Both the roots are imaginary. So,

$D < 0$

$\Rightarrow \quad (m-1)(m-9) < 0$

$\Rightarrow \quad m \in (1, 9)$

(iv)

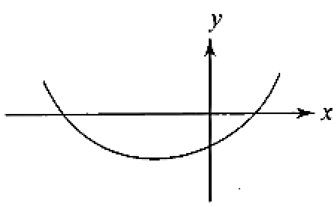


Fig. 1.53

The roots are opposite in sign. Hence, the product of roots is negative. So,

$m < 0 \Rightarrow m \in (-\infty, 0)$

(v)

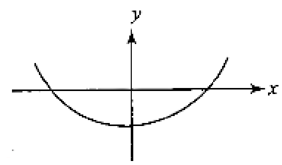


Fig. 1.54

Roots are equal in magnitude but opposite in sign. Hence, sum of roots is zero as well as $D \geq 0$. So,

$m \in (-\infty, 1) \cup (9, \infty)$ and $m - 3 = 0$, i.e., $m = 3$

$\Rightarrow$ no such m exists, so $m \in \phi$.

(vi)

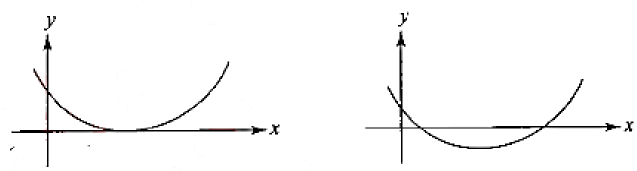


Fig. 1.55

Both the roots are positive. Hence, $D \geq 0$ and both the sum and the product of roots are positive. So,

$m - 3 > 0, m > 0$ and $m (-\infty, 1] \cup [9, \infty)$

$m \in [9, \infty)$

(vii)

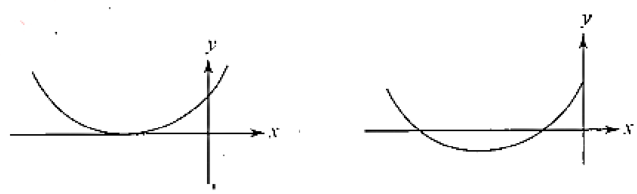


Fig. 1.56

Both the roots are negative. Hence, $D \geq 0$, and sum is negative but product is positive. So,

$m - 3 < 0, m > 0, m \in (-\infty, 1] \cup [9, \infty)$

$\Rightarrow \quad m \in (0, 1]$

(viii) At least one root is positive. Hence, either one root is positive or both roots are positive. So,

$m \in (-\infty, 0) \cup [9, \infty)$

(ix)

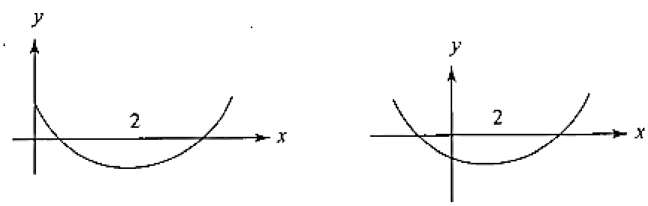


Fig. 1.57

One root is smaller than 2 and the other root is greater than 2, i.e., 2 lies between the roots. So,

$f(2) < 0$

$\Rightarrow \quad 4 - 2(m - 3) + m < 0$

$\Rightarrow \quad m > 10$

(x)

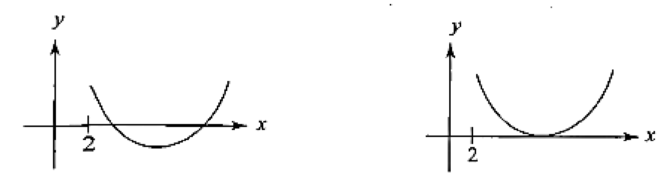


Fig. 1.58

Both the roots are greater than 2. So,

$$f(2) > 0,\ D \geq 0,\ -\frac{b}{2a} > 2$$

$\Rightarrow \quad m < 10$ and $m \in (-\infty, 1] \cup [9, \infty)$ and $m - 3 > 4$

$\Rightarrow \quad m \in [9, 10)$

(xi)

Fig. 1.59

Both the roots are smaller than 2. So,

$$f(2) > 0,\ D \geq 0,\ -\frac{b}{2a} < 2$$

$\Rightarrow \quad m \in (-\infty, 1]$

(xii)

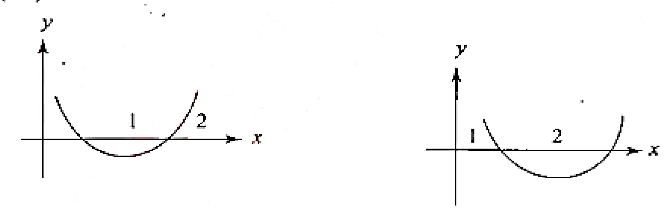


Fig. 1.60

Exactly one root lies in (1, 2). So,

$f(1)f(2) < 0$

$\Rightarrow \quad 4(10 - m) < 0$

$\Rightarrow \quad m \in (10, \infty)$

(xiii) Both the roots lie in the interval (1, 2). Then,

$$D \geq 0 \Rightarrow (m - 1)(m - 9) \geq 0 \Rightarrow m \leq 1 \text{ or } m \geq 9 \quad (1)$$

Also

$$f(1) > 0 \text{ and } f(2) > 0 \Rightarrow 10 > m \quad (2)$$

and

$$1 < -\frac{b}{2a} < 2 \Rightarrow 5 < m < 7 \quad (3)$$

Thus, no such m exists.

(xiv) **Case I:** Exactly one root lies in (1, 2). So,

$$f(1)f(2) < 0 \Rightarrow m > 10$$

Case II: Both the roots lie in (1, 2). So, from (xiii), $m \in \phi$. Hence, $m \in (10, \infty)$.

(xv) For one root greater than 2 and the other root smaller than 1,

$$f(1) < 0 \quad (1)$$

$$f(2) < 0 \quad (2)$$

From (1), $f(1) < 0$, but $f(1) = 4$, which is not possible. Thus, no such m exists.

Example 1.146 **Find the values of a for which the equation $\sin^4 x + a \sin^2 x + 1 = 0$ will have a solution.**

Sol. Let

$$t = \sin^2 x \Rightarrow t \in [0, 1]$$

Hence, $t^2 + at + 1 = 0$ should have at least one solution in [0, 1]. Since product of roots is positive and equal to one, $t^2 + at + 1 = 0$ must have exactly one root in [0, 1]. Hence,

$f(1) < 0$

$\Rightarrow \quad 2 + a < 0$

$\Rightarrow \quad a \in (-\infty, -2)$

Example 1.147 **If $(x^2 + x + 2)^2 - (a - 3)(x^2 + x + 1)(x^2 + x + 2) + (a - 4)(x^2 + x + 1)^2 = 0$ has at least one root, then find the complete set of values of a.**

Sol. Let,

$$t = x^2 + x + 1 \Rightarrow t \in \left[\frac{3}{4}, \infty\right)$$

Hence,

$(t + 1)^2 - (a - 3)\, t(t + 1) + (a - 4)\, t^2 = 0$

$\Rightarrow \quad t^2 + 2t + 1 - (a - 3)(t^2 + t) + (a - 4)t^2 = 0$

$\Rightarrow \quad t(2 - a + 3) + 1 = 0$

$\Rightarrow \quad t = \frac{1}{(a - 5)}$

$\Rightarrow \quad \frac{1}{a - 5} \geq \frac{3}{4}$

$\Rightarrow \quad \frac{19 - 3a}{(a - 5)} \geq 0$

$\Rightarrow \quad a \in \left(5, \frac{19}{3}\right]$

Example 1.148 **If α is a real root of the quadratic equation $ax^2 + bx + c = 0$ and β is a real root of $-ax^2 + bx + c = 0$,**

then show that there is a root γ of the equation $(a/2)x^2 + bx + c = 0$ which lies between a and β.

Sol. Let,

$$f(x) = \frac{a}{2}x^2 + bx + c$$

$$\Rightarrow \quad f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c$$

$$= a\alpha^2 + b\alpha + c - \frac{a}{2}\alpha^2$$

$$= -\frac{a}{2}\alpha^2 \qquad (\because \alpha \text{ is a root of } ax^2 + bx + c = 0)$$

$$f(\beta) = \frac{a}{2}\beta^2 + b\beta + c$$

$$= -a\beta^2 + b\beta + c + \frac{3}{2}a\beta^2$$

$$= \frac{3}{2}a\beta^2 \qquad (\because \beta \text{ is a root of } -ax^2 + bx + c = 0)$$

Now,

$$f(\alpha)f(\beta) = \frac{-3}{4}a^2\alpha^2\beta^2 < 0$$

Hence, $f(x) = 0$ has one real root between α and β.

Example 1.149 **For what real values of a do the roots of the equation $x^2 - 2x - (a^2 - 1) = 0$ lie between the roots of the equation $x^2 - 2(a + 1)x + a(a - 1) = 0$.**

Sol. $x^2 - 2x - (a^2 - 1) = 0$ (1)

$x^2 - 2(a + 1)x + a(a - 1) = 0$ (2)

From Eq. (1),

$$x = \frac{2 \pm \sqrt{4 + 4(a^2 - 1)}}{2} = 1 \pm a$$

Now, roots of Eq. (1) lie between roots of Eq. (2). Hence, graphs of expressions for Eqs. (1) and (2) are as follows:

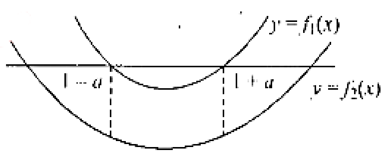


Fig. 1.61

$$f_1(x) = x^2 - 2x - (a^2 - 1)$$

$$f_2(x) = x^2 - 2(a + 1)x + a(a - 1)$$

From the graph, we have

$$f_2(1 - a) < 0 \text{ and } f_2(1 + a) < 0$$

$$\Rightarrow \quad (1 - a)^2 - 2(a + 1)(1 - a) + a(a - 1) < 0$$

$$\Rightarrow \quad (1 - a)[(1 - a) - 2a - 2 - a] < 0$$

$$\Rightarrow \quad (1 - a)(-4a - 1) < 0$$

$$\Rightarrow \quad (a - 1)(4a + 1) < 0$$

$$\Rightarrow \quad -\frac{1}{4} < a < 1 \qquad (3)$$

and

$$\Rightarrow \quad (1 + a)^2 - 2(a + 1)(a + 1) + a(a - 1) < 0$$

$$\Rightarrow -(a + 1)^2 + a(a - 1) < 0$$

$$\Rightarrow \quad -a^2 - 2a - 1 + a^2 - a < 0$$

$$\Rightarrow \quad 3a + 1 > 0$$

$$\Rightarrow \quad a > -\frac{1}{3} \qquad (4)$$

From (3) and (4), the required values of a lies in the range $-1/4 < a < 1$.

SOLVING INEQUALITIES USING LOCATION OF ROOTS

Example 1.150 **Find the value of a for which $ax^2 + (a - 3)x + 1 < 0$ for at least one positive real x.**

Sol. Let $f(x) = ax^2 + (a - 3)x + 1$

Case I:

If $a > 0$, then $f(x)$ will be negative only for those values of x which lie between the roots. From the graphs, we can see that $f(x)$ will be less than zero for at least one positive real x, when $f(x) = 0$ has distinct roots and at least one of these roots is a positive real root.

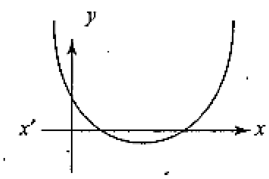


Fig. 1.62

Since $f(0) = 1 > 0$, the favourable graph according to the question is shown in the figure given above. From the graph, we can see that both the roots are non-negative. For this,

(i) $D > 0 \Rightarrow (a - 3)^2 - 4a > 0$

$$\Rightarrow \quad a < 1 \text{ or } a > 9 \qquad (1)$$

(ii) sum > 0 and product ≥ 0

$$\Rightarrow \quad -(a - 3) > 0 \text{ and } 1/a > 0$$

$$\Rightarrow \quad 0 < a < 3 \qquad (2)$$

From (1) and (2), we have

$$a \in (0, 1)$$

Case II: $a < 0$

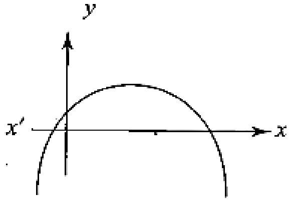


Fig. 1.63

Since $f(0) = 1 > 0$, then graph is as shown in the figure, which shows that $ax^2 + (a-3)x + 1 < 0$, for at least one positive x.

Case III: $a = 0$

If $a = 0$,

$$f(x) = -3x + 1$$

$$\Rightarrow f(x) < 0, \forall x > 1/3$$

Thus, from all the cases, the required set of values of a is $(-\infty, 1)$.

Example 1.151 **If $x^2 + 2ax + a < 0\ \forall x \in [1, 2]$, then find the values of a.**

Sol. Given,

$$x^2 + 2ax + a < 0, \forall x \in [1, 2]$$

Hence, 1 and 2 lie between the roots of the equation $x^2 + 2ax + a = 0$,

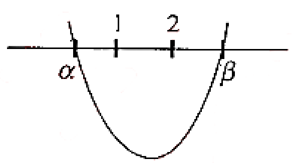


Fig. 1.64

$$\Rightarrow f(1) < 0 \text{ and } f(2) < 0$$

$$\Rightarrow 1 + 2a + a < 0, 4 + 4a + a < 0$$

$$\Rightarrow a < -\frac{1}{3}, a < -\frac{4}{5}$$

$$\Rightarrow a \in \left(-\infty, -\frac{4}{5}\right)$$

Example 1.152 **If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.**

Sol. $(y^2 - 5y + 3)(x^2 + x + 1) < 2x, \forall x \in R$

$$\Rightarrow y^2 - 5y + 3 < \frac{2x}{x^2 + x + 1} \quad (\because x^2 + x + 1 > 0\ \forall x \in R)$$

L.H.S. must be less than the least value of R.H.S. Now lets find the range of R.H.S.

Let

$$\frac{2x}{x^2 + x + 1} = p$$

$$\Rightarrow px^2 + (p-2)x + p = 0$$

Since x is real,

$$(p-2)^2 - 4p^2 \geq 0$$

$$\Rightarrow -2 \leq p \leq \frac{2}{3}$$

The minimum value of $2x/(x^2 + x + 1)$ is -2. So,

$$y^2 - 5y + 3 < -2$$

$$\Rightarrow y^2 - 5y + 5 < 0$$

$$\Rightarrow y \in \left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$$

Example 1.153 **Find the values of a for which $4^t - (a-4)2^t + (9/4)a < 0, \forall t \in (1, 2)$.**

Sol. Let $2^t = x$ and $f(x) = x^2 - (a-4)x + (9/4)a$. We want $f(x) < 0, \forall x \in (2^1, 2^2)$, i.e., $\forall x \in (2, 4)$.

(i) Since coefficient of x^2 in $f(x)$ is positive, $f(x) < 0$ for some x only when roots of $f(x) = 0$ are real and distinct. So,

$$D > 0$$

$$\Rightarrow a^2 - 17a + 16 > 0 \quad (1)$$

(ii) Since we want $f(x) < 0\ \forall x \in (2, 4)$, one of the roots of $f(x) = 0$ should be smaller than 2 and the other must be greater than 4, i.e.,

$$f(2) < 0 \text{ and } f(4) < 0$$

$$\Rightarrow a < -48 \text{ and } a > 128/7$$

which is not possible. Hence, no such a exists.

Concept Application Exercise 1.10

1. Find the values of a if $x^2 - 2(a-1)x + (2a+1) = 0$ has positive roots.
2. If the equation $(a-5)x^2 + 2(a-10)x + a + 10 = 0$ has roots of opposite sign, then find the values of a.
3. If both the roots of $x^2 - ax + a = 0$ are greater than 2, then find the values of a.
4. If both the roots of $ax^2 + ax + 1 = 0$ are less than 1, then find exhaustive range of values of a.
5. If both the roots of $x^2 + ax + 2 = 0$ lies in the interval (0, 3), then find exhaustive range of values of a.
6. If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.
7. If α is the root (having the least absolute value) of the equation $x^2 - bx - 1 = 0$ ($b \in R^+$), then prove that $-1 < \alpha < 0$.
8. If $a < b < c < d$, then show that the quadratic equation $\mu(x-a)(x-c) + \lambda(x-b)(x-d) = 0$ has real roots for all real μ and λ.