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PHYSICS

MOMENT OF INERTIA

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MOMENT OF INERTIA

UNIT:VI CHAP:02

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Moment Of Inertia

A Quantity that measures the inertia of rotational motion of the body is called rotational inertia (or moment of inertia). Just like mass of a body is a measure of inertia of the body in linear motion.

Moment of inertia: According to Newton's first law of motion, everybody continues in its state of rest or of uniform linear motion, unless an external force acts on it to change that state. **This inability of a body to change by itself its state of rest or of linear uniform motion is called inertial.**

Similarly, a body rotating about a given axis tends to maintain its state of uniform rotation, unless and external torque is applied on it to change that state. ***This property of a body by virtue of which it opposes the torque tending to change its state of rest or of uniform rotation about an axis is called rotational inertia or moment of inertia.***

"Moment of inertia of a rigid body about a given axis of rotation is defined as the sum of to the product of masses Of the particles constituting the body and the square of their respective distance from the axis of rotation".

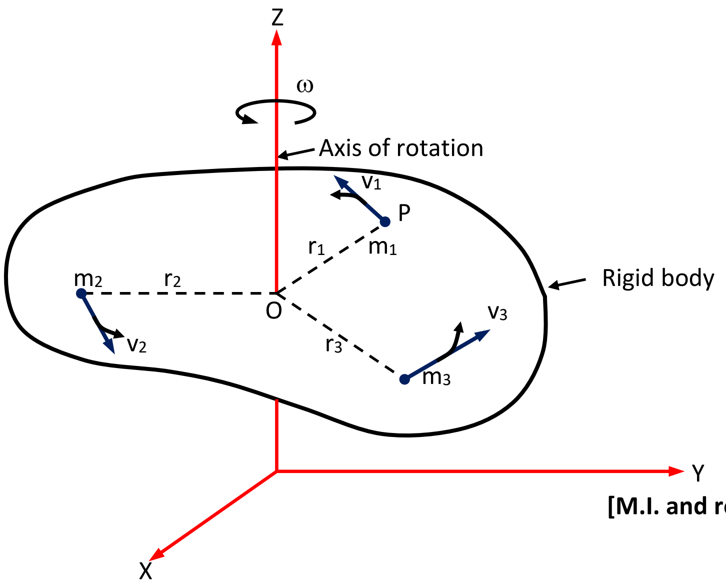


fig-[i]

[M.I. and rotational K.E. of a rigid body]

Consider a rigid body rotating with uniform angular velocity ω about a vertical axis through O, as shown in Fig. Suppose the body consists of n particles of masses $m_1, m_2, m_3, \ldots, m_n$ situated at distances $r_1, r_2, r_3, \ldots, r_n$ respectively from the axis of rotation. The moment of inertia of the body about the axis OZ is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2$$

or $$I = \sum_{i=1}^{n} m_i r_i^2$$

- The **dimensional formula** of moment of inertia is **$[ML^2 T^0]$.**
- The **SI unit of moment of inertia** is **kg m²**
- **CGS** unit is **g cm².**

Physical significance of moment of inertia: The mass of a body resists a change in its state of linear motion, it is a measure of its inertia in linear motion. Similarly, the moment of inertia of a body about an axis of rotation resists a change in its rotational motion. The greater the moment of inertia of a body, the greater is the torque required to change its state of rotation. Thus, moment of inertia of a body can be regarded as the measure of rotational inertia of the body. The moment of inertia of a body plays the same role in the rotational motion as the mass plays in linear motion. ***That is why moment of inertia is called the rotational analogue of mass in linear motion.***

Factors on which the moment of inertia depends: The moment of inertia of a body is the measure of the manner in which its different parts are distributed at different distance from the axis of rotation. Unlike mass, it is not a fixed quantity as it depends on the position and orientation of the axis of rotation with respect to the body as a whole.

The moment of inertia of a body depends on factors:

-------(i) Mass of a body.
-------(ii) Size and shape of the body.
-------(iii) Distribution of mass about the axis of rotation.
-------(iv) Position and orientation of the axis of rotation w.r.t. the body.

Practical applications of moment of inertia:

(i) A heavy wheel, called flywheel, is attached to the shaft of steam engine, automobile engine, etc. Because of its large moment of inertia, the flywheel opposes the sudden increases or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motions and hence ensures smooth ride for the passengers.

(ii) In a bicycle bullock-cart, etc., the moment of inertia is increased by concentrating most of the mass at the rim of the wheel and connecting the rim to the axle through the spokes. Even after we stop paddling, the wheels of a bicycle continue to rotate for some time due to their large moment of inertia.

Conclusion

------- The M.I not only depends on the mass of the body but also depends on how the masses are distributed . Also it depends on the distance of the particle from the axis of rotation.

------- ***Moment of inertia plays the same role in rotational motion as mass does in linear motion.***

------- When the body is rotating in X-Y plane $\vec{r}$ can be expressed in term of X, Y component.

i.e., $I = \Sigma \; m [x^2 + y^2]$

-------When the body has a continuous uniform distribution of mass then, $I = \int r^2 \, dm$; where, dm = mass of small element

RADIUS OF GYRATION

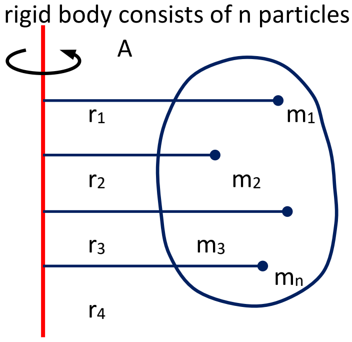
For anybody capable of rotation about a given axis, it is possible to find a radial distance from the axis where, if whole mass of the body is concentrated, its moment of inertia will remain unchanged. This radial distance is called radius of gyration and is denoted **by k.**

The radius of gyration of a body about its axis of rotation may be defined as the distance from the axis of rotation at which, if the whole mass of the body were concentrated, its moment of inertia about the given axis would be the same as with the actual distribution of mass.

---The radius of gyration k is a geometrical property of the body and the axis of rotation.

---It gives a measure of the manner in which the mass of a rotating body is distributed with respect to the axis of rotation.k has the dimensions of length L and is measured in meter or cm.

Expression for k: Suppose a rigid body consists of n particles of mass m each, situated at distances $r_1, r_2, r_3, \ldots r_n$ from the axis of rotation AB.



[Axis of gyration]

The moment of inertia of the body about the axis AB is

$$I = mr_1^2 + mr_2^2 + mr_3^2 + \ldots + mr_n^2$$
$$= m(r_1^2 + r_2^2 + \ldots + r_n^2)$$
$$= m \times n \frac{(r_1^2 + r_2^2 + r_3^2 + \ldots + r^2 n)}{n}$$

or $I = M \frac{(r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2)}{n}$ where $M = m \times n$ = total mass of the body.

---If the total mass of the body M were concentrated at a point 'P' at a perpendicular Distance 'K' from the axis of rotation then, The M. I of a body of mass M and radius of gyration K is

$$I` = MK^2$$

But , $I = I`$

$\therefore \quad Mk^2 = M \left(\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n} \right)$

or $k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}}$ **= Root mean square distance of the particles from the axis of rotation.**

" Radius of gyration of a body about an axis of rotation is defined as the root mean square distance of the particles from the same axis of rotation".

Factors on which radius of gyration of a body depend:

---(i) Position and direction of the axis of rotation.

---(ii) Distribution of mass about the axis of rotation.

◙ **When a body rotates about a given axis and the axis of rotation also moves .The motion is combination of translatory and rotatory motion.**

Total K.E of the body = K.E of translation + K.E of rotational motion.

◙ **UNIT OF GYRATION**

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + r_4^2 + \ldots + r_n^2}{N}} = \text{meter (S.I)} = \text{cm (cgs)}$$

◙ **Dimension ---- $[M^0 L^1 T^0]$**

RELATION BETWEEN ROTATIONAL K.E. AND MOMENT OF INERTIA

Fig [i], consider a rigid body rotating about an axis OZ with uniform angular velocity ω. The body may be assumed to consists of n particles of masses $m_1, m_2, m_3, \ldots m_n$; situated at distances $r_1, r_2, r_3, \ldots r_n$ from the axis of rotation. As the angular velocity ω of all the n particles is same, so their linear velocities are

$v_1 = r_1 \omega, \quad v_2 = r_2 \omega, \quad v_3 = r_3 \omega, \ \ldots \ldots, vn = r_n \omega$

Hence the total kinetic energy of rotation of the body about the axis OZ is

Rotational K.E $= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \ldots + \frac{1}{2} m_n v_n^2$

$= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + m_3 r^2_3 \omega^2 + \ldots + \frac{1}{2} m_n r^2_n \omega^2$

$= \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2 + m^2 r^2_3 + \ldots + m_n r_n^2) \omega^2 = \frac{1}{2} (\Sigma mr^2) \omega^2$

But $\Sigma mr^2 = I$, the moment of inertial of the body about the axis of rotation.

∴ Rotational K.E. $= \frac{1}{2} I \omega^2$

When $\omega = 1$, rotational K.E. $= \frac{1}{2} I$

or $I = 2 \times$ Rotational K.E.

Hence **the moment of inertia of a rigid body about an axis of rotation is numerically equal to twice the rotational kinetic energy of the body when it is rotating with unit angular velocity about that axis.**

◙ Rotational KE , $E_k = \frac{1}{2} \omega^2 I$

◙ Linear KE $= \frac{1}{2} m v^2$

◙ If **$\omega = 1$, then KE = $\frac{1}{2} (1)^2 I \Rightarrow I = 2 E_k$** , **thus, MI of a body about the given axis of rotation is equal to Twice the kinetic energy of rotation of the body rotating with unit angular velocity.**

RELATION BETWEEN ANGULAR MOMENTUM & MOMENT OF INERTIA

We know that ,

Angular momentum $L = r p$

Or, $L = r m v = r v m$ [since, $P = mv$]

$L = r \omega r m = m r^2 \omega$ [since, $v = r \omega$]

$\boxed{L = I \omega}$ [since, $I = mr^2$]

If, $\omega = 1$ then, $L = I \times 1$ $\boxed{L = I}$ ◙ In vector form $\vec{L} = \vec{I} \omega$

Hence, "**moment of Inertia of a body is numerically equal to its angular momentum when rotating with unit angular Velocity.**"

RELATION BETWEEN TORQUE & M I

We know that,

$\vec{\tau} = \vec{r} \times \vec{F} \Rightarrow \tau = r F \sin \theta$

$\tau = r F$ [$\vec{F}$ is ar⊥ $\vec{r}$]

$\tau = m \times a \times r$ [since, $F = m a$]

$\tau = r \times m \times (r \alpha)$ [since, $a = r \alpha$]

$\tau = m r^2 \alpha$

$\boxed{\tau = I \alpha}$ [since, $m r^2 = I$]

If $\alpha = 1$

Then, $\tau = I$

Thus, "M **I of a body about a given axis is numerically equal to the external torque required to produce unit α.**"

PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

A law of conservation of angular momentum **"The momentum of a system of particles is constant when the net external torque acting in the system is zero"**

simply, the angular momentum of a system is conserved if no external torque acts on it.

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

if $\vec{\tau} = 0$ then, $\frac{d\vec{L}}{dt} = 0$ ∴ $\vec{L}$ = constant .-----------(i)

But, $\vec{L} = I\vec{\omega}$ therefore, I ω = constant. ----------(ii)

From (i) $\vec{L}$ = constant

∴, $\vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \text{- - - - - - - -} + \vec{L}_n$ = constant.

Form (ii) since, $I\,\omega$ = constant.

i.e **ω** can be increased decreasing by I vice – versa .

- If M.I of the body changes from I_1 to I_2 due to the change of the distribution of mass of the body , then angular velocity Of the body changes $\vec{\omega}_1$ **to** $\vec{\omega}_2$ such that, $I_1\,\vec{\omega}_1 = I_2\,\vec{\omega}_2$ **i.e.,** $\boxed{I_1\,\omega_1 = I_2\,\omega_2}$

- Since, $I_1\,\omega_1 = I_2\,\omega_2$

$$I_1 \times \frac{2\pi}{T_1} = I_2\,\frac{2\pi}{T_2} \qquad \left[\text{since, } \omega = \frac{2\pi}{T}\right]$$

$$\frac{I_1}{T_1} = \frac{I_2}{T_2} \qquad \text{where 'T' is time period.}$$

Or, $\boxed{T_2 = \frac{I_2}{I_2} \times T_1}$.

Examples of conservation of momentum

1.) A man is sitting in a rotating table with his arm stretched his chest, his angular speed increases due to the decrease of moment of Inertia.

2.) When a speed planet revolving around the Sun in an elliptical this is because as the planet comes near the sun, its moment of Inertia decreases and hence according to the law of conservation of angular momentum (I **ω** = constant) its angular velocity Increases.

3.) A diver from a spring board, performs somersaults in air.
Examples: -- When a diver jumps from the spring board, he curls his body by rolling his arms and legs by doing so, he decreases his M.I and hence angular speed increase to conserve the angular momentum.

4.) A ballet dancer can vary her angular speed by out stretching her arms & legs.
Examples: -- A ballet dancer makes use of law of conservation of angular momentum to vary her angular speed. As she stretches her legs and arms out her moment of inertia increases and angular speed decreases. However, as she withdraws legs and arms towards her body, moment of inertia decreases and hence her angular Speed increases.

5) While falling, a cat stretches its body along with its tail so that its moment of inertia increases. As no external torque is acting L = I **ω** = constant. Since I increase, **ω** decreases and the cat lands gently on its feet.

- In all these illustrations, the angular momentum is conserved. However, the rotational KE is not conserved.
A/Law of conservation of angular momentum,

$$I_1\,\omega_1 = I_2\,\omega_2$$

Squaring both side ,

$$I_1^2\,\omega_1^2 = I_2^2\,\omega_2^2$$

Multiplying both side by ½

$$\tfrac{1}{2}\,I_1^2\,\omega_1^2 = \tfrac{1}{2}\,I_2^2\,\omega_2^2$$

$$I_1\,[\tfrac{1}{2}\,I_1\,\omega_1^2] = I_2\,[\tfrac{1}{2}\,I_2\,\omega_2^2]$$

$$I_1\,E_{k1} = I_2\,E_{k2} \qquad \text{Where } E = [\tfrac{1}{2}\,I\,\omega^2]$$

If $I_1 > I_2$ then $E_{k1} < E_{k2}$

i.e., **When the moment of inertia decreases, the rotational KE increases and vice – versa.**

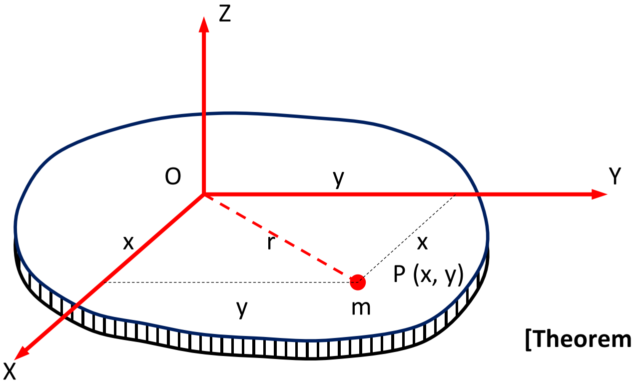
TERMS	LINEAR MOTION	TERMS	ROTATIONAL MOTION
1. Position	x	Position	θ
2. Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
3. Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
4. Mass	m	Moment of inertia	I
5. Linear momentum	$p = mv$	Angular momentum	$L = I\omega$
6. Force	$F = ma$	Torque	$\tau = I\alpha$
7. N's law	$F = dP/dt$	Consequence of N's law	$\tau = dL/dt$
8. K.E	$E = \frac{1}{2} mv^2 = p^2/2m$	Rotational KE	$E = \frac{1}{2}\omega^2 I = L^2/2I$
9. Work	$W = F.dS$	Work	$W = \tau . d\theta$
10. Power	$P = F.v$	power	$P = \tau . \omega$
◙ If external force acts, then linear momentum is conserved.		◙ If no ext. torque acts, L is conserved.	

THEOREMS OF PARALLEL AND PERPENDICULAR AXES

● **Theorem of perpendicular axes:** ***The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its own plane and intersecting each other at the point where the perpendicular axis passes through the lamina.***

In other words, "The **moment of inertia of a plane about an axis (say OZ) perpendicular to the plane lamina (2-D body) is the sum of the moment of inertia about any two mutually perpendicular axis OX & OY Both lying in the same plane".**

Proof: Consider a plane lamina lying in the XOY plane. It can be assumed to be made up of large number of particles. Consider one such particles of mass m situated at point p(x, y). The distances of the particle from X-, Y- and Z-axes are y, x and r respectively such that $r^2 = y^2 + x^2$



[Theorem of perpendicular axis]

Moment inertia of the particle about X-axis = my^2

∴ **Moment of inertia of whole lamina about X-axis is $I_x = \Sigma my^2$**

Moment of inertia of whole lamina about Y-axis is $I_y = \Sigma mx^2$

Moment of inertia of whole lamina about Z-axis is

$$I_z = \Sigma mr^2 = \Sigma m(y^2 + x^2) = \Sigma my^2 + \Sigma mx^2$$

or $\boxed{I_z = I_x + I_y}$ This proves the theorem of perpendicular axis.

Theorem of parallel axes: *The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.*

In other words"M.I Of plane lamina about any axis in its plane is equal to its M.I about a parallel axis passing through its centre of mass (C) of the lamina plus the product of mass of the lamina and the square of the distance between two axis".

Proof: Let I be the moment of inertia of a body of mass M about an axis PQ. Let RS be a parallel axis passing through the centre of mass C of the body and at distance d from PQ. Let I_{CM} be the moment of inertia of the body about the axis RS.

Consider a particle P of mass m at distance x from RS and so at distance (x + d) from PQ.

Moment of inertia of the particle about axis PQ = $m(x + d)^2$

∴ Moment of inertia of the whole body about the axis PQ is

$$I = \Sigma m (x + d)^2 = \Sigma m (x^2 + d^2 + 2xd)$$

$$= \Sigma mx^2 + \Sigma md^2 + \Sigma 2mxd$$

Now, $\Sigma mx^2 = I_{CM}$

$\Sigma md^2 = (\Sigma m) d^2 = Md^2$

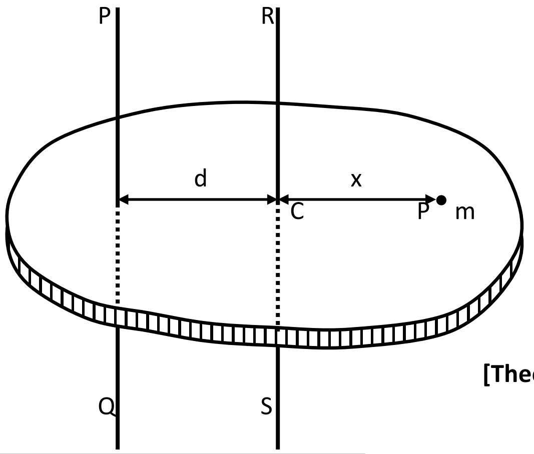
$\Sigma 2mxd = 2d \Sigma mx = 2d \times 0 = 0$

This is because a body can balance itself about its centre of mass, so the algebraic sum of moments (Σmx) of masses of all its particles about the axis RS is zero. i.e., $\sum (mg) x = 0$ or, $g \sum m x = 0$

Since $g \neq 0$, therefore $\sum m x = 0$

Hence, $I = I_{CM} + Md^2$

This proves the theorem of parallel axes.



[Theorem of parallel axes]

MOMENT OF INERTIA OF A THIN CIRCULAR RING

(a) M. I. of a ring about an axis through its centre and perpendicular to its plane: Consider a thin uniform circular ring of radius R and mass M. we have to determine its moment O and perpendicular to it. The ring can be imagined to be made of a large number of small elements. Consider one such element of length dx.

Length of the ring = circumference = $2\pi R$

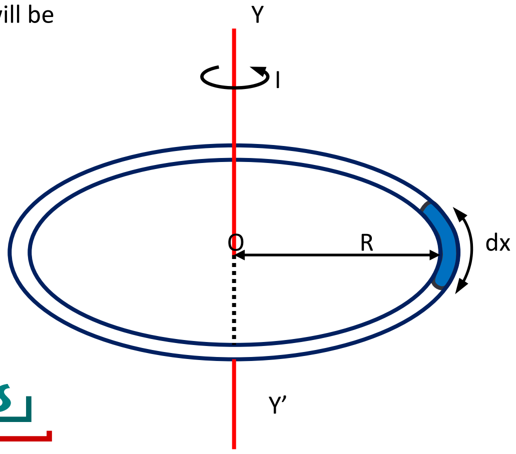
Mass per unit length of ring = $\dfrac{M}{2\pi R}$

Mass of the small element = $\dfrac{M}{2\pi R} dx$

Moment of inertia of the small element about the axis YY'

$$dI = \left(\frac{M}{2\pi R} dx\right) R^2 = \frac{MR}{2\pi} dx$$

The small elements lie along the entire circumference of the ring i.e., from x = 0 to x = $2\pi R$. Hence the moment of inertia of the whole ring about the axis YY' will be

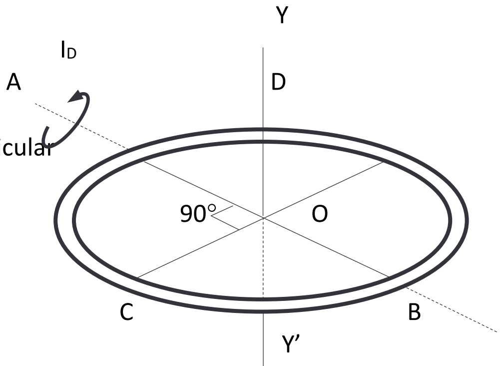


[M.I. of a ring about central axis]

$$I = \int_0^{2\pi R} \frac{MR}{2\pi}\, dx = \frac{MR}{2\pi}\int_0^{2\pi R} dx$$

$$= \frac{MR}{2\pi}\left[x\right]_0^{2\pi R} = \frac{MR}{2\pi}(2\pi R - 0)$$

or $\mathbf{I = MR^2}$



[M.I. of a ring about a tangent in its plane]

(b) M. I. of a ring about any diameter: According to the theorem of perpendicular axes, the moment of inertia about an axis YY' through O and perpendicular to the ring is equal to sum of its moment of inertia about **two perpendicular diameters** AB and CD,

$I_{AB} + I_{CD} = I_{YY'}$

$I_D + I_D = MR^2$ or $\mathbf{I_D = \frac{1}{2} MR^2}$

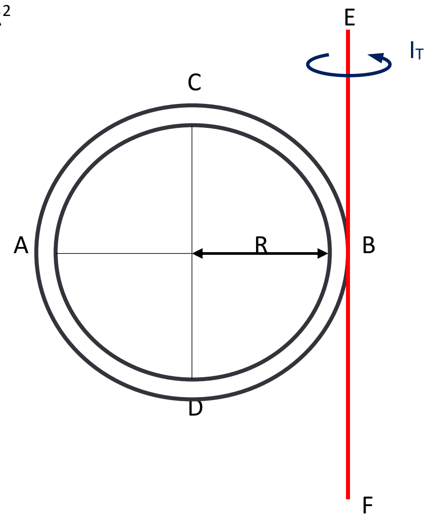
Here I_D is the M.I. of the ring about any diameter.

(c) M. I. of a ring about a tangent in its plane: Let I_T be the moment of inertia of the ring about the tangent EBF. Applying the theorem of parallel axes, we get

I_T = M.I. about diameter CD + MR^2

$= \frac{1}{2} MR^2 + MR^2$

or $\mathbf{I_T = \frac{3}{2} MR^2}$



[M.I. of a ring about a tangent in its plane]

(d) M. I. of a ring about a tangent perpendicular to its plane: Let I_T' be the moment of inertia of the ring about the axis PAQ tangent to the plane of the ring. Applying the theorem of parallel axes,

$I_{PQ} = I_{YY'} + MR^2$

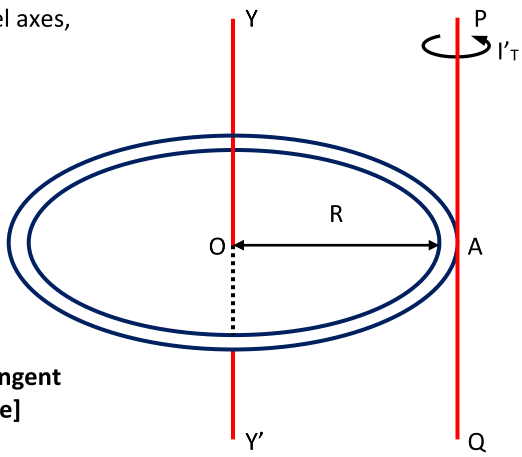
$= MR^2 + MR^2$

or $\mathbf{I_T' = 2\, MR^2}$

We can determine the radius of gyration (k) of the ring about any axis be equating its M.I. about that axis to Mk^2. For example, the radius of gyration of a thin ring about any diameter is given by

$I_D = \frac{1}{2} MR^2 = Mk^2$

or $k = R/\sqrt{2}$



[M.I. of a ring about a tangent perpendicular to its plane]

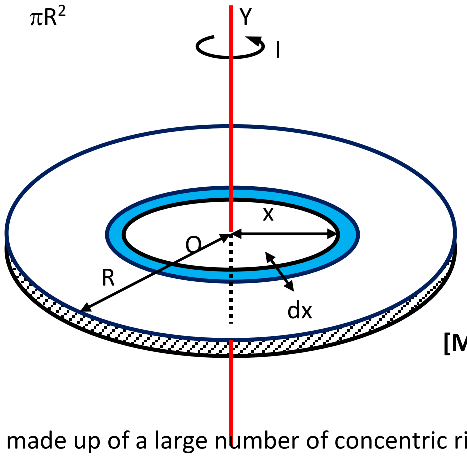
MOMENT OF INERTIA OF A UNIFORM CIRCULAR DISC

The moment of inertia of a disc about (a) an axis through the centre and perpendicular to its plane, (b) its diameter, (c) a tangent in its own plane, (d) a tangent perpendicular to its plane.

(a) M.I. of a circular disc about an axis through its centre and perpendicular to its plane: consider a uniform disc of mass M and radius R. Suppose YY' is an axis passing through the centre O of the disc and perpendicular to its plane.

Area of the disc = πR^2

Mass per unit area of the disc = $\frac{M}{\pi R^2}$



[M.I. of a disc about a central axis]

We can imagine the disc to be made up of a large number of concentric rings, whose radii vary from O to R. Let us consider one such concentric ring of radius x and width dx.

Area of the ring = Circumference × Width = $2\pi x \times dx$

Mass of concentric ring, $m = \frac{M}{\pi R^2} 2\pi x\, dx = \frac{2Mx\, dx}{R^2}$

Moment of inertia of the concentric ring about the axis YY' $dI = mx^2 = \frac{2\,Mx\,dx}{R^2} \times x^2 = \frac{2\,Mx^3\,dx}{R^2}$

The moment of inertia of the whole disc about the axis YY' can be obtained by integrating the above expression between the limits O to R.

$$I = \int_0^R \frac{2\,Mx^3\,dx}{R^2} = \frac{2M}{R^2}\int_0^R x^3\,dx$$

$$= \frac{2M}{R^2}\left(\frac{x^4}{4}\right)_0^R = \frac{2\,M}{4\,R^2}[R^4 - 0] = \frac{M}{2\,R^2} \times R^4$$

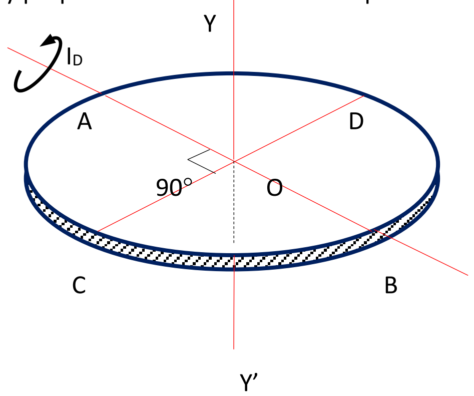
or **$I = ½ MR^2$**

(b) M.I. of a disc any diameter: In Fig. AB and CD are two mutually perpendicular diameters in the plane of the disc. Applying the theorem of perpendicular axes, we get

$I_{AB} + I_{CD} = I_{YY'}$

or $I_D + I_D = ½ MR^2$

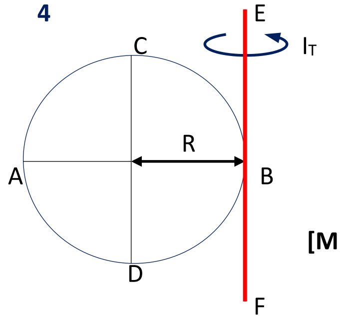
or **$I_D = ¼ MR^2$**



(c) M. I. of a disc about a tangent in its plane: Let I_T the moment of inertia of the disc about a tangent EBF in the plane of the disc. This tangent is parallel to the diameter CD of the disc. Applying the theorem of parallel axes, we get

I_T = Moment of inertia of disc about CD + MR^2

or **$I_T = ¼ MR^2 + MR^2 = \frac{5}{4} MR^2$**



[M.I. of a disc about a tangent in its plane]

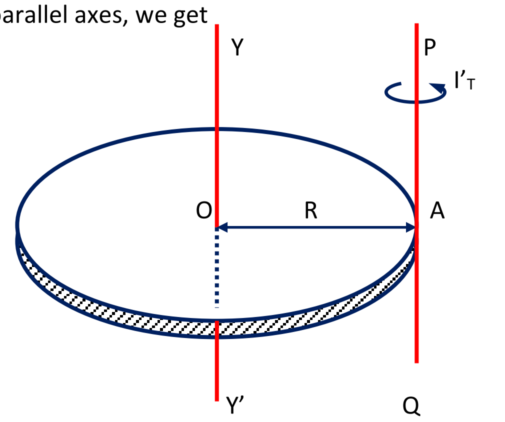
(d) M.I. of a disc about a tangent perpendicular to its plane: In Fig., let I_T' be the moment of inertia of disc about the tangent PAB perpendicular to the plane of the disc. Applying the theorem of parallel axes, we get

$$I_{PQ} = I_{YY'} + MR^2 = \frac{1}{2} MR^2 + MR^2$$

or $$I_{T'} = \frac{3}{2} MR^2$$

Moreover, the radius of gyration (k) in this case is given by

$$Mk^2 = \frac{3}{2} MR^2 \quad \text{or} \quad k = \sqrt{\frac{3}{2}}\, R$$

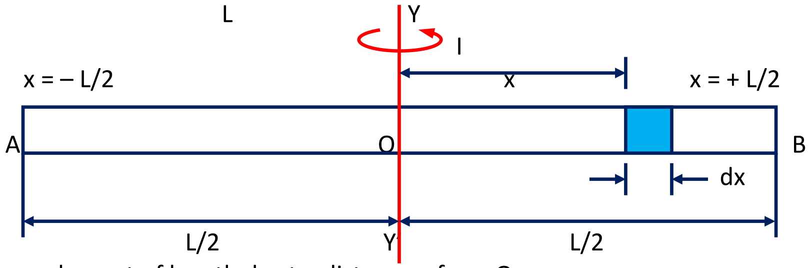


[M.I. of a disc about a tangent perpendicular to its plane]

MOMENT OF INERTIA OF A THIN UNIFORM ROD

M.I. of a thin uniform rod about a perpendicular axis through its centre: Consider a thin uniform rod AB of length L and mass M, free to rotate about an axis YY' through its centre O and perpendicular to its length.

∴ Mass per unit length or rod $= \frac{M}{L}$



[M.I. of a rod about an axis through its centre]

Consider a small mass element of length dx at a distance x from O.

Mass of the small element $= \frac{M}{L} dx$

Moment of inertia of the small element about YY'.

$$dI = \text{Mass} \times (\text{distance})^2 = \frac{M}{L} dx \times x^2$$

The moment of inertia of the whole rod about the axis YY' can be obtained by integrating the above expression between the limits $x = -L/2$ and $x = +L/2$.

$$\therefore \quad I = \int dI = \int_{-L/2}^{+L/2} \frac{M}{L} x^2 dx$$

$$= \frac{M}{L} \int_{-L/2}^{+L/2} x^2 dx = \frac{M}{L} \left[\frac{x^3}{3}\right]_{-L/2}^{+L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{8} + \frac{L^3}{8}\right) = \frac{M}{3L} \times \frac{L^3}{4}$$

or $$I = \frac{ML^2}{12}$$

☑ **Radius of gyration:** Let k be the radius of gyration of the rod about the axis YY'. Then

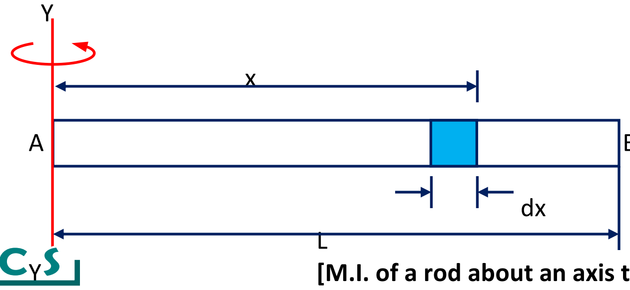
$$I = Mk^2$$

$$\therefore \quad Mk^2 = \frac{ML^2}{12} \quad \text{or} \quad k^2 = \frac{L^2}{12}$$

or $$k = \frac{L}{2\sqrt{3}}.$$

Thus, the radius of gyration of a uniform thin rod rotating about an axis passing through its centre and perpendicular to its length is $L/2\sqrt{3}$.

M.I. of a thin uniform rod about a perpendicular axis through its one end: Consider a thin uniform rod AB of length L and mass M, free to rotate about an axis YY' passing through its one end A and perpendicular to its length, as shown in Fig.



[M.I. of a rod about an axis through its one end]

Mass per unit length of rod = $\frac{M}{L}$

Consider a small element of length dx of the rod at a distance x from the end A.

Mass of the small element = $\frac{M}{L}dx$

Moment of inertia of the small element about the axis YY',

$$dI = \text{Mass} \times (\text{distance})^2 = \frac{M}{L}dx.\, x^2$$

The moment of inertia of the whole rod about the axis YY' can be obtained by integrating the above expression between the limits x = 0 and x = L.

$$I = \int dI = \int_0^L \frac{M}{L} dx\,.\,x^2 = \frac{M}{L}\int_0^L x^2\,dx$$

$$= \frac{M}{L}\left(\frac{x^3}{3}\right)_0^L = \frac{M}{3L}\left(x^3\right)_0^L = \frac{M}{3L}[L^3 - 0] = \frac{ML^3}{3L}$$

or $\quad$ **$I = \frac{ML^2}{3}$**

☑ **Radius of gyration:** Let k be the radius of gyration of the rod about the axis YY'. Then

$$\frac{ML^2}{3} = Mk^2 \quad \text{or} \quad k^2 = \frac{L^2}{3}$$

or, $k = \frac{L}{\sqrt{3}}$ Thus the radius of gyration of the rod about an axis passing through its end and perpendicular to its length is $L/\sqrt{3}$.

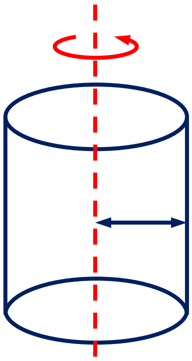
MOMENT OF INERTIA OF A CYLINDER

M.I. of a hollow cylinder about its own axis: Consider a hollow cylinder of mass M and radius R. In Fig., every element of the cylinder is at the same perpendicular distance R from its axis. Hence the moment of inertia of the hollow cylinder about its own axis is

$$I = \int r^2\,dm = \int R^2\,dm$$

$$= R^2 \int dm = R^2 \times M$$

or $\quad$ **$I = MR^2$**

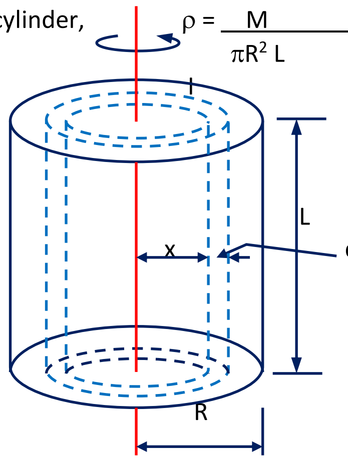


[M.I. of a hollow cylinder about its own axis]

Moment of inertia of uniform solid cylinder about its own axis: Consider a solid cylinder of mass M, radius R and length L. We wish to determine its moment of inertia about its own axis YY'.

Volume of the cylinder = $\pi R^2 L$

Mass of unit volume of the cylinder, $\rho = \frac{M}{\pi R^2 L}$



[M.I. of a solid cylinder about its own axis]

We can imagine the cylinder to be made of a large number of coaxial cylinder cylindrical shells. Consider one such cylindrical shell of inner radius x and outer radius x + dx, as shown in Fig. The cross-section of the shell is a ring of radius x and thickness dx.

∴ $\quad$ Cross-sectional area of the cylindrical shell = Circumference × thickness = $2\pi x\,dx$

Volume of the cylindrical shell = Cross-sectional area × length = $2\pi x\,dx \times L$

Mass of the cylindrical shell, $\quad m = \text{Volume} \times \rho = 2\pi xL\,dx \times \frac{M}{\pi R^2 L} = \frac{2M\,x\,dx}{R^2}$

As the mass of the shell is distributed at the same distance x from its axis, so its moment of inertia about the axis YY' is

$$dI = mx^2 = \frac{2M}{R^2}\, x\, dx \times x^2 = \frac{2M}{R^2}\, x^3\, dx$$

The moment of inertia of the solid cylinder can be obtained by integrating the above expression between the limits x = 0 and x = R.

$$\therefore \quad I = \int dI = \int_0^R \frac{2M}{R^2}\, x^3\, dx$$

$$= \frac{2M}{R^2}\int_0^R x^3\, dx = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R$$

$$= \frac{2M}{R^2}\,[R^4 - 0]$$

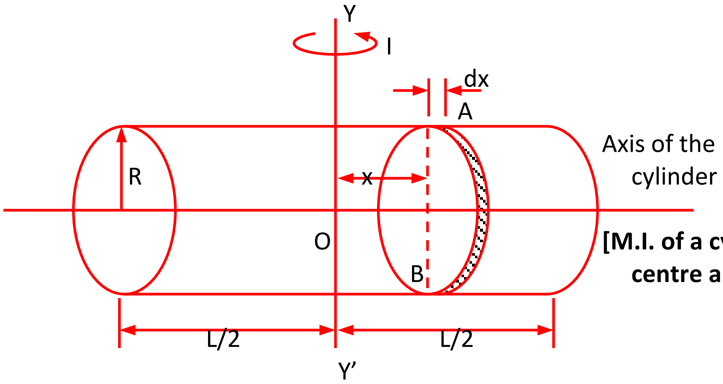
or $\quad$ **$I = \frac{1}{2} MR^2$**

Obviously, the moment of inertia of a cylinder about its own axis does not depend on its length.

M.I. of a solid cylinder about an axis through its centre and perpendicular to its axis: Consider a uniform solid cylinder of mass M, radius R and length L. We wish to determine its moment of inertia about an axis YY' passing through its centre O and perpendicular to its length.

Mass per unit length $= \frac{M}{L}$

We can imagine the cylinder to be made up of a large number of thin 'circular discs placed perpendicular to the axis of the cylinder. As shown in Fig., consider one such thin disc of thickness dx and at distance x from the centre O.



[M.I. of a cylinder about an axis through its centre and perpendicular to its axis]

Mass of the elementary disc $= \frac{M}{L}\, dx$

Radius of the elementary disc = R

Moment of inertia of the elementary disc about the diameter AB $= \frac{1}{4}\text{ Mass} \times \text{radius}^2 = \frac{1}{4} \cdot \frac{M}{L}\, dx \times R^2 = \frac{MR^2}{4L}\, dx$

Applying the theorem of parallel axes, the moment of inertial of the elementary disc about the axis YY',

$$dI = \text{M.I. about the diameter AB} + \text{Mass} \times x^2$$

$$= \frac{MR^2}{4L}\, dx + \frac{M}{L}\, dx \times x^2 = \frac{M}{L}\left(\frac{R^2}{4} + x^2\right) dx$$

The moment of inertia of the cylinder about the axis YY' can be obtained by integrating the above expression between the limits x = 0 and x = L/2 and multiplying the result by 2 to cover both halves of the cylinder. Thus

$$I = 2\int dI = 2\int_0^{L/2} \frac{M}{L}\left(\frac{R^2}{4} + x^2\right) dx$$

$$= \frac{2M}{L}\left(\frac{R^2}{4}\int_0^{L/2} dx + \int_0^{L/2} x^2\, dx\right)$$

$$= \frac{2M}{L}\left(\frac{R^2}{4}\,|\,x\,|_0^{L/2} + \left|\frac{x^3}{3}\right|_0^{L/2}\right)$$

$$= \frac{2M}{L}\left(\frac{R^2}{4}\left(\frac{L}{2} - 0\right) + \left(\frac{(L/2)^3}{3} - 0\right)\right)$$

$$= \frac{2M}{L}\left(\frac{R^2}{4} \cdot \frac{L}{2} + \frac{L^3}{24}\right)$$

or $\quad I = M\left(\frac{R^2}{4} + \frac{L^2}{12}\right)$

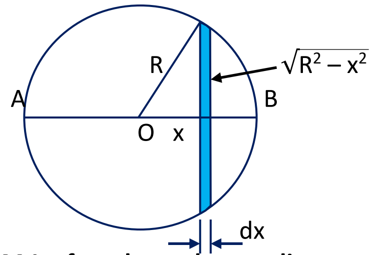
MOMENT OF INERTIA OF A SOLID SPHERE

Moment of inertia of a solid sphere about its diameter: Consider a uniform solid sphere of mass M and radius R. We wish to determine its moment of inertia about diameter AB.

Volume of the sphere $= \frac{4}{3}\pi R^3$

Mass per unit volume, $\rho = \frac{3M}{4\pi R^3}$

We can imagine the sphere to be made up of a large number of thin slices placed perpendicular to the diameter AB. Consider one such slice of thickness dx placed at distance x from the centre O.



[M.I. of a sphere about a diameter]

We **can imagine the sphere to be made up of a large number of thin slices placed perpendicular to the diameter AB**. Consider one such slice of thickness dx placed at distance x from the centre O.

Radius of the elementary slice $= \sqrt{R^2 - x^2}$

Volume of the elementary slice = Area × thickness

$= \pi(\sqrt{R^2 - x^2})^2 \times dx = \pi(R^2 - x^2)dx$

Mass of the elementary slice = Volume × ρ $= \pi(R^2 - x^2)\,dx \times \frac{3M}{4\pi R^3}$

$= \frac{3M(R^2 - x^2)\,dx}{4R^3}$

Moment of inertia of the thin slice about the axis AB passing through its centre and perpendicular to its plane,

$dI = \frac{1}{2}\text{Mass} \times (\text{radius})^2$

$= \frac{1}{2} \cdot \frac{3M(R^2 - x^2)\,dx}{4R^3} \cdot (R^2 - x^2)$

$= \frac{3M(R^2 - x^2)^2\,dx}{8R^3}$

The moment of inertia of the whole sphere about the diameter AB can be obtained by integrating the above expression between the limits x = 0 and x = R and multiplying the result by 2 to include both halves of the sphere.

$\therefore \quad I = 2\int dI = 2\int_0^R \frac{3M(R^2 - x^2)^2\,dx}{8R^3}$

$= \frac{2 \times 3M}{8R^3}\int_0^R (R^2 - x^2)^2\,dx$

$= 3M\int_0^R (R^4 - 2R^2x^2 - x^4)\,dx$

$= \frac{3M}{4R^3}\left(R^4\int_0^R dx - 2R^2\int_0^R x^2\,dx + \int_0^R x^4\,dx\right)$

$= \frac{3M}{4R^3}\left(R^4\,|x|_0^R - 2R^2\left|\frac{x^3}{3}\right|_0^R + \left|\frac{x^5}{5}\right|_0^R\right)$

$= \frac{3M}{4R^3}\left(R^4(R - 0) - 2R^2\left(\frac{R^3}{3} - 0\right) + \left(\frac{R^5}{5} - 0\right)\right)$

$= \frac{3M}{4R^3}\left(R^5 - \frac{2}{3}R^5 + \frac{R^5}{5}\right) = \frac{3M}{4R^3} \times \frac{8}{15}R^5$

$I = \frac{2}{5}MR^2$

Moment of inertia of a solid sphere about a tangent: Applying the theorem of parallel axes, the moment of inertia of a solid sphere about a tangent is given by

I_T = M.I. about a diameter + Mass × (radius)^2

$= \frac{2}{5}MR^2 + MR^2 \quad$ or $\quad I_T = \frac{7}{5}MR^2$

Examples based on Moment of Inertia, Radius of Gyration and Rotational K.E.

◆ **FORMULA USED** : 1. Moment of inertia of a body about the given axis of rotation,

$I = m_1r_1^2 + m_2r_2^2 + \ldots + m_nr_n^2 = \sum_{i=1}^{n} m_ir_i^2$

2. Radius of gyration K is given by

$I = MK^2 \quad$ or $\quad K = \sqrt{\frac{I}{M}}$

When all the particles are of some mass, $K = \sqrt{\frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}}$

3. Theorem of parallel axis: $I_z = I_x + I_y$

4. Theorem of perpendicular axis, $I = I_{CM} + Md^2$

5. M.I. of a circular ring about an axis through its centre and perpendicular to its plane, $I = MR^2$
6. M.I. of a thin ring about any diameter, $I = \frac{1}{2} MR^2$
7. M.I. of a thin ring about any tangent in its plane, $I = \frac{3}{2} MR^2$
8. M.I. of a circular disc about an axis through its centre and perpendicular to its plane, $I = \frac{1}{2} MR^2$
9. M.I. of a circular disc about any diameter, $I = \frac{1}{4} MR^2$
10. M.I. of a circular disc about a tangent in its plane, $I = \frac{5}{4} MR^2$
11. M.I. of a thin rod about an axis through its middle point and perpendicular to rod, $I = \frac{1}{12} ML^2$
12. M.I. of a thin rod about an axis through its one end and perpendicular to rod, $I = \frac{1}{3} ML^2$
13. M.I. of a rectangular lamina of sides l and b about an axis through its centre and perpendicular to its plane,

$$I = M\left(\frac{l^2 + b^2}{12}\right)$$

14. M.I. of a right circular solid cylinder about its symmetry axis, $I = \frac{1}{2} MR^2$
15. M.I. of a right circular hollow cylinder about its axis $I = MR^2$
16. M.I. of a solid sphere about an axis through its centre, I = 2 MR
17. M.I. of a solid about any tangent, $I = \frac{7}{5} MR^2$
18. M.I. of a hollow sphere about an axis through its centre, $I = \frac{2}{3} MR^2$
19. M.I. of a hollow sphere about any tangent, $I = \frac{5}{3} MR^2$
20. Rotational K.E. = $\frac{1}{2} I \omega^2 + \frac{1}{2} Mv^2$

◆ **UNITS USED**

Mass M is in kg, radius R in m, moment of inertia I in kg m^2 and radius of gyration K in metre, rotational K.E. in joule and angular velocity ω in rad s^{-1}.

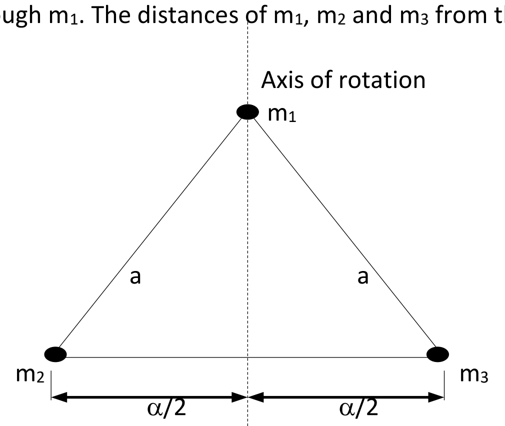
Q. 1. A wheel of mass 8 kg and radius of gyration 25 cm is rotating at 300 rpm. What is its moment of inertia?

Sol. Here $M = 8$ kg, $K = 25$ cm $= 0.25$ m

$\therefore$ $I = MK^2 = 8 \times (0.25)^2 = 0.5$ kgm^2.

Q. 2. Three mass points m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of length a. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1?

Sol. the axis of rotation passes through m_1. The distances of m_1, m_2 and m_3 from the axis of rotation are 0, a/2 and a/2 respectively.



$\therefore$ M.I. of the system about the altitude through m_1 is

$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$

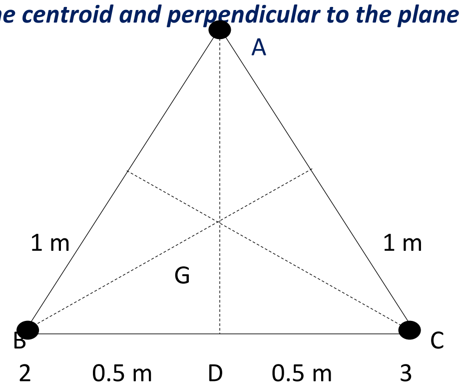
$= m_1 (0)^2 + m_2 \left(\frac{a}{2}\right)^2 + m_3 \left(\frac{a}{2}\right)^2$ or $I = \frac{a^2}{4}(m_2 + m_3)$

Q. 3. Three balls of masses 1, 2 and 3 kg respectively are arranged at the corners of an equilateral triangle of side 1 m. What will be the moment of inertia of the system about an axis through the centroid and perpendicular to the plane of the triangle?

Sol. Median $AD = \sqrt{AB^2 - BD^2}$

$= \sqrt{1^2 - (0.5)^2} = \sqrt{0.75}$

$\therefore$ $AG = BG = CG = \frac{2}{3} AD = \frac{2}{3}\sqrt{0.75}$

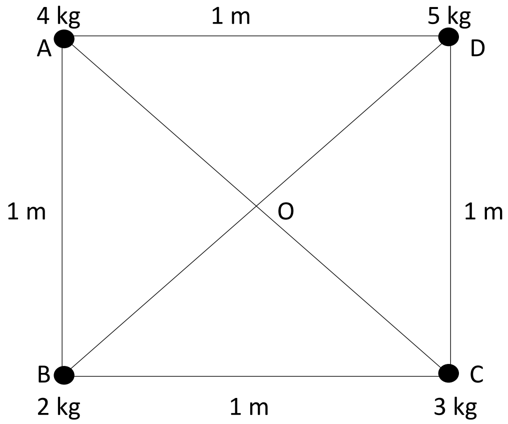


M.I. of the system about an axis through centroid G and perpendicular to the plane of the triangle is

$I = 1 \times AG^2 + 2 \times BG^2 + 3 \times CG^2$

$= (1 + 2 + 3) \times \left(\frac{2\sqrt{0.75}}{3}\right) = \frac{6 \times 4 \times 0.75}{9} = 2 \text{ kg m}^2$

Q. 4. ***Four particles of masses 4 kg, 2 kg, 3 kg and 5 kg are respectively located at the four corners A, B, C and D of the square of side 1 m, as shown in Fig. Calculate the moment of inertia of the system about***
(i) An axis passing through the point of intersection of the diagonals and perpendicular to the plane of the square,
(ii) The side AB, and
(iii) The diagonal BD.



Sol. Here $AB = BC = CD = DA = 1$ m

$OA = OB = OC = OD = \frac{1}{2}\sqrt{1^2 + 1^2} = 1/\sqrt{2}$ m

(i) M.I. of the system about an axis through O and perpendicular to the plane of the square,

$I = 4\,(OA)^2 + 2\,(OB)^2 + 3\,(OC)^2 + 5\,(OD)^2$

$= (4 + 2 + 3 + 5) \times \left(\frac{1}{\sqrt{2}}\right)^2 = 14 \times \frac{1}{2} = 7 \text{ kg m}^2$

(ii) M.I. of the system about the side AB,

$I = 3\,(BC)^2 + 5\,(AD)^2 = 3 \times 1 + 5 + 1 =$ **8 kg m^2**.

(iii) M.I. of the system about the diagonal BD,

$I = 4\,(OA)^2 + 3\,(OC)^2 = 4 \times \frac{1}{2} + 3 \times \frac{1}{2} =$ **3.5 kg m^2**.

Q. 5. ***The moment of inertia of a uniform circular disc about its diameter is 100 g cm^2. What is its moment of inertia (i) about the tangent (ii) about an axis perpendicular to its plane?***

Sol. M.I. of disc about its diameter,

$I_d = \frac{1}{4} MR^2 = 100 \text{ g cm}^2$

(i) By theorem of parallel axes, M.I. about a tangent parallel to the diameter,

$I = I_d + MR^2 = \frac{1}{4} MR^2 + MR^2 = \frac{5}{4} MR^2$

$= 5 \times 100 = 500 \text{ g cm}^2$

(ii) By theorem of perpendicular axes, M.I. of the disc about an axis perpendicular to its plane,

I = Sum of the moments of inertia about two perpendicular diameters

$I = I_d + I_d = 2 \times \frac{1}{4} MR^2 = 2 \times 100 = 200 \text{ g cm}^2$

Q. 6. ***Calculate the moment of inertia of a cylinder of length 1.5 m, radius 0.05 m and density 8×10^3 kg m^{-3} about the axis of the cylinder.***

Sol. Here, $R = 0.05$ m, $l = 1.5$ m, $\rho = 8 \times 10^3$ kg m^{-3}

Mass of cylinder, $M = \text{Volume} \times \text{density} = \pi R^2 l \,.\, \rho$

$= 3.14 \times (0.05)^2 \times 1.5 \times 8 \times 10^3 = 94.2$ kg

M.I. of the cylinder about its own axis,

$I = \frac{1}{2} MR^2 = \frac{1}{2} \times 94.2 \times (0.5)^2 = 0.1175 \text{ kg m}^2$.

Q. 7. ***Calculate the moment of inertia of the earth about its diameter, taking it to be a sphere of 10^{25} kg and diameter 12800 km.***

Sol. Here $M = 10^{25}$ kg, $R = 6400 \text{ km} = 6.4 \times 10^6$ m

M.I. of the earth about its diameter

$I = \frac{2}{5} MR^2 = \frac{2}{5} \times 10^{25} \times (6.4 \times 10^6)^2 = 1.64 \times 10^{38} \text{ kg m}^2$

Q. 8. ***Four spheres of diameter 2a and mass M each are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia of the system about one side of the square taken as its axis.***

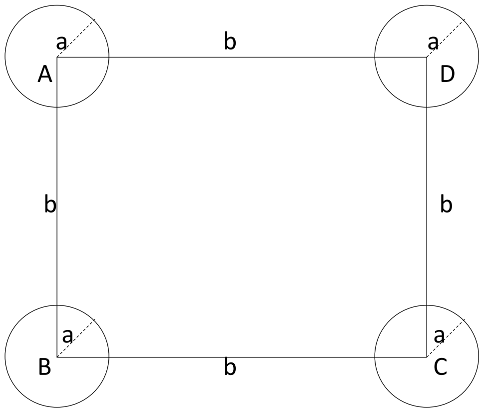
Sol. The situation is shown in fig. Let us calculate the moment of inertia of the system about the side CD.

I = M.I. of A about CD + M.I. of B about CD + M.I. of C about CD + M.I. of D about CD

$= \left(\frac{2}{5} Ma^2 + Mb^2\right) + \left(\frac{2}{5} Ma^2 + Mb^2\right) + \frac{2}{5} Ma^2 + \frac{2}{5} Ma^2$ or $I = 2\,M\,(4a^2 + 5b^2)$

$= 8\,Ma^2 + 2\,Mb^2$

or $I = 2/5\,M\,(4a^2 + 5b^2)$



Q. 9. ***Two-point masses of 2 kg and 10 kg are connected by a weightless rod of length 1.2 m. Calculate the M.I. of the system about an axis passing through the centre of mass and perpendicular to the system.***

Sol. Here $m_1 = 2$ kg, $m_2 = 10$ kg, length of rod = 1.2 m.

Suppose the centre of mass lies at distance x from mass m_1. Then

$m_1 x = m_2\,(1.2 - x)$ Hint ; both $m_1 x$ & $m_2\,(1.2 - x)$ lies on the opposite side of X –axis.

or $2x = 10 \times (1.2 - x)$ or $x = 1$ m

As the rod is weightless, its moment of inertia about any axis is zero.

M.I. of m_1 about CM $= m_1 x^2 = 2 \times (1)^2 = 2$ kg m^2

M.I. of m_2 about CM

$= m_2\,(1.2 - x)^2 = 10 \times (1.2 - 1)^2 = 1.4$ kgm^2 ∴ Total M.I. = 2 + 0.4 = 2.4 kg m^2

Q. 10. ***Find the moment of inertia of a rectangular bar magnet about an axis passing through its centre and parallel to its thickness. Mass of the magnetic is 100 g, length is 12 cm breadth is 3 cm and thickness is 2 cm.***

Sol. Here M = 100 g, l = 12 cm, b = 3 cm, t = 2 cm

M.I. of the bar magnet about the axis through its centre and parallel to its thickness is

$I = M\left(\frac{l^2 + b^2}{12}\right) = 100\left(\frac{12^2 + 3^2}{12}\right) = \frac{100 \times 153}{12} = 1275$ g cm^2

Q. 11. ***Calculate the ratio of radii of gyration of a circular ring and a disc of the same radius about the axis passing through their centres and perpendicular to their planes.***

Sol. Let K_1 and K_2 be the radii of gyration of the ring and the disc about the axis passing through their centres and perpendicular to their planes. Then

M.I. of the ring $= MR^2 = MK_1^2$ or $K_1 = R$

M.I. of the disc $= ½\,MR^2 = MK_2^2$ or $K_2 = 1/\sqrt{2}\,R$ ∴ $\frac{K_1}{K_2} = \frac{R}{R/\sqrt{2}} = \sqrt{2} : 1$

Q. 12. ***Find the radius of gyration of a rod of mass 100 g and length 100 cm about an axis passing through its centre and perpendicular to its length.***

Sol. Here M = 100 g = 0.1 kg, l = 100 cm = 1 m, K = ?

M.I. of the rod about an axis through its centre and perpendicular to its length is

$I = \frac{Ml^2}{12} = MK^2$ or $K^2 = \frac{l^2}{12}$

∴ $K = \frac{l}{\sqrt{12}} = \frac{1\text{ m}}{3.464} = 0.289$ m.

Q. 13. ***A wheel is rotating at a rate of 1000 rpm and its kinetic energy is 10^6 J. Determine the moment of inertia of the wheel about its axis of rotation.***

Sol. Here $v = 1000$ rpm $= \frac{1000}{60}$ rps $= \frac{50}{3}$ rps, $\omega = 2\pi v = \frac{100\,\pi}{3}$ rad s^{-1}

As rotational K.E. $= ½\,I\omega^2$

∴ $10^6 = ½ \times I \times \left(\frac{100\,\pi}{3}\right)^2$

or $I = \frac{2 \times 10^6 \times 9}{(100)^2\,\pi^2} = \frac{200 \times 9}{9.87} = 182.4$ kg m^2

Q. 14. ***Calculate the kinetic energy of rotation of a circular disc of mass 1 kg and radius 0.2 m rotating about an axis passing through its centre and perpendicular to its plane. The disc is making 30/π rotations per minute.***

Sol. Here M = 1 kg, R = 0.2 m

$$v = \frac{30}{\pi} \text{ rpm} = \frac{30}{\pi \times 60} \text{ rps} = \frac{1}{2\pi} \text{ rps}$$

$$\omega = 2\pi v = 2\pi \times \frac{1}{2\pi} = 1 \text{ rad s}^{-1}$$

M.I. of the disc about an axis through its centre and perpendicular to its plane,

$$I = ½ MR^2 = ½ \times 1 \times (0.2)^2 = 0.02 \text{ kgm}^2$$

∴ Rotational K.E. $= ½ I\omega^2 = ½ \times 0.02 \times (1)^2 = 0.01$ J

Q. 15. ***Energy of 484 J is spent in increasing the speed of a flywheel from 60 rpm to 360 rpm. Find the moment of inertia of the wheel.***

Sol. Energy spent = 484 J

$v_1 = 60$ rpm = 1 rps, $v_2 = 360$ rpm = 6 rps

$\omega_1 = 2\pi v_1 = 2\pi$ rad s^{-1}, $\omega_2 = 2\pi v_2 = 12\pi$ rad s^{-1}

Let I be the moment of inertia of the wheel .

Initial K.E. of rotation $= ½ I\omega_1^2 = ½ I \times (2\pi)^2 = 2\pi^2 I$

Final K.E. of rotation $= ½ I\omega_2^2 = ½ I \times (12\pi)^2 = 72\pi^2 I$

Increase in K.E. of rotation of wheel = Energy spent on the wheel ∴ $72\pi^2 I - 2\pi^2 I = 484$

or $$I = \frac{484}{70\pi^2} = \frac{484 \times 7 \times 7}{70 \times 22 \times 22} = 0.7 \text{ kg m}^2$$

Q. 16. ***Calculate the rotational K.E. of the earth about its own axis. Mass of the earth = 6 × 10^{24} kg and radius of the earth = 6400 km.***

Sol. Here R = 6400 km = 6.4×10^6 m, M = 6×10^{24} kg,

M.I. of the earth about its own axis, $$I = \frac{2}{5} MR^2 = \frac{2}{5} \times 6 \times 10^{24} \times (6.4 \times 10^6)^2$$

$$= 9.83 \times 10^{37} \text{ kg m}^2$$

Angular velocity of the earth, $$\omega = \frac{2\pi}{T} = \frac{2\pi \text{ rad}}{1 \text{ day}} = \frac{2\pi \text{ rad}}{24 \times 60 \times 60 \text{ s}}$$

Rotational K.E. of the earth

$$½ I\omega^2 = ½ \times 9.83 \times 10^{37} \times \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2 = 2.6 \times 10^{29} \text{ J.}$$

Q. 17. ***A metre scale AB is held vertically with its one end A on the floor and is then allowed to fall. Find the speed of the other and B when it strikes the floor, assuming that the end on the floor does not slip.***

Sol. Let M be the mass and b the length of the metre scale. When the upper end of the rod strikes the floor, its centre of gravity

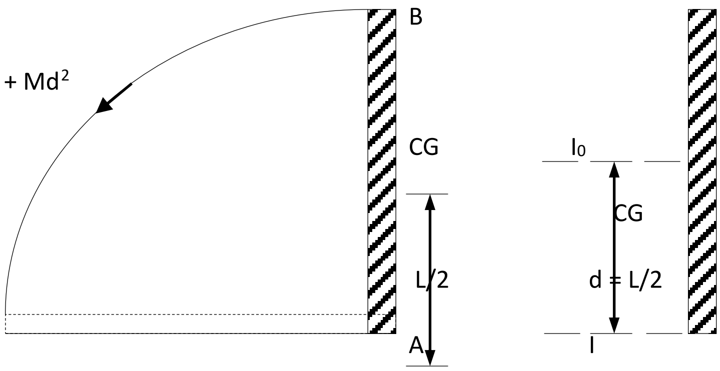
falls through height L/2.

∴ Loss in P.E. $= Mg.\frac{L}{2}$

M.I. of the scale about the lower end A,

I = M.I. of the scale about the parallel axis through CG + Md^2

$$= I_0 + Md^2 = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3} \qquad \left(\because d = \frac{L}{2}\right)$$



Also, $$\omega = \frac{v}{r} = \frac{v}{L}$$

Gain in rotational K.E. $$= ½ I\omega^2 = ½ . \frac{ML^2}{3} . \frac{v^2}{L^2} = \frac{Mv^2}{6}$$

Now, Gain in rotational K.E. = Loss of in P.E.

$$\frac{Mv^2}{6} = Mg . \frac{L}{2} \quad \text{or} \quad v^2 = 3gl$$

or $v = \sqrt{3gL} = \sqrt{3 \times 9.8 \times 1} = 5.4$ ms^{-1}

Q. 18. ***A uniform circular disc of mass m is set rolling on a smooth horizontal table with a uniform linear velocity v. Find the total K.E. of the disc.***

Sol. M.I. of the disc about its own axis, $I = ½ mr^2$

As $v = r\omega$ ∴ $\omega^2 = \frac{v^2}{r^2}$

Rotational K.E. $$= ½ I\omega^2 = ½ \times ½ mr^2 \times \frac{v^2}{r^2} = \frac{1}{4} mv^2$$

Translational K.E. $= ½ mv^2$

Total K.E. = Rotational K.E. + Translational K.E. $= ¼ mv^2 + ½ mv^2 = 3/4 \, mv^2$

Q. 19. ***A solid sphere is rolling on a frictionless plane surface about its axis of symmetry. Find the rotational energy and the ratio of its rotational energy to its total energy.***

Sol. Suppose the sphere has mass M and rolls with a uniform speed v,

M.I. of the sphere, $I = \frac{2}{5} MR^2$

Angular velocity, $\omega = \frac{v}{R}$

Rotational K.E. $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{2}{5} MR^2 \times \frac{v^2}{R^2} = \frac{1}{5} Mv^2$

Total energy = Translational K.E. + Rotational K.E.

$= \frac{1}{2} Mv^2 + \frac{1}{5} Mv^2 = \frac{7}{10} Mv^2$

$\frac{\text{Rotational K.E.}}{\text{Translational K.E}} = \frac{1/5\ Mv^2}{7/10\ mv^2} = \frac{2}{7} = 2 : 7$

Q. 20. ***A wheel of mass 5 kg and radius 0.40 m is rolling on a road without sliding with angular velocity 10 rad s^{-1}. The moment of inertia of the wheel about the axis of rotation is 0.65 kgm^2. What is the percentage of kinetic energy of rotation in the total kinetic energy of the wheel?***

Sol. Here M = 5 kg, R = 0.40 m, $\omega = 10$ rad s^{-1}, I = 0.65 kg m^2

Linear velocity, $v = R\omega = 0.40 \times 10 = 4.0\ ms^{-1}$

Translational K.E. $= \frac{1}{2} Mv^2 = \frac{1}{2} \times 5 \times 16 = 40$ J

Rotational K.E. $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.65 \times 100 = 32.5$ J

Total K.E. = Translational K.E. + Rotational K.E.

= 40 + 32.5 = 72.5 J

$\frac{\text{Rotational K.E.}}{\text{Total K.E.}} = \frac{32.5}{72.5} = 0.448 = 44.8\ \%$

Q. 21. ***The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m^2 about an axis through its centre perpendicular to the line joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translational. Find the average angular velocity of the molecule.***

Sol. Rotational K.E. $= \frac{2}{3}$ Translational K.E.

or $\frac{1}{2} I\omega^2 = \frac{2}{3} \cdot \frac{1}{2} mv^2$

or $\omega = v \times \sqrt{\frac{2m}{3I}} = 500 \times \sqrt{\frac{2 \times 5.30 \times 1^{-26}}{3 \times 1.94 \times 10^{-46}}}$

$= 500 \times \sqrt{1.82 \times 10^{20}} = 500 \times 1.35 \times 10^{10}$ rad $s^{-1} = 6.75 \times 16^{12}$ rad s^{-1}

Q. 22. ***A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it?***

Sol. Here R = 2m, M = 100 kg, $v_{cm} = 20\ cms^{-1} = 0.20\ ms^{-1}$

Work required to stop of hoop = Total K.E. of the hoop

= Rotational K.E. + Translational K.E.

$= \frac{1}{2} I\omega^2 + \frac{1}{2} Mv^2_{cm}$

$= \frac{1}{2} \times MR^2 \times \left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2} M (v_{cm})^2$

$= Mv^2_{cm} = 100 \times (0.20)^2 = 4$ J

Q. 23. ***A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5 m/s.***

(a) How far will the cylinder go up the plane? ***(b) How long will it take to return to the bottom?***

Sol. (a) Total initial kinetic energy of the cylinder,

$K_i = \frac{1}{2} Mv^2_{CM} + \frac{1}{2} I_{CM}\ \omega^2$

$= \frac{1}{2} Mv^2_{CM} + \frac{1}{2} \times \frac{1}{2} MR^2 \times \frac{v^2_{CM}}{R^2}$

$= \frac{1}{2} Mv^2_{CM} + \frac{1}{4} Mv^2_{CM} = \frac{3}{4} Mv^2_{CM}$

Initial potential energy, $U_i = 0$ Final kinetic energy, $K_f = 0$

Final potential energy, $U_f = Mgh = Mgs \sin 30° = \frac{1}{2} Mgs$

Where s is the distance travelled up the incline and h is the vertical height covered above the bottom.

Gain in P.E. = Loss in K.E.

$\frac{1}{2} Mgs = \frac{3}{4} Mv^2_{CM}$

$s = \frac{3v^2_{CM}}{2g} = \frac{3 \times (5)^2}{2 \times 9.8} = 3.8$ m

(b) Using equation of motion for the motion up the incline, we get

$0 = v_{CM} + at$ or $a = -\frac{v_{CM}}{t}$

Also, $0^2 - v^2_{CM} = 2as$ or $a = -\frac{v^2_{CM}}{2s}$

∴ $\frac{v_{CM}}{t} = \frac{v^2_{CM}}{2s}$

or $t = \frac{2s}{v_{CM}} = \frac{2 \times 3.8}{5} = 1.5$ s

Total time taken in returning to the bottom = 2 × 1.5 = 3..0 s

Q. 24. A solid cylinder rolls down an inclined plane. Its mass is 2 kg and radius 0.1 m. If the height of the inclined plane is 4 m, what is its rotational K.E. when it reaches the foot of the plane?

Sol. Here M = 2 kg, R = 0.1 m

Height of inclined plane, h = 4 m

At the top of the inclined plane, the cylinder has P.E. = mgh

At the bottom of the inclined plane, the cylinder has translational K.E. $[= \frac{1}{2} Mv^2]$ and rotational K.E. $[= \frac{1}{2} I\omega^2]$

By conservation of energy = $\frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 = Mgh$

But $v = R\omega$ and $I = = \frac{1}{2} MR^2$

∴ $\frac{1}{2} M (R\omega)^2 + \frac{1}{2} \times \frac{1}{2} MR^2 \omega^2 = Mgh$

or $\frac{3}{4} Mr^2 \omega^2 = Mgh$ or $\omega^2 = \frac{4\,gh}{3\,R^2}$

Rotational K.E. = $\frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{1}{2} MR^2 \times \frac{4\,gh}{3\,R^2}$

$= \frac{Mgh}{3} = \frac{2 \times 9.8 \times 4}{3} = 26.13$ J

Q. 25. A bucket of mass 8 kg is supported by a light rope wound around a solid wooden cylinder of mass 12 kg and radius 20 cm free to rotate about its axis. A man holding the free end of the rope, with the bucket and the cylinder at rest initially, lets go the bucket freely downwards in a well 50 m deep. Neglecting friction, obtain the speed of the bucket and the angular speed of the cylinder just before the bucket enters water. Take g = 10 ms⁻².

Sol. Mass of bucket, $m_1 = 8$ kg

Mass of cylinder, $m_2 = 12$ kg.

Radius of the cylinder, R = 20 cm = 0.20 m.

When the bucket just enters water,

P.E. lost by bucket = Linear K.E. of the bucket + Rotational K.E. of the cylinder

or $m_1 gh = \frac{1}{2} m_1 v^2 + \frac{1}{2} I\omega^2$

$= \frac{1}{2} m_1 v^2 + \frac{1}{2} \left(\frac{1}{2} m_2 R^2\right) \frac{v^2}{R^2}$ $\left(\because I = \frac{1}{2} m_2 R^2, \omega = \frac{v}{R}\right)$

$= \frac{1}{2} v^2 \left(m_1 + \frac{1}{2} m^2\right) = \frac{1}{2} v^2 (8 + 6) = 7v^2$

or $v = \sqrt{\frac{m_1 gh}{7}} = \sqrt{\frac{8 \times 10 \times 50}{7}}$

$= \sqrt{\frac{4000}{7}} = 23.9$ ms⁻¹

The angular speed of cylinder before the bucket touches water,

$\omega = \frac{v}{R} = \frac{23.9}{0.20} = 11.9.5$ rads⁻¹

Examples based on Relations between Torque, Angular momentum and Moment of inertia

◆ **FORMULA USED**

1. Torque = M.I. × angular acceleration

or $\tau = I\alpha$

2. Work done by a torque, $W = \tau\theta$

3. Angular momentum = M.I. × angular velocity or $L = I\omega$

◆ **UNITS USED** :Torque τ is in Nm, moment of inertia I in kgm² and angular momentum L in kgm² s⁻¹.

Q. 1. A torque of 2.0 × 10⁻⁴ Nm is applied to produce an angular acceleration of 4 rad s⁻² in a rotating body. What is the moment of inertia of the body?

Sol. Here $\tau = 2.0 \times 10^{-4}$ N, $\alpha = 4$ rad s⁻², I = ?

As $\tau = I\,\alpha$

∴ $I = \frac{\tau}{\alpha} = \frac{2.0 \times 10^{-4}}{4} = 0.5 \times 10^{-4}$ kg m².

Q. 2. ***An automatic move on a road with a speed of 54 kmh^{-1}. The radius of its wheels is 0.35 m. What is the average negative torque transmitted by its brakes to a wheel if the vehicle is bought to rest in 15 s? The moment of inertia of the wheel about the axis of rotation is 3 kgm^2.***

Sol. Here u = 54 kmh^{-1} = 15 ms^{-1}, r = 0.35 m, t = 15 s, I = 3 kg m^2

$$\omega_0 = \frac{u}{R} = \frac{15}{0.35} \text{ rad s}^{-1}, \ \omega = 0$$

Average angular acceleration, $\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 15/0.35}{15} = -\frac{1}{0.35} \text{ rad s}^{-2}$

Average torque transmitted by the brakes, $\tau = I \,.\, \alpha = -3 \times \frac{1}{0.35} = -8.57 \text{ kgm}^2 \text{ s}^{-2}$

Q. 3. ***A flywheel of mass 25 kg has a radius of 0.2 m. What force should be applied tangentially to the rim of the flywheel so that it acquires an angular acceleration of 2 rad s^{-2}?***

Sol. Here M = 25 kg, R = 0.2 m, α = 2 rad s^{-2}

M.I. of the flywheel about its axis,

$$I = \tfrac{1}{2} MR^2 = \tfrac{1}{2} \times 25\,(0.2)^2 = 0.5 \text{ kg m}^2$$

As torque, $\tau = F \,.\, R = I\,\alpha$

∴ Force, $F = \frac{I\alpha}{R} = \frac{0.5 \times 2}{0.2} = 5 \text{ N}$

Q. 4. ***A torque of 10 Nm is applied to a flywheel of mass 10 kg and radius of gyration 50 cm. What is the resulting angular acceleration?***

Sol. Here τ = 10 Nm, M = 10 kg, K = 0.50 m, α = ?

As $\tau = I\,\alpha = MK^2\,\alpha$

∴ $\alpha = \frac{\tau}{MK^2} = \frac{1}{10 \times (0.50)^2} = 4 \text{ rads}^{-2}$.

Q. 5. ***A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.***

Sol. Here M = 3 kg, R = 40 cm = 0.40 m, F = 30 N

Torque, $\tau = F \times R = 30 \times 0.40 = 12$ Nm

M.I. of the hollow cylinder about its own axis,

$$I = MR^2 = 3 \times (0.40)^2 = 0.48 \text{ kg m}^2$$

Angular acceleration, $\alpha = \frac{\tau}{I} = \frac{12}{0.48} = 25 \text{ rad s}^{-2}$. Linear acceleration, $a = R\alpha = 0.40 \times 25 = 10 \text{ ms}^{-2}$.

Q. 6. ***A flywheel of mass 25 kg has a radius of 0.2 m. It is making 240 r.p.m. What is the torque necessary to bring it to rest in 20 s? If the torque is due to a force applied tangentially on the rim of the flywheel, what is the magnitude of the force?***

Sol. M = 25 kg, R = 0.2 m, v_0 = 240 rpm = 4 rps

$\omega_0 = 2\,\pi\, v_0 = 2\pi \times 4 = 8\,\pi$ rad s^{-1}, ω = 0, t = 20 s

As $\omega = \omega_0 + \alpha\, t$ ∴ $0 = 8\pi + \alpha \times 20$

or $\alpha = -\frac{8\,\pi}{20} = -\frac{2}{5}\,\pi \text{ rad s}^{-2}$

M.I. of the flywheel about its own axis, $I = \tfrac{1}{2} MR^2 = \tfrac{1}{2} \times 25 \times (0.2)^2 = \tfrac{1}{2} \text{ kg m}^2$

Torque acting on the flywheel, $\tau = I\,\alpha = -\tfrac{1}{2} \times \frac{2}{5}\,\pi = -\frac{\pi}{5}$ Nm

The negative sign indicates that the torque is of retarding nature.

Now Torque = Force × perpendicular distance i.e., $\tau = F \times R$

∴ $F = \frac{\tau}{R} = \frac{\pi}{5 \times 0.2} = \pi$ N

Q. 7. ***A cord is wound around the circumference of a wheel of diameter 0.3 m. The axis of the wheel is horizontal. A mass of 0.5 kg is attached at the end of the cord and it is allowed to fall from rest. If the weight falls 1.5 m in 4 s, what is the angular acceleration of the wheel? Also find out the moment of inertia of the wheel.***

Sol. Radius of the wheel, $R = \frac{0.3}{2} = 0.15$ m

For the attached mass:

m = 0.5 kg, u = 0, s = 1.5 m, t = 4 s

Let a be the linear acceleration of the attached mass.

As $s = ut + \tfrac{1}{2} at^2$

∴ $1.5 = 0 \times 4 + \tfrac{1}{2}\, a \times (4)^2$

or $a = \frac{1.5}{8} = \frac{3}{16} \text{ ms}^{-2}$

If α is angular acceleration of the wheel, then $a = R\alpha$

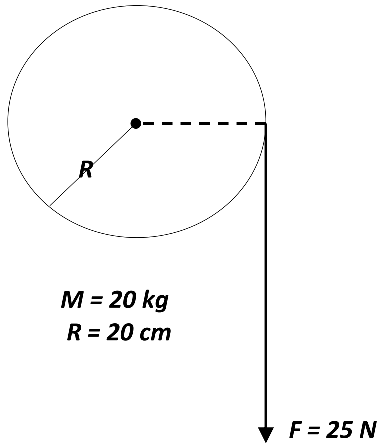
or $\alpha = \frac{a}{R} = \frac{3}{16 \times 0.15} = 1.25 \text{ rad s}^{-2}$

Torque applied by the attached mass, $\tau = F \times R = mgR = 0.5 \times 9.8 \times 0.15$ Nm

Now, $\tau = I\,\alpha$

∴ $I = \frac{\tau}{\alpha} = \frac{0.5 \times 9.8 \times 0.15}{1.25} = 0.588 \text{ kg m}^2$

Q. 8. ***A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Fig. The flywheel is mounted on a horizontal axle with frictionless bearings.***



(a) Compute the angular acceleration of the wheel. (b) Find the work done by the pull, when 2 m of the cord is unwound. (c) Find also the kinetic energy of the wheel at this point. (d) Compare answers to parts (b) and (c).

Sol. (a) Torque, $\tau = FR = 25 \text{ N} \times 0.20 \text{ m} = 50$ Nm

Moment of inertia of the wheel about its axis, $I = \frac{MR^2}{2} = \frac{20 \times (0.20)^2}{2} = 0.4 \text{ kg m}^2$

As $\tau = I\alpha$

∴ Angular acceleration,

$\alpha = \frac{\tau}{I} = \frac{5.0 \text{ Nm}}{0.4 \text{ kg m}^2} = 12.5 \text{ rad s}^{-2}$

(b) Work done by the pull unwinding 2m of the cord = 25 N × 2 m = 50 J

(c) Angular displacement of the wheel,

$\theta = \frac{\text{Length of unwound string}}{\text{Radius of the wheel}}$

$= \frac{2\text{m}}{0.20 \text{ m}} = 10$ rad

As the wheel starts from rest, $\omega_0 = 0$

Final angular velocity ω is given by

$\omega^2 = \omega_0^2 + 2\alpha\theta = 0 + 2 \times 12.5 \times 10$

$= 250 \text{ (rad s}^{-1})^2$

∴ K.E. gained $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.4 \times 250 = 50$ J

(d) The answers are the same, i.e., the kinetic energy gained by the wheel = work done by the force. There is no loss of energy due to friction.

Q. 9. ***A body whose moment of inertia is 3 kgm², is at rest. It is rotated for 20 s with a moment of force 6 Nm. Find the angular displacement of the body. Also calculate the work done.***

Sol. Here $I = 3 \text{ kg m}^2$, $t = 20$ s, $\tau = 6$ Nm, $\theta = ?$, $W = ?$

As $\tau = I\,\alpha$

∴ $\alpha = \frac{\tau}{I} = \frac{6}{3} = 2 \text{ rad s}^{-2}$

Angular displacement in 20 s is

$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 = 0 + \frac{1}{2} \times 2 \times (20)^2 = 400$ rad

Work done, $W = \tau\,\theta = 6 \times 400 = 2400$ J.

Q. 10. ***How much tangential force would be needed to stop the earth in one year, if the were rotating with angular of velocity of 7.3×10^{-5} rad s⁻¹? Given the moment of inertia of the earth = 9.3×10^{37} kg m² and radius of the earth = 6.4×10^{6} m.***

Sol. Here $I = 9.3 \times 10^{37} \text{ kg m}^2$, $R = 6.4 \times 10^6$ m, $\omega_0 = 7.5 \times 10^{-5} \text{ rad s}^{-1}$, t = 1 year = 365 × 24 × 3600 s

As $\omega = \omega_0 + \alpha t$

∴ $\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 7.3 \times 10^{-5}}{365 \times 24 \times 3600}$

$= \dfrac{7.3 \times 10^{-5}}{365 \times 24 \times 3600}$ rad s^{-2}

Torque, $\tau = I\alpha = 9.3 \times 10^{37} \times \dfrac{7.3 \times 10^{-5}}{365 \times 24 \times 3600}$ Nm [Omitting –ve sign]

Let F be the tangential force needed to stop the earth. Then

$\tau = FR$

or $F = \dfrac{\tau}{R} = \dfrac{9.3 \times 10^{37} \times 7.3 \times 10^{-5}}{365 \times 24 \times 3600 \times 6.4 \times 10^{6}} = 3.363 \times 10^{19}$ N.

Q. 11. ***The angular momentum of a body is 31.4 Js and its rate of revolution is 10 cycles per second. Calculate the moment of inertia of the body about the axis of rotation.***

Sol. Here $L = 31.4$ Js, $v = 10$ rps, $\omega = 2\pi v = 3 \times 3.14 \times 10$ rad^{-1}

As $L = I\omega$

∴ $I = \dfrac{L}{\omega} = \dfrac{31.4}{2 \times 3.14 \times 10} = 0.5$ kgm^2

Q. 12. ***A 40 kg flywheel in the form of a uniform circular disc of 1 m radius is making 120 rpm. Calculate the angular momentum.***

Sol. Here $M = 40$ kg, $R = 1$ m,

$v = 120$ rpm $= \dfrac{120}{60}$ rps $= 2$ rps

∴ $I = ½ MR^2 = ½ \times 40 \times (1)^2 = 20$ kg m^2

and $\omega = 2\pi v = 2\pi \times 2 = 4\pi$ rad s^{-1}

Angular momentum, $L = I\omega = 20 \times 4\pi = 80 \times 3.14 = 251.2$ kg m^2 s^{-1}

Q. 13. ***A solid cylinder of mass 20 kg rotates about its axis with angular sped 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?***

Sol. Here $M = 20$ kg, $\omega = 100$ rad s^{-1}, $R = 0.25$ m

M.I. of the cylinder about its own axis,

$I = ½ Mr^2 = ½\ 20 \times (0.25)^2 = 0.625$ kgm^2

Rotational K.E.= $½ I\omega^2 = ½ \times 0.625 \times (100)^2 = 3125$ J

Angular momentum, $L = I\omega = 0.625 \times 100 = 62.5$ kg m^2 s^{-1}.

Q. 14. ***A ring of diameter 0.4 m and of mass 10 kg is rotating about its axis at the rate of 2100 rpm. Find (i) moment of inertia (ii) angular momentum and (iii) rotational K.E. of the ring.***

Sol. Here R = 0.4 = 0.2 m, M = 10 kg

$v = 2100$ rpm $= \dfrac{2100}{60}$ rps $= 35$ s^{-1}

∴ $\omega = 2\pi v = 2 \times \dfrac{22}{7} \times 35 = 220$ rad s^{-1}

(i) M.I. of the ring about its axis,

$I = MR^2 = 10 \times (0.2)^2 = 0.4$ kg m^2

(ii) Angular momentum,

$L = I\omega = 0.4 \times 220 = 88$ kg m^2 s^{-1}

(iii) Rotational K.E.= $½ I\omega^2 = ½ \times 0.4 \times (220)^2 = 9680$ J

Q. 15. ***Calculate the angular momentum of the earth rotating about its own axis. Mass of the earth = 5.98 × 10^{27} kg, mean radius of the earth = 6.37 × 10^6 m, M.I. of the earth = $\dfrac{2}{5}$ MR^2.***

Sol. Here $M = 5.98 \times 10^{24}$ kg, $R = 6.37 \times 10^6$ m

$I = \dfrac{2}{5} MR^2 = \dfrac{2}{5} \times 5.98 \times 10^{24} \times (6.37 \times 10^6)^2$

$= 2.1 \times 10^{38}$ kgm^2

$\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{1 \text{ day}} = \dfrac{2\pi}{24 \times 60 \times 60}$ rad s^{-1}

Angular momentum, $L = I\omega = 2.1 \times 10^{38} \times \dfrac{2\pi}{24 \times 60 \times 60} = 1.53 \times 10^{34}$ kg m^2 s^{-1}

Q. 16. ***A cylinder of mass 5 kg and radius 30 cm, and free to rotate about its axis, receives an angular impulse of 3 kg m^2 s^{-1} initially followed by a similar impulse after every 4 s. What is the angular speed of the cylinder 30 s after the initial impulse? The cylinder, is at rest initially.***

Sol. Here $M = 5$ kg, $R = 30$ cm $= 0.30$ m, $\omega = 0$, $\omega = ?$

Angular impulse = Change in angular momentum

∴ $3 = I(\omega_2 - \omega_1) = ½ MR^2 (\omega - \omega_0)$

or $3 = ½ \times 5 \times (0.30)^2 (\omega - 0)$

or $\omega = \dfrac{3 \times 2}{5 \times 0.09} = \dfrac{40}{3}$ rad s^{-1}

Now, $\omega = \omega_0 + \alpha t$

∴ $\dfrac{40}{3} = 0 + \alpha \times 4$ or $\alpha = 10$ rad s^{-}

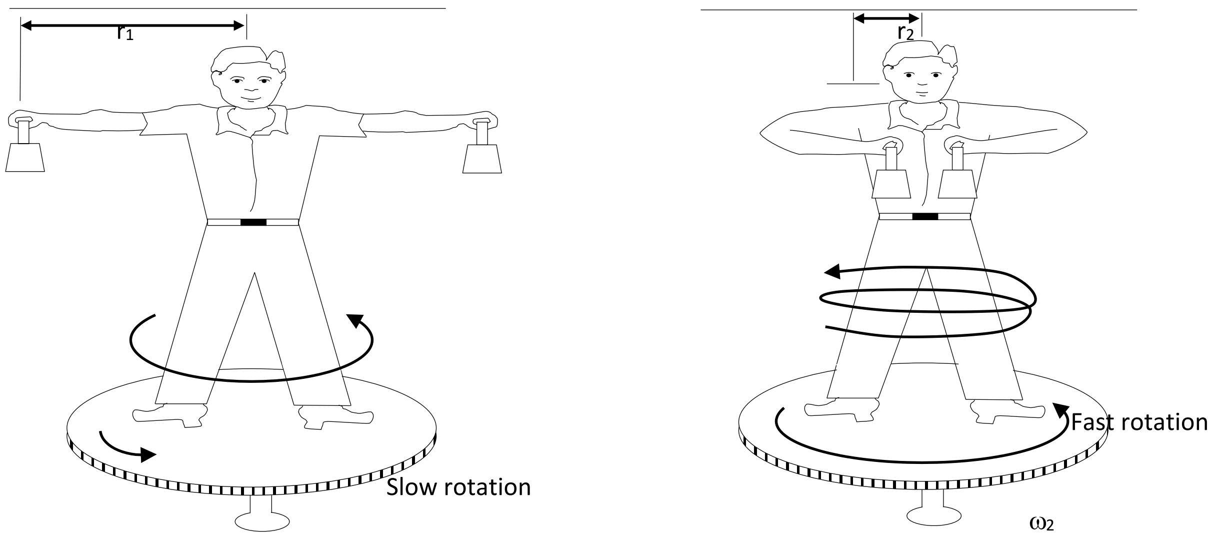
The angular impulse is imparted after every 4 seconds. So, the pulses are imparted at t = 0, 4, 8, 12, 16, 20, 24 and 28 s. But last impulse continues to act up to 32 s, before the next impulse is imparted. So

$$\omega = \omega_0 + \alpha t = 0 + \frac{10}{3} \times 32 = 106.67 \text{ rad s}^{-1}.$$

Illustrations of the law of conservation of angular momentum:

(i) Planetary motion: The angular velocity of a planet revolving in an elliptical orbit around the sun increases, when it comes closer to the sun because its moment of inertia about the axis through the sun decreases. When it goes far away from the sun, its moment of inertia increases and hence angular velocity decreases so as to conserve angular momentum.

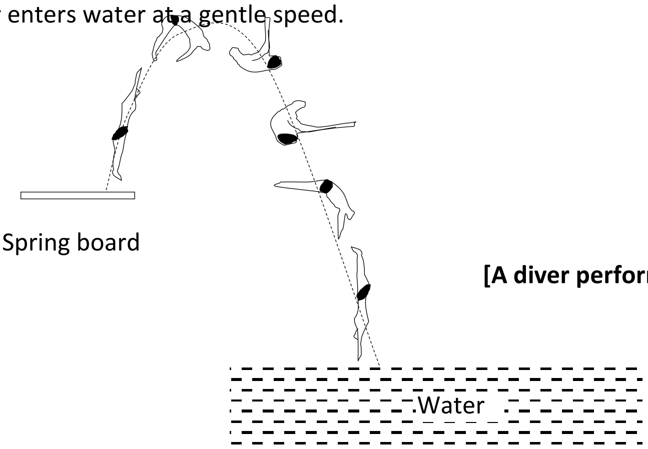
(ii) A man carrying heavy weights in his hands and standing on a rotating turn-table can change the angular speed of the turn-table. As shown in Fig.



$\omega_2 > \omega_1$

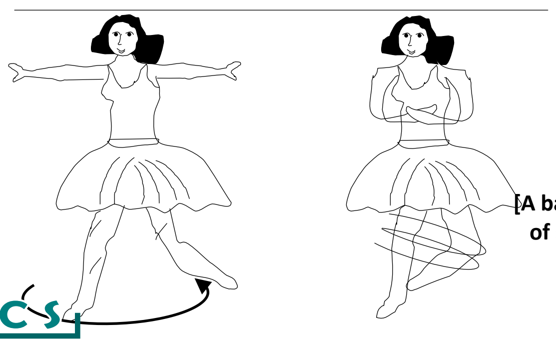
If a person stands on a turn-table with some heavy weights in his hands stretched out and the table is rotated slowly, his angular speed at once increases, as he draws his hands to his chest. The moment of inertia of man and weights taken together decreases, as he draws his arms inward. As moment of inertia decreases, the angular speed increases so as to conserve total angular momentum.

(iii) A diver jumping from a spring board exhibits somersaults in air before touching the water surface: After leaving the spring board, a diver curls his body by pulling his arms and legs towards the centre of his body. This decreases his moment of inertia and he spins fast in midair. Just before hitting the water surface, he stretches out his arms. This decreases his moment of inertia and the diver enters water at a gentle speed.



[A diver performing somersaults]

(iv) An ice-skater of a ballet dancer can increase her angular velocity by folding her arms and bringing the stretched leg close to the other leg: When she stretches her hands and a leg outward [Fig (a)], her moment of inertia increases and hence angular speed decreases to conserve angular momentum. When she folds her arms and brings the stretched leg close to the other let [Fig. (b)]



[A ballet dance increases ω by changing to a posture of smaller M.I.]

(v) The speed of the inner layers of the whirlwind in a tornado is alarmingly high: The angular velocity or air in the tornado increases as it goes towards the centre. This is because as the air moves towards the centre, its moment of inertia (I) decreases and to conserve angular momentum ($L = I\omega$), the angular velocity ω increases.

Examples based on Law of Conservation of Angular Momentum

◆ **FORMULA USED**

In the absence of any external torque, $L = I\omega$ = a constant

or $I_1\omega_1 = I_2\omega_2$ or $I_1 . \frac{2\pi}{T_1} = I_2 . \frac{2\pi}{T_2}$

◆ **UNITS USED** Moment of inertia I is in kg m^2 and angular velocity ω in rad s^{-1}.

Q. 1. ***A small block is rotating in a horizontal circle at the end of a thread which passes down through a hole at the centre of table top. If the system is rotating at 2.5 rps in a circle of 30 cm radius, what will be the speed of rotation when the thread is pulled inwards to decrease the radius to 10 cm? Neglect friction.***

Sol. Here v_1 = 2.5 rps, r_1 = 30 cm, r_2 = 10 cm, v_2 = ?

By law of conservation of angular momentum,

$L_1 = L_2$ or $I_1\omega_1 = I_2\omega_2$

or $mr_1^2 . 2\pi v_1 = mr_2^2 . 2\pi v_2$

$\therefore$ $v_2 = \frac{r_1^2 v_1}{r_2^2} = \frac{30 \times 30 \times 2.5}{10 \times 10} = 22.5$ rps

Q. 2. ***A star of mass twice the solar mass and radius 10^6 km rotates about its axis with an angular speed or 10^{-6} rad s^{-1}. What is the angular speed of the star when it collapses (due to inward gravitational force) to a radius of 10^4 km? Solar mass 1.99×10^{30} kg.***

Sol. During collapse, the total angular momentum of an isolated star is conserved, hence

$I_1\omega_1 = I_2\omega_2$

or $\frac{2}{5}MR_1^2\omega_1 = \frac{2}{5}MR_2^2\omega_2$ $\left(\because I = \frac{2}{5}MR^2\right)$

or $R_1^2\omega_1 = R_2^2\omega_2$ $\therefore$ $\omega_2 = \frac{R_1^2}{R_2^2}\omega_1$

But $R_1 = 10^6$ km, $R_2 = 10^4$ km, $\omega_1 = 10^{-6}$ s^{-1}. $\therefore$ $\omega_2 = \frac{(10^6)^2}{(10^4)^2} \times 10^{-6} = 0.01$ rad s^{-1}

Q. 3. ***(i) A child stands at the centre of turntable with his two arms out stretched. The turntable is set rotating with an angular speed of 40 rpm. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/3 times the initial value? Assume that the turntable rotates without friction.***
(ii) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

Sol. Here ω_1 = 40 rpm, $I_2 = \frac{2}{5}I_1$

By the principle of conservation of angular momentum,

$I_1\omega_1 = I_2\omega_2$ or $I_1 \times 40 = \frac{2}{5}I_1\omega_2$ or ω_2 = 100 rpm

(ii) Initial kinetic energy of rotation $= ½ I_1\omega_1^2 = ½ I_1 (40)^2 = 800 I_1$

New kinetic energy of rotation

$= ½ I_2\omega_2^2 = ½ \times \frac{2}{3} I_1 \times (100)^2 = 2000 I_1$

$\therefore$ $\frac{\text{New K.E.}}{\text{Initial K.E.}} = \frac{2000 I_1}{800 I_1} = 2.5$

Thus, the child's new kinetic energy of rotation is 2.5 times its initial kinetic energy of rotation. This increase in kinetic energy is due to the internal energy of the child which he uses in folding his hands back from the out stretched position.

Q. 4. ***A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90 cm to 20 cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m².***
(a) What is his new angular speed? (Neglect friction)
(b) Is kinetic energy conserved in the process? If not, from where does the change come about?

Sol. (a) Total initial moment of inertia, I_1 M.I. of man and platform + M.I. of two 5 kg weights

$= 7.6 + 2 \times 5 \times (0.90)^2 = 7.6 + 8.1 = 15.7$ kg m^2

Initial angular speed, ω_1 = 30 rpm

Total final moment of inertia, $I_2 = 7.6 + 2 \times 5 \times (0.20)^2 = 7.6 + 0.4 = 8.0$ kg m^2

By the principle of conservation of angular momentum,

$I_1\omega_1 = I_1\omega_2$

or $15.7 \times 30 = 8.0 \times \omega_2$ or $\omega_2 = \frac{15.7 \times 30}{8.0} = 58.875 \approx 59$ rpm.

(b) $\frac{\text{Final K.E.}}{\text{Initial K.E}} = \frac{½ I_1\omega_1^2}{½ I_2\omega_2^2} = \frac{8.0 \times (59)^2}{15.7 \times (30)^2} = 1.97$

Thus, the final K.E. is about twice the initial K.E. i.e., K.E. is not conserved in the process. The increase in K.E. is due to the internal energy the man uses in bringing his arms closer to his body.

Q. 5. ***A bullet of mass 10 g and speed m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.***
[Hint: the moment of inertia of the door about the vertical axis at one end is $ML^2/3$.]

Sol. By the Principle of conservation of angular momentum,

Initial angular momentum of the bullet = Final angular momentum of the door

or $pr = I\omega$

or $mvr = \frac{ML^2}{3} \times \omega$

or $\omega = \frac{3mvr}{ML^2}$

Here $m = 10\text{ g} = 10^{-2}$ kg, $v = 500\text{ ms}^{-1}$,

$r = \frac{1.0}{2} = 0.5$ m

$L = 1.0$ m, $M = 12$ kg

∴ $\omega = \frac{3 \times 10^{-2} \times 500 \times 0.5}{12 \times (1.0)^2} = 0.625\text{ rad s}^{-1}$

Q. 6. ***If the earth were to suddenly contract to half of its present radius (without any external torque on it), by what duration would be day be decreased? Assume earth to be a perfect solid sphere of moment of inertia $\frac{2}{5}MR^2$.***

Sol. Present radius of the earth, $R_1 = R$

New radius of the earth after contraction, $R_2 = R/2$

$T_1 = 24$ H, $T_2 = ?$

By conservation of angular momentum,

$I_1\omega_1 = I_2\omega_2$

or $\frac{2}{5}MR_1^2 \cdot \frac{2\pi}{T_1} = \frac{2}{5}MR_2^2 \cdot \frac{2\pi}{T_2}$

or $T_2 = \left(\frac{R_2}{R_1}\right)^2 \cdot T_1 = \left(\frac{R/2}{R}\right)^2 \times 24 = \frac{1}{4} \times 24 = 6$ h

∴ Decreases in the duration of the day = 24 – 6 = 18 h

Q. 7. ***What will be the duration of the day, if earth suddenly shrinks to 1/64 of its original volume, mass remaining the same?***

Sol. Original volume of the earth,

$V = \frac{4}{3}\pi R^3$

Volume of the earth after shrinking, $V' = \frac{V}{64}$

or $\frac{4}{3}\pi R'^3 = \frac{1}{64} \times \frac{4}{3}\pi R^3$

or $R' = R/4$

By conservation of angular momentum $I'\omega' = I\omega$

or $\frac{2}{5}MR'^2 \times \frac{2\pi}{T'} = \frac{2}{5}MR^2 \times \frac{2\pi}{T}$

or $T' = \left(\frac{R'}{R}\right)^2 . T = \left(\frac{R/4}{R}\right)^2 \times 24 = \frac{1}{16} \times 24 = 1.5$ h

Q. 8. ***The maximum and minimum distances of a comet from the sun are 1.4×10^{12} m and 7×10^{10} m. If its velocity nearest to the sun is $6 \times 10^4\text{ ms}^{-1}$, what is the velocity in the farthest position? Assume that path of the comet in both the instantaneous position is circular.***

Sol. At minimum distance, $r_1 = 7 \times 10^{10}$ m; velocity, $v_1 = 6 \times 10^4\text{ ms}^{-1}$

At maximum distance, $r_2 = 1.4 \times 10^{12}$ m; velocity, $v_2 = ?$

By conservation of angular momentum,

$I_1\omega_1 = I_2\omega_2$

or $mr_1^2 \times \frac{v_1}{r_1} = mr_2^2 . \frac{v^2}{r_2}$ or $v_1 r_1 = v_2 r_2$

or $v_2 = \frac{v_1 r_1}{r_2} = \frac{6 \times 10^4 \times 7 \times 10^{10}}{1.4 \times 10^{12}} = 3000\text{ ms}^{-1}$

Q. 9. ***A horizontal disc rotating about a vertical axis passing through its centre makes 180 rpm. A small piece of wax of mass 10 g falls vertically on the disc and adheres to it at a distance of 8 cm from its axis. If the frequency is thus reduced to 150 rpm, calculate the moment of inertia of the disc.***

Sol. Here $v_1 = 180$ rpm = 3 rps, $v_2 = 150\text{ rpm} = \frac{150}{60}\text{ rps} = \frac{5}{2}$ rps

∴ $\omega_1 = 2\pi v_1 = 2\pi \times 3 = 6\pi\text{ rad s}^{-1}$, $\omega_2 = 2\pi \times \frac{5}{2} = 5\pi\text{ rad s}^{-1}$

Let I be the M.I. of the disc about the given axis and I_2 be the M.I. when mass m sticks to it at distance r. Then, $I_2 = I + mr^2$

By conservation of angular momentum,

$I_1\omega_1 = I_2\omega_2$

$I \times 6\pi = (I + mr^2) . 5\pi$ or $6I = 5I + 5mr^2$ or $I = 5mr^2 = 5 \times 10 \times 10^{-3} \times (8 \times 10^{-2})^2 = 3.2 \times 10^{-8}\text{ kgm}^2$

ANALOGY BETWEEN TRANSLATIONAL AND ROTATIONAL MOTIONS.

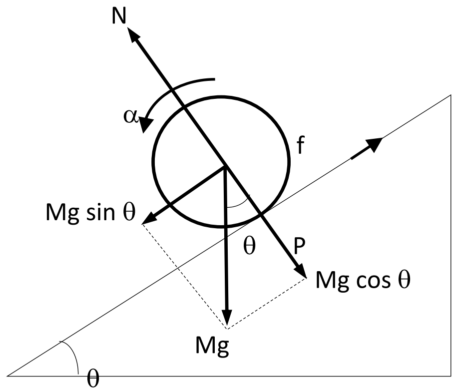
quantities that describe linear motion and the corresponding quantities that describe rotational motion.

Linear motion		Rotational motion	
Quantities			
Displacement	s	Angular displacement	θ
Velocity	v	Angular velocity	ω
Acceleration	a	Angular acceleration	α or a_θ
Force	F	Torque	τ
Mass	M	Moment of inertia	I
Expressions:			
Velocity	$v = \frac{ds}{dt}$	Angular velocity	$\omega = \frac{d\theta}{Dt}$
Acceleration	$a = \frac{dv}{dt}$	Angular acceleration	$\alpha = \frac{d\omega}{dt}$
Force	$F = ma = \frac{d}{dt}(mv)$	Torque	$\tau = I\alpha = \frac{d}{Dt}(I\omega)$
Work done	$W = Fs$	Work done	$W = \tau\,\theta$
Linear K.E.	$E = ½\, mv^2$	Rotational K.E.	$E = ½\, I\omega^2$
Power	$P = Fv$	Power	$P = \tau\omega$
Linear momentum	$P = mv$	Angular momentum	$L = I\omega$
Impulse	$F\,\Delta t = mv - mu$	Angular impulse	$\tau\,\Delta t = I\omega_f - I\omega_i$
Equations of motion:			
(i) $v = u + at$ (ii) $s = ut + ½\, at^2$ (iii) $v^2 - u^2 = 2as$		(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \theta_0 t + ½\,\alpha t^2$ (iii) $\omega^2 - \omega_0^2 = 2\,\alpha\theta$	
Dimensions:			
Velocity	$[LT^{-1}]$	Angular velocity	$[T^{-1}]$
Acceleration	$[LT^{-2}]$	Angular acceleration	$[T^{-2}]$
Mass	[M]	Moment of inertia $I = \Sigma\, mr^2$	$[ML^2]$
Force	$[MLT^{-2}]$	Torque $\tau = Fr$	$[ML^2T^{-2}]$
Linear K.E.	$[ML^2\,T^{-2}]$	Rotational K.E.	$[ML^2T^{-2}]$
Momentum	$[MLT^{-1}]$	Angular momentum	$[ML^2T^{-1}]$
Power	$[ML^2\,T^{-3}]$	Power	$[ML^2T^{-3}]$

SOLID CYLINDER ROLLING WITHOUT SLIPPING DOWN AN INCLINED PLANE

Consider a solid cylinder of mass M and radius R rolling down a plane inclined at an angle θ to the horizontal. Suppose the cylinder rolls down without slipping.

- **The condition for rolling without slipping is that at each instant the line of contact of the cylinder with the surface at P is momentarily at rest and the cylinder rotates about this line as axis.**
- **The centre of mass of the cylinder moves in a straight line parallel to the inclined plane. Notably, it is the friction which prevents slipping.**



[Cylinder rolling without slipping]

The external forces acting on the cylinder are :

--(i) The weight Mg of the cylinder acting vertically downwards through the centre of mass of the cylinder.
--(ii) The normal reaction N of the inclined plane acting perpendicular to the plane at P.
--(iii) The frictional force f acting upwards and parallel to the inclined plane.

The weight Mg can be resolved into two rectangular components:

--(i) Mg cos θ perpendicular to the inclined plane.
--(ii) Mg sin θ acting down the inclined plane.

As there is no motion in a direction normal to the inclined plane, so **$N = Mg\cos\theta$**

Applying Newton's second law to the linear motion of centre of mass, the net force on the cylinder rolling down the inclined plane is $F = Ma = Mg\sin\theta - f$ ... (1)

It is only the force of friction f which exerts torque τ on the cylinder and makes it rotate with angular acceleration α. It acts tangentially at the point of contact P and has lever arm equal to R.

$\therefore \quad \tau$ = Force × force arm = f . R

Also, τ = M.I. × angular acceleration = $I\alpha$

$\therefore \quad fR = I\alpha$

or $\quad f = \frac{I\alpha}{R} = \frac{Ia}{R^2} \qquad \left(\because \alpha = \frac{a}{R}\right)$

Putting the value of f in equation (1), we get

$$Ma = Mg\sin\theta - \frac{Ia}{R^2}$$

$$a = g\sin\theta - \frac{Ia}{MR^2}$$

or $\quad a + \frac{Ia}{MR^2} = g\sin\theta$

or $\quad a\left(1 + \frac{I}{MR^2}\right) = g\sin\theta$

$$a = \frac{g\sin\theta}{1 + \frac{I}{MR^2}}$$

Moment of inertia of the solid cylinder about its axis = ½ MR^2

$\therefore \quad a = \frac{g\sin\theta}{1 + \frac{1}{2}MR^2/MR^2}$

or $\quad$ **$a = \frac{2}{3} g\sin\theta$**

The linear acceleration a of solid cylinder rolling down and inclined plane is less than the acceleration due to gravity g **(a < g).** The linear acceleration is constant for a given inclined plane (or given θ) and is independent of its mass M and radius R. However, for a hollow cylinder, $I = MR^2$, the value of a would decrease to ½ g sin θ.

From equation (1), the value of force of friction is $\quad f = Mg\sin\theta - Ma = Mg\sin\theta - M.\ 2/3\ g\sin\theta = 1/3\ Mg\sin\theta$

If μ_s is the coefficient of friction between the cylinder and the inclined plane, then

$$\mu_s = \frac{f}{N} = \frac{1/3\ Mg\sin\theta}{Mg\cos\theta} = 1/3\tan\theta$$

To prevent slipping, the coefficient of static friction must be equal to or greater than the above value. That is

$$\mu_s \geq 1/3\tan\theta \quad \text{or} \quad \tan\theta \leq 3\mu_s.$$

Expression for the kinetic energy of a body rolling without slipping.

The kinetic energy of a body rolling without slipping is the sum of kinetic energies of translation and rotation.

K = K.E. of the translational motion of CM + K.E. of rotational motion of CM

$= \frac{1}{2} mv^2_{CM} + \frac{1}{2} I\omega^2$

where v_{CM} is the velocity of CM and I is the moment of inertia about the symmetry axis of the rolling body. If R is the radius and k the radius of gyration of the rolling body, then

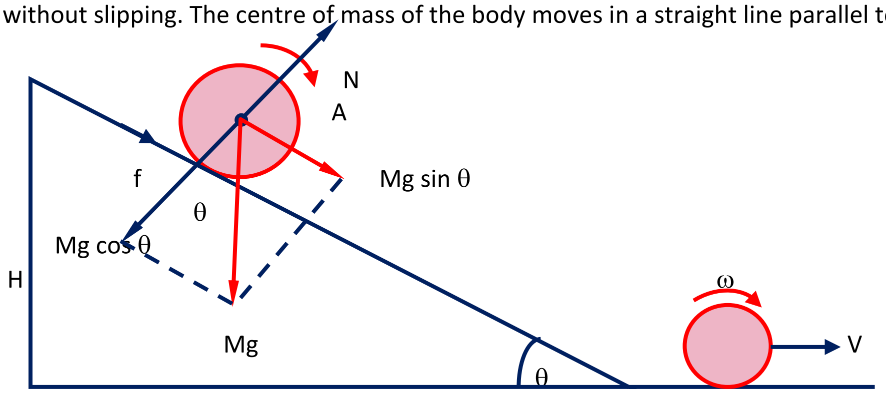
$v_{CM} = R\omega$ and $I = mk^2$

$\therefore \quad K = \frac{1}{2} mv^2_{CM} + \frac{1}{2} mk^2 \left(\frac{v_{CM}}{R}\right)^2$

or $\quad K = \frac{1}{2} mv^2_{cm} \left(1 + \frac{k^2}{R^2}\right)$

VELOCITY ATTAINED BY A BODY ROLLING DOWN AN INCLINED PLANE:

Consider a body of mass M and radius R rolling down a plane inclined at an angle θ with the horizontal. It is only due to friction at the line of contact that body can roll without slipping. The centre of mass of the body moves in a straight line parallel to the inclined plane.



The external forces on the body are:

--(i) The weight Mg acting vertically downwards.
--(ii) The normal reaction N of the inclined plane.
--(iii) The force of friction acting up the inclined plane.

Let a be the downward acceleration of the body. The equations of motion for the body can be written as

$N - Mg\cos\theta = 0$

$F = Ma = Mg\sin\theta - f$

where k is the radius of gyration of the body about its axis of rotation. Clearly

$Ma = Mg\sin\theta - M\frac{k^2}{R^2} . a$

or $a = \frac{g\sin\theta}{(1 + k^2/R^2)}$

Let h be height of the inclined plane and s the distance travelled by the body down the plane. The velocity v attained by the body at the bottom of the inclined plane can be obtained as follows:

$v^2 - u^2 = 2as$

or $v^2 - 0^2 = 2 . \frac{g\sin\theta}{(1 + k^2/R^2)} . s$

or $v^2 = \frac{2gh}{1 + k^2/R^2}$ $\left(\because \frac{h}{s} = \sin\theta\right)$

or $v = \sqrt{\frac{2gh}{(1 + k^2/R^2)}}$

Examples based on Motion of a Cylinder Rolling without Slipping on an Inclined Plane

◆ **FORMULA USED**

For a cylinder of mass M and radius R rolling without slipping down plane inclined at angle θ with the horizontal,

1. Force of friction between the plane and cylinder,

$f = \frac{1}{3} Mg\sin\theta$

2. Linear acceleration, $a = \frac{2}{3} g\sin\theta$

3. Condition for rolling without slipping is

$\mu_s \geq \frac{1}{3}\tan\theta$

◆ **UNITS USED** Acceleration a and g are in ms^{-2} and coefficient of friction μ_s has no units.

Q. 1. ***A cylinder of mass 5 kg and radius 30 cm is rolling down an inclined plane at an angle of 45° with the horizontal. Calculate (i) force of friction, (ii) acceleration with which the cylinder rolls down and (iii) the minimum value of static friction so that cylinder does not slip while rolling down the plane.***

Sol. Here M = 5 kg, R = 30cm = 0.30 m, θ = 45°

(i) Force of friction,

$f = \frac{1}{3} Mg\sin\theta = \frac{1}{3} \times 5 \times 9.8 \sin 45° = 11.55$ N.

(ii) Acceleration,

$a = \frac{2}{3} g\sin\theta = \frac{2}{3} \times 9.8 \sin 45° = 4.62\ ms^{-2}$.

(iii) Minimum value of coefficient of static friction,

$\mu_s = \frac{1}{3}\tan\theta = \frac{1}{3}\tan 45° = \frac{1}{3}$

Q. 2. ***A solid cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The coefficient of static friction, μ_s = 0.25. (i) Find the force of friction acting on the cylinder. (ii) What is the work done against friction during rolling? (iii) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly?***

Sol. Here M = 10 kg, R = 0.15 m μ_s = 0.25, θ = 30°

(i) Force of friction,

$F = \frac{1}{3} Mg\sin\theta = \frac{1}{3} \times 10 \times 9.8 \times \sin 30°$

(ii) Work done against friction during rolling = 0 J

(iii) Condition for skidding (or no rolling) is

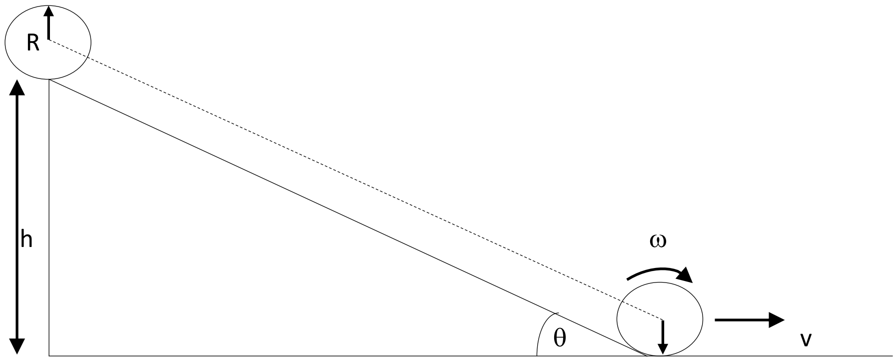
$\frac{f}{N} \leq \mu_s$ or $\frac{1/3\ Mg\sin\theta}{Mg\cos\theta} \leq$

or $\frac{1}{3}\tan\theta \leq \mu_s$

Thus, the cylinder will start skidding at an angle of inclination θ given by ; $\tan\theta = 3\mu_s = 3 \times 0.25 = 0.75$ or θ = 36° 52′

Q. 3. ***Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity?***

Sol. Suppose a body of mass m starting form rest rolls down an inclined plane. We assume there is no loss of energy due to friction.



By conservation of energy,

P.E. lost by the body in rolling down the inclined plane = K.E. gained by the body

= Translational K.E. + Rotational K.E.

$= \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} mv^2 + \frac{1}{2} mk^2 . \left(\frac{v}{R}\right)^2$

or $mgh = \frac{1}{2} mv^2 \left(1 + \frac{k^2}{R^2}\right)$

or $v = \sqrt{\frac{2gh}{1 + k^2/R^2}}$

Clearly, the velocity v attained by the rolling body at the bottom of the inclined plane is independent of its mass.

For a ring, $k^2 = R^2$

∴ $v_{ring} = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh}$

For a solid cylinder, $k^2 = R^2/2$

∴ $c_{ylinder} = \sqrt{\frac{2gh}{1 + 1/2}} = \sqrt{\frac{4gh}{3}}$

For a solid sphere, $k^2 = 2R^2/5$

∴ $v_{sphere} = \sqrt{\frac{2gh}{1 + 2/5}} = \sqrt{\frac{10\,gh}{7}}$;Clearly, among the three bodies the sphere has the greatest and the ring has the least velocity of the centre of mass at the bottom of the inclined plane.

Q. 4.A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination. (a) Will it reach the bottom with the same speed in each case? (b) Will it take longer to roll down one plane than the other? (c) If so, which one and why?

Sol. Acceleration of the rolling sphere,

$a = \frac{g \sin\theta}{(1 + k^2/R^2)}$

Velocity of the sphere at the bottom of the inclined plane,

$v = \sqrt{\frac{2gh}{(1 + k^2/R^2)}}$

(a) Yes, the sphere will reach the bottom with the same speed v because h is same in both cases.

(b) Yes, the sphere will take longer time to roll down one plane than the other.

(c) The sphere will take larger time in case of the plane with smaller inclination because the acceleration, $a \propto \sin\theta$

Q. 5. ***A solid cylinder of radius 4 cm and mass 250 g rolls down an inclined plane (1 in 10). Calculate the acceleration and the total energy of the cylinder after 5 s.***

Sol. Here M = 250 g = 0.25 kg,

R = 4 cm = 0.04 m, $\sin\theta = 1/10$, t = 5 s

Acceleration with which the cylinder rolls down,

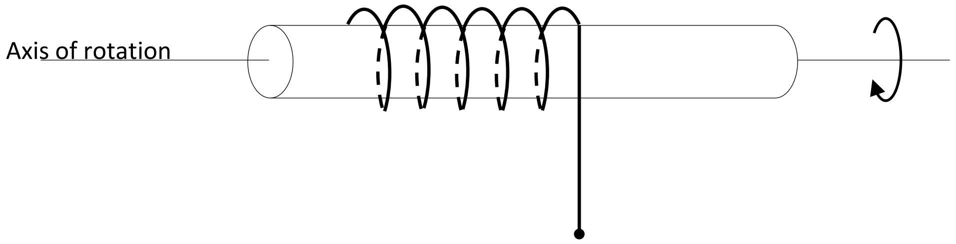
$a = \frac{g\sin\theta}{1 + \frac{I}{MR^2}} = \frac{g\sin\theta}{1 + \frac{\frac{1}{2}MR^2}{MR^2}} = \frac{2}{3} g\sin\theta = \frac{2}{3} \times 9.8 \times \frac{1}{10} = 0.653\ ms^{-2}$

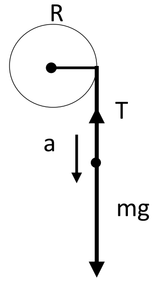
Using first equation of motion,

$= \text{Translational K.E.} + \text{Rotational K.E.} = \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 + \frac{1}{2} Mv^2 + \frac{1}{2} \times \frac{1}{2} MR^2 \times \frac{v^2}{R^2} = \frac{3}{4} Mv^2 = \frac{3}{4} \times 0.25 \times (3.26)^2 = 2.0\ J$

MASS POINT ON STRING WOUND ON A CYLINDER

consider a solid cylinder of mass m and radius R. It is mounted on a frictionless horizontal axle so that it can freely rotate about its axis. A light string is wound round the cylinder and mass m is suspended from it. When the mass m is released from rest, if moves down with acceleration a. Let T be the tension in string.





(a) Linear acceleration of the point mass: The forces acting on the point mass:

(i) Its weight mg acting vertically downwards. (ii) Tension T in the string acting upwards.

According to Newton's second law, the net downward force on the point mass is

$$Ma = mg - T \quad \text{... (1)}$$

The tension T is the string acts tangentially on the cylinder and produces a torque τ given by

$$\tau = \text{Force} \times \text{lever arm} = T.R \quad \text{... (2)}$$

If *I* is the moment of inertia of the cylinder and α, the angular acceleration produced in it, then

$$\tau = I\alpha \quad \text{... (3)}$$

From equations (2) and (3),

$$TR = I\alpha$$

or $T = \frac{I}{R}\alpha = \frac{Ia}{R^2}$ $\left(\because \alpha = \frac{a}{R}\right)$ ... (4)

From equation (1), we have

$$Ma = mg - \frac{Ia}{R^2}$$

or $ma + \frac{Ia}{R^2} = mg$

or $ma\left(1 + \frac{I}{mR^2}\right) = mg$

or $\mathbf{a = \frac{g}{1 + I/mR^2}}$ **... (5)**

This gives the linear downward acceleration of the point mass.

(b) Angular acceleration of the point mass: As I, m and R are positive quantities, so a is always less than 'g'.

Angular acceleration, $\alpha = \frac{a}{R} = \frac{g/R}{1 + I/mR^2}$ **... (6)**

(c) Tension in the string: From equations (4) and (5), we have

$$T = Ia = \frac{Ig}{R^2\left(1 + \frac{I}{mR^2}\right)} = \frac{Ig}{R^2 \cdot \frac{I}{mR^2}\left(\frac{mR^2}{I} + 1\right)}$$

or $\mathbf{t = \frac{mg}{1 + \frac{mR^2}{I}}}$ **... (7)**

Clearly, T is less than the weight mg of the point mass.

Q. 1. A body of mass 5 kg is attached to a weightless string wound round a cylinder of mass 8 kg and radius 0.3 m. The body is allowed to fall. Calculate (i) tension in the string (ii) acceleration with which the body falls and (iii) the angular acceleration of the cylinder.

Sol. Here m = 5 kg, M = 8 kg, R = 0.3 m

(i) M.I. of the cylinder, I = ½ MR²

∴ Tension in the string,

$$T = \frac{mg}{1 + \frac{mR^2}{I}} = \frac{mg}{1 + \frac{2\,mR^2}{MR^2}} = \frac{mg}{1 + \frac{2m}{M}}$$

$$= \frac{5 \times 9.8}{1 + \frac{2 \times 5}{8}} = \frac{49.0}{2.25} = 21.78\text{ N}$$

(ii) Linear acceleration, $a = \frac{g}{1 + I/mR^2} = \frac{g}{1 + MR^2/2\,mR^2}$

$$= \frac{g}{1 + M/2m} = \frac{9.8}{1 + 8/2 \times 5} = \frac{9.8}{1.8} = 5.44\text{ ms}^{-2}$$

(iii) Angular acceleration,

$$\alpha = a/R = 5.44/0.3 = 18.13\text{ rad s}^{-2}$$

... **END.**