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INDEFINITE INTEGRATION

1. Definition :

Integration is the inverse process of differentiation. The process of finding f(x), when its derivative $f'(x)$ is given is known as integration.

2. Integral as Anti Derivative :

If f(x) is a differentiable function such that $f'(x) = g(x)$, then integration of g(x) w.r.t. x is f(x) + c.

Symbolically it is written as $\int g(x)dx = f(x) + c$, here c is known as constant of integration and it can take any real value. For example $\frac{d}{dx}(\tan x) = \sec^2 x$, so $\int \sec^2 x\, dx = \tan x + c$.

3. List of Standard Formula :

Based upon the about method and the previous knowledge of differentiation of standard functions, here is the list of integration of standard functions.

Function f(x) (Integrand) | **Integration $\int f(x)dx$**

Constant k	Kx + c
x^n	$\frac{x^{n+1}}{n+1} + c\ (n \neq -1)$
$\frac{1}{x}\ (x \neq 0)$	$\ell n\,\lvert x \rvert + c$
$a^x\ (a > 0)$	$\frac{a^x}{\ell n\ a} + c$
e^x	$e^x + c$
sin x	– cos x + c
cos x	sin x + c
$\sec^2 x$	tan x + c
$\text{cosec}^2 x$	– cot x + c
sec x tan x	sec x + c
cosec x cot x	– cosec x + c
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$
$\frac{1}{1+x^2}$	$\tan^{-1} x + c$
$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$	$\sec^{-1} x + c$

Theorem 1 :

Two integrals of the same function can differ only by a constant.

Proof :

Let $f_1(x)$ and $f_2(x)$ be two integrals of g(x). Then by definition $f_1'(x) = g(x)$ and $f_2'(x) = g(x)$ for all possible value of x.

$\Rightarrow$ $f_1'(x) = f_2'(x) \forall x$ Let $h(x) = f_1(x) - f_2(x)$ $\Rightarrow$ $h'(x) = 0 \forall x$

Now consider the interval [a, b] (a < b) then by Lagrange's Mean value's theorem, there exists some $c \in (a, b)$ such that $h'(c) = \dfrac{h(b) - h(a)}{b - a}$

Since $h'(x) = 0 \forall \times \text{so } h'(c) = 0$

$\Rightarrow$ h(b) = h(a) $\Rightarrow$ h(x) is a constant function

Let h(x) = c $\Rightarrow$ $f_1(x) - f_2(x) = c$

Hence two integral of the same function can differ only by a constant.

Theorem 2 :

(i) $\int (af(x) + bg(x)dx = a\int (f(x)dx + b\int g(x)dx$, where a and b are constants.

(ii) $\int f(x)dx = g(x) + c$, then $\int f(ax + b)dx = \dfrac{1}{a} g(ax + b) + c$, where a and b are constants and $a \neq 0$.

Illustration 1 :

Evaluate: $\int \left(\sqrt{3}\sin x - \cos x\right)dx$.

Solution:

$\int \left(\sqrt{3}\sin x - \cos x\right)dx$

$= \sqrt{3}\int \sin x\, dx - \int \cos dx$ $= -\sqrt{3}\cos x - \sin x + c$

$= -2\left[\cos x.\cos\dfrac{\pi}{6} + \sin x.\sin\dfrac{\pi}{6}\right] + c$ $= -2\cos\left(x - \dfrac{\pi}{6}\right) + c$

If in illustration 1, we write $\sqrt{3}\sin x - \cos x$ as $-2\cos\left(x + \dfrac{\pi}{3}\right)$, then what will be integral ?

Illustration 2 :

Evaluate: $\int \sec^2(3x + 5)dx$

Solution:

We know that $\int \sec^2 x\ dx = \tan x + c$

so $\int \sec^2(3x + 5)dx = \dfrac{1}{3}\tan(3x + 5) + c$

4. Integration by Substitution :

It is not always possible to find the integral of a complicated function only by observation, so we need some methods of integration and integration by substitution is one of them. This methods has 3 parts :

(i) Direct substitution **(ii)** Standard substitution
(iii) Indirect substitution

4.1 DIRECT SUBSTITUTION

If $\int f(x)dx = g(x)+c$, then in I = $\int f(h(x))h'(x)dx$,

We put h(x) = t $\Rightarrow$ $h'(x)dx = dt$ So I = $\int f(t)dt = g(t)+c = g(h(x))+c$

***Illustration* 3 :**

Evaluate: $\int \cot x\,dx$.

***Solution*:**

I = $\int \cot x\,dx$ = $\int \frac{\cos x\,dx}{\sin x}$

Put sin x = t $\Rightarrow$ cos x dx = dt

So I = $\int \frac{dt}{t}$ = $\ell n|t|+c$ = $\ell n|\sin x|+c$

***Illustration* 4 :**

Evaluate: $\int \frac{dx}{2\sqrt{x}(x+1)}$.

***Solution*:**

Put $x = t^2$ $\Rightarrow$ dx = 2t dt

So I = $\int \frac{dx}{2\sqrt{x}(x+1)}$ = $\int \frac{2t\,dt}{2t(t^2+1)}$ = $\int \frac{dt}{1+t^2}$ = $\tan^{-1} t + c$ = $\tan^{-1}(\sqrt{x})+c$

4.2 STANDARD SUBSTITUTION

In some standard integrand or a part of it, we have standard substitution. List of standard substitution is as follows :

Integrand	Substitution
$x^2 + a^2$ or $\sqrt{x^2+a^2}$	$x = a\tan\theta$
$x^2 - a^2$ or $\sqrt{x^2-a^2}$	$x = a\sec\theta$
$a^2 - x^2$ or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a+x}$ and $\sqrt{a-x}$	$x = a\cos 2\theta$
$\left(x\pm\sqrt{x^2\pm a^2}\right)^n$	expression inside the bracket = t
$\frac{2x}{a^2-x^2}, \frac{2x}{a^2+x^2}, \frac{a^2-x^2}{a^2+x^2}$	$x = a\tan\theta$
$2x^2 - 1$	$x = \cos\theta$
$\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}} (n\in N,\ n>1)$	$\frac{x+a}{x+b} = t$

Illustration 5 :

Evaluate: $\int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}}$.

Solution:

$$I = \int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}} = \int \frac{dx}{\left(\frac{x+3}{x-4}\right)^{15/16}(x-4)^2}$$

Put $\frac{x+3}{x-4} = t \Rightarrow \left(\frac{(x-4)-(x+3)}{(x-4)^2}\right)dx = dt \Rightarrow \frac{dx}{(x-4)^2} = \frac{dt}{-7}$

So $I = \frac{-1}{7}\int \frac{dt}{t^{15/16}} = \frac{-1}{7}\int t^{-15/16}dt = \frac{-16}{7}t^{1/16}+c = \frac{-16}{7}\left(\frac{x+3}{x-4}\right)^{1/16}+c$

Illustration 6 :

Evaluate: $\int \frac{dx}{\left(x+\sqrt{x^2-4}\right)^{5/3}}$.

Solution:

$I = \int \frac{dx}{\left(x+\sqrt{x^2-4}\right)^{5/3}}$ Put $x+\sqrt{x^2-4} = t$

$\Rightarrow \left(1+\frac{x}{\sqrt{x^2-4}}\right)dx = dt \Rightarrow x+\sqrt{x^2-4} = t \Rightarrow \sqrt{x^2-4} = t-x$

$\Rightarrow x = \frac{t^2+4}{2t} \Rightarrow x^2-4 = \left(\frac{t^2+4}{2t}\right)^2-4 = \frac{t^4+16+8t^2-16t^2}{4t^2} = \left(\frac{t^2-4}{2t}\right)^2$

so $I = \int\left(\frac{t^2-4}{2t^2}\right)\frac{1}{t^{5/3}}dt = \frac{1}{2}\int t^{-5/3}dt - 2\int t^{-11/3}dt$

$= \frac{1}{2}\frac{t^{-2/3}}{-2/3} - 2\frac{t^{-8/3}}{-8/3}+c = \frac{3}{4}t^{-8/3}[1-t^2]+c$

Where $t = \left(x+\sqrt{x^2-4}\right)$

4.3 INDIRECT SUBSTITUTION

If integrand f(x) can be rewritten as product of two functions. $f(x) = f_1(x)\,f_2(x)$, where $f_2(x)$ is a function of integral of $f_1(x)$, then put integral of $f_1(x) = t$.

Illustration 7 :

Evaluate: $\int \sqrt{\frac{x}{4-x^3}}\,dx$.

Solution:

$$I = \int \sqrt{\frac{x}{4-x^3}}\,dx = \int \frac{\sqrt{x}\,dx}{\sqrt{4-x^3}}$$

Here integral of $\sqrt{x} = \frac{2}{3}x^{3/2}$ and $4 - x^3 = 4 - (x^{3/2})^2$

Put $x^{3/2} = t \Rightarrow \sqrt{x}\,dx = \frac{2}{3}dt$

So $I = \frac{2}{3}\int \frac{dt}{\sqrt{4-t^2}}$ $= \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{2}\right) + c$

Illustration 8 :

Evaluate: $\int (\cos x - \sin x)(3 + 4\sin 2x)dx$.

Solution:

$I = \int (\cos x - \sin x)(3 + 4\sin 2x)dx$

Here integration of cos x – sin x = sin x + cos x

and 3 + 4 sin 2x = 3 + 4((sin x + cos x)^2–1)

Put sin x + cos x = 1

= (cos x – sin x)dx = dt

So $I = \int (3 + 4(t^2 - 1)dt$ $= \frac{t}{3}[4t^2 - 3] + c$

$= \left(\frac{\sin x + \cos x}{3}\right)[4(\sin x + \cos x)^2 - 3]$ $= \left(\frac{\sin x + \cos x}{3}\right)(1 + 4\sin 2x) + c$

Practice Problems # 01

Integrate the following functions with respect to x :

1. $\sqrt{x} + \frac{1}{x}$

2. $\frac{1}{x\,\ell n\,x}$

3. $\frac{\sin x - \cos x}{\sin x + \cos x}$

4. $\frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2} + 3}$

5. $\sec^2(4x - 7)$

6. $\frac{\cos(\tan^{-1}x)}{(1+x^2)\sqrt{\sin(\tan^{-1}x)}}$

5. Integration by Parts :

If integrand can be expressed as product of two functions, then we use the following formula.
$\int f_1(x)f_2(x)dx = f_1(x)\int f_2(x) - \int f_1'(x)\int f_2(x)dx)dx$, where $f_1(x)$ and $f_2(x)$ are known as first and second function respectively.

Remarks :

(i) We do not put constant of integration in 1st integral, we put this only once in the end.

(ii) Order of $f_1(x)$ and $f_2(x)$ is normally decided by the rule ILATE, where I $\rightarrow$ Inverse, L $\rightarrow$ Logarithms, A $\rightarrow$ Algebraic, T $\rightarrow$ Trigonometric and E $\rightarrow$ Exponential.

***Illustration* 9 :**

Evaluate: $\int x^2 \sin x\, dx$.

***Solution*:**

$$\int x^2 \sin x\, dx$$

$$= x^2 \int \sin x\, dx - \int \left(2x \int \sin x\, dx\right) dx$$

$$= -x^2 \cos x + 2[x \int \cos x\, dx - \int (1 \int \cos x\, dx) dx$$

$= -x^2 \cos x + 2x \sin x - 2 \cos x + c$

***Illustration* 10 :**

Evaluate: $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$.

***Solution*:**

$$I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$$

Here $\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) = \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+9}}\right)$

Put $2x + 2 = 3\tan\theta$

$$dx = \frac{3}{2}\sec^2\theta\, d\theta$$

Also $\frac{2x+2}{\sqrt{(2x+2)^2+9}} = \frac{3\tan\theta}{3\sec\theta} = \sin\theta$

So $I = \frac{3}{2}\int \theta \sec^2\theta\, d\theta$

$$= \frac{3}{2}\left[\theta \int \sec^2\theta - \int \left(1 \int \sec^2\theta\, d\theta\right) d\theta\right] = \frac{3}{2}[\theta \tan\theta + \ell n(\cos\theta)] + c$$

$$= \frac{3}{2}\left[\frac{2x+3}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) + \ell n\left(\frac{3}{\sqrt{4x^2+8x+13}}\right)\right] + c$$

5.1 SPECIAL USE OF INTEGRATION BY PARTS

(i) $\int f(x) dx = \int (f(x)).1\, dx$

Now integrate taking f(x) as 1st function and 1 as 2nd function.

(ii) $\int \frac{f(x)}{g(x)^n} dx = \int \frac{f(x)}{g'(x)} \cdot \frac{g'(x)}{g(x_1)^n} dx$

Now integrate taking $\frac{f(x)}{g'(x)}$ as 1st function and $\frac{g'(x)}{g(x_1)^n}$ as 2nd function.

(iii) If integrand is of the form e^x f(x), then rewrite f(x) as sum of two functions in which one is derivative of other.

$$\int e^x f(x)dx = \int e^x(g(x)+g'(x))dx = e^x g(x) + c$$

Illustration 11 :

Evaluate: $\int \ell n\, x\, dx$.

Solution:

$$I = \int \ell n\, x\, dx = \int (\ell n\, x.1)\, dx = \ell n\, x.x - \int \frac{1}{x}.x\, dx$$

$$= x\,\ell n\, x - x + c = x(\ell n\, x - 1) + c$$

Illustration 12 :

Evaluate: $\int \frac{x^2}{(x\sin x+\cos x)^2}$.

Solution:

$$I = \int \frac{x^2}{(x\sin x+\cos x)^2} = \int x.\sec x\left(\frac{x\cos x}{(x\sin x+\cos x)^2}\right)dx = \frac{-x\sec x}{x\sin x+\cos x}+\tan x+c$$

Illustration 13 :

Evaluate: $\int \left(\frac{x-1}{x^2+1}\right)^2 e^x\, dx$.

Solution:

$$I = \left(\frac{x-1}{x^2+1}\right)^2 = \frac{x^2-2x+1}{(x^2+1)^2}$$

$$= \frac{1}{(x^2+1)}+\left(\frac{-2x}{(x^2+1)}\right)$$

Here derivative of $\frac{1}{x^2+1}$ is $\frac{-2x}{(x^2+1)^2}$

So $\int e^x\left(\frac{x-1}{x^2+1}\right)^2 dx = \frac{e^x}{(x^2+1)}+c$

Practice Problems # 02

Integrate the following functions with respect to x :

1. x e^x.

2. $x\,\ell n\sqrt{x}$.

3. $\tan^{-1}$ x.

4. $\ell n\,(x^2 + 1)$

5. x $\sin^{-1}$ x

6. Integration by Partial Fraction :

When integrand is a rational function i.e. of the form $\frac{f(x)}{g(x)}$, where f(x) and g(x) are the polynomials functions of x, we use the method of partial fraction.

For example we can rewrite $\frac{1}{(3x-1)(3x+2)}$ and $\frac{1}{3(3x-1)}-\frac{1}{3(3x+2)}$.

If degree of f(x) is less then degree of g(x) and g(x) = $(x-a_1)^{\alpha_1}.........(x^2+b_1x+c_1)^{\beta_1}.........,$

then we can put $\frac{f(x)}{g(x)} = \frac{A_1}{(x-a_1)}+\frac{A_2}{(x-a_1)^2}+........+\frac{A_{\alpha_1}}{(x-a_1)^{\alpha_1}}........$

$$+\frac{B_1x+C_1}{(x^2+b_1x+c_1)}+\frac{B_2x+C_2}{(x^2+b_1x+c_1)^2}+........+\frac{B_{\beta_1}+C_{\beta_1}}{(x^2+b_1x+c_1)^{\beta_1}}.........$$

Here A_1, A_2,........, A_{α_1} ...., B_1, B_2........... B_{β_1} ....C_1, C_2........ C_{β_1} ....are the real constants and these can be calculated by reducing both sides of the above equation as identity in polynomial form and then by comparing the coefficients of like powers. The constants can also be obtained by putting some suitable numerical values of x in both sides of the identity.
If degree of f(x) is more than or equal to degree of g(x), then divide f(x) by g(x) so that the remainder has degree less than of g(x).

Illustration 14 :

Evaluate: $\int\frac{dx}{(x-1)(x-2)(x-3)}$.

Solution:

Put $\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$\Rightarrow$ 1 = A (x – 2) (x – 3) + B (x – 1) (x – 3) + C (x – 1) (x – 2)

Put x = 1, we get, A = $\frac{1}{2}$ x = 2, we get, B = – 1 x = 3, we get, C = $\frac{1}{2}$

So integral = $\frac{1}{2}\int\frac{dx}{x-1}-\int\frac{dx}{x-2}+\frac{1}{2}\int\frac{dx}{x-3}$ $= \ell n\left(\frac{\sqrt{x^2-4x+3}}{|x-2|}\right)+c$

Illustration 15 :

Evaluate: $\int\frac{dx}{(x+2)(x^2+1)}$.

Solution:

Let $\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2}+\frac{Bx+C}{(x^2+1)}$ $\Rightarrow$ 1 = A(x² + 1) + (Bx + C) (x + 2)

Put x = – 2, we get A = $\frac{1}{5}$

Now compare the coefficients of x² and constant term we get 0 = A + B and 1 = A + 2C

$\Rightarrow$ $B=\frac{1}{5}, C=\frac{2}{5}$

So $\quad I = \frac{1}{5}\int\frac{dx}{x+2} - \frac{1}{5}\int\frac{x}{x^2+1}dx + \frac{2}{5}\int\frac{dx}{x^2-1}$

$= \frac{1}{5}\ell n|x+2| - \frac{1}{10}\ell n(x^2+1) + \frac{2}{5}\tan^{-1}x + C$

***Illustration* 16 :**

Evaluate: $\int\frac{x^4dx}{(x-1)(x+1)^2}$.

Solution:

Here degree of numerator is more than the degree of denominator so first we have to divide it to reduce it to proper fraction.

$$\frac{x^4}{(x-1)(x+1)^2} = (x-1) + \frac{2x^2-1}{(x-1)(x+1)^2}$$

Put $\quad \frac{2x^2-1}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$

$\Rightarrow \quad 2x^2 - 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

Put $\quad x = 1$, we get $A = \frac{1}{2}$ $\qquad$ Put $\quad x = -1$, we get $C = -\frac{1}{2}$

Comparing the coefficient of x^2, we get $\qquad 2 = A + B \Rightarrow B = \frac{3}{2}$

So $\quad I = \int(x-1)dx + \frac{1}{2}\int\frac{dx}{(x-1)} + \frac{3}{2}\int\frac{dx}{(x+1)} - \frac{1}{2}\int\frac{dx}{(x+2)^2}$

$= \frac{x^2}{2} - x + \frac{1}{2}\ell n|x-1| + \frac{3}{2}\ell n|x+1| + \frac{1}{2(x+2)} + C$

Practice Problems # 03

Integrate the following functions with respect to x :

1. $\frac{1}{(x+1)(x+2)}$

2. $\frac{1}{x^2-9}$

3. $\frac{2}{(1-x)(1+x^2)}$.

4. $\frac{x}{(x-2)(x+5)}$.

5. $\frac{2x^2}{(x^2+1)^2}$.

6. $\frac{2\tan x\sec^2 x}{\tan^2 x + 3\tan x + 2}$.

7. Algebraic Integration :

Using the technique of standard substitution and integration by parts, we can derive the following formula :

(i) $\dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + c$

(ii) $\dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\ell n\dfrac{x-a}{x+a} + c$

(iii) $\dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a} + c$

(iv) $\dfrac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right] + c$

(v) $\dfrac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right] + c$

(vi) $\sqrt{a^2-x^2}\,dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + c$

(vii) $\sqrt{x^2+a^2}\,dx = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\ell n\; x+\sqrt{x^2-a^2}$

(viii) $\sqrt{x^2-a^2}\,dx = \dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\ell n\; x+\sqrt{x^2-a^2}$

7.1 INTEGRAL OF THE FORM

$$\int\frac{dx}{ax^2+bx+c},\quad \int\frac{dx}{\sqrt{ax^2+bx+c}},\quad \int\sqrt{ax^2+bx+c}\;dx$$

Here in each case write $ax^2 + bx + c = a\left(x+\dfrac{b}{2a}\right)^2 + \dfrac{4ac-b^2}{4a}$ put $x+\dfrac{b}{2a} = t$ and use the standard formulae.

Illustration 17 :

Evaluate: $\displaystyle\int\frac{dx}{\sqrt{-x^2+4x+6}}$.

Solution:

$-x^2 + 4x + 6 = -(x^2 - 4x + 4) + 10 = 10 - (x-2)^2$

$I = \displaystyle\int\frac{dx}{\sqrt{10-(x-2)^2}}$ Put $x - 2 = t \Rightarrow dx = dt$

$I = \displaystyle\int\frac{dt}{\sqrt{10-t^2}} = \sin^{-1}\frac{t}{\sqrt{10}} + c = \sin^{-1}\left(\frac{x-2}{\sqrt{10}}\right) + c$

Illustration 18 :

Evaluate: $\displaystyle\int\sqrt{3x^2-6x+10}\,dx$.

Solution:

$3x^2 - 6x + 10 = 3(x-1)^2 + 7$

Put $x - 1 = t \Rightarrow dx = dt$

I = $\sqrt{3}\int\sqrt{t^2+\frac{7}{3}}\,dt$ = $\sqrt{3}\left[\frac{t}{2}\sqrt{t^2+\frac{7}{3}}+\frac{7}{6}\ell n\left|t+\sqrt{t^2+\frac{7}{3}}\right|\right]+c$

where t = x – 1

7.2 INTEGRALS OF THE FORM

$$\int\frac{(ax+b)dx}{\sqrt{cx^2+ex+f}},\ \int\frac{(ax+b)dx}{cx^2+ex+f},\ \int(ax+b)\sqrt{cx^2+ex+f}\,dx$$

Here write ax + b = A(2cx + e) + B

Find A and B by comparing, the coefficients of x and constant term.

***Illustration* 19 :**

Evaluate: $\int\frac{(3x+5)dx}{\sqrt{x^2+4x+3}}$.

***Solution*:**

Write 3x + 5 = A(2x + 4) + B

$\Rightarrow$ A = $\frac{3}{2}$, B = – 1 So I = $\frac{3}{2}\int\frac{2x+4}{\sqrt{x^2+4x+3}}-\int\frac{dx}{\sqrt{x^2+4x+3}}$

In 1st integral put $x^2 + 4x + 3 = t$ $\Rightarrow$ (2x + 4)dx = dt

$$I = \frac{3}{2}\int\frac{dt}{\sqrt{t}}-\int\frac{dx}{\sqrt{(x+2)^2-1}}$$

$$= 3\sqrt{x^2+4x+3}-\ell n\left|(x+2)+\sqrt{x^2+4x+3}\right|+c$$

7.3 INTEGRALS OF THE FORM

$$\int\frac{(ax^2+bx+c)dx}{\sqrt{(ex^2+fx+g)}},\ \int\frac{(ax^2+bx+c)dx}{(ex^2+fx+g)},\ \int(ax^2+bx+c)\sqrt{(ex^2+fx+g)}\,dx$$

Here put $ax^2 + bx + c = A(ex^2 - fx + g) + B(2ex + f) + c$ find the values of A, B and C by comparing the coefficients of x^2, x and constant term.

***Illustration* 20 :**

Evaluate: $\int\frac{(x^2+4x+7)}{\sqrt{x^2+x+1}}$.

***Solution*:**

Let $x^2 + 4x + 7 = A(x^2 + x + 1) + B(2x + 1) + C$

Comparing the coefficients of x^2, x and constant term, we get

A = 1, A + 2B = 4, A + B + C = 7

$\Rightarrow$ A = 1, B = $\frac{3}{2}$, C = $\frac{9}{2}$

=

So $\quad I = \int \sqrt{x^2+x+1}\,dx + \frac{3}{2}\int \frac{(2x+1)dx}{\sqrt{x^2+x+1}} + \frac{9}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$

Now $\quad x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

$$I = \left(\frac{x+\frac{1}{2}}{2}\right) + \sqrt{x^2+x+1} + \frac{3}{8}\ell n\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right) + 3\sqrt{x^2+x+1} + \frac{9}{2}\ell n\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right) + c$$

7.4 INTEGRALS OF THE FORM

$\int \frac{dx}{(ax+b)\sqrt{ex^2+fx+g}}$. Here $ax+b=\frac{1}{t}$.

Illustration 21 :

Evaluate: $\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$.

Solution:

Put $x+2=\frac{1}{t}$ $\quad\Rightarrow\quad dx = \frac{-dt}{t^2}$

Now $\quad x^2 + 4x + 8 = (x + 2)^2 + 4$

So $\quad I = \int \frac{-dt}{t\sqrt{\frac{1}{t^2}+4}} = -\int \frac{dt}{\sqrt{1+4t^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t^2+\frac{1}{4}}} = -\frac{1}{2}\ell n\left|t+\sqrt{t^2+\frac{1}{4}}\right| + c$

$\quad = -\frac{1}{2}\ell n\left|\frac{1}{x+2}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right| + c$

7.5 INTEGRALS OF THE FORM

$\int \frac{(ax+b)dx}{(cx+e)\sqrt{ex^2+fx+g}}$. Here put (ax + b) = A(cx + e) + B, find the values of A and B by comparing the coefficients of x and constant term.

Illustration 22 :

Evaluate: $\int \frac{(4x+7)}{(x+2)\sqrt{x^2+4x+8}}$.

Solution:

Let $\quad 4x + 7 = A(x + 2) + B \quad\Rightarrow\quad A = 4, B = -1$

So $\quad I = 4\int \frac{dx}{\sqrt{x^2+4x+8}} - \int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

$\quad = 4\,\ell n\left(x+2+\sqrt{x^2+4x+8}\right) + \frac{1}{2}\ell n\left|\frac{1}{x+2}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right| + c$

7.6 INTEGRALS OF THE FORM

$\int \frac{(ax^2+bx+c)dx}{(ex+f)\sqrt{gx^2+hx+i}}$. Here put ax² + bx + c = A(ex + f) (2gx + h) + B(ex + 1) + C, find the values of A, B and C by comparing the coefficients of x^2, x and constant term.

Illustration 23 :

Evaluate: $\int \frac{2x^2+7x+11}{(x+2)\sqrt{x^2+4x+8}}$.

Solution:

Put $2x^2 + 7x + 11 = A(x + 2)(2x + 4) + B(x + 2) + C$

Compare the coefficient of x^2, x and constant term, we get

A = 1, 7 = 8 A + B, C + 2B + 8A = 11 $\Rightarrow$ B = – 1, C = 5

So I = $\int \frac{2x+4}{\sqrt{x^2+4x+8}} - \int \frac{dx}{\sqrt{x^2+4x+8}} + 5\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

$$= 2\sqrt{x^2+4x+8} - \ell n\left|(x+2)+\sqrt{x^2+4x+8}\right| - \frac{5}{2}\ell n\left|\frac{1}{(x+2)} + \sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right| + c$$

7.7 INTEGRALS OF THE FORM

$\int \frac{x\,dx}{(ax^2+b)\sqrt{(cx^2+e)}}$, here put $cx^2 + e = t^2$.

Illustration 24 :

Evaluate: $\int \frac{x\,dx}{(2x^2+3)\sqrt{x^2-1}}$.

Solution:

Put $x^2 - 1 = t^2$ $\Rightarrow$ x dx = t dt

So I = $\int \frac{t\,dt}{(2t^2+5)t} = \int \frac{dt}{2t^2+5} = \frac{1}{2}\int \frac{dt}{t^2+\frac{5}{2}} = \frac{1}{10}\tan^{-1}\left(\sqrt{\frac{2}{5}}\sqrt{x^2-1}\right) + c$

7.8 INTEGRALS OF THE FORM

$\int \frac{dx}{(ax^2+b)\sqrt{(cx^2+e)}}$. Here 1st put x = $\frac{1}{t}$ and then the expression inside the square root as y^2.

Illustration 25 :

Evaluate: $\int \frac{dx}{(x^2+5)\sqrt{2x^2-3}}$.

Solution:

Put x = $\frac{1}{t}$ $\Rightarrow$ dx = $-\frac{dt}{t^2}$

So $I = \int \frac{-dt}{t^2\left(\frac{1}{t^2}+5\right)\sqrt{\frac{2}{t^2}-3}}$ $= \int \frac{-t\,dt}{(1+5t^2)\sqrt{2-3t^2}}$

Put $2 - 3t^2 = y^2$ $\Rightarrow$ $-t\,dt = \frac{y\,dy}{3}$

So $I = -\frac{1}{3}\int \frac{y\,dy}{\left(\frac{13-5y^2}{3}\right)y}$ $= \frac{1}{5}\ell n\left|\frac{y-\sqrt{13/5}}{y+\sqrt{13/5}}\right| + C$

7.9 INTEGRALS OF THE TYPE

$\int x^m (a+bx^n)^p dx (p \neq 0)$. Here we have the following cases.

Case I : If p is a natural number, then expand $(a + bx^n)^p$ by binomial theorem and integrate.

Case II : If p is a negative integer and m and n are rational number, put $x = t^k$, when k is the LCM of denominators of m and n.

Case III : If $\frac{m+1}{n}$ is an integer and p is rational number, put $(a + bx^n) = t^k$, when k is the denominator of p.

Case IV : If $\frac{m+1}{n}$ is an integer, put $\frac{a+bx^n}{x^n} = t^k$, where k is the denominator of p.

Illustration 26 :

Evaluate: $\int x^{-\frac{2}{3}}\left(1+x^{\frac{2}{3}}\right)^{-1}$.

Solution:

Here p = – 1, is a negative integer and m and n are rational numbers.

Put $x = t^3$ $\Rightarrow$ dx $3t^2$ dt

So $I = \int t^{-2}(1+t^2)^{-1}3t^2dt = \int \frac{3\,dt}{1+t^2}$ $= 3\tan^{-1}(x^{1/3}) + c$

Illustration 27 :

Evaluate: $\int x^{\frac{-1}{3}}\left(1+x^{\frac{1}{3}}\right)^{1/4} dx$.

Solution:

Here $m = -\frac{1}{3}$, $n = \frac{1}{3}$, $p = \frac{1}{4}$ $\frac{m+1}{n} = 2$, which is an integer

So $\left(1+x^{1/3}\right) = t^4$ $\Rightarrow$ $\frac{dx}{3x^{2/3}} = 4t^3dt$

$I = 12\int (t^4-1)t^4dt$ $= -\frac{4}{15}\left(1+x^{1/3}\right)^{5/4}[4+9x^{1/3}] + c$

Illustration 28 :

Evaluate: $\int x^{-11}(1+x^4)^{-1/2}dx$.

Solution:

Here m = – 11, m = 4, p = $-\frac{1}{2}$

$\frac{m+1}{n}+p=-\frac{10}{4}-\frac{1}{2}=-3$, which is an integer.

So put $\frac{1+x^4}{x^4}=t^2$ $\Rightarrow$ $1+\frac{1}{x^4}=t^2$ $\Rightarrow$ $\frac{-4}{x^5}dx=2t\,dt$

So I = $\int\frac{dx}{x^{13}\left(1+\frac{1}{x^4}\right)^{1/2}}$ = $-\frac{1}{4}\int(t^2-1)^2.\frac{1}{t}.2t\,dt$

= $-\frac{1}{2}\int(t^4-2t^2+1)dt$ = $\frac{t^5}{-10}+\frac{t^3}{3}-\frac{t}{2}+c$ Where t = $\sqrt{1+\frac{1}{x^4}}$.

Practice Problems # 04

Integrate the following functions with respect to x :

1. $\sqrt{1-4x^2}$

2. $\frac{1}{\sqrt{(2-x)^2+1}}$

3. $\frac{1}{x^5(1+x^4)}$.

4. $\frac{1}{\sqrt{x^2+4x+5}}$.

5. $\frac{2x-3}{\sqrt{x^2+x+1}}$.

6. $\frac{1}{\sqrt{4+x^2}}$

7. $\frac{1}{x\sqrt{x^2+x+1}}$.

8. Trigonometric Integrals :

8.1 INTEGRALS OF THE FORM :

$\int\left(\frac{f(\sin x,\cos x)}{g(\sin x,\cos x)}\right)dx=\int R(\sin x,\cos x)dx$, where f and g both are polynomials in sin x and cos x. Here we can convert them in algebraic by putting $\tan\frac{x}{2}=t$ after writing ,

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \quad \text{and} \quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$

Some time instead of putting the above substitution we go for below procedure.

(i) If R(– sin x, cos x) = – R(sin x, cos x), put cos x = t

(ii) If R(sin x, – cos x) = R(sin x, cos x) put tan x = t

(iii) If R(–sin x, cos x) = R(sin x, cos x) put tan x = t

Illustration 29 :

Evaluate: $\int \frac{dx}{\sin x(2\cos^2 x-1)}$.

Solution:

Here R(sin x, cos x) = $\frac{1}{\sin x(2\cos^2 x-1)}$

R(sin x, cos x) = $\frac{1}{-\sin x(2\cos^2 x-1)}$ = R – (sin x, cos x)

So we put cos = t $\Rightarrow$ – sin x dx = dt

$$I = \int \frac{\sin dx}{(1-\cos^2 x)(2\cos^2 x-1)} = \int \frac{dt}{(t^2-1)(2t^2-1)} = \int \frac{dt}{t^2-1} - 2\int \frac{dt}{2t^2-1}$$

$$= \frac{1}{2}\ell n\left|\frac{\cos x-1}{\cos x+1}\right| - \frac{1}{\sqrt{2}}\ell n\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right| + C$$

Illustration 30 :

Evaluate: $\int \frac{\cos x\, dx}{\sin^2 x(\sin x+\cos x)}$.

Solution:

Here R(sin x, cos x) = $\frac{\cos x\, dx}{\sin^2 x(\sin x+\cos x)}$

R(– sin x, – cos x) = R(sin x , cos x)

So put tan x = t $\Rightarrow$ $\sec^2 x\, dx = dt$

$$I = \int \frac{\cos x \sec^2 x\, dx}{\sec^2 x\ \sin^2 x(\sin x+\cos x)} = \int \frac{dt}{t^2(1+t)}$$

Let $\frac{1}{t^2(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{(1+t)}$ or $1 = At(1+t) + B(1+t) + ct^2$

Put t = 0, we get B = 1, put t = –1, we get C = 1

compare the coefficients of t^2, we get 0 = A + C $\Rightarrow$ A = – 1

So $I = -\int \frac{dt}{t} + \int \frac{dt}{t^2} + \int \frac{dt}{1+t}$ $= \ell n\left|\frac{1+\tan x}{\tan x}\right| - \cot x + c$

8.2 INTEGRALS OF THE FORM :

$$\int\left(\frac{p\sin x+q\cos x+r}{a\sin x+b\cos x+c}\right)dx,$$

here put p sin x + q cos x + r = A(a sin x + b cos x + c) + B(a cos x – b sin x) + C values of A, B and C can be obtained by comparing the coefficients of sin x, cos x and constant term by this technique. The given integral becomes sum of 3 integrals in which 1st two are very easy in 3rd we can put $\tan\frac{x}{2}=t$.

Illustration 31 :

Evaluate: $\int\frac{(5\sin x+6)dx}{\sin x+2\cos x+3}$.

Solution:

Let 5 sin x + 6 = A(sin x + 2 cos x + 3) + B(cos x – 2 sin x) + C

Equating the coefficients of sin x, cos x and constant term, we get

$$\left.\begin{aligned}A-2B&=5\\2A+B&=0\\3A+C&=6\end{aligned}\right\}\Rightarrow A=1,\ B=-2,\ C=3$$

$$I=\int dx-2\int\frac{(\cos x-2\sin x)dx}{\sin x+2\cos x+3}+3\int\frac{dx}{\sin x+\cos x+3}\ x-2\ell n|\sin x+2\cos x+3|+3\ell_1$$

Put $\tan\frac{x}{2}=t\Rightarrow\sec^2\frac{x}{2}dx=2dt$

So $\ell_1=\int\frac{2dt}{t^2+2t+5}=\int\frac{2dt}{(t+1)^2+4}=\tan^{-1}\left(\frac{t+1}{2}\right)+C=\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{2}\right)+C$

8.3 INTEGRALS OF THE FORM :

$\int\sin^p x\cos^q x\,dc$, Where p and q are rational number such that $\frac{p+q-2}{2}$ is a negative integer, then put tan x = t or cot x = t.

Illustration 32 :

Evaluate: $\int\sin^{-7/5}x\cos^{-3/5}dx$.

Solution:

Here $p=-\frac{7}{5}$, $q=-\frac{3}{5}$ $\qquad\frac{p+q-2}{2}=-2$

$$I=\int\sin^{-7/5}\cos^{-3/5}x\,dx=\int\frac{\cos^{-3/5}x}{\sin^{-3/5}x\sin^2x}dx$$

$$=\int(\cot x)^{-3/5}\operatorname{cosec}^2x\,dx$$

Put cot x = t ⇒ cosec² x = – dt

So $I=-\int t^{-3/5}dt=-\frac{5}{2}(\cot x)^{2/5}+c$

9. Successive Reduction in Integration :

Sometimes the integrand is a function of x as well as on $n(n \in N)$, then we use this method.

***Illustration* 33:**

If $I_n = \int \tan^n x\, dx$, then prove that $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

***Solution*:**

Here $I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx$

$= \int \tan^{n-2} x (\sec^2 x - 1)\, dx$

$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x\, dx$

$= \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$

$$I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

Hence $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

Practice Problems # 05

Integrate the following functions with respect to x :

1. $\frac{1}{1+2\cos x}$.

2. $\frac{\cos^2 x}{\sin x}$.

3. $\frac{\sin x}{\sin x + \cos x}$.

4. $\frac{1}{12+12\cos\theta}$

5. $\frac{1}{\cos(x+a)\cos(x+b)}$

6. $\frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}$

***Illustration* 34:**

Prove that $\frac{dx}{1+x^2\ \ n} = \frac{1}{2\ \ -1}\ \frac{}{1+x^2\ \ n\ 1} + 2n-3\ \ \frac{dx}{1+x^2\ \ n\ 1}$, n N. Hence, compute the value of $\int \frac{1}{(1+x^2)^2} dx$.

***Solution*:**

If $I_n = \int \frac{dx}{(1+x^2)^n} = \frac{1}{(1+x^2)^n}.x - \int \frac{-n}{(1+x^2)^{n+1}}.2x.x\ dx$

$= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2+1-1}{(1+x^2)^{n+1}} dx = \frac{x}{(1+x^2)^n} + 2n\ [I_n - I_{n+1}]$

$= 2n\ I_{n+1} = (2n-1)\ I_n + \frac{x}{(1+x^2)^n}$

replace $n \to n-1$

$2(n-1)\ I_n = (2n-3)\ I_{n-1} + \frac{x}{(1+x^2)^{n-1}}$

$\Rightarrow In = \frac{1}{2\ n-1}\ \frac{x}{1+x^2\ ^{n\ 1}} + 2n-3\ \frac{dx}{1+x^2\ ^{n\ 1}}$, n N

Now, put n = 2

$\frac{dx}{1+x^2\ ^{2}} = \frac{1}{2}\left[\frac{x}{1+x^2} + \tan^{-1} x\right] + c$

Practice Problems # 06

1. $I_n = \frac{x^n dx}{\sqrt{x^2+a}}$ $\frac{n-1}{n} a.I_{n-2} = \frac{1}{n}.x^{n-1}.\sqrt{x^2+a}$.

2. If $I_n =$ $x.\text{cosec}^n x.dx,\ n \geq 3$. Prove that

$I_n = -\frac{1}{n-1}.\text{cosec}^{n-2}.x\ \ x \cot x + \frac{1}{n-2}\ \ \frac{n-2}{n-1}.I_{n-2}$.

3. If $I_n =$ $\log_e\ x\ _n dx$

4. If $I_n = \int \frac{dx}{(x^2+a^2)^n}$, then prove that $2(n-1)\ a^2\ I_n = \frac{x}{(x^2+a^2)^{n-1}} + (2n-3)_{n-1}$.

ANSWER SHEET

Practice Problems # 01

1. $\frac{2}{3}x^{3/2}+\ell n|x|+C$
2. $\ell n|\ell n\ x|+C$
3. $\ell n\left|\frac{1}{\sin x+\cos x}\right|+C$
4. $\tan^{-1}\left(x+\frac{1}{x}\right)+C$
5. $\frac{1}{4}\tan(4x-7)+C$
6. $2\sqrt{\sin(\tan^{-1}x)}+C$

Practice Problems # 02

1. $e^x(x-1)+C$
2. $\frac{x^2}{4}(2\ell n\ x-1)+C$
3. $x\tan^{-1}x-\frac{1}{2}\ell n(1+x^2)+C$
4. $x\log(x^2+1)-2x+2\tan^{-1}x+C$
5. $\frac{1}{4}\sin^{-1}(x)(2x-1)+\frac{x\sqrt{1-x^2}}{4}+C$

Practice Problems # 03

1. $\log\left|\frac{x+1}{x+2}\right|+C$
2. $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right|+C$
3. $\frac{1}{2}\ell n(1+x^2)-\ell n|1-x|+\tan^{-1}x+C$
4. $\frac{2}{7}\ell n|x-2|+\frac{5}{7}\ell n|x+5|+C$
5. $\frac{-x}{x^2+1}+\tan^{-1}x+C$
6. $\ell n\left(\frac{\tan x+2}{(\tan x+1)^2}\right)^4+C$

Practice Problems # 04

1. $\frac{1}{4}\sin^{-1}2x+\frac{1}{2}x\sqrt{1-4x^2}+C$
2. $\ell n\left|\frac{1}{2-x+\sqrt{x^2-4x+5}}\right|+C$
3. $\frac{1}{4}\left[\ell n\left|1+\frac{1}{x^4}\right|-\left(1+\frac{1}{x^4}\right)\right]+C$
4. $\ell n\left|(x+2)+\sqrt{x^2+4x+5}\right|+C$
5. $2\sqrt{x^2+x+1}-4\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$
6. $\ell n\left|x+\sqrt{x^2+4}\right|+C$
7. $-\ell n\left|\frac{1}{x}+\frac{1}{2}+\sqrt{\frac{1}{x^2}+\frac{1}{x}+1}\right|+C$

Practice Problems # 05

1. $\frac{1}{\sqrt{3}}\ell n\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|+C$
2. $\cos x+\frac{1}{2}\ell n\left|\frac{\cos x-1}{\cos x+1}\right|+C$
3. $\frac{1}{2}(1-\ell n|\sin x+\cos x|)+C$
4. $\frac{1}{12}\tan\frac{\theta}{2}+C$
5. $\frac{1}{\sin(a-b)}\ell n\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$
6. $-\frac{1}{2}\sin 2x+C$

Solved Examples (Subjective)

Example 1 :

Evaluate : $I=\int x\ln\left(1+\frac{1}{x}\right)dx$.

Solution :

$$\ln\left(1+\frac{1}{x}\right)=\ln\frac{x+1}{x}=\ln(x+1)-\ln x$$

$$\therefore \quad I=\int x\ln(x+1)\,dx-\int x\ln x\,dx=I_1-I_2$$

Let us integrate I_1 and I_2 by parts. Put $I_1=\int x\ln(x+1)dx=\ln(x+1)\frac{x^2}{2}-\frac{1}{2}\int\frac{x^2-1+1}{x+1}dx$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\int\frac{1}{1+x}dx-\frac{1}{2}\int(x-1)\,dx \qquad =\frac{x^2}{2}\ln(1+x)-\frac{1}{2}\ln(1+x)-\frac{x^2}{4}+\frac{x}{2}+C$$

$$=\frac{x^2-1}{2}\ln(x+1)-\frac{1}{4}x^2+\frac{1}{2}x+C$$

Similarly $I_2=\int x\ln x\,dx=\frac{x^2}{2}\ln x-\frac{1}{4}x^2+C$

$\therefore \quad I=I_1-I_2 \qquad I=\frac{1}{2}(x^2-1)\ln(x+1)-\frac{x^2}{2}\ln x+\frac{x}{2}+C$

Example 2 :

Evaluate $\int\frac{1}{1+\sin x+\cos x}dx$

Solution :

Put $\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}$ and $\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, we have

$$I=\int\frac{1}{1+\sin x+\cos x}dx=\int\frac{dx}{1+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$=\int\frac{1+\tan^2 x/2}{1+\tan^2 x/2+2\tan x/2+1-\tan^2 x/2}dx$$

$=\frac{1}{2}\int\frac{\sec^2 x/2}{1+\tan x/2}$ Put $1+\tan x/2=t$ then $\sec^2 x/2\,dx=2dt$

$\therefore \quad I=\int\frac{dt}{t}=\log t=\log\left(1+\tan\frac{x}{2}\right)+C_1$

Example 3 :

Integrate $\int e^x \left(\frac{x+2}{x+4}\right)^2$

Solution :

$$I=\int e^x\left(1-\frac{2}{x+4}\right)^2 dx \qquad \therefore \qquad I=\int e^x dx+\int e^x\left\{-\frac{4}{x+4}+\frac{4}{(x+4)^2}\right\}dx$$

$$= e^x-4\int e^x\left\{\frac{1}{x+4}+\frac{-1}{(x+4)^2}\right\}dx \qquad = e^x-4\frac{e^x}{x+4}=\frac{xe^x}{(x+4)}$$

Example 4 :

If $I_{m,n}=\int \cos^m x\sin nx\,dx$ then prove that $I_{m,n}=\frac{m}{m+n}I_{m-1,n-1}-\frac{\cos^m x\cos nx}{m+n}$

Solution :

Let $I_{m,n}=\int \cos^m x\sin nx\,dx$

Apply by parts taking $\cos^m x$ as the first part and sin nx as the second part.

$$\Rightarrow \qquad I_{m,n}=-\frac{\cos^m x\cos nx}{n}-\frac{m}{n}\int \cos^{m-1}x(\sin x\cos nx)\,dx$$

Now sin (n – 1) x = sin nx cos x – cosnx sin x

or cos nx sin x = sin nx cosx – sin (n – 1) x

$$\Rightarrow \qquad I_{m,n}=-\frac{\cos^m x\cos nx}{n}-\frac{m}{n}\int \cos^{m-1}x[\sin nx\cos x-\sin(n-1)x]\,dx$$

$$\Rightarrow \qquad I_{m,n}=-\frac{\cos^m x\cos nx}{n}-\frac{m}{n}\int \cos^m x\sin nx\,dx+\frac{m}{n}\int \cos^{m-1}x\sin(n-1)x\,dx$$

$$\Rightarrow \qquad \left[1+\frac{m}{n}\right]I_{m,n}=-\frac{\cos^m x\cos nx}{n}+\frac{m}{n}I_{m-1,n-1} \qquad \Rightarrow \qquad I_{m,n}=\frac{m}{m+n}I_{m-1,n-1}-\frac{\cos^m x\cos nx}{m+n}$$

Example 5 :

Evaluate $I=\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$

Solution :

Let $I=\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \int \frac{dx}{\sqrt[4]{\left(\frac{x-1}{x+2}\right)^3(x+2)^8}} = \int \frac{dx}{(x+2)^2\left(\frac{x-1}{x+2}\right)^{3/4}}$

Now put $\frac{x-1}{x+2}=t \qquad \therefore \qquad \frac{3}{(x+2)^2}dx=dt \qquad$ or $\qquad \frac{dx}{(x+2)^2}=\frac{dt}{3}$

$$I=\int\frac{dt}{3(t)^{3/4}}=\frac{1}{3}\int(t)^{-3/4}dt \quad =\frac{1}{3}\int\frac{(t)^{1/4}}{(1/4)}+C \quad =\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$$

Example 6 :

Evaluate $\int \frac{1}{\sqrt{(x+1)}-\sqrt[4]{(x+1)}}dx$

Solution :

We put 1 + x = y^4 $\Rightarrow$ dx = $4y^3$ dy

$$\therefore \quad \int \frac{1}{\sqrt{(x+1)}-\sqrt[4]{(x+1)}}dx = \int \frac{4y^3}{y^2-y}dy = \int \frac{4y^2}{y-1}dy$$

$$= 4\int\left[(y+1)+\frac{1}{y-1}\right]dy = 4\left[\frac{y^2}{2}+y+\log(y-1)\right]$$ = $2y^2 + 4y + 4\log(y-1)$, where $x + 1 = y^4$

Example 7 :

Evaluate $\int \sqrt[3]{\tan x}\,dx$

Solution :

Let tanx = $z^{3/2}$, then $\sec^2 x dx = \frac{3}{2} z^{1/2} dz$

$$\therefore \quad I = \int \sqrt[3]{z^{3/2}}\,\frac{3}{2}z^{1/2}\frac{1}{\sec^2 x}dz = \frac{3}{2}\int\frac{z}{1+z^3}dz = \frac{3}{2}\int\frac{zdz}{(1+z)(1-z+z^2)}$$

Let $\frac{z}{(1+z)(1-z+z^2)} = \frac{A}{1+z}+\frac{Bz+C}{z^2-z+1} \Rightarrow z \equiv A(z^2-z+1)+(Bz+C)(1+z)$

put z = – 1. Then –1 = A(1 + 1 + 1) so, $A = -\frac{1}{3}$

Equating coefficients of z^2 on both sides, O = A + C, $\therefore$ C = $\frac{1}{3}$ and B = 1/3

$$\therefore \quad I = \frac{3}{2}\int -\frac{1}{3}.\frac{dz}{1+z}+\frac{3}{2}\int\frac{\frac{1}{3}z+\frac{1}{3}}{z^2-z+1}dz = -\frac{1}{2}\log(1+z)+\frac{1}{2}\int\frac{z+1}{z^2-z+1}dz$$

$$= -\frac{1}{2}\log(1+z)+\frac{1}{4}\int\frac{(2z-1)+3}{z^2-z+1}dz$$

$$= -\frac{1}{2}\log(1+z)+\frac{1}{4}\int\frac{(2z-1)\,dz}{z^2-z+1}+\frac{3}{4}\int\frac{dz}{\left(z-\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{2}\log(1+z)+\frac{1}{4}\log(z^2-z+1)+\frac{3}{4}.\frac{1}{\sqrt{3}}\tan^{-1}\frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c$$

$$= -\frac{1}{2}\log(1+\tan^{2/3}x)+\frac{1}{4}\log(\tan^{4/3}x-\tan^{2/3}x+1)+\frac{\sqrt{3}}{2}\tan^{-1}\frac{2\tan^{2/3}x-1}{\sqrt{3}}+c$$

Example 8 :

Evaluate $I = \int \frac{x^2}{(x^4-1)\sqrt{x^4+1}} dx$

Solution :

$= \frac{1}{4}\int \frac{(x^2+1)^2-(x^2-1)^2}{(x^4-1)\sqrt{x^4+1}}$ $\quad = \frac{1}{4}\left\{\int \frac{(x^2+1)}{(x^2-1)\sqrt{x^4+1}} dx - \int \frac{(x^2-1)}{(x^2+1)\sqrt{x^4+1}} dx\right\}$

$\therefore \quad I = \frac{1}{4}[I_1 - I_2] \quad \ldots (1)$

Now $I_1 = \int \frac{x^2(1+1/x^2)dx}{x^2\left(x-\frac{1}{x}\right)\sqrt{\left(x-\frac{1}{x}\right)^2+2}}$ Putting $x - 1/x = t$, we get $I_1 = \int \frac{dt}{t\sqrt{t^2+2}}$

Now putting $t^2 + 2 = u^2$

$I_1 = \int \frac{u\,du}{(u^2-2)u} = \frac{1}{2\sqrt{2}}\log\frac{u-\sqrt{2}}{u+\sqrt{2}}$ $\quad = \frac{1}{2\sqrt{2}}\log\frac{(u-\sqrt{2})^2}{u^2-2} = \frac{1}{\sqrt{2}}\log\left(\frac{u-\sqrt{2}}{t}\right)$

$= \frac{1}{\sqrt{2}}\log\frac{\sqrt{x^2+\frac{1}{x^2}}-\sqrt{2}}{x-\frac{1}{x}} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{x^4+1}-x\sqrt{2}}{x^2-1}$

$I_2 = \int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}} dx \quad = \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}} dx$

putting $x + 1/x = v$, we have $I_2 = \int \frac{dv}{v\sqrt{v^2-2}}$, $v^2 - 2 = w^2$ gives

$= \int \frac{w\,dw}{(w^2+2)w} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{w}{\sqrt{2}}$ $\quad = \frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{x^2+\frac{1}{x^2}}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{x^4+1}}{x\sqrt{2}}$

$\therefore \quad I = \frac{1}{4\sqrt{2}}\left[\log\left(\frac{\sqrt{x^4+1}-x\sqrt{2}}{x^2-1}\right) - \tan^{-1}\frac{\sqrt{x^4+1}}{x\sqrt{2}}\right]$

Example 9 :

Evaluate $\int \frac{\sqrt{x}}{\sqrt{x}+2}dx$

Solution :

[Here integrand = $\frac{\sqrt{x}}{\sqrt{x}+2} = \frac{x}{\sqrt{x}+2}\cdot\frac{1}{\sqrt{x}} = \left(\frac{\sqrt{x^2}}{\sqrt{x}+2}\right)\frac{1}{\sqrt{x}}$

and $\frac{\sqrt{x^2}}{\sqrt{x}+2}$ is a function of $\sqrt{x}$ and d.c. of $\sqrt{x} = \frac{1}{2\sqrt{x}} = \frac{1}{2}$. Second function. Hence put $z=\sqrt{x}$.

Let $z=\sqrt{x}$, and $dz=\frac{1}{2\sqrt{x}}dx$

Now $\int \frac{\sqrt{x}}{\sqrt{x}+2}dx = \int \frac{x}{\sqrt{x}+2}\cdot\frac{dx}{\sqrt{x}} = \int\left(\frac{z^2}{z+2}\right)2dz = 2\int\frac{z^2}{z+2}dz$

Let y = z + 2, then dy = dz Now $\int\frac{z^2}{z+2}dz = \int\frac{(y-2)^2}{y}dy = \int\frac{y^2-4y+4}{y}dy$

$= \int(y-4+\frac{4}{y})dy = \frac{y^2}{2}-4y+4\log|y|+c \quad = \frac{(z+2)^2}{2}-4(z+2)+4\log|z+2|+c$

$= \frac{z^2+4+4z}{2}-4z-8+4\log|z+2|+c \quad = \frac{x+4+4\sqrt{x}}{2}-4\sqrt{x}-8+4\log|\sqrt{x}+2|+c$

Example 10 :

Evaluate : $\int\frac{1}{x^4+1+5x^2}dx$

Solution :

$I = \frac{1}{2}\int\frac{2}{x^4+1+5x^2}dx = \frac{1}{2}\int\frac{1+x^2}{x^4+1+5x^2}dx - \frac{1}{2}\int\frac{x^2-1}{x^4+1+5x^2}dx$

$= \frac{1}{2}\int\frac{1+(1/x^2)}{x^2+(1/x^2)+5}dx - \frac{1}{2}\int\frac{1-(1/x^2)}{x^2+(1/x^2)+5}dx = (I_1-I_2)/2$

For I_1 = we write $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$\Rightarrow \quad I_1 = \int\frac{dt}{t^2+(\sqrt{7})^2} = \frac{1}{\sqrt{7}}\tan^{-1}\frac{t}{\sqrt{7}} = \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right)$ For I_2, we write $x + \frac{1}{x} = t$

$\Rightarrow \quad \left(1-\frac{1}{x^2}\right)dx = dt \qquad I_2 = \frac{dt}{t^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right)$

Combining the two results, we get $I = (I_1 - I_2)/2$

$= \frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right) - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right) + c$

Example 11 :

Evaluate: $\int x^2 e^{3x}\,dx$

Solution :

$$\int x^2 e^{3x}\,dx = \frac{x^2 e^{3x}}{3} - \int\left(2x\frac{e^{3x}}{3}\right)dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,dx$$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[x\frac{1}{3}e^{3x} - \int\frac{1}{3}e^{3x}\,dx\right] = \frac{x^2}{3}e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x}$$

$$= e^{3x}\left(\frac{x^2}{3} - \frac{2}{9}x + \frac{2}{27}\right) + c$$

Example 12 :

Evaluate: $\int e^x\left[\frac{2+\sin 2x}{1+\cos 2x}\right]dx$

Solution :

$$I = e^x\left[\frac{2+\sin 2x}{1+\cos 2x}\right]dx \Rightarrow I = \int e^x\left[\frac{2}{1+\cos 2x} + \frac{\sin 2x}{1+\cos 2x}\right]dx$$

$$\Rightarrow I = e^x\left[\frac{2}{2\cos^2 x} + \frac{2\sin x\cos x}{2\cos^2 x}\right]dx = \int e^x\left[\sec^2 x + \tan x\right]dx$$

$$\Rightarrow I = \int e^x[\tan x + \sec^2 x]\,dx = e^x\tan x + c$$

Example 13 :

Evaluate : $\int \frac{dx}{x^3+1}$

Solution :

Let $f(x) = \frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$

$$\Rightarrow f(x) = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$\Rightarrow$ $1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$

Comparing the coefficients of x^2, x, and constants

$0 = A + B,\ 0 = -A + B + C,\ 1 = A + C$

$\Rightarrow$ $A = 1/3,\ B = -1/3\ \&\ C = 2/3$ $\Rightarrow$ $f(x) = \frac{\frac{1}{3}}{x+1} + \frac{-\frac{x}{3}+\frac{2}{3}}{x^2-x+1}$

Let $I_1 = \frac{1}{3}\int\frac{dx}{x+1} = \frac{1}{3}\log|x+1| + C_1$ Let $I_2 = \int\frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}dx = \frac{1}{3}\int\frac{2-x}{x^2-x+1}dx$

Express the numerator in terms of derivative of denominator.

$\Rightarrow \quad I_2 = -\frac{1}{6}\int \frac{2x-4}{x^2-x+1}dx \quad \Rightarrow \quad I_2 = -\frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$

$\Rightarrow \quad I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\int \frac{dx}{x^2-x+1}$

$\Rightarrow \quad I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$

$\Rightarrow \quad I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$

$\Rightarrow \quad I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2 \quad \Rightarrow \quad \int \frac{dx}{x^3+1} = \int f(x)\,dx = I_1 + I_2$

$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

$= \frac{1}{3}\log\left|\frac{x+1}{\sqrt{x^2-x+1}}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

Example 14 :

Evaluate: $\int \frac{3x+1}{\sqrt{x^2+4x+1}}dx$

Solution :

The linear expression in the numerator can be expressed as

$3x + 1 = l\,\frac{d}{dx}(x^2 + 4x + 1) + m \quad \Rightarrow \quad 3x + 1 = l\,(2x + 4) + m$

Comparing the coefficients of x and constants both sides.

$3 = 2l \;\;\&\;\; 1 = 4l + m$

$\Rightarrow \quad l = 3/2 \;\;\&\;\; m = -5 \quad \Rightarrow \quad I = \int \frac{3x+1}{\sqrt{x^2+4x+1}} = \int \frac{3/2(2x+4)-5}{\sqrt{x^2+4x+1}}dx$

$= \frac{3}{2}\int \frac{2x+4}{\sqrt{x^2+4x+1}} - 5\int \frac{dx}{\sqrt{x^2+4x+1}}$

Let $\quad I_1 = \frac{3}{2}\int \frac{2x+4}{\sqrt{x^2+4x+1}} = \frac{3}{2}\int \frac{dt}{\sqrt{t}}$ (where $t = x^2 + 4x + 1$) $\quad = 3\sqrt{t} + C = 3\sqrt{x^2+4x+1} + C$

Let $\quad I_2 = 5\int \frac{dx}{\sqrt{x^2+4x+1}} = 5\int \frac{dx}{\sqrt{(x+2)^2-3}} \quad = 5\log|x+2+\sqrt{(x+2)^2-3}| + C$

$\Rightarrow \quad I = I_1 - I_2 = 3\sqrt{x^2+4x+1} - 5\log|x+2+\sqrt{x^2+4x+1}| + C$

Example 15 :

Evaluate : $\int \frac{3\cos x+2}{\sin x+2\cos x+3}\,dx$

Solution :

3cosx + 2 = A(sinx + 2cosx + 3) + $B\frac{d}{dx}$ (sinx + 2cosx + 3) + λ

3cosx + 2 = sinx (A –2B) + cosx (2A + B) + 3A + λ

Comparing the coefficients of sinx, cosx and constant terms on both sides, we get

A – 2B = 0, 2A + B = 3, 3A + λ = 2

Solving these equation we get $A=\frac{6}{5},\ B=\frac{3}{5},\ \lambda=-\frac{8}{5}$

$$= \int \frac{3\cos x+2}{\sin x+2\cos x+3}dx=\int \frac{\frac{6}{5}(\sin x+2\cos x+3)+\frac{3}{5}(\cos x-2\sin x)-\frac{8}{5}}{\sin x+2\cos x+3}$$

$$= \int \frac{3\cos x+2}{\sin x+2\cos x+3}dx=\int \frac{A(\sin x+2\cos x+3)+B(\cos x-2\sin x)+\lambda}{\sin x+2\cos x+3}$$

$$= \frac{6}{5}x+\frac{3}{5}\log|\sin x+2\cos x|-\frac{8}{5}\int \frac{1}{\sin x+2\cos x+3}$$

Now $\int \frac{1}{\sin x+2\cos x+3}dx$

Put sinx = $\frac{2\tan x/2}{1+\tan^2 x/2}\cos x=\frac{1-\tan^2 x/2}{1+\tan x/2}$ and $\tan\frac{x}{2}=t$

$\Rightarrow$ $\tan\frac{x}{2}$ = t $\Rightarrow$ $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$

$$\int \frac{1}{\sin x+2\cos x+3}dx=\int \frac{2dt}{t^2+2t+5} = 2\int \frac{dt}{(t+1)^2+2^2} = \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+C$$

$$\int \frac{3\cos x+2}{\sin x+2\cos x+3}dx=\frac{6}{5}x+\frac{3}{5}\log|\sin x+2\cos x|-\frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+C$$

Example 16 :

Evaluate $\int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx$

Solution :

Let $x^2 - 3x + 1 = A(1 - x^2) + B\frac{d}{dx}(1-x^2)+\lambda$

Comparing the coefficients like powers of x

$A = -1$, $B = 3/2$, $\lambda = 2$ $\quad \int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx = \int \frac{-(1-x^2)+\frac{3}{2}(-2x)+2}{\sqrt{1-x^2}} dx$

$$= - \int \sqrt{1-x^2}\,dx - \int \frac{3x}{\sqrt{1-x^2}} dx + 2\int \frac{1}{\sqrt{1-x^2}} dx$$

$$= - \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right] - 3\left(-\frac{1}{2}\right)\int \frac{-2x}{\sqrt{1-x^2}} dx + 2\int \frac{1}{\sqrt{1-x^2}} dx$$

$$= -\frac{x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1} x + 3\sqrt{1-x^2} + C \qquad = \frac{6-x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1} x + C$$

Example 17 :

Evaluate $\int \frac{dx}{(x+1)\sqrt{x+2}}$

Solution :

Let $I = \int \frac{dx}{(x+1)\sqrt{x+2}}$ $\qquad$ Substitute: $x + 2 = t^2 \Rightarrow dx = 2t\,dt$

$$\Rightarrow \int \frac{dx}{x+1\left(\sqrt{x+2}\right)} = \int \frac{2t\,dt}{(t^2-1)\sqrt{t^2}} = 2\int \frac{dt}{t^2-1} \quad = \log\left|\frac{t-1}{t+1}\right| + C = \log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$$

Example 18 :

Evaluate $\int \frac{dx}{(x^2+1)\sqrt{x^2+2}}$

Solution :

Let $I = \int \frac{dx}{(x^2+1)\sqrt{x^2+2}}$ $\qquad$ Substitute $x = \frac{1}{t} \Rightarrow dx = \frac{1}{t^2} dt$

$$\Rightarrow \quad I = \int \frac{-\frac{1}{t^2} dt}{\left(\frac{1}{t^2}+1\right)\sqrt{\frac{1}{t^2}+2}} = \int \frac{-t\,dt}{(1+t^2)\sqrt{1+2t^2}}$$

Let $1 + 2t^2 = z^2 \Rightarrow 4t\,dt = 2z\,dz$

$$\Rightarrow \quad I=\frac{-1}{2}\int\frac{zdz}{\left(1+\frac{z^2-1}{2}\right)\sqrt{z^2}}=\int\frac{dz}{z^2+1}=-\tan^{-1}z+C$$

$$\Rightarrow \quad I=-\tan^{-1}\sqrt{1+2t^2}+C=-\tan^{-1}\sqrt{1+\frac{2}{x^2}}+C$$

Example 19 :

Evaluate $\int\frac{dx}{(x+2)\sqrt{x^2+6x+7}}$

Solution :

Let $I=\int\frac{dx}{(x+2)\sqrt{x^2+6x+7}}$ Substitute $x + 2 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}$

$$\Rightarrow \quad x^2+6x+7=\left(\frac{1}{t}-2\right)^2+6\left(\frac{1}{t}-2\right)+7=\frac{1+2t-t^2}{t^2}$$

$$\Rightarrow \quad I=\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1+2t-t^2}{t^2}}}=-\int\frac{dt}{\sqrt{1+2t-t^2}}$$

$$\Rightarrow \quad I=\int\frac{dt}{\sqrt{2-(t-1)^2}}=-\sin^{-1}\left(\frac{t-1}{\sqrt{2}}\right)+C \quad \Rightarrow \quad I=\sin^{-1}\left[\frac{x+1}{(x+2)\sqrt{2}}\right]+C$$

Example 20 :

Evaluate $\int\frac{dx}{\sqrt[3]{x+1}+\sqrt{x+1}}$

Solution :

Let $I=\int\frac{dx}{\sqrt[3]{x+1}+\sqrt{x+1}} \Rightarrow I=\int\frac{dx}{(x+1)^{1/3}+(x+1)^{1/2}}$

The least common multiple of 2 and 3 is 6.
So substitute $x+1 = t^6 \Rightarrow dx = 6t^5\,dt$

$$\Rightarrow \quad I=\int\frac{6t^5dt}{t^2+t^3}=6\int\frac{t^3dt}{1+t} \quad \Rightarrow \quad I=6\int t^2-t+1-\frac{1}{1+t})dt$$

$$\Rightarrow \quad I=6\left(\frac{t^3}{3}-\frac{t^2}{2}+t-\log(t+1)\right)+C$$

On substituting $t = (1 + x)^{1/6}$, we get

$$I=6\left(\frac{(1+x)^{1/2}}{3}-\frac{(1+x)^{1/3}}{2}+(1+x)^{1/6}-\log\left((x+1)^{1/6}+1\right)\right)+C$$

Example 21 :

Evaluate: $\int x^{13/2}\left(1+x^{5/2}\right)^{1/2} dx$

Solution :

Let $I=\int x^{13/2}\left(1+x^{5/2}\right)^{1/2} dx$

Comparing with integral of type $x^m(a + bx^n)^p$, we can see that p = 1/2 which is not an integer. So check the sign of (m + 1)/n

$$\frac{m+1}{n}=\frac{\frac{13}{2}+1}{\frac{5}{2}}=\frac{15}{5}=3 \quad \Rightarrow \quad \text{(m + 1)/n is an integer.}$$

To solve this integral, substitute $1 + x^{5/2} = t^2$ then

$\Rightarrow \quad \frac{5}{2} x^{3/2}\, dx = 2t\, dt \quad \Rightarrow \quad I=\frac{2}{5}\int (t^2-1)^2 (t^2)^{1/2}\, 2t\, dt$

$\Rightarrow \quad I=\frac{4}{5}\int t^2(t^2-1)^2\, dt \quad \Rightarrow \quad I=\frac{4}{5}\int t^6+t^2-2t^4\, dt \quad \Rightarrow \quad I=\frac{4}{5}\left(\frac{t^7}{7}+\frac{t^3}{3}-2\frac{t^5}{5}\right)+C$

On substituting $t = (1 + x^{5/2})^{1/2}$, we get

$$I=\frac{4}{5}\left(\frac{(1+x^{5/2})^{7/2}}{7}+\frac{(1+x^{5/2})^{3/2}}{3}-\frac{2(1+x^{5/2})^{5/2}}{5}\right)+C$$

Solved Examples (Objective)

Example 1 :

$I=\int \frac{1}{1-\cos^4 x}dx$ is equal to

(A) $2\sqrt{2}(\cot x+\sqrt{2}\tan^{-1}\sqrt{2}\cot x)$

(B) $-\frac{1}{2\sqrt{2}}[\sqrt{2}\cot x+\tan^{-1}(\sqrt{2}\cot x)]+c$

(C) $2\sqrt{2}\{\cot x+\tan^{-1}(\cos x)\}$

(D) none of these

Solution :

$I=\int \frac{1}{(1+\cos^2 x)\sin^2 x}dx \qquad =\int \frac{(1+\cot^2 x)\operatorname{cosec}^2 x dx}{(1+2\cot^2 x)}$ Let P = cot x $\Rightarrow$ dp = $-\operatorname{cosec}^2 x dx$

$$=-\int \frac{(1+p^2)dp}{1+2p^2}=-\frac{1}{2}\int \frac{(2+2p^2)dp}{1+2p^2}=-\frac{1}{2}\int dp-\frac{1}{2}\int \frac{pdp}{1+2p^2}$$

$$=-\frac{1}{2}p-\frac{1}{4}\int \frac{dp}{1/2+p^2}=-\frac{1}{2}p-\frac{\sqrt{2}}{4}\tan^{-1}\left(p\sqrt{2}\right)$$

$$=-\frac{1}{2}\cot x-\frac{1}{2\sqrt{2}}\tan^{-1}\left(\cot x\sqrt{2}\right)+c$$

Hence (B) is the correct answer.

Example 2 :

$\int \frac{dx}{x(x^n+1)}$ is equal to

(A) $\frac{1}{n}\ln\left(\frac{x^n}{x^n+1}\right)+c$

(B) $-\frac{1}{n}\ln\left(\frac{x^n+1}{x^n}\right)+c$

(C) $\ln\left(\frac{x^n}{x^n+1}\right)+c$

(D) None of these

Solution :

Let $I=\int \frac{dx}{x(x^n+1)}=\int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)}$

If $\left(1+\frac{1}{x^n}\right)=p$, then $\frac{-n}{x^{n+1}}dx=dp$

$$I=-\frac{1}{n}\int \frac{dp}{p}=\frac{-1}{n}\ln p=-\frac{1}{n}\ln\left(\frac{x^n+1}{x^n}\right)$$

Hence (B) is the correct answer.

Example 3 :

$\int \frac{dx}{(1+\sqrt{x})\sqrt{(x-x^2)}}$ is equal to

(A) $\frac{1+\sqrt{x}}{(1-x)^2}+c$ (B) $\frac{1+\sqrt{x}}{(1+x)^2}+c$ (C) $\frac{1-\sqrt{x}}{(1-x)^2}+c$ (D) $\frac{2(\sqrt{x}-1)}{\sqrt{(1-x)}}+c$

Solution :

Let $I=\int \frac{dx}{(1+\sqrt{x})\sqrt{(x-x^2)}}$

If $\sqrt{x}=\sin p$, then $\frac{1}{2\sqrt{x}}dx=\cos p dp$

$$I=\int \frac{2\sin p\cos p dp}{(1+\sin p)\sin p\cos p}=2\int \frac{dp}{(1+\sin p)}=2\int \frac{(1-\sin p)dp}{\cos^2 p}$$

$$=2\left\{\int \sec^2 p dp-\int (\tan p\sec p)dp\right\}$$

$$= 2(\tan p - \sec p) = 2\left(\sqrt{\frac{x}{(1-x)}}-\frac{1}{\sqrt{(1-x)}}\right) = 2\frac{[\sqrt{x}-1]}{\sqrt{1-x}}+C$$

Hence (D) is the correct answer.

Example 4 :

$\int \frac{dx}{\cos^6 x+\sin^6 x}$ is equal to

(A) $\log_e (\tan x - \cot x) + c$ (B) $\log_e (\cot x - \tan x) + c$
(C) $\tan^{-1} (\tan x - \cot x) + c$ (D) $\tan^{-1} (\cot x - \tan x) + c$

Solution :

Let $I=\int \frac{dx}{\cos^6 x+\sin^6 x} = \int \frac{\sec^6 x}{1+\tan^6 x}dx$

$\Rightarrow \quad I=\int \frac{(1+\tan^2 x)^2 \sec^2 x dx}{1+\tan^6 x}$ If $\tan x = p$, then $\sec^2 x dx = dp$

$$=\int \frac{p^2\left(1+\frac{1}{p^2}\right)}{p^2\left(p^2+\frac{1}{p^2}-1\right)}dp$$

$$= \int \frac{dk}{k^2+1}=\tan^{-1}(k)+c \quad \left(p-\frac{1}{p}=k, \left(1+\frac{1}{p^2}\right)dp=dk\right)$$

$$= \tan^{-1}\left(p-\frac{1}{p}\right)+c = \tan^{-1} (\tan x - \cot x) + c = \tan^{-1} (-2 \cot 2x) + c$$

Hence (C) is the correct answer.

Example 5 :

$\int \frac{3+2\cos x}{(2+3\cos x)^2}dx$ is equal to

(A) $\int \frac{\sin x}{(2+3\cos x)}+c$ (B) $\left(\frac{2\cos x}{3\sin x+2}\right)+c$ (C) $\left(\frac{2\cos x}{3\cos x+2}\right)+c$ (D) $\left(\frac{2\sin x}{3\sin x+2}\right)+c$

Solution :

Let $I=\int \frac{3+2\cos x}{(2+3\cos x)^2}dx$

Multiplying Nr. & Dr. by $\text{cosec}^2 x$

$$\Rightarrow \quad I=\int \frac{(3\,\text{cosec}^2 x+2\cot x\,\text{cosec}x)}{(2\,\text{cosec}x+3\cot x)^2}dx$$

$$=-\int \frac{-3\,\text{cosec}^2 x-2\cot x\,\text{cosec}x}{(2\,\text{cosec}x+3\cot x)^2}dx = \frac{1}{2\,\text{cosec}x+3\cot x}=\left(\frac{\sin x}{2+3\cos x}\right)+c$$

Hence (A) is the correct answer.

Example 6 :

$\int \left(x+\frac{1}{x}\right)^{n+5}\left(\frac{x^2-1}{x^2}\right)dx$ is equal to

(A) $\frac{\left(x+\frac{1}{x}\right)^{n+6}}{n+6}+c$ (B) $\left(\frac{x^2+1}{x^2}\right)^{n+6}(n+6)+c$ (C) $\left(\frac{x}{x^2+1}\right)^{n+6}(n+6)+c$

(D) none of these

Solution :

$I=\int \left(x+\frac{1}{x}\right)^{n+5}\left(\frac{x^2-1}{x^2}\right)dx$, put $x+\frac{1}{x}=p$ then, $\left(1-\frac{1}{x^2}\right)dx=dp$

$$\Rightarrow \quad \int p^{n+5}\,dp = \frac{p^{n+6}}{n+6}+c = \frac{\left(x+\frac{1}{x}\right)^{n+6}}{n+6} + c.$$

Hence (A) is the correct answer.

Example 7 :

If $\int \frac{dx}{x^{22}(x^7-6)}$ = A{ln (p)^6 + 9p^2 – 2p^3 – 18p} + c, then

(A) $A=\frac{1}{9072},\ p=\left(\frac{x^7-6}{x^7}\right)$ (B) $A=\frac{1}{54432},\ p=\left(\frac{x^7-6}{x^7}\right)$

(C) $A=\frac{1}{54432},\ p=\left(\frac{x^7}{x^7-6}\right)$ (D) $A=\frac{1}{9072},\ p=\left(\frac{x^7-6}{x^7}\right)^{-1}$

Solution :

$$I=\int \frac{dx}{x^{29}\left(1-\frac{6}{x^7}\right)}$$

Let $\left(1-\frac{6}{x^7}\right)=p \Rightarrow \frac{42}{x^8}dx=dp$ and $x^7=\left(\frac{6}{1-p}\right)$

$$I=\frac{1}{42}\int\frac{(1-p)^3}{(6)^3p}dp=\frac{1}{(42)(216)}\int\frac{1-p^3-3p+3p^2}{p}dp$$

$= \frac{1}{54432}$ $[\ln p^6 + 9p^2 - 2p^3 - 18p] + c$ Hence (B) is the correct answer.

Example 8 :

$I=\int\frac{(x+x^{2/3}+x^{1/6})}{x(1+x^{1/3})}dx$ is equal to

(A) $\frac{3}{2}x^{2/3}+6\tan^{-1}\left(x^{1/6}\right)+c$

(B) $\frac{3}{2}x^{2/3}-6\tan^{-1}\left(x^{1/6}\right)+c$

(C) $-\frac{3}{2}x^{2/3}+\tan^{-1}\left(x^{1/6}\right)+c$

(D) none of these

Solution :

Substituting $x = p^6$, $dx = 6p^5\,dp$, we have

$$I=\int\frac{6p^5(p^6+p^4+p)}{p^6(1+p^2)}dp=\int\frac{6(p^5+p^3+1)}{(p^2+1)}dp=\int 6p^3dp+\int\left(\frac{6}{p^2+1}\right)dp$$

$= \frac{6p^4}{4} + 6\tan^{-1} p=\frac{3}{2}x^{2/3}+6\tan^{-1}\left(x^{1/6}\right)+c$

Hence (A) is the correct answer.

Example 9 :

If $\int f(x)\cos x\,dx=\frac{1}{2}f^2(x)+c$, then f(x) can be

(A) x (B) 1 (C) cosx (D) sinx

Solution :

Given equation is satisfied if cosx dx = d (f(x)) $\Rightarrow$ f(x) = sinx

Hence (D) is the correct answer.

Example 10 :

$\int\left(\frac{\ln x-1}{(\ln x)^2+1}\right)^2 dx$ is equal to

(A) $\frac{x}{x^2+1}+c$ (B) $\frac{\ln x}{(\ln x)^2+1}$ (C) $\frac{x}{(\ln x)^2+1}+c$ (D) $e^x\left(\frac{x}{x^2+1}\right)+c$

Solution :

Put ln x = t

$$I=\int e^t\left(\frac{t-1}{t^2+1}\right)^2 dt=\int e^t\left(\frac{1}{t^2+1}-\frac{2t}{(t^2+1)^2}\right)dt \quad =\frac{e^t}{t^2+1}+c=\frac{x}{(\ln x)^2+1}+c.$$

Hence (c) is the correct answer.

Example 11 :

If $\int \frac{x+(\cos^{-1}3x)^2}{\sqrt{1-9x^2}}dx=A\sqrt{1-9x^2}+B(\cos^{-1}3x)^3+c$, where c is integration constant, then the value of A and B are,

(A) $A=-\frac{1}{9};\ B=-\frac{1}{9}$ (B) $A=-\frac{1}{9};\ B=\frac{1}{9}$ (C) $A\ \frac{1}{9};\ B=\frac{1}{9}$

(D) none of these

Solution :

Let 3x = cos θ $\Rightarrow$ 3dx = – sin θ d θ

$$\therefore -\frac{1}{3}\int \frac{\frac{\cos\theta}{3}+\theta^2}{\sin\theta}\sin\theta d\theta=-\frac{1}{3}\int\left(\frac{1}{3}\cos\theta+\theta^2\right)d\theta$$

$$=-\frac{1}{9}\sin\theta-\frac{1}{9}\theta^3+c=-\frac{1}{9}\sqrt{1-9x^2}-\frac{1}{9}(\cos^{-1}3x)^3+c$$

Hence (A) is the correct answer.

Example 12 :

$\int \frac{10x^9+10^x\log_e 10}{10^x+x^{10}}dx$ is equal to

(A) $\log_e|10^x+x^{10}|+c$ (B) $\log_e|10^x-x^{10}|+c$

(C) $\log_e|10^x-2x^{10}|+c$ (D) none of these

Solution :

Put $10^x+x^{10}=t$ $\Rightarrow$ $(10^x\log_e 10+10x^9)\,dx=dt$

$$\int\frac{1}{t}dt=\log_e(t)=\log_e|10^x+x^{10}|+c$$

Hence (A) is the correct answer.

Example 13 :

$\int\frac{dx}{\sqrt{(x-a)(b-x)}}$ is equal to

(A) $2\sin^{-1}\sqrt{\left(\frac{x+a}{b+a}\right)}+c$ (B) $2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)}+c$ (C) $2\cos^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)}+c$

(D) none of these

Solution :

Put x = $a\cos^2\theta + b\sin^2\theta$. The given integral becomes,

$$I = \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{\{(a\cos^2\theta + b\sin^2\theta - a)\ (b - a\cos^2\theta - b\sin^2\theta)}$$

$$= \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{(b-a)\sin\theta\cos\theta} = 2\int d\theta$$

$$= 2\theta + c = 2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)} + c$$

Hence (B) is the correct answer.

Example 14 :

$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$ is equal to

(A) $\frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$ (B) $\frac{2}{6}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$ (C) $\frac{2}{3}\cos^{-1}\left(\frac{x}{a}\right)^{3/2} + c$

(D) none of these

Solution :

Integral of the numerator = $\frac{x^{3/2}}{3/2}$

Put $x^{3/2} = t$

We get $I = \frac{2}{3}\int \frac{dt}{\sqrt{a^3 - t^2}} = \frac{2}{3}\sin^{-1}\frac{t}{a^{3/2}} + c = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$

Hence (A) is the correct answer.

SUMMARY

INDEFINITE INTEGRATION

Substitution

(i) If $\int f(x)dx = g(x) + c$, then $\int f(h(x))h'(x)dx = g(h(x)) + c$

(ii) In $\sqrt{x^2+a^2}$ put $x = a\tan\theta$

In $\sqrt{x^2-a^2}$ put $x = a\sec\theta$

In $\sqrt{a^2-x^2}$ put $x = a\sin\theta$

In $\sqrt{a+x}, \sqrt{a-x}$ put $x = a\cos\theta$

In $\left(x \pm \sqrt{x^2 \pm a^2}\right)^n$ put $x \pm \sqrt{x^2 \pm a^2} = t$

(iii) If $f_1(x)$ is a function of integral of $f_2(x)$ then in $\int f_1(x)f_2(x)dx$ put integral of $f_2(x) = t$.

Integration as anti derivative

If $f'(x) = g(x)$, then $\int g(x)dx = f(x) + c$

Partial function

$$\frac{f(x)}{(x-a)^n} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^n}$$

$$\frac{g(x)}{(x^2+ax+b)^m} = \frac{A_1x+B}{(x^2+ax+b)} + ... + \frac{A_mx+B_m}{(x^2+ax+b)^m}$$

Algebraic Integrals

(i) In $\sqrt{ax^2+bx+c}$ make the perfect square

(ii) In $\int \frac{(ax+b)dx}{cx^2+ex+f}$ write $ax+b = A(2cx+e)+B$

(iii) In $\int \frac{(ax^2+bx+c)dx}{\sqrt{ex^2+fx+g}}$ write ax^2+b+c

$= A(ex^2+fx+g) + B(2ex+f) + C$

(iv) In $\int \frac{dx}{(ax+b)\sqrt{(cx^2+ex+f)}}$.

Put $ax+b = 1/t$

(v) In $\int \frac{(ax+b)dx}{(cx+e)\sqrt{fx^2+gx+h}}$.

Put $ax+b = A(cx+e)+B$

(vi) In $\int \frac{(ax^2+bx+c)dx}{(ex+f)\sqrt{gx^2+hx+i}}$.

Put ax^2+bx+c

$= A(gx^2+hx+i) + B(2gx+h)(ex+f) + C$

(vii) In $\int \frac{x\,dx}{(ax^2+b)\sqrt{(cx^2+e)}}$.

Put $cx^2+e = t^2$

By partial fraction

$\int f_1(x)f_2(x)dx$

$= f_1(x)\int f_2(x)dx - \int\left(f_1'(x)\int f_2(x)dx\right)dx$

and order of $f_1(x)$ and $f_2(x)$ are normally decided by ILATE.

Trigonometric integration

(i) In $R(\sin x, \cos x)$

If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

Put $\sin x = t$

If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

Put $\cos x = t$

If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

Put $\tan x = t$

(ii) In $\int \frac{(a\sin x + b\cos x + c)dx}{e\sin x + f\cos x + g}$

Put $a\sin x + b\cos x + c$

$= A(e\sin x + f\cos x + g)$

$+ B(e\cos x - f\sin x) + C$

PRACTICE SET

A SINGLE CORRECT CHOICE TYPE

Each of these questions has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. Antiderivative of $\dfrac{x-1}{(x+1)\sqrt{x^3+x^2+x}}$ is

(a) $\tan^{-1}\left(x+\dfrac{1}{x}+1\right)$ (b) $\tan^{-1}\sqrt{x+\dfrac{1}{x}+1}$

(c) $2\tan^{-1}\sqrt{x+\dfrac{1}{x}+1}$ (d) $\sqrt{x+\dfrac{1}{x}+1}$

2. The integral of $\int e^{\sin x}(x\cos x-\sec x\tan x)\,dx$ is

(a) $xe^{\sin x}-e^{\sin x}\sec x+C$

(b) $(x+\sec x)e^{\sin x}+C$

(c) $e^{\sin x}\cos x+C$

(d) $e^{\sin x}(\cos x-\sec x)+C$

3. $\int \dfrac{\sin^3 x\,dx}{(\cos^3 x+3\cos^2 x+1)\tan^{-1}(\sec x+\cos x)}=$

(a) $\tan^{-1}(\sec x+\cos x)+c$

(b) $\log\tan^{-1}(\sec x+\cos x)+c$

(c) $\dfrac{1}{(\sec x+\cos x)^2}+c$

(d) None of these

4. $\int e^x\dfrac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}dx=$

(a) $e^x\left(\dfrac{1-x^n}{1+x^n}\right)+C$ (b) $e^x\left(\dfrac{1+x^n}{1-x^n}\right)+C$

(c) $e^x\left(\sqrt{\dfrac{1+x^n}{1-x^n}}\right)+C$ (d) $e^x\left(\sqrt{\dfrac{1-x^n}{1+x^n}}\right)+C$

5. If $f(x)=\lim\limits_{n\to\infty} n^2(x^{1/n}-x^{1/(n+1)})$, x > 0 then $\int xf(x)\,dx$ is equal to

(a) $\dfrac{x^2}{2}+\ln x+C$ (b) $-\dfrac{x^2}{4}\ln x+\dfrac{x^2}{2}+C$

(c) $\dfrac{x^3}{3}+x\ln x+C$ (d) $\dfrac{x^2}{2}\ln x-\dfrac{x^2}{4}+C$

6. $\int \dfrac{1+x^4}{(1-x^4)^{3/2}}dx=$

(a) $\dfrac{1}{\sqrt{x^2-\dfrac{1}{x^2}}}+c$ (b) $\dfrac{1}{\sqrt{\dfrac{1}{x^2}-x^2}}+c$

(c) $\dfrac{1}{\sqrt{\dfrac{1}{x^2}+x^2}}+c$ (d) None of these

7. Let $f(x)=\dfrac{x+2}{2x+3}$, $x>0$. If $\int\left(\dfrac{f(x)}{x^2}\right)^{1/2}dx$

$$=\frac{1}{\sqrt{2}}g\left(\frac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right)-\sqrt{\frac{2}{3}}h\left(\frac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right)+C$$

Where C is the constant of intergation, then

(a) $g(x)=\tan^{-1}(x), h(x)=\ln|x|$

(b) $g(x)=\ln|x|, h(x)=\tan^{-1}(x)$

(c) $g(x)=\tan^{-1}(x), h(x)=\tan^{-1}(x)$

(d) $g(x)=\ln|x|, h(x)=\ell n|x|$

MARK YOUR RESPONSE					
	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ
	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ			

8. $\int \sqrt{1+\operatorname{cosec} x}\, dx =$

(a) $\pm\sin^{-1}(\tan x - \sec x) + c$

(b) $2\sin^{-1}(\cos x) + c$

(c) $2\sin^{-1}\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) + c$

(d) $\pm 2\sin^{-1}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + c$

9. If $f(x) = \lim\limits_{n\to\infty} [2x + 4x^3 + \ldots\ldots + 2nx^{2n-1}]$; $(0 < x < 1)$, then $\int (f(x))\,dx$ is equal to

(a) $-\sqrt{1-x^2} + c$ (b) $\frac{1}{\sqrt{1-x^2}} + c$

(c) $\frac{1}{x^2-1} + c$ (d) $\frac{1}{1-x^2} + c$

10. If $f(x) = \int \frac{dx}{\sin^{1/2} x \cos^{7/2} x}$, then $f\left(\frac{\pi}{4}\right) - f(0) =$

(a) $\frac{7}{5}$ (b) $\frac{5}{2}$

(c) $\frac{12}{5}$ (d) 5

11. If $f\left(\frac{3x-4}{3x+4}\right) = x + 2$, then $\int f(x)dx$ is equal to

(a) $e^{x+2} \ln\left|\frac{3x-4}{3x+4}\right| + c$

(b) $-\frac{8}{3}\ln|(1-x)| + \frac{2}{3}x + c$

(c) $\frac{8}{3}\ln|1-x| + \frac{x}{3} + c$

(d) None of these

12. The integral $\int \frac{\sec^{3/2}\theta - \sec^{1/2}\theta}{2+\tan^2\theta} \tan\theta d\theta$ is equal to

(a) $\sqrt{2}\tan^{-1}\left(\frac{\sec\theta + 1}{\sqrt{2\sec\theta}}\right) + C$

(b) $\frac{1}{\sqrt{2}}\log_e\left|\frac{\sec\theta - \sqrt{2\sec\theta} + 1}{\sec\theta + \sqrt{2\sec\theta} + 1}\right| + C$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left|\frac{\sec\theta + 1}{\sqrt{2\sec\theta}}\right| + C$

(d) $\sqrt{2}\log_e\left|\frac{\sec\theta - \sqrt{2\sec\theta} + 1}{\sec\theta + \sqrt{2\sec\theta} + 1}\right| + C$

13. If $f(x) = \lim\limits_{n\to\infty} \frac{x^n - x^{-n}}{x^n + x^{-n}}, 0 < x < 1$, $n \in N$ then $\int (\sin^{-1} x) f(x) dx$ is equal to

(a) $-[x\sin^{-1} x + \sqrt{1-x^2}] + C$

(b) $x\sin^{-1} x + \sqrt{1-x^2} + C$

(c) $\frac{x^2}{2} + C$

(d) $\frac{1}{2}(\sin^{-1} x)^2 + C$

14. If $f(x) = \lim\limits_{n\to\infty} \frac{x^n - x^{-n}}{x^n + x^{-n}}, x > 1$; then

$\int \frac{xf(x)\log(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx$ is equal to

(a) $\log(x + \sqrt{1+x^2}) - x + C$

(b) $\frac{1}{2}[x^2\log(x + \sqrt{1+x^2}) - x^2] + C$

(c) $x\log(x + \sqrt{1+x^2}) - \log(x + \sqrt{1+x^2}] + C$

(d) $\sqrt{1+x^2}\log(x + \sqrt{1+x^2}) - x + C$

MARK YOUR RESPONSE	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ	12. ⓐⓑⓒⓓ
	13. ⓐⓑⓒⓓ	14. ⓐⓑⓒⓓ			

15. If $I_n = \int \cot^n x\,dx$, then $I_0 + I_1 + 2(I_2 + I_3 + ...I_8) + I_9 + I_{10} =$

(a) $-\sum_{k=1}^{9} \frac{\cot^k x}{k}$ (b) $\sum_{k=1}^{9} \frac{\cot^k x}{k!}$

(c) $\sum_{k=1}^{10} \frac{\cot^k x}{10}$ (d) $-\sum_{k=1}^{10} k\cot^k x$

16. The value of integral $\int \frac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}} d\theta$ is

(a) $\frac{7}{11}\left(\tan\frac{\theta}{2}\right)^{\frac{11}{7}} + C$ (b) $\frac{7}{11}\left(\cos\frac{\theta}{2}\right)^{\frac{11}{7}} + C$

(c) $\frac{7}{11}\left(\sin\frac{\theta}{2}\right)^{\frac{11}{7}} + C$ (d) none of these

17. If $I_{m,n} = \int \cos^m x \sin nx\,dx$, then $7I_{4,3} - 4I_{3,2} =$

(a) constant (b) $-\cos^2 x + C$

(c) $-\cos^4 x\cos 3x + C$ (d) $\cos 7x - \cos 4x + C$

18. Anti derivative of the fucntion $x^{\sin x - 1}\sin x + x^{\sin x}.\cos x.ln\,x$ is

(a) $x^{\sin x}$ (b) $x^{\sin x} + \sin x$

(c) $(\sin x)^{1/x}$ (d) $x^{\sin x} + x^{ln\,x}$

19. Let $f(xy) = f(x).f(y), \forall x > 0, y > 0$, where $f(x)$ is not constant and $f(x+1) = 1 + x\{1 + g(x)\}$ where $\lim_{x\to 0} g(x) = 0$, then $\int \frac{f(x)}{f'(x)} dx$ is

(a) $\frac{x^2}{2} + c$ (b) $\frac{x^3}{3} + c$

(c) $\frac{x^2}{3} + c$ (d) $\ell n|x| + c$

20. If $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = -[f(x)]^{1/n} + C$, then f(x) is

(a) $(1+x^n)$ (b) $1 + x^{-n}$

(c) $x^n + x^{-n}$ (d) none of these

21. If $\int \frac{dx}{(x-1)(-x^2+3x-2)} = -2f(x) + C$, then $f(x) =$

(a) $\sqrt{\frac{2-x}{x-1}}$ (b) $\sqrt{\frac{x-2}{x-1}}$

(c) $\sqrt{\frac{x+2}{x-1}}$ (d) $\sqrt{\frac{x-1}{x-2}}$

22. $\int \frac{(x+\sqrt{1+x^2})^{15}}{\sqrt{1+x^2}} dx = A\{x + \sqrt{1+x^2}\}^n + C$, then

(a) $A = \frac{1}{15}, n = 15$ (b) $A = \frac{1}{14}, n = 14$

(c) $A = \frac{1}{16}, n = 16$ (d) none of these

23. $\int \frac{e^x(2-x^2)dx}{(1-x)\sqrt{1-x^2}} =$

(a) $\frac{e^x}{\sqrt{1-x^2}} + c$ (b) $e^x\sqrt{1-x^2} + c$

(c) $\frac{e^x(2-x^2)}{\sqrt{1-x^2}} + c$ (d) $\frac{e^x(1+x)}{\sqrt{1-x^2}} + c$

24. If $\int \frac{ln(1+\sin^2 x)}{\cos^2 x} dx =$

$\frac{1}{\sqrt{2}} f(x)\,ln\,g(x) - 2x + \sqrt{2}\tan^{-1} f(x) + c$, then

(a) $f(x) = \tan x, g(x) = 1 + \sin^2 x$

(b) $f(x) = \sqrt{2}\tan x, g(x) = 1 + \sin^2 x$

(c) $f(x) = \sin x, g(x) = 1 + \tan^2 x$

(d) $f(x) = \sqrt{2}\cos x, g(x) = 1 + \sin^2 x$

MARK YOUR RESPONSE					
	15. ⓐⓑⓒⓓ	16. ⓐⓑⓒⓓ	17. ⓐⓑⓒⓓ	18. ⓐⓑⓒⓓ	19. ⓐⓑⓒⓓ
	20. ⓐⓑⓒⓓ	21. ⓐⓑⓒⓓ	22. ⓐⓑⓒⓓ	23. ⓐⓑⓒⓓ	24. ⓐⓑⓒⓓ

25. If $\int \frac{(\sin^{3/2}\theta + \cos^{3/2}\theta)d\theta}{\sqrt{\sin^3\theta\cos^3\theta\sin(\theta+\alpha)}} =$

$a\sqrt{\cos\alpha\tan\theta + \sin\alpha} + b\sqrt{\cos\alpha + \sin\alpha\cot\theta} + c$, then

(a) $a = 2\sec\alpha, b = 2\operatorname{cosec}\alpha, c \in R$

(b) $a = 2\sec\alpha, b = -2\operatorname{cosec}\alpha, c \in R$

(c) $a = -2\sec\alpha, b = 2\operatorname{cosec}\alpha, c \in R$

(d) $a = 2\operatorname{cosec}\alpha, b = 2\sec\alpha, c \in R$

26. $\int \frac{x^2 dx}{(x\sin x + \cos x)^2} =$

(a) $\frac{\sin x - x\cos x}{x\sin x + \cos x} + c$ (b) $\frac{x\sin x - \cos x}{x\sin x + \cos x} + c$

(c) $\frac{\sin x + x\cos x}{x\sin x + \cos x} + c$ (d) none of these

27. If $\int \frac{\cos^2 x + \sin 2x}{(2\cos x - \sin x)^2} dx =$

$\frac{\cos x}{2\cos x - \sin x} + ax + b\ln|2\cos x - \sin x| + c$, then

(a) $a = \frac{1}{5}, b = \frac{2}{5}$ (b) $a = \frac{1}{5}, b = -\frac{2}{5}$

(c) $a = -\frac{1}{5}, b = \frac{2}{5}$ (d) $a = -\frac{1}{5}, b = -\frac{2}{5}$

28. Let $I_n = \int \frac{dx}{(x^2+a^2)^n}$, where $n \in N$ and $n > 1$. If I_n and I_{n-1} are related by the relation

$PI_n = \frac{x}{(x^2+a^2)^{n-1}} + QI_{n-1}$. Then P and Q are respectively given by

(a) $(2n-1)a^2, 2n-3$ (b) $2a^2(n-1), 2n-3$

(c) $a^2(n+1), 2n+3$ (d) $a^2, a^2(n+1)$

29. If $U_n = \int x^n\sqrt{a^2-x^2}\,dx$, then $(n+2)u_n - (n-1)a^2u_{n-2} =$

(a) $x^n\sqrt{a^2-x^2}$ (b) $-x^{n-1}\sqrt{a^2-x^2}$

(c) $-x^{n-1}(a^2-x^2)^{3/2}$ (d) none of these

30. If $\int f(x)\sin x\cos x\,dx = \frac{1}{2(b^2-a^2)}\ln f(x) + c$, then $f(x)$ is

(a) $\frac{1}{a\sin x + b\cos x}$ (b) $\frac{1}{a^2\sin^2 x + b^2\cos^2 x}$

(c) $\frac{1}{a^2\sin x + b^2\cos x}$ (d) $\frac{1}{a\sin^2 x + b\cos^2 x}$

31. $\int \frac{(ax^2-b)dx}{x\sqrt{c^2x^2-(ax^2+b)^2}} =$

(a) $\sin^{-1}\left(\frac{ax+bx^2}{c}\right)+k$ (b) $\tan^{-1}\left(\frac{ax+bx^2}{cx}\right)+k$

(c) $\sin^{-1}\left(\frac{ax^2+b}{cx}\right)+k$ (d) $\tan^{-1}(ax^2+bx+c)+k$

32. If $f(x) = \tan^{-1}x + \ln\sqrt{1+x} - \ln\sqrt{1-x}$, then the integral of $\frac{1}{2}f'(x)$ with respect to x^4 is

(a) $e^{-x^4}+c$ (b) $-\ln(1-x^4)+c$

(c) $e^{\sqrt{1-x^2}}+c$ (d) $\ln(1+x^4)+c$

33. Let $f:\left[0,\frac{\pi}{2}\right] \longrightarrow R$ be such that $f(0) = 3$ and $f'(x) = \frac{1}{1+\cos x}$. If $a < f\left(\frac{\pi}{2}\right) < b$, then a and b can be

(a) $\frac{\pi}{2}, \pi$ (b) 3, 4

(c) $3+\frac{\pi}{4}, 3+\frac{\pi}{2}$ (d) $3+\frac{\pi}{2}, 3+\frac{3\pi}{4}$

MARK YOUR RESPONSE					
	25. ⓐⓑⓒⓓ	26. ⓐⓑⓒⓓ	27. ⓐⓑⓒⓓ	28. ⓐⓑⓒⓓ	29. ⓐⓑⓒⓓ
	30. ⓐⓑⓒⓓ	31. ⓐⓑⓒⓓ	32. ⓐⓑⓒⓓ	33. ⓐⓑⓒⓓ	

34. $\int x^3 d(\tan^{-1} x)$ equals

(a) $\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C$

(b) $x^2 + \ln(1+x^2) + C$

(c) $x^2 \tan^{-1} x + \ln(1+x^2) + C$

(d) $x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

35. If $0 < x < \frac{\pi}{2}$ then $\int \sqrt{1+2\tan x(\tan x + \sec x)}\, dx$ equals to

(a) $\frac{1}{2}$ ln sec $x + C$

(b) ln tan x (sec x + tan x) + C

(c) ln sec x (sec x + tan x) + C

(d) $\frac{1}{2}$ $\{\tan x(\sec x + \tan x)\}^{-1/2} + C$

36. The integral $\int \left(\frac{\sec 6\alpha}{\text{cosec} 2\alpha} + \frac{\sec 18\alpha}{\text{cosec} 6\alpha} + \frac{\sec 54\alpha}{\text{cosec} 18\alpha}\right) d\alpha$ is equal to

(a) $\frac{\ln|\sec 54\alpha|}{108} - \frac{\ln|\sec 2\alpha|}{4} + c$

(b) $\frac{\ln|\sec 6\alpha|}{6} + \frac{\ln|\sec 18\alpha|}{18} + \frac{\ln|\sec 54\alpha|}{54} + c$

(c) $\frac{1}{6}\ln\left|\frac{\sec 6\alpha}{\text{cosec} 2\alpha}\right| + \frac{1}{18}\ln\left|\frac{\sec 18\alpha}{\text{cosec} 6\alpha}\right| + \frac{1}{54}\ln\left|\frac{\sec 54\alpha}{\text{cosec} 18\alpha}\right| + c$

(d) none of these

37. $\int \left(\frac{\ell n x - 1}{(\ell n x)^2 + 1}\right)^2 dx$ is equal to

(a) $\frac{x}{x^2+1} + c$

(b) $\frac{\ell n x}{(\ell n x)^2 + 1} + c$

(c) $\frac{\ell n x}{(\ell n x)^2 + 1} + c$

(d) $e^x\left(\frac{x}{x^2+1}\right) + c$

38. $\int \left(\ell n(1+\cos x) - x\tan\frac{x}{2}\right) dx$ is equal to

(a) $x\,\ell n(1+\cos x) + c$

(b) $x\,\ell n(1+\sec x) + c$

(c) $x^2\,\ell n(1+\cos x) + c$

(d) $x\,\ell n \tan x + c$

39. The integral $\int \left(3x^2 \tan\frac{1}{x} - x\sec^2\frac{1}{x}\right) dx$ equals to

(a) $x\left(\tan\frac{1}{x} + \sec\frac{1}{x}\right) + c$

(b) $x^2 \tan\frac{1}{x} - \sec\frac{1}{x} + c$

(c) $x^3 \tan\frac{1}{x} + c$

(d) $(x^3 - 1)\tan\frac{1}{x} + c$

40. The value of $\int e^{\tan\theta}(\sec\theta - \sin\theta)d\theta$ is

(a) $e^{\tan\theta}\sec\theta + c$

(b) $e^{\tan\theta}\sin\theta + c$

(c) $e^{\tan\theta}(\sec\theta + \sin\theta) + c$

(d) $e^{\tan\theta}\cos\theta + c$

41. If $\int \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}} dx. = -2\tan^{-1} u + c$ then u is equal to

(a) $\sqrt{1+\tan x + \cot x}$

(b) $\sqrt{1+\tan x + \tan^2 x}$

(c) $\sqrt{\tan x + \cot x}$

(d) $\tan^{-1}(\tan x + \cot x)$

42. The value of integral $\int \frac{1+(\sin x)^{2/3}}{1+(\sin x)^{4/3}} . d(\sqrt[3]{\sin x})$ is equal to

(a) $\sin^{-1}(1+\sqrt[3]{\sin x}) + c$

(b) $\sin^{-1}\sqrt[3]{\sin x} + c$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt[3]{\sin x} - 1}{\sqrt{2}.\sqrt[3]{\sin x}}\right) + c$

(d) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt[3]{\sin x}}{\sqrt{2}} + c$

MARK YOUR RESPONSE					
	34. ⓐⓑⓒⓓ	35. ⓐⓑⓒⓓ	36. ⓐⓑⓒⓓ	37. ⓐⓑⓒⓓ	38. ⓐⓑⓒⓓ
	39. ⓐⓑⓒⓓ	40. ⓐⓑⓒⓓ	41. ⓐⓑⓒⓓ	42. ⓐⓑⓒⓓ	

43. The integral $\int \frac{x^2(x\sec^2 x+\tan x)}{(x\tan x+1)^2}dx$ is equal to

(a) $-\frac{x^2}{x\tan x+1}+c$

(b) $2\log_e|x\sin x+\cos x|+c$

(c) $-\frac{x^2}{x\tan x+1}+2\log_e|x\sin x+\cos x|+c$

(d) $\frac{x^2}{x^2\tan x-1}-2\log_e|x\sin x+\cos x|+c$

44. If $\int \frac{dx}{(a+bx^2)\sqrt{b-ax^2}}$ =K $\tan^{-1}(L\tan\theta)+M$, M being constant of integration then KL is equal to

(a) 1 (b) $\frac{1}{a}$

(c) $\frac{1}{\sqrt[a]{a}}$ (d) $\sqrt{a}$

45. If $xf(x)=3f^2(x)+2$ then

$\int \frac{2x^2-12xf(x)+f(x)}{(6f(x)-x)(x^2-f(x))^2}dx$ equals

(a) $\frac{1}{x^2-f(x)}+c$ (b) $\frac{1}{x^2+f(x)}+c$

(c) $\frac{1}{x-f(x)}+c$ (d) $\frac{1}{x+f(x)}+c$

MARK YOUR RESPONSE	43. ⓐⓑⓒⓓ	44. ⓐⓑⓒⓓ	45. ⓐⓑⓒⓓ		

B COMPREHENSION TYPE

This section contains groups of questions. Each group is followed by some multiple choice questions based on a paragraph. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

PASSAGE-1

In general if we have an integral of type $\int f(g(x))g'(x)dx$, we substitute $g(x)=t \Rightarrow g'(x)dx=dt$ and the integral becomes $\int f(t)dt$.

Some of the substitution can be guessed by keen observation of the nature of given integrand. For example, we have $\frac{d}{dx}\left(x+\frac{1}{x}\right)=1-\frac{1}{x^2}$. So if the integrand is of the type $f\left(x+\frac{1}{x}\right).\left(1-\frac{1}{x^2}\right)$, we can substitute $x+\frac{1}{x}=t$

Some more similar forms are given below

for integral $\int f\left(x-\frac{a}{x}\right).\left(1+\frac{a}{x^2}\right)dx$, put $x-\frac{a}{x}=t$

for integral $\int f\left(x+\frac{a}{x}\right).\left(1-\frac{a}{x^2}\right)dx$, put $x+\frac{a}{x}=t$

for integral $\int f\left(x^2-\frac{a}{x^2}\right).\left(x+\frac{a}{x^3}\right)dx$, put $x^2-\frac{a}{x^2}=t$

for integral $\int f\left(x^2+\frac{a}{x^2}\right).\left(x-\frac{a}{x^3}\right)dx$, put $x^2+\frac{a}{x^2}=t$

Many integrands can be brought into above forms by suitable reductions or transformations

1. $\int \frac{x^2+1}{x^4+1}dx=$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x}+C$ (b) $\sin^{-1}\frac{x^2+1}{\sqrt{2}x}+C$

(c) $\frac{1}{2}\log\frac{\sqrt{2}x+1}{\sqrt{2}x-1}+c$ (d) $x^2+\frac{1}{x^2}+C$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ				

2. $\int \frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}dx =$

(a) $\tan^{-1}\left(x+\frac{1}{x}\right)+C$

(b) $\left(x+\frac{1}{x}\right)\tan^{-1}\left(x+\frac{1}{x}\right)+C$

(c) $\ln\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right|+C$

(d) $\frac{1}{2}\ln\left|x+\frac{1}{x}\right|+C$

3. $\int \frac{x^4-2}{x^2\sqrt{x^4+x^2+2}}dx =$

(a) $\sqrt{x^2+1+\frac{1}{x^2}}+C$ (b) $\sqrt{x^2+1+\frac{2}{x^2}}+C$

(c) $\sqrt{x^2+\frac{1}{x^2}}+C$ (d) $\sqrt{x^2+\frac{2}{x^2}}+C$

4. The derivative of $x^{-4}+x^{-5}$ is $-(4x^{-5}+5x^{-6})$. So,

$\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx =$

(a) $x^5+x+1+C$ (b) $\frac{1}{x^5+x+1}+C$

(c) $x^{-4}+x^{-5}+C$ (d) $\frac{x^5}{x^5+x+1}+C$

PASSAGE-2

Integrals of the form $\int f(x,\sqrt{ax^2+bx+c})dx$ can be evaluated with the help of the Euler's substitutions. There are normally three Euler's substitutions :

I. **First Euler's substitution**

If $a>0$, we put $\sqrt{ax^2+bx+c}=t\pm x\sqrt{a}$

or $ax^2+bx+c=t^2+ax^2\pm 2tx\sqrt{a}$

or $bx+c=t^2\pm 2tx\sqrt{a}$

II. **Second Euler's Substitutions**

If $c>0$, we put $\sqrt{ax^2+bx+c}=tx\pm\sqrt{c}$

or $ax+b=t^2x\pm 2t\sqrt{c}$

III. **Third Euler's substitution**

If the trinomial ax^2+bx+c has real roots α and β that is $ax^2+bx+c=a(x-\alpha)(x-\beta)$ then we put

$\sqrt{ax^2+bx+c}=(x-\alpha)t$ or $(x-\beta)t$

5. $\int \frac{\left(x+\sqrt{1+x^2}\right)^{15}}{\sqrt{1+x^2}}dx =$

(a) $\frac{(x+\sqrt{1+x^2})^{16}}{16}+C$ (b) $\frac{(x+\sqrt{1+x^2})^{15}}{15}+C$

(c) $\frac{1}{15(\sqrt{1+x^2}+x)^{15}}+C$ (d) $\frac{15}{(\sqrt{1+x^2}-x)^{15}}+C$

6. $\int \frac{dx}{(x-1)\sqrt{-x^2+3x-2}}$ is equal to

(A) $-2\sqrt{\frac{x-2}{1-x}}+c$ (B) $-2\sqrt{\frac{x-2}{x-1}}+c$

(C) $2\sqrt{\frac{1-x}{x-2}}+c$ (D) $\sqrt{\frac{1-x}{x-2}}+c$

7. If $f(x)$ is the antiderivative obtained in Q. No.6 then the limit $\lim_{x\to 2}\frac{\sin f(x)}{\sqrt{2-x}}$ $(x<2)$ is

(a) 0 (b) 1

(c) –2 (d) not finite

PASSAGE-3

In some of the cases we can split the integrand into the sum of the two functions such that the integration of one of them by parts produces an integral which cancels the other integral.

Suppose we have an integral of the type $\int [f(x)\,h(x)+g(x)]\,dx$

Let $\int f(x)\,h(x)\,dx = I_1$ and $\int g(x)\,dx = I_2$

Integrating I_1 by parts we get

$I_1 = f(x)\int h(x)\,dx - \int\{f'(x)\int h(x)\,dx\}dx$

Suppose $\int\{f'(x)\int h(x)\,dx\}$ converts to I_2, then we get

Mark Your Response	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ	6. ⓐⓑⓒⓓ
	7. ⓐⓑⓒⓓ				

$I_1 + I_2 = f(x)\int h(x)dx + C$, which is the desired integral.

In particular consider the integral of the kind

$I = \int e^x\{f(x) + f'(x)\}\,dx = \int e^x f(x)\,dx + \int e^x\, f'(x)\,dx$

Integrating first integral by parts, we get (e^x is second function)

$I = e^x f(x) - \int e^x f'(x)\,dx + \int e^x f'(x)\,dx = e^x f(x) + C$

8. The integral of $f(x) = \dfrac{1}{\ln x} - \dfrac{1}{(\ln x)^2}$ is

(a) $\ln(\ln x) + C$ (b) $x \ln x + C$

(c) $\dfrac{x}{\ln x} + C$ (d) $x + \ln x + C$

9. $\int \dfrac{x + \sin x}{1 + \cos x}dx =$

(a) $\tan\dfrac{x}{2} + C$ (b) $x\tan\dfrac{x}{2} + C$

(c) $x + \cos x + C$ (d) $e^x \tan\dfrac{x}{2} + C$

10. $\int \dfrac{xe^x}{(1+x)^2}\,dx =$

(a) $xe^x + C$ (b) $\dfrac{e^x}{(x+1)^2} + C$

(c) $e^x - \dfrac{1}{x+1} + C$ (d) $\dfrac{e^x}{x+1} + C$

11. Antiderivative of $f(x) = \log(\log x) + \dfrac{1}{(\log x)^2}$ is

(a) $\log(\log x)$ (b) $x\log(\log x) - \dfrac{x}{\log x}$

(c) $\dfrac{x}{\log x} - \log x$ (d) $\log(\log x) - \dfrac{x}{\log x}$

MARK YOUR RESPONSE	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ	

C REASONING TYPE

In the following questions two Statements (1 and 2) are provided. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. Mark your responses from the following options:

(a) Both Statement-1 and Statement-2 are true and Statement-2 is the correct explanation of Statement-1.

(b) Both Statement-1 and Statement-2 are true and Statement-2 is not the correct explanation of Statement-1.

(c) Statement-1 is true but Statement-2 is false.

(d) Statement-1 is false but Statement-2 is true.

1. **Statement - 1 :** If $I_n = \int \tan^n x\,dx$ then $5(I_4 + I_6) = \tan^5 x$

Statement - 2 : If $I_n = \int \tan^n x\,dx$ then

$I_n = \dfrac{\tan^{n+1} x}{n+1} - I_{n-2}$ where $n \in N$

2. **Statement - 1 :** If $\int \dfrac{1}{f(x)}dx = \log_e (f(x))^2 + c$ then

$f(x) = \dfrac{1}{2}x$.

Statement - 2 : $\int \dfrac{f'(x)}{f(x)}dx = \log_e |f(x)| + c$

3. **Statement - 1 :** $\int \dfrac{3 + 4\cos x}{(4 + 3\cos x)^2}dx = \left(\dfrac{\sin x}{4 + 3\cos x}\right) + C$

Statement - 2 : $\int \{f(x)\}^n f'(x)dx = \dfrac{\{f(x)\}^{n+1}}{n+1} + C,\ n \neq -1$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ		

D MULTIPLE CORRECT CHOICE TYPE

Each of these questions has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

1. If $f(x)=\lim\limits_{n\to\infty} e^{x\tan(1/n)\ln(1/n)}$ and $\int \frac{f(x)}{\sqrt[3]{\sin^{11}x\cos x}}dx = g(x)+C$ (C being the constant of integration), then

(a) $g\left(\frac{\pi}{4}\right)=\frac{3}{2}$

(b) $g(x)$ is continuous for all x

(c) $g\left(\frac{\pi}{4}\right)=-\frac{15}{8}$

(d) $g(x)$ is not differentiable at infinitely many points

2. $\int e^x\left\{\frac{2\tan x}{1+\tan x}+\cot^2\left(x+\frac{\pi}{4}\right)\right\}dx$ is equal to

(a) $e^x\tan\left(\frac{\pi}{4}-x\right)+C$ (b) $e^x\cot\left(\frac{3\pi}{4}-x\right)+C$

(c) $e^x\tan\left(x-\frac{\pi}{4}\right)+C$ (d) $e^x\cot\left(x+\frac{\pi}{4}\right)+C$

3. If $I=\int_0^1\frac{dx}{1+x^{\pi/2}}$ then

(a) $I>\ln 2$ (b) $I<\ln 2$

(c) $I<\frac{\pi}{4}$ (d) $I>\frac{\pi}{4}$

4. If $\forall\, x\in[-1,0)$, $\int\left(\cos^{-1}x+\cos^{-1}\sqrt{1-x^2}\right)dx$

$=Ax+f(x)\sin^{-1}x-2\sqrt{1-x^2}+C$, then

(a) $A=\frac{\pi}{4}$ (b) $A=\frac{\pi}{2}$

(c) $f(x)=x$ (d) $f(x)=-2x$

5. If $\int x^{-1/2}(2+3x^{1/3})^{-2}dx=$

$A\tan^{-1}\left\{\sqrt{\frac{3}{2}}x^{1/6}\right\}+B\frac{x^{1/6}}{2+3x^{1/3}}+C$ then

(a) $A=\frac{1}{\sqrt{6}}$ (b) $A=-\frac{1}{\sqrt{6}}$

(c) $B=1$ (d) $B=-1$

6. If $\int\frac{x^4+1}{x^6+1}dx=\tan^{-1}f(x)-\frac{2}{3}\tan^{-1}g(x)+C$ then

(a) $f(x)=x+\frac{1}{x}$ (b) $f(x)=x-\frac{1}{x}$

(c) $g(x)=x^{-3}$ (d) $g(x)=x^3$

7. The value of the integral $\int e^{\sin^2 x}(\cos x+\cos^3 x)\sin x\,dx$ is

(a) $\frac{1}{2}e^{\sin^2 x}(3-\sin^2 x)+c$

(b) $e^{\sin^2 x}\left(1+\frac{1}{2}\cos^2 x\right)+c$

(c) $e^{\sin^2 x}(3\cos^2 x+2\sin^2 x)+c$

(d) $e^{\sin^2 x}(2\cos^2 x+3\sin^2 x)+c$

8. If $I=\int\frac{(x^2+n)(n-1)x^{2n-1}}{(x\sin x+n\cos x)^2}dx=f(x)+g(x)+c$ then

(a) $f(x)=\frac{x^n}{x^n\sin x+n\cos x}$

(b) $f(x)=-\frac{x^n\sec x}{x^n\sin x+nx^{n-1}\cos x}$

(c) $g(x)=\tan x$

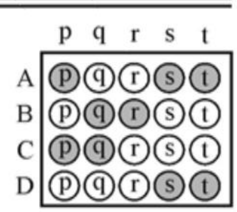
(d) $g(x)=\sec x$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ
	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ		

E

MATRIX-MATCH TYPE

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labeled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column -I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example:
If the correct matches are A–p, s and t; B–q and r; C–p and q; and D–s and t; then the correct darkening of bubbles will look like the given.



1. Observe the following Columns :

Column-I		Column-II
(A) $\int \frac{e^x}{x+2}[(1+(x+2)\log(x+2)]dx$	p.	$\sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$
(B) $\int \sin^2 x \cos^3 x\, dx$	q.	$\frac{x^4}{4}-\frac{x^2}{2}+\frac{1}{2}\log(x^2+1)+c$
(C) $\int \frac{dx}{\sqrt{2-3x-x^2}}$	r.	$e^x \log(x+2)+c$
(D) $\int \frac{x^5}{x^2+1}dx$	s.	$\frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+c$

2. Observe the following Columns :

Column-I		Column-II
(A) $\int \frac{dx}{\sqrt{x}(x+9)}=$	p.	$\log\left\|1-\tan\left(\frac{x}{2}\right)\right\|+c$
(B) $\int e^x(1-\cot x+\cot^2 x)\,dx=$	q.	$\log\left\|1-\cot\left(\frac{x}{2}\right)\right\|+c$
(C) $\int \frac{\sin^3 x+\cos^3 x}{\cos^2 x \sin^2 x}dx=$	r.	$\sec x-\operatorname{cosec} x+c$
(D) $\int \frac{dx}{1-\cos x-\sin x}=$	s.	$\frac{2}{3}\tan^{-1}\left(\frac{\sqrt{x}}{3}\right)+c$
	t.	$-e^x . \cot x + c$

3. Observe the following Columns :

Column-I		Column-II
(A) $\int (e^{a\log x}+e^{x\log a})\,dx$	p.	$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+c$
(B) $\int \frac{e^{\log\left(1+\frac{1}{x^2}\right)}dx}{x^2+\frac{1}{x^2}}$	q.	$\frac{1}{4}\tan^{-1}\left(\tan x+\frac{1}{2}\right)+c$
(C) $\int \frac{dx}{4\sin^2 x+4\sin x\cos x+5\cos^2 x}$	r.	$\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$
(D) $\int (\sqrt{\sin x}+\sqrt{\cos x})^{-4}dx$	s.	$-\frac{1}{(1+\sqrt{\tan x})^2}+\frac{2}{3(1+\sqrt{\tan x})^3}+c$

MARK YOUR RESPONSE

1.

	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

2.

	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

3.

	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

4. If $\int \frac{\log_e(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$ dx = fog(x) + C, Now match the entries from the following columns:

Column-I		**Column-II**
(A) $f(2)$ is equal to	p.	0
(B) g (0) is equal to	q.	1
(C) If $\int f(x)g(x)dx = ax^3g(x)+b(1+x^2)^{3/2}+c(1+x^2)^{1/2}+d$, then $a+c$ is equal to	r.	2
(D) If $\int e^{g(x)}dx = ax\left(x+\sqrt{1+x^2}+ag(x)+c\right)$ then a is equal to	s.	$\frac{1}{2}$
	t.	$\frac{1}{3}$

5. **Observe the following Column :**

Column-I		**Column-II**
(A) If $\int x^2 d(\tan^{-1}x) = x+f(x)+c$, then f(1) is equal to	p.	0
(B) If $\int \sqrt{1+2\tan x(\tan x+\sec x)}dx = a\log\left\|\cos\frac{x}{2}-\sin\frac{x}{2}\right\|+c$ then a is equal to $(0<x<\frac{\pi}{2})$	q.	–2
(C) If $\int x^2e^{2x}dx = e^{2x}f(x)+c$, then the minimum value of $f(x)$ is equal to	r.	$\frac{\pi}{4}$
(D) If $\int \frac{x^4+1}{x(x^2+1)^2}dx = a\log\|x\|+\frac{b}{x^2+1}+c$ then a – b is equal to	s.	1
	t.	$\frac{1}{8}$

6. **Observe the following columns :**

Column I		**Column II**
(A) If $\int \frac{x+(\cos^{-1}3x)^2}{\sqrt{1-9x^2}}dx$ $= A\sqrt{1-9x^2}+B(\cos^{-1}3x)^3+C$, then	p.	$A=-\frac{1}{2}$
(B) If $\int \frac{(2x+1)dx}{x^4+2x^3+x^2-1}dx = A\ell n\left\|\frac{x^2+x+1}{x^2+x-1}\right\|+C$, then	q.	$A=-\frac{1}{9}$
(C) If $\int x^5(x^{10}+x^5+1)(2x^{10}+3x^5+6)^{1/5}dx$ $=-\frac{A}{4}(2x^{15}+3x^{10}+6x^5)^{6/5}+C$, then	r.	$B=-\frac{1}{9}$
(D) If $\int \frac{x^3-x}{\sqrt{1-x^2}}\frac{1}{(1+\sqrt{1-x^2})}dx$ $=-f(x)+Ax^2+B\ln f(x)+C$, then	s.	B = 1
	t.	A = B

MARK YOUR RESPONSE

4.

	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

5.

	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

6.

	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

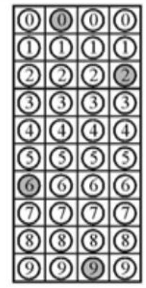
F NUMERIC/INTEGER ANSWER TYPE

The answer to each of the questions is either numeric (eg. 304, 40, 3010 etc.) or a single-digit integer, ranging from 0 to 9.

The appropriate bubbles below the respective question numbers in the response grid have to be darkened.

For example, if the correct answers to a question is 6092, then the correct darkening of bubbles will look like the given.

For single digit integer answer darken the extreme right bubble only.



1. If $\int \dfrac{\sin^3 \frac{\theta}{2}\, d\theta}{\cos\frac{\theta}{2}\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}} = \tan^{-1}\sqrt{f(\theta)}+c$ then the least value of $f(\theta)$ for allowable values of θ is equal to

2. If $P=\int e^{ax}\cos bx\,dx$ and $Q=\int e^{ax}\sin bx\,dx$, without constant of integration then $e^{-2ax}(P^2+Q^2)(a^2+b^2)$ is equal to

3. If $\int \sin 4x\, e^{\tan^2 x}dx = -A\cos^4 x e^{\tan^2 x}+B$, then A is equal to

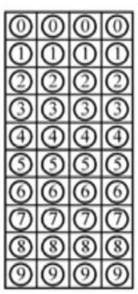
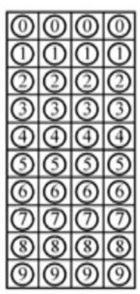
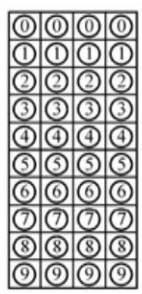
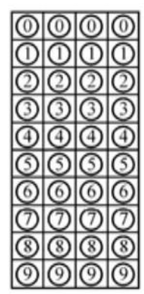
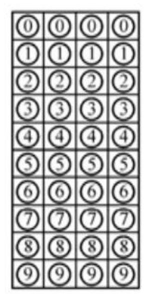
4. If $\int \operatorname{cosec}^2 x\, ln\left(\cos x+\sqrt{\cos 2x}\right)dx$

$= f(x)\,ln\left(\cos x+\sqrt{\cos 2x}\right)+g(x)+f(x)-x+c,$

then $f^2(x)-g^2(x)$ is equal to $\left(0<x\le\dfrac{\pi}{2}\right)$.

5. If $\int \dfrac{\sqrt{\cot x}-\sqrt{\tan x}}{4+3\sin 2x}dx = \dfrac{1}{2\sqrt{2}}\ell n\left|\dfrac{f(x)+a}{f(x)-a}\right|$, where $0<x<\dfrac{\pi}{2}$ then a is equal to

MARK YOUR RESPONSE

1. 
2. 
3. 
4. 
5. 

Solutions

A SINGLE CORRECT CHOICE TYPE

1. **(c)** $I=\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx$

$=\int \frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}}dx$

$=\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}dx$

Put $x+\frac{1}{x}+1=t^2 \Rightarrow \left(1-\frac{1}{x^2}\right)dx=2tdt$

$\therefore\ I=\int \frac{2tdt}{(t^2+1)t}=2\int \frac{dt}{t^2+1}=2\tan^{-1}t+C$

$=2\tan^{-1}\sqrt{x+\frac{1}{x}+1}+C$

2. **(a)** $I=\int e^{\sin x}.x\cos x\, dx-\int e^{\sin x}\sec x\tan x\, dx$

$=\left\{xe^{\sin x}-\int e^{\sin x}\, dx\right\}-\left\{e^{\sin x}.\sec x-\int e^{\sin x}\, dx\right\}$

$=xe^{\sin x}-e^{\sin x}\sec x+C$

3. **(b)** $I=\int \frac{\sin^3 xdx}{(\cos^4 x+3\cos^2 x+1)\tan^{-1}(\sec x+\cos x)}$

Let $\tan^{-1}(\sec x+\cos x)=t$

$\Rightarrow\ \frac{1}{1+(\sec x+\cos x)^2}(\sec x\tan x-\sin x)dx=dt$

or $\frac{\sin^3 x\,dx}{\cos^4 x+3\cos x^2+1}=dt$

$\therefore\ \ I=\int \frac{dt}{t}=ln|t|+c$

$=ln|\tan^{-1}(\sec x+\cos x)|+c$

4. **(c)** We have $\int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}dx$

$=\int e^x\left[\sqrt{\frac{1+x^n}{1-x^n}}+\frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right]dx$

$=e^x\sqrt{\frac{1+x^n}{1-x^n}}+C$

Here, $f(x)=\sqrt{\frac{1+x^n}{1-x^n}}$, then

$f'(x)=\frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}$

and $\int e^x(f(x)+f'(x))dx=e^x f(x)+C$

5. **(d)** $\lim\limits_{n\to\infty} n^2\left(x^{1/n}-x^{1/(n+1)}\right)$

$=\lim\limits_{h\to 0}\frac{1}{h^2}\left(x^h-x^{\frac{h}{h+1}}\right)$

$=\lim\limits_{h\to 0}\frac{1}{h^2}\left(x^{\left(h-\frac{h}{h+1}\right)}-1\right)x^{\frac{h}{h+1}}$

$=\lim\limits_{h\to 0}\left(\frac{x^{\frac{h^2}{h+1}}-1}{\frac{h^2}{h+1}}\right).\left(\frac{1}{h+1}\right)x^{\frac{h}{h+1}}$

$=\ln x.1.1\qquad \left(\because \lim\limits_{x\to 0}\frac{a^x-1}{x}=\ln a\right)$

$\therefore\ f(x)=\ln x$

So, $I=\int xf(x)dx=\int x\ln xdx$

$=\frac{x^2}{2}\ln x-\int \frac{x^2}{2}.\frac{1}{x}dx$

$=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C$

6. **(b)** $I=\int \frac{1+x^4}{(1-x^4)^{3/2}}dx$

$=\int \frac{x^3(x+1/x^3)dx}{(1-x^4)^{3/2}}=\int \frac{(x+1/x^3)dx}{\left(\frac{1}{x^2}-x^2\right)^{\frac{3}{2}}}$

Let $\frac{1}{x^2}-x^2=t \Rightarrow \left(\frac{-2}{x^3}-2x\right)dx=dt$

$\Rightarrow \left(x+\frac{1}{x^3}\right)dx=-\frac{1}{2}dt$

$\therefore\ I=-\frac{1}{2}\int\frac{dt}{t^{3/2}}=\frac{1}{\sqrt{t}}+c=\frac{1}{\sqrt{\frac{1}{x^2}-x^2}}+C$

7. **(d)** $I=\int\left(\frac{f(x)}{x^2}\right)^{\frac{1}{2}}dx=\int\left(\frac{x+2}{2x+3}\right)^{\frac{1}{2}}\frac{dx}{x}$

Put $\frac{x+2}{2x+3}=y^2 \Rightarrow x=\frac{3y^2-2}{1-2y^2}$

and $dx=\frac{-2ydy}{(1-2y^2)^2}$

$\therefore\ I=-\int y.\frac{2y}{(1-2y^2)^2}.\frac{1-2y^2}{3y^2-2}dy$

$=2\int\frac{y^2dy}{(2y^2-1)(3y^2-2)}$

$=2\int\left(\frac{2}{3y^2-2}-\frac{1}{2y^2-1}\right)dy$

$=\frac{4}{3}\int\frac{dy}{y^2-\frac{2}{3}}-\int\frac{dy}{y^2-\frac{1}{2}}$

$=\frac{4}{3}.\frac{1}{2\sqrt{\frac{2}{3}}}ln\left|\frac{y-\sqrt{\frac{2}{3}}}{y+\sqrt{\frac{2}{3}}}\right|-\frac{1}{2\times\frac{1}{\sqrt{2}}}ln\left|\frac{y-\frac{1}{\sqrt{2}}}{y+\frac{1}{\sqrt{2}}}\right|+C$

$=\frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}y}{1-\sqrt{2}y}\right|-\sqrt{\frac{2}{3}}\log\left|\frac{\sqrt{3}y+\sqrt{2}}{\sqrt{3}y-\sqrt{2}}\right|+C$

$=\frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right|-\sqrt{\frac{2}{3}}\log\left|\frac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right|+C$

Thus $g(x)=\log|x|$ and $h(x)=\log|x|$

8. **(d)** $I=\int\sqrt{1+\text{cosec}\,x}dx=\int\frac{\sqrt{1+\sin x}}{\sqrt{\sin x}}$

$=\int\pm\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sqrt{2\sin\frac{x}{2}\cos\frac{x}{2}}}=\pm\int\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sqrt{1-\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}dx$

Put $\sin\frac{x}{2}-\cos\frac{x}{2}=t \Rightarrow \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)dx=2dt$

$I=\pm\int\frac{2dt}{\sqrt{1-t^2}}=\pm2\sin^{-1}t=\pm2\sin^{-1}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+c$

9. **(d)** Let $g_n(x)=1+x^2+x^4+....+x^{2n}=\frac{x^{2n+2}-1}{x^2-1}$

So, $2x+4x^3+.....+2nx^{2n-1}=$

$h_n(x)=g'_n(x)=\frac{2x(nx^{2n+2}-(n+1)x^{2n}+1)}{(x^2-1)^2}$

Now $f(x)=\lim\limits_{n\to\infty}h_n(x)=\frac{2x}{(x^2-1)^2}$ as $0<x<1$

Thus $\int f(x)dx=\int\frac{2x}{(x^2-1)^2}dx$

$=-\frac{1}{x^2-1}=\frac{1}{1-x^2}+c$

10. **(c)** Here $-\frac{1}{2}-\frac{7}{2}=-4=$ negative even number

Put $\tan x=t \Rightarrow \sec^2x\,dx=dt$

$f(x)=\int\frac{dx}{\frac{\sin^{1/2}}{\cos^{1/2}}.\cos^4x}=\int\frac{(1+\tan^2x)\sec^2xdx}{\sqrt{\tan x}}$

$=\int\left(\frac{1+t^2}{\sqrt{t}}\right)dt=\int(t^{-1/2}+t^{3/2})dt$

$=2t^{1/2}+\frac{2}{5}t^{5/2}=2\sqrt{\tan x}+\frac{2}{5}(\tan x)^{5/2}$

$\therefore\ f\left(\frac{\pi}{4}\right)-f(0)=2+\frac{2}{5}=\frac{12}{5}$

11. (b) $f\left(\dfrac{3x-4}{3x+4}\right)=x+2$. Let, $\dfrac{3x-4}{3x+4}=t$

$3x-4=3xt+4t$

$x=\dfrac{4t+4}{3(1-t)}$ $\quad\therefore\ f(t)=\dfrac{4t+4}{3(1-t)}+2$

$\therefore\ f(x)=\dfrac{4x+4}{3(1-x)}+2=\dfrac{4(x-1)+8}{3(1-x)}+2$

$=2-\dfrac{4}{3}-\dfrac{8}{3(x-1)}$

$\int f(x)dx=\dfrac{2}{3}x-\dfrac{8}{3}ln|x-1|+c$

12. (b) The given integral is $I=\int\dfrac{(\sec\theta-1)\sqrt{\sec\theta}}{1+\sec^2\theta}\tan\theta d\theta$

$=\int\dfrac{(\sec\theta-1)\sec\theta\tan\theta}{(1+\sec^2\theta)\sqrt{\sec\theta}}d\theta$

Put $\sqrt{\sec\theta}=t\ \Rightarrow\ \dfrac{1}{2\sqrt{\sec\theta}}\sec\theta\tan\theta d\theta=dt$

$\therefore\ I=\int\dfrac{(t^2-1)}{1+t^4}.2dt$

$=2\int\dfrac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=2\int\dfrac{\left(1-\frac{1}{t^2}\right)dt}{\left(1+\frac{1}{t}\right)^2-2}$

Put $t+\dfrac{1}{t}=u\ \Rightarrow\ \left(1-\dfrac{1}{t^2}\right)dt=du$

$\therefore\ I=2\int\dfrac{du}{u^2-2}=\dfrac{2}{2\sqrt{2}}\log_e\left|\dfrac{u-\sqrt{2}}{u+\sqrt{2}}\right|+C$

$=\dfrac{1}{\sqrt{2}}\log_e\left|\dfrac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|+C$

$=\dfrac{1}{\sqrt{2}}\log_e\left|\dfrac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right|+C$

$=\dfrac{1}{\sqrt{2}}\log_e\left|\dfrac{\sec\theta-\sqrt{2\sec\theta}+1}{\sec\theta+\sqrt{2\sec\theta}+1}\right|+C$

13. (a) $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n}-1}{x^{2n}+1}=-1\quad(\because 0<x<1)$, so

$\int\sin^{-1}x(f(x))dx=-\int\sin^{-1}x\,dx$

$=-[x\sin^{-1}x+\sqrt{1-x^2}]+C$

14. (d) Since, $x>1$ so $x^{-n}\longrightarrow 0$ as $n\longrightarrow\infty$.

Hence $f(x)=\lim\limits_{n\to\infty}\dfrac{1-x^{-2n}}{1+x^{-2n}}=1$

So, $\int\dfrac{xf(x)\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx$

$=\dfrac{1}{2}\int\dfrac{2x}{\sqrt{1+x^2}}\log(x+\sqrt{1+x^2})dx$

$=\sqrt{1+x^2}\log(x+\sqrt{1+x^2})$

$-\int\dfrac{\sqrt{1+x^2}}{x+\sqrt{1+x^2}}\times\dfrac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$

$=\sqrt{1+x^2}\log(x+\sqrt{1+x^2})-x+C$

15. (a) $I_n=\int\cot^n x\,dx=\int\cot^{n-2}x\cot^2x\,dx$

$=\int\cot^{n-2}x(\mathrm{cosec}^2x-1)dx$

$=\int\cot^{n-2}x\mathrm{cosec}^2x\,dx-I_{n-2}$

Thus $I_n+I_{n-2}=\dfrac{\cot^{n-1}x}{n-1}$ ......(i)

$I_0+I_1+2(I_2+......+I_8)+I_9+I_{10}$

$=(I_2+I_0)+(I_3+I_1)+(I_4+I_2)+(I_5+I_3)$

$+(I_6+I_4)+(I_7+I_5).+(I_8+I_6)+(I_9+I_7)$

$+(I_{10}+I_8)$

$=-\left(\dfrac{\cot x}{1}+\dfrac{\cot^2x}{2}+.......+\dfrac{\cot^9x}{9}\right)$ [using (i)]

$=-\sum\limits_{k=1}^{9}\dfrac{\cot^kx}{k}$

16. (a) Let $I=\int\dfrac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}}d\theta$

$=\int\dfrac{(2\sin^2\theta/2)^{2/7}}{(2\cos^2\theta/2)^{9/7}}d\theta$

$=\dfrac{1}{2}\int\dfrac{(\sin\theta/2)^{4/7}}{(\cos\theta/2)^{18/7}}d\theta$

Put $\frac{\theta}{2}=t$, $\therefore \frac{d\theta}{2}=dt$

$\therefore I=\int\frac{(\sin t)^{4/7}}{(\cos t)^{18/7}}dt$ (Here $m+n=-2$)

$=\int(\tan t)^{4/7}\sec^2 t dt$

Put $\tan t=u$ $\therefore \sec^2 tdt=du$

$\therefore I=\int u^{4/7}du=\frac{u^{11/7}}{11/7}+c$

$=\frac{7}{11}(\tan t)^{11/7}+c=\frac{7}{11}\left(\tan\frac{\theta}{2}\right)^{11/7}+c$

17. (c) $I_{4,3}=\int\cos^4 x\sin 3x dx$

Integrating by parts, we have

$I_{4,3}=-\frac{\cos 3x\cos^4 x}{3}-\frac{4}{3}\int\cos^3 x\sin x\cos 3x\, dx$

Now,

$\sin 2x=\sin(3x-x)=\sin 3x\cos x-\cos 3x\sin x$, so,

$I_{4,3}=-\frac{\cos 3x\cos^4 x}{3}+\frac{4}{3}\int\cos^3 x\sin 2x$

$-\frac{4}{3}\int\cos^4 x\sin 3x\,dx+C$

$=-\frac{\cos 3x\cos^4 x}{3}+\frac{4}{3}I_{3,2}-\frac{4}{3}I_{4,3}+C$

Therefore, $\frac{7}{3}I_{4,3}-\frac{4}{3}I_{3,2}=-\frac{\cos 3x\cos^4 x}{3}+C$

or $7I_{4,3}-4I_{3,2}=-\cos 3x\cos^4 x+$ const.

18. (a) Let $I=\int(x^{\sin x-1}.\sin x+x^{\sin x}.\cos x\ln x)dx$

Put $x^{\sin x}=t$

$\Rightarrow e^{\sin x\ln x}=t$

$\Rightarrow e^{\sin x\ln x}\left\{\sin x.\frac{1}{x}+\ln x.\cos x\right\}dx=dt$

$\Rightarrow x^{\sin x}\left\{\frac{\sin x}{x}+\cos x\ln x\right\}dx=dt$

$\Rightarrow (x^{\sin x-1}.\sin x+x^{\sin x}.\cos x\ln x)dx=dt$

$\therefore \mathrm{I}=\int \mathrm{dt}=\mathrm{t}+\mathrm{c}=\mathrm{x}^{\sin \mathrm{x}}+\mathrm{c}$

19. (a) $f(xy)=f(x)f(y)$, Put $x=y=1$, we get $f(1)=f^2(1)$

$\Rightarrow f(1)=1$

Differentiation w.r.t x (partially),

we get $yf'(xy)=f'(x).f(y)$

Putting $x=1$, $yf'(y)=f'(1).f(y)$

$\Rightarrow \frac{f(y)}{f'(y)}=\frac{y}{f'(1)}$

$\therefore \int\frac{f(x)}{f'(x)}dx=\int\frac{x}{f'(1)}dx=\frac{1}{f'(1)}\left(\frac{x^2}{2}+c\right)$

$\because f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$=\lim_{h\to 0}\frac{1+h+hg(h)-1}{h}=1$

$\therefore \int\frac{f(x)}{f'(x)}dx=\frac{x^2}{2}+c$

20. (b) We have $\int\frac{dx}{x^2(x^n+1)^{(n-1)/n}}$

$=\int\frac{dx}{x^2.x^{n-1}\left(1+\frac{1}{x^n}\right)^{(n-1)/n}}=\int\frac{dx}{x^{n+1}(1+x^{-n})^{(n-1)/n}}$

Put $1+x^{-n}=t$

$\therefore -nx^{-n-1}dx=dt$ or $\frac{dx}{x^{n+1}}=-\frac{dt}{n}$ then

$\int\frac{dx}{x^2(x^n+1)^{(n-1)/n}}=-\frac{1}{n}\int\frac{dt}{t^{(n-1)/n}}=-\frac{1}{n}\int t^{\frac{1}{n}-1}dt$

$=-\frac{1}{n}\frac{t^{1/n-1+1}}{1/n-1+1}+c=-t^{1/n}+c=-(1+x^{-n})^{1/n}+c$

21. (a) Let $I=\int\frac{dx}{(x-1)\sqrt{(-x^2+3x-2)}}$

The trinomial $-x^2+3x-2=-(x-2)(x-1)$

Put $\sqrt{(-x^2+3x-2)}=(x-2)t$

$\therefore t=\sqrt{\frac{-(x-1)}{(x-2)}}$ $(\because 1<x<2)$

We get $x=\left(\frac{2t^2+1}{t^2+1}\right)$ $\therefore dx=\frac{2tdt}{(t^2+1)^2}$

Also, $\sqrt{(-x^2+3x-2)}=\frac{t}{(t^2+1)}$

$\therefore I=\int\frac{\frac{2tdt}{(t^2+1)^2}}{\left(\frac{t^2}{t^2+1}\right).\frac{t}{(t^2+1)}}=\int\frac{2}{t^2}dt=-\frac{2}{t}+c$

$=-2\sqrt{\left\{-\left(\frac{x-2}{x-1}\right)\right\}}+c$

22. (a) Let $I=\int \frac{(x+\sqrt{1+x^2})^{15}}{\sqrt{1+x^2}}dx$

Put $\sqrt{(1+x^2)}=t-x$ (Euler's substitution)

$\Rightarrow x+\sqrt{(1+x^2)}=t \quad \therefore \left(1+\frac{x}{\sqrt{(1+x^2)}}\right)dx=dt$

$\Rightarrow \frac{tdx}{\sqrt{(1+x^2)}}=dt$ or $\frac{dx}{\sqrt{(1+x^2)}}=\frac{dt}{t}$

then $I=\int \frac{t^{15}dt}{t}$

$=\int t^{14}dt=\frac{t^{15}}{15}+c=\frac{(x+\sqrt{(1+x^2)})^{15}}{15}+c$

23. (d) Let $I=\int \frac{e^x(2-x^2)}{(1-x)\sqrt{(1-x^2)}}dx$

$I=\int \frac{e^x(1+1-x^2)}{(1-x)\sqrt{(1-x^2)}}dx$

$=\int e^x\left\{\frac{1}{(1-x)\sqrt{(1-x^2)}}+\sqrt{\left(\frac{1+x}{1-x}\right)}\right\}dx$

Let $f(x)=\sqrt{\left(\frac{1+x}{1-x}\right)}$; $f'(x)=\frac{1}{(x-1)\sqrt{(1-x^2)}}$

$\because \int e^x(f(x)+f'(x))dx=e^x f(x)+c$

$\therefore I=e^x\sqrt{\left(\frac{1+x}{1-x}\right)}+c=\frac{e^x(1+x)}{\sqrt{1-x^2}}+C$

24. (b) Let $I=\int \frac{\ln(1+\sin^2 x)}{\cos^2 x}dx$

$=\int ln(1+\sin^2 x)\sec^2 x\,dx$

Intergrating by parts taking $\sec^2 x$ as the second fucntion, we have

$=ln(1+\sin^2 x).\tan x-2\int \frac{(\sin^2 x+1)-1}{(1+\sin^2 x)}dx$

$=\tan x\,ln(1+\sin^2 x)-2x+2\int \frac{dx}{(1+\sin^2 x)}$

$=\tan x\,ln(1+\sin^2 x)-2x+2\int \frac{\sec^2 x\,dx}{1+2\tan^2 x}$

Put $\sqrt{2}\tan x=t \qquad \therefore \sec^2 x dx=\frac{dt}{\sqrt{2}}$

$\therefore I=\tan x\,ln(1+\sin^2 x)-2x+\frac{2}{\sqrt{2}}\int \frac{dt}{1+t^2}$

$=\tan x\,ln(1+\sin^2 x)-2x+\sqrt{2}\tan^{-1}t+c$

$=\tan x\,ln(1+\sin^2 x)-2x+\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)+c$

25. (b) Let $I=\frac{(\sin^{3/2}\theta+\cos^{3/2}\theta)d\theta}{\sqrt{\sin^3\theta\cos^3\theta\sin(\theta+\alpha)}}$

$=\int \frac{\sin^{3/2}\theta d\theta}{\sqrt{\sin^3\theta\cos^3\theta\sin(\theta+\alpha)}}+\int \frac{\cos^{3/2}\theta d\theta}{\sqrt{\sin^3\theta\cos^3\theta\sin(\theta+\alpha)}}$

$=\int \frac{d\theta}{\sqrt{\cos^3\theta(\sin\theta\cos\alpha+\cos\theta\sin\alpha)}}$

$+\int \frac{d\theta}{\sqrt{\sin^3\theta(\sin\theta\cos\alpha+\cos\theta\sin\alpha)}}$

$=\int \frac{\sec^2\theta d\theta}{\sqrt{(\cos\alpha\tan\theta+\sin\alpha)}}+\int \frac{\text{cosec}^2\theta d\theta}{\sqrt{(\cos\alpha+\cot\theta\sin\alpha)}}$

Put $\cos\alpha\tan\theta+\sin\alpha=t^2$, in the first integral and $\cos\alpha+\cot\theta\sin\alpha=u^2$ in second integral

$\Rightarrow \sec^2 d\theta=\frac{2tdt}{\cos\alpha}$ and $\text{cosec}^2\theta d\theta=-\frac{2u\,du}{\sin\alpha}$

$\therefore I=\int \frac{2tdt}{(\cos\alpha)t}-\int \frac{2udu}{\sin\alpha.u}=\frac{2}{\cos\alpha}t-\frac{2}{\sin\alpha}u+c$

$=\frac{2}{\cos\alpha}\sqrt{(\cos\alpha\tan\theta+\sin\alpha)}$

$-\frac{2}{\sin\alpha}\sqrt{(\cos\alpha+\cot\theta\sin\alpha)}+c$

26. (a) Let, $I=\int \frac{x^2dx}{(x\sin x+\cos x)^2}$

$\because \frac{d}{dx}(x\sin x+\cos x)=x\cos x$, we write

$I=\int\left(\frac{x}{\cos x}\right).\left(\frac{x\cos x}{(x\sin x+\cos x)^2}\right)dx$

Integrating by part taking $\frac{x\cos x}{(x\sin x+\cos x)^2}$ as second function, we get

$$I = \left(\frac{x}{\cos x}\right)\left(-\frac{1}{x\sin x + \cos x}\right)$$

$$+\int \frac{(x\sin x + \cos x)}{\cos^2 x}\cdot\frac{1}{(x\sin x+\cos x)}dx$$

$$= -\frac{x}{\cos x(x\sin x + \cos x)} + \int \sec^2 x dx$$

$$= -\frac{x}{\cos x(x\sin x + \cos x)} + \tan x + c$$

$$= -\frac{x}{\cos x(x\sin x + \cos x)} + \frac{\sin x}{\cos x} + c$$

$$= \frac{-x + \sin x(x\sin x + \cos x)}{\cos x(x\sin x + \cos x)} + c$$

$$= \frac{(\sin x - x\cos x)}{(x\sin x + \cos x)} + c$$

27. (d) Let $I = \int \frac{(\cos^2 x + \sin 2x)}{(2\cos x - \sin x)^2}dx$

$$= \int \frac{(\cos x + 2\sin x)\cos x}{(2\cos x - \sin x)^2}dx$$

Integrating by part , taking cos x as the first and $\frac{(\cos x + 2\sin x)}{(2\cos x - \sin x)^2}$ as the second function, we have

$$= \cos x\left\{\frac{1}{2\cos x - \sin x}\right\} - \int \frac{-\sin x dx}{(2\cos x - \sin x)}$$

$$= \cos x\left\{\frac{1}{2\cos x - \sin x}\right\} + \int \frac{-\sin x\ dx}{(2\cos x - \sin x)}$$

$$= \frac{\cos x}{(2\cos x - \sin x)}$$

$$+\int \frac{-\frac{1}{5}(2\cos x - \sin x) - \frac{2}{5}(-2\sin x - \cos x)}{(2\cos x - \sin x)}dx$$

$$\left[N^{\gamma} = \lambda Dr + \mu \frac{d}{dx}Dr\right]$$

$$= \frac{\cos x}{(2\cos x - \sin x)} - \frac{1}{5}\int dx - \frac{2}{5}\int \frac{(-2\sin x - \cos x)}{2\cos x - \sin x}dx$$

$$= \frac{\cos x}{(2\cos x - \sin x)} - \frac{1}{5}x - \frac{2}{5}ln|2\cos x - \sin x| + c$$

28. (b) $I_n = \int \frac{dx}{(x^2+a^2)^n}$

$$= \frac{1}{(x^2+a^2)^n}.x - \int \frac{(-n)2x}{(x^2+a^2)^{n+1}}.xdx$$

[Integrating by parts using 1 as second function]

$$\therefore\ I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n(I_n - a^2 I_{n+1})$$

$$\Rightarrow 2na^2 I_{n+1} = \frac{x}{(x^2+a^2)} + (2n-1)I_n$$

Replace n by $n-1$, then

$$2(n-1)a^2 I_n = \frac{x}{(x^2+a^2)^{n-1}} + (2n-3)I_{n-1}$$

29. (c) $\because\ u_n = \int x^n\sqrt{(a^2-x^2)}dx = \int x^{n-1}\{x\sqrt{(a^2-x^2)}\}dx$

Integrating by parts taking x^{n-1} as first function, we have

$$= x^{n-1}\left\{\frac{-(a^2-x^2)^{3/2}}{3}\right\}$$

$$+\int (n-1)\frac{x^{n-2}(a^2-x^2)^{3/2}}{3}dx$$

$$\Rightarrow u_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3}$$

$$+\frac{(n-1)}{3}\int x^{n-2}(a^2-x^2)\sqrt{(a^2-x^2)}dx$$

$$\Rightarrow u_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3}u_{n-2} - \frac{(n-1)}{3}u_n$$

$$\Rightarrow \frac{(n+2)u_n}{3} = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n-1)a^2u_{n-2}}{3}$$

$$\Rightarrow u_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{(n+2)} + \frac{(n-1)a^2u_{n-2}}{(n+2)}$$

$$\Rightarrow\ (n+2)u_n - (n-1)a^2u_{n-2} = -x^{n-1}(a^2-x^2)^{3/2}$$

30. (b) Given $\int f(x)\sin x\cos x dx = \frac{1}{2(b^2-a^2)} \ln f(x) + c$

Differentiating both sides w.r.t.x then

$$f(x)\sin x\cos x = \frac{1}{2(b^2-a^2)}.\frac{f'(x)}{f(x)}$$

$$\Rightarrow\ 2(b^2-a^2)\sin x\cos x = \frac{f'(x)}{\{f(x)\}^2}$$

$$\Rightarrow 2b^2\sin x\cos x - 2a^2\sin x\cos x = \frac{f'(x)}{\{f(x)\}^2}$$

Integrating both side w.r.t. x we get

$$-b^2\cos^2 x - a^2\sin^2 x = -\frac{1}{f(x)}$$

or $f(x) = \frac{1}{(a^2\sin^2 x + \mathbf{b}^2\cos^2 x)}$

31. (c) Let $I = \int \frac{(ax^2-b)dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$

$$= \int \frac{(ax^2-b)dx}{x^2\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}} = \int \frac{\left(a-\frac{b}{x^2}\right)dx}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}$$

Put $ax+\frac{b}{x} = c\sin\theta \quad \therefore \left(a-\frac{b}{x^2}\right)dx = c\cos\theta\, d\theta$

then $I = \int \frac{c\cos\theta d\theta}{\sqrt{c\cos\theta}} = \int d\theta = \theta + k$, k is constant of integration. So, $I = \sin^{-1}\left(\frac{ax^2+b}{cx}\right) + k$

32. (b) Since $f(x) = \tan^{-1}x + ln\sqrt{(1+x)} - ln\sqrt{(1-x)}$

$$\therefore\ f'(x) = \frac{1}{(1+x^2)} + \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

$$= \frac{1}{(1+x^2)} + \frac{1}{(1-x^2)} = \frac{2}{(1-x^4)}$$

$$\Rightarrow\ \frac{1}{2}f'(x) = \frac{1}{(1-x^4)}$$

$$\therefore\ \int \frac{1}{2}f'(x)dx^4 = \int \frac{dx^4}{(1-x^4)}$$

Put $1-x^4 = t \ \therefore -dx^4 = dt \ or \ dx^4 = -dt$ then

$$\int \frac{1}{2}f'(x)dx^4 = -\int \frac{dt}{t} = -\ln t + c = ln(1-x^4) + c$$

33. (c) $\because\ f'(x) = \frac{1}{1+\cos x} = \frac{1}{2\cos^2(x/2)} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

Integrating both sides with respect to x, we have

$$f(x) = \tan\left(\frac{x}{2}\right) + c$$

$\therefore\ f(0) = 0 + c = 3$ then $f(x) = \tan\left(\frac{x}{2}\right) + 3$

$$\therefore\ f\left(\frac{\pi}{2}\right) = \tan\left(\frac{\pi}{4}\right) + 3 = 4$$

Now $3+\frac{\pi}{4} = 3 + \frac{22}{7\times 4} = 3 + \frac{11}{14} = \frac{53}{14} = 3.78$

and $3+\frac{\pi}{2} = 3 + \frac{22}{14} = 3 + \frac{11}{14} = \frac{32}{7} = 4.57$

$\because 3.78 < 4 < 4.57$

Hence, $3+\frac{\pi}{4} < f\left(\frac{\pi}{2}\right) < 3 + \frac{\pi}{2}$

It can be checked that other options do not satisfy the conditions.

34. (a) $\int x^2 d(\tan^{-1}x) = \int x^3.\frac{1}{1+x^2}dx$

$$= \int\left(x - \frac{x}{1+x^2}\right)dx = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C$$

35. (c) $I = \int \sqrt{1 + 2\tan^2 x + 2\tan x\sec x}\, dx$

$$= \int \sqrt{\sec^2 x + \tan^2 x + 2\tan x\sec x}\, dx$$

$$= \int (\sec x + \tan x)\, dx = \ln(\sec x + \tan x) + \ln\sec x + C$$

36. (a) We have $\frac{\sec 6\alpha}{\text{cosec}2\alpha} + \frac{\sec 18\alpha}{\text{cosec}6\alpha} + \frac{\sec 54\alpha}{\text{cosec}18\alpha}$

$$= \frac{\sin 2\alpha}{\cos 6\alpha} + \frac{\sin 6\alpha}{\cos 18\alpha} + \frac{\sin 18\alpha}{\cos 54\alpha}$$

Now, $\frac{\sin 2\alpha}{\cos 6\alpha} = \frac{1}{2}\frac{\sin 4\alpha}{\cos 2\alpha\cos 6\alpha}$

$$= \frac{1}{2}\left[\frac{\sin 6\alpha\cos 2\alpha - \cos 6\alpha\sin 2\alpha}{\cos 2\alpha\cos 6\alpha}\right]$$

$= \frac{1}{2}(\tan 6\alpha - \tan 2\alpha)$

Similarly, $\frac{\sin 6\alpha}{\cos 18\alpha} = \frac{1}{2}(\tan 18\alpha - \tan 6\alpha)$

and $\frac{\sin 18\alpha}{\cos 54\alpha} = \frac{1}{2}(\tan 54\alpha - \tan 18\alpha)$

Thus integral $= \frac{1}{2}\int(\tan 54\alpha - \tan 2\alpha)\,dx$

$= \frac{1}{2}\left[\frac{\ell n|\sec 54\alpha|}{54} - \frac{\ell n|\sec 2\alpha|}{2}\right] + c$

37. (c) Put $\ell n\, x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$I = \int e^t\left(\frac{t-1}{t^2+1}\right)^2 dt = \int e^t\left(\frac{1}{t^2+1} - \frac{2t}{(t^2+1)^2}\right)dt$

$= \frac{e^t}{t^2+1} + c = \frac{x}{(\ell n\, x)^2+1} + c.$

38. (a) $I = \int\left(\ell n(1+\cos x) - x\tan\frac{x}{2}\right)dx$

$= \int \ell n(1+\cos x).1dx - \int x\tan\frac{x}{2}dx$

$= \ell n(1+\cos x).x + \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}.x\,dx - \int x\tan\frac{x}{2}dx$

$= \ell n(1+\cos x).\,x + \int x\tan\frac{x}{2}dx - \int x\tan\frac{x}{2}dx + c$

$= \ell n(1+\cos x).\,x + \text{c}.$

39. (c) $\int\left(3x^2\tan\frac{1}{x} - x\sec^2\frac{1}{x}\right)dx$

$= \int 3x^2\tan\frac{1}{x}dx - \int x\sec^2\frac{1}{x}dx$

$= \tan\frac{1}{x}x^3 - \int\left(\sec^2\frac{1}{x}\right)\left(-\frac{1}{x^2}\right).x^3dx - \int x\sec^2\frac{1}{x}dx$

$= x^3.\tan\frac{1}{x} + c$

40. (d) Let $I = \int e^{\tan\theta}(\sec\theta - \sin\theta)d\theta$

Put $\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt \Rightarrow d\theta = \frac{dt}{1+t^2}$

$\Rightarrow I = \int e^t\left(\sqrt{1+t^2} - \frac{t}{\sqrt{1+t^2}}\right)\frac{dt}{1+t^2}$

$= \int e^t\left(\frac{1}{\sqrt{1+t^2}} - \frac{t}{(1+t^2)^{3/2}}\right)dt$

Integrating first part by parts we have,

$\frac{1}{\sqrt{1+t^2}}e^t + \int\frac{t}{(1+t^2)^{3/2}}.e^t dt - \int\frac{t}{(1+t^2)^{3/2}}e^t dt + c$

$= \frac{e^t}{\sqrt{1+t^2}} + c = e^{\tan\theta}\cos\theta + c$

41. (a) $I = -\int\frac{(\tan x - 1)\sec^2 x dx}{(\tan x+1)\sqrt{\tan^3 x + \tan^2 x + \tan x}}dx$

Put $\tan x = t$

$I = -\int\frac{(t-1)}{(t+1)\sqrt{t^3+t^2+t}}dt$

$= -\int\frac{t^2-1}{(t^2+2t+1)\sqrt{t^3+t^2+t}}dt$

$= -\int\frac{1-\frac{1}{t^2}}{\left(t+2+\frac{1}{t}\right)\sqrt{t+1+\frac{1}{t}}}dt$, put $1+t+\frac{1}{t} = u^2$

$\therefore\ I = -\int\frac{2du}{1+u^2} = -2\tan^{-1}u + c,$

where $u = \sqrt{1+\tan x + \frac{1}{\tan x}}$

42. (c) $\int\frac{1+(\sin x)^{2/3}}{1+(\sin x)^{4/3}}.d(\sqrt[3]{\sin x})$

$= \int\frac{1+t^2}{1+t^4}dt = \int\frac{1+1/t^2}{(t-1/t)^2+2}dt = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1/t}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt[3]{\sin x} - \frac{1}{\sqrt[3]{\sin x}}}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt[3]{\sin^2 x} - 1}{\sqrt{2}\sqrt[3]{\sin x}}\right) + c$

43. (c) We note that $\frac{d}{dx}(x\tan x+1)=x\sec^2 x+\tan x$

$\therefore$ integrating by parts with x^2 as first function, we get

$$I=\int x^2\frac{x\sec^2 x+\tan x}{(x\tan x+1)^2}dx$$

$$=x^2\left(-\frac{1}{x\tan x+1}\right)-\int 2x\left(-\frac{1}{x\tan x+1}\right)dx$$

$$=-\frac{x^2}{x\tan x+1}+2\int\frac{x\cos x}{x\sin x+\cos x}dx$$

$$=-\frac{x^2}{x\tan x+1}+2\log_e|x\sin x+\cos x|+c$$

$\left(\because \frac{d}{dx}(x\sin x+\cos x)=x\cos x\right)$

44 (c) Let $ax^2=b\sin^2\theta \Rightarrow x=\sqrt{\frac{b}{a}}\sin\theta$

$2ax\,dx=2b\sin\theta\cos\theta d\theta$

$$\Rightarrow dx=\frac{2b\sin\theta\cos\theta}{2ax}d\theta=\sqrt{\frac{b}{a}}\cos\theta d\theta$$

$$I=\sqrt{\frac{b}{a}}\int\frac{\cos\theta d\theta}{\left(a+b.\frac{b}{a}\sin^2\theta\right).\sqrt{b}\cos\theta}$$

$$=\sqrt{a}\int\frac{d\theta}{(a^2+b^2\sin^2\theta)}$$

$$=\sqrt{a}\int\frac{\sec^2\theta d\theta}{(a^2\sec^2\theta+b^2\tan^2\theta)}$$

Putting $\tan\theta=z$

$\Rightarrow \sec^2\theta d\theta=dz$ then

$$I=\sqrt{a}\int\frac{dz}{a^2(1+z^2)+b^2z^2}$$

$$=\sqrt{a}\int\frac{dz}{(a^2+b^2)z^2+a^2}$$

$$=\frac{\sqrt{a}}{(a^2+b^2)}\int\frac{dz}{z^2+\frac{a^2}{(a^2+b^2)}}$$

$$=\frac{\sqrt{a}}{a^2+b^2}.\frac{\sqrt{a^2+b^2}}{a}\tan^{-1}\left(\frac{z\sqrt{a^2+b^2}}{a}\right)+c$$

$$\therefore I=\frac{1}{\sqrt{a(a^2+b^2)}}\tan^{-1}\left(\frac{z\sqrt{a^2+b^2}}{a}\right)+c,$$ where

$z=\tan\theta$.

45. (a) Given $x\,f(x)=3f^2(x)+2$

$\Rightarrow f(x)+xf'(x)=6f(x)f'(x)$

$$\Rightarrow f'(x)=\frac{f(x)}{6f(x)-x}$$

Now $I=\int\frac{2x(x-6f(x))+f(x)}{(6f(x)-x)(x^2-f(x))^2}dx$

$$\Rightarrow I=-\int\frac{2x-f'(x)}{(x^2-f(x))^2}dx=\frac{1}{x^2-f(x)}+c$$

B COMPREHENSION TYPE

1. (a) $\int\frac{x^2+1}{x^4+1}dx=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}$

Put $x-\frac{1}{x}=t\Rightarrow\left(1+\frac{1}{x^2}\right)dx=dt$

$$\therefore I=\int\frac{dt}{t^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x}+C$$

2. (c) $I=\int\frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$

$$=\int\frac{1-\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}+3\right)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$$

Put $x+\frac{1}{x}=t\Rightarrow\left(1-\frac{1}{x^2}\right)dx=dt$ and $x^2+\frac{1}{x^2}+2=t^2$

$$\therefore I=\int\frac{dt}{(t^2+1)\tan^{-1}t}=\ell n|\tan^{-1}t|+C$$

$$=\ell n\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right|+C$$

3.(b) $I=\int \frac{x^4-2}{x^2\sqrt{x^4+x^2+2}}dx=\int \frac{x^4-2}{x^3\sqrt{x^2+1+\frac{2}{x^2}}}dx$

$$=\int \frac{x-\frac{2}{x^3}}{\sqrt{x^2+1+\frac{2}{x^2}}}dx$$

Put $x^2+\frac{2}{x^2}+1=t \Rightarrow \left(x-\frac{2}{x^3}\right)dx=\frac{dt}{2}$,

we get $I=\int \frac{dt}{2\sqrt{t}}=\sqrt{t}+C=\sqrt{x^2+1+\frac{2}{x^2}}+C$

4.(d) Divide numerator and demoninator by x^{10} we get

$I=\int \frac{5x^{-6}+4x^{-5}}{(1+x^{-4}+x^{-5})^2}dx$

Put $1+x^{-4}+x^{-5}=t$ and evaluate

5.(b) Put $\sqrt{1+x^2}=(t-x)$

[Here $a=1>0$, we can put $\sqrt{1+x^2}=t+x$ also]

$\Rightarrow x+\sqrt{1+x^2}=t$

or $\left(1+\frac{x}{\sqrt{1+x^2}}dx=dt\right) \Rightarrow \frac{dx}{\sqrt{1+x^2}}=\frac{dt}{t}$

$\therefore I=\int t^{15}.\frac{dt}{t}=\frac{t^{15}}{15}+C=\frac{(x+\sqrt{1+x^2})^{15}}{15}+C$

$$=\frac{1}{15(\sqrt{1+x^2}-x)^{15}}+C$$

6. (a) Here $a<0$ and $c<0$, but $-x^2+3x-2=-(x-1)(x-2)$

So, we put $\sqrt{-x^2+3x-2}=(x-2)t$ or $(x-1)t$ or $t(1-x)$.

Put $-x^2+3x-2=t(x-2) \Rightarrow t=\sqrt{\frac{1-x}{x-2}}$ $\quad (\because 1<x<2)$

We get, $x=\frac{2t^2+1}{t^2+1}$ and $dx=\frac{2t\ dt}{(t^2+1)^2}$

So, the integral becomes

$$I=\int \frac{\frac{2t\ dt}{(t^2+1)^2}}{\left(\frac{t^2}{t^2+1}\right).\frac{t}{t^2+1}}=\int \frac{2}{t^2}dt$$

$$=-\frac{2}{t}+C=-2\sqrt{\frac{x-2}{1-x}}+C$$

7.(c) The antiderivative is $-2\sqrt{\frac{x-2}{1-x}}$. So, the limit is

$$\lim_{x\to 2}\frac{\sin\left(-2\sqrt{\frac{x-2}{1-x}}\right)}{\sqrt{2-x}}$$

$$=\lim_{x\to 2}\frac{\sin\left(-2\sqrt{\frac{x-2}{1-x}}\right)}{\left(-2\sqrt{\frac{x-2}{1-x}}\right)}\times -2\sqrt{\frac{x-2}{1-x}}\times\frac{1}{\sqrt{2-x}}$$

$=1\times -2=-2$

8.(c) $\int\left[\frac{1}{\ln x}-\frac{1}{(\ln x)^2}\right]dx$

$=\frac{1}{\ln x}.x-\int -\frac{1}{(\ln x)^2}.\frac{1}{x}.x\ dx-\int \frac{1}{(\ln x)^2}\ dx$

$=\frac{x}{\ln x}+C$

9.(b) $\int \frac{x+\sin x}{1+\cos x}dx=\int \frac{x+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}$

$=\int \frac{1}{2}x\sec^2\frac{x}{2}dx+\int \tan\frac{x}{2}dx$

Integrating second integral by parts taking 1 as second function, we get

$I=\int \frac{1}{2}x\sec^2\frac{x}{2}dx+x\tan\frac{x}{2}-\int x.\frac{1}{2}\sec^2\frac{x}{2}\ dx$

$=x\tan\frac{x}{2}+C$

10.(d) $I=\int \frac{xe^x}{(x+1)^2}dx=\int \frac{(x+1-1)e^x}{(x+1)^2}\ dx$

$$=\int e^x\left\{\frac{1}{x+1}+\frac{-1}{(x+1)^2}\right\}dx=\frac{e^x}{x+1}+C$$

$\left[\because \frac{d}{dx}\left(\frac{1}{x+1}\right)=-\frac{1}{(x+1)^2}\right]$

11.(b) $I=\int\left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx=\int\left(\log t+\frac{1}{t^2}\right)e^t\ dt$

[Putting $\log x=t$]

$=\int e^t\left\{\left(\log t+\frac{1}{t}\right)-\left(\frac{1}{t}-\frac{1}{t^2}\right)\right\}dt$

$=e^t\left(\log t-\frac{1}{t}\right)=x\log(\log x)-\frac{x}{\log x}$

C REASONING TYPE

1. **(c)** $I_n = \int \tan^n x\,dx$

$= \int \tan^{n-2} x \sec^2 x - \int \tan^{n-2} x\,dx$

$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

Put $n = 6$, $5(I_6 + I_4) = \tan^5 x$

Statement -1 is true and statement -2 is false.

2. **(b)** $\int \frac{1}{f(x)}dx = \log(f(x))^2 + C \Rightarrow \frac{1}{f(x)} = \frac{2f'(x)}{f(x)}$

$\Rightarrow f'(x) = \frac{1}{2} \Rightarrow f(x) = \frac{x}{2}$

Statement -2 is clearly true.

3. **(a)** $I = \int \frac{3+4\cos x}{(4+3\cos x)^2}dx$

$= \int \frac{3\operatorname{cosec}^2 x + 4\cot x \operatorname{cosec} x}{(4\operatorname{cosec} x + 3\cot x)^2}dx$

(Mulitplying N_r and D_r by $\operatorname{cosec}^2 x$)

$\therefore I = -\int \{f(x)\}^{-2} f'(x)dx,$

where $f(x) = 4\operatorname{cosec} x + 3\cot x$

$= \frac{1}{f(x)} = \frac{1}{4\operatorname{cosec} x + 3\cot x} = \frac{\sin x}{4+3\cos x} + C$

Statement -2 is clearly true

D MULTIPLE CORRECT CHOICE TYPE

1. **(c,d)** $f(x) = \lim_{n\to\infty} e^{x\tan(1/n)\log(1/n)}$

But $\lim_{n\to\infty} \tan 1/n \log 1/n$

$= \lim_{n\to\infty}\left(\frac{\log n}{n}\frac{\tan 1/n}{1/n}\right) = 0$

So, $f(x) = e^0 = 1$.

Hence, $\int \frac{f(x)}{\sqrt[3]{\sin^{11} x\cos x}}dx = \int \frac{\sec^4 x}{(\tan x)^{11/3}}dx$

$= \int \frac{1+t^2}{t^{11/3}}dt$ (Putting $t = \tan x$)

$= \int (t^{-11/3} + t^{-5/3})dt = -\frac{3}{8}t^{-8/3} - \frac{3}{2}t^{-2/3} + C$

$= -\frac{3}{8}\frac{(1+4\tan^2 x)}{\tan^2 x\sqrt[3]{\tan^2 x}} + C$

Thus, $g(x) = -\frac{3}{8}\frac{(1+4\tan^2 x)}{\tan^2 x\sqrt[3]{\tan^2 x}}$

and $g\left(\frac{\pi}{4}\right) = -\frac{15}{8}$

Clearly g is not defined at $x = 0$ and odd multiples of $\frac{\pi}{2}$. So (b) is not correct.

2. **(b,c)** $I = \int e^x\left\{\frac{2\tan x}{1+\tan x} + \cot^2\left(x+\frac{\pi}{4}\right)\right\}dx$

$= \int e^x\left\{\frac{2}{1+\cot x} - 1 + \operatorname{cosec}^2\left(x+\frac{\pi}{4}\right)\right\}dx$

$= \int e^x\left\{-\cot\left(x+\frac{\pi}{4}\right) + \operatorname{cosec}^2\left(x+\frac{\pi}{4}\right)\right\}dx$

$= -e^x\cot\left(x+\frac{\pi}{4}\right) + C = e^x\cot\left(\frac{3\pi}{4} - x\right) + C$

Again,

$I = e^x\cot\left(\frac{3\pi}{4} - x\right) + C = e^x\cot\left(\frac{\pi}{2} + \frac{\pi}{4} - x\right) + C$

$= e^x\tan\left(x - \frac{\pi}{4}\right) + C$

3. **(a,c)** As $0 < x < 1 \Rightarrow x^2 < x^{\frac{\pi}{2}} < x$

$\Rightarrow \frac{1}{1+x} < \frac{1}{1+x^{\pi/2}} < \frac{1}{1+x^2}$

or $\int_0^1 \frac{dx}{1+x} < \int_0^1 \frac{dx}{1+x^{\pi/2}} < \int_0^1 \frac{dx}{1+x^2}$

$\Rightarrow \ln 2 < I < \frac{\pi}{4}$

4. **(b,d)** $\cos^{-1}\sqrt{1-x^2} = -\sin^{-1}x, \quad \because x<0$

$$\therefore \int\left(\cos^{-1}x+\cos^{-1}\sqrt{1-x^2}\right)dx$$

$$=\int\left(\cos^{-1}x-\sin^{-1}x\right)dx$$

$$=\int\left(\frac{\pi}{2}-2\sin^{-1}x\right)dx$$

$$=\frac{\pi}{2}x-2x\sin^{-1}x+\int\frac{2x}{\sqrt{1-x^2}}dx$$

$$=\frac{\pi}{2}x-2x\sin^{-1}x-2\sqrt{1-x^2}+C$$

5. **(a, d)** Let $I=\int x^{-1/2}(2+3x^{1/3})^{-2}dx$

Put $x=t^6$ $\qquad \therefore dx=6t^5dt$

then $I=\int t^{-3}(2+3t^2)^{-2}.6t^5dt$

$$=6\int\frac{t^2}{(2+3t^2)^2}dt=\frac{6}{9}\int\frac{t^2dt}{\left(\frac{2}{3}+t^2\right)^2}$$

Now put $t=\sqrt{\left(\frac{2}{3}\right)}\tan\theta$

$$\therefore dt=\sqrt{\left(\frac{2}{3}\right)}\sec^2\theta d\theta$$

$$\therefore I=\frac{6}{9}\int\frac{\frac{2}{3}\tan^2\theta.\sqrt{\frac{2}{3}}\sec^2\theta d\theta}{\frac{4}{9}\sec^4\theta}=\sqrt{\frac{2}{3}}\int\sin^2\theta d\theta$$

$$=\frac{1}{\sqrt6}\int(1-\cos2\theta)d\theta=\frac{1}{\sqrt6}\left\{\theta-\frac{\sin2\theta}{2}\right\}+C$$

$$=\frac{1}{\sqrt6}\left\{\theta-\frac{\tan\theta}{1+\tan^2\theta}\right\}+C$$

$$=\frac{1}{\sqrt6}\left\{\tan^{-1}\left\{\sqrt{\frac{3}{2}}t\right\}-\frac{t\sqrt{\frac{3}{2}}}{1+\frac{3}{2}t^2}\right\}+C$$

$$=\frac{1}{\sqrt6}\left\{\tan^{-1}\left\{\sqrt{\frac{3}{2}}x^{1/6}\right\}-\frac{\sqrt6x^{1/6}}{2+3x^{1/3}}\right\}+C$$

$$\left\{\because \tan\theta=\sqrt{\frac{3}{2}}t\right\}$$

6. **(b,d)** Let $I=\int\frac{(x^4+1)}{(x^6+1)}dx=\int\frac{(x^2+1)^2-2x^2}{(x^2+1)(x^4-x^2+1)}dx$

$$=\int\frac{(x^2+1)dx}{(x^4-x^2+1)}-2\int\frac{x^2dx}{(x^6+1)}$$

$$=\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x^2-1+\frac{1}{x^2}\right)}-2\int\frac{x^2dx}{(x^3)^2+1}$$

$$=\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+1}-2\int\frac{x^2dx}{(x^3)^2+1}$$

In first integral put $x-\frac{1}{x}=t$

$\therefore \left(1+\frac{1}{x^2}\right)dx=dt$ and in second integral put $x^3=u$

$\therefore x^2dx=\frac{du}{3}$ then $I=\int\frac{dt}{1+t^2}-\frac{2}{3}\int\frac{du}{1+u^2}$

$$=\tan^{-1}t-\frac{2}{3}\tan^{-1}u+c$$

$$=\tan^{-1}\left(x-\frac{1}{x}\right)-\frac{2}{3}\tan^{-1}(x^3)+c$$

7. **(a, b)** Put $t=\sin^2x$

The integral reduces to

$$I=\frac{1}{2}\int e^t(2-t)dt=\frac{3}{2}e^t-\frac{te^t}{2}+c$$

$$=\frac{1}{2}e^{\sin^2x}(3-\sin^2x)+c$$

$$=e^{\sin^2x}\left(1+\frac{1}{2}\cos^2x\right)+c$$

8. **(b,c)** $I=\int\frac{x^2+n(n-1)}{(x\sin x+n\cos x)^2}dx$

Multiplying and dividing by x^{2n-2}

$$I=\int\frac{(x^2+n(n-1)).x^{2n-2}}{(x\sin x+n\cos x)^2.x^{2n-2}}dx$$

$$I=\int\frac{(x^2+n(n-1))x^{2n-2}}{(x^n\sin x+nx^{n-1}\cos x)^2}dx$$

Let $x^n\sin x+nx^{n-1}\cos x=t$

$\Rightarrow (nx^{n-1}\sin x+x^n\cos x+n(n-1))x^{n-2}$

$\cos x - nx^{n-1}\sin x)dx = dt$

$\Rightarrow x^{n-2}\cos x.(x^2 + n(n-1))dx = dt$

$$I = \int \frac{(x^2 + n(n-1)).x^{n-2}\cos x}{(x^n \sin x + nx^{n-1}\cos x)^2}.x^n.\sec x\,dx$$

Integrating by parts; we get

$$I = x^n \sec x.\left(-\frac{1}{x^n \sin x + nx^{n-1}\cos x}\right)$$

$$+\int \frac{x^n \sec x \tan x + nx^{n-1}.\sec x}{(x^n \sin x + nx^{n-1}\cos x)}dx$$

$$= -\frac{x^n \sec x}{x^n \sin x + nx^{n-1}\cos x} + \int \sec^2 x dx$$

$$\therefore \quad I = -\frac{x^n \sec x}{x^n \sin x + nx^{n-1}\cos x} + \tan x + c.$$

E MATRIX-MATCH TYPE

1. **A-r; B-s; C-p; D-q**

(A) $\int e^x\left[\frac{1}{x+2} + \log(x+2)\right]dx = e^x \log(x+2) + c$

(B) $\int \sin^2 x(1-\sin^2 x)\cos x dx$

$= \int \sin^2 x \cos x dx - \int \sin^4 x.\cos x\, dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$

(C) $\int \frac{dx}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x+\frac{3}{2}\right)^2}} = \sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c$

(D) $\int \frac{x^5}{x^2+1}dx = \int\left(x^3 - x + \frac{x}{x^2+1}\right)dx$

$= \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ell n(x^2+1) + C$

2. **A-s; B-t; C-r; D-q**

(A) Since $I = \int \frac{dx}{\sqrt{x}(x+9)}$

Put $\sqrt{x} = t \Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

$\therefore\ I = \int \frac{2dt}{t^2+9} = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$

$= \frac{2}{3}\tan^{-1}\left(\frac{\sqrt{x}}{3}\right) + C$

(B) $\because \int e^x(1-\cot x + \cot^2 x)dx$

$\Rightarrow \int e^x(\operatorname{cosec}^2 x + (-\cot x))dx = e^x(-\cot x) + c$

(C) $\int(\tan x \sec x + \cot x \operatorname{cosec} x)dx = \sec x - \operatorname{cosec} x + c$

(D) $\int \frac{dx}{2\sin^2\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$

$= \int \frac{\operatorname{cosec}^2\left(\frac{x}{2}\right)\frac{1}{2}}{1-\cot\left(\frac{x}{2}\right)}dx = \int \frac{f'(x)}{f(x)}dx = \log\left|1-\cot\left(\frac{x}{2}\right)\right| + c$

3. **A-r; B-p; C-q ; D-s**

(A) $\int(x^a + a^x)dx = \frac{x^{a+1}}{a+1} + \frac{a^x}{(\log a)} + C$

(B) $\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$ put $x - \frac{1}{x} = t, \left(x^2 + \frac{1}{x^2}\right)dx = dt$

$\Rightarrow \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + C$

(C) $\frac{1}{4}\int \frac{\sec^2 x\, dx}{\tan^2 x + \tan x + \frac{5}{4}} = \frac{1}{4}\int \frac{\sec^2 x\, dx}{\left(\tan x + \frac{1}{2}\right)^2 + 1}$

$= \frac{1}{4}\tan^{-1}\left(\tan x + \frac{1}{2}\right) + C$

(D) $\int \frac{1}{(\sqrt{\sin x} + \sqrt{\cos x})^4}dx = \int \frac{\sec^2 x}{(\sqrt{\tan x}+1)^4}dx$

(Using $\tan x = t^2 \Rightarrow \sec^2 x\,dx = 2t\,dt$)

$$= \int \frac{2t\,dt}{(t+1)^4} = 2\int \frac{1}{(t+1)^3} - \frac{1}{(t+1)^4}\,dt$$

$$= -\frac{1}{(t+1)^2} + \frac{1}{3(t+1)^3} + c$$

$$= \frac{-1}{(1+\sqrt{\tan x})^2} + \frac{2}{3(1+\sqrt{\tan x})^3} + c.$$

A-r; B-p; C-t; D-s

$$\int \frac{\ell n(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx = I$$

Put $\log(x+\sqrt{1+x^2}) = t \Rightarrow \frac{dx}{\sqrt{1+x^2}} = dt$

So, $I = \int t\,dt = \frac{t^2}{2} + C = \frac{1}{2}\left\{\log(x+\sqrt{1+x^2})\right\}^2 + C$.

Thus,

(A) $f(x) = \frac{x^2}{2}$

(B) $g(x) = \log(x+\sqrt{x^2+1})$

(C) Now, $\int \frac{x^2}{2}\log(x+\sqrt{x^2+1})dx = \frac{x^3}{6}\log(x+\sqrt{x^2+1})$

$$-\frac{1}{2}\int \frac{x^3}{3} \times \frac{1}{x+\sqrt{x^2+1}}\left\{1+\frac{2x}{2\sqrt{x^2+1}}\right\}dx$$

$$= \frac{x^3}{6}\log(x+\sqrt{x^2+1}) - \frac{1}{6}\int \frac{x^3\,dx}{\sqrt{x^2+1}}$$

Putting $x^2 + 1 = t^2$

$$= \frac{x^3}{6}\log(x+\sqrt{x^2+1}) - \frac{1}{6}\int (t^2-1)\,dt$$

$$= \frac{x^3}{6}\log(x+\sqrt{x^2+1}) - \frac{1}{18}(1+x^2)^{3/2} + \frac{1}{6}(1+x^2)^{1/2} + C$$

(D) $\int e^{g(x)}dx = \int (x+\sqrt{1+x^2})dx$

$$= \frac{x^2}{2} + \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\,(x+\sqrt{1+x^2}) + C$$

$$= \frac{1}{2}x(x+\sqrt{1+x^2}) + \frac{1}{2}g(x) + C$$

5. **A-r, B-q, C-t, D-p**

(A) $\int x^2 d(\tan^{-1}x) = \int \frac{x^2}{1+x^2}\,dx = \int\left(1-\frac{1}{1+x^2}\right)dx$

$$= x + \tan^{-1}x + C \Rightarrow f(x) = \tan^{-1}x$$

(B) $\int \sqrt{1+2\tan x(\tan x+\sec x)}\,dx$

$$= \int \sqrt{(\sec x+\tan x)^2}\,dx$$

$$= \ln|\sec x+\tan x| + \ln|\sec x| + C$$

$$= \ln\left|\frac{1+\sin x}{\cos^2 x}\right| + C = \ln\left|\frac{1}{1-\sin x}\right| + C$$

$$= \ln\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|^{-2} + C = -2\ln\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right| + C$$

(C) $\int x^2e^{2x}dx = x^2\frac{e^{2x}}{2} - \int 2x\frac{e^{2x}}{2}\,dx$

$$= \frac{x^2}{2}e^{2x} - \left[x.\frac{e^{2x}}{2} - \int 1.\,\frac{e^{2x}}{2}\,dx\right]$$

$$= \left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right)e^{2x} + C = \frac{1}{4}(2x^2-2x+1)e^{2x} + C$$

Now, $2x^2 - 2x + 1 \geq \frac{1}{2}$

$$\Rightarrow f(x) = \frac{1}{4}(2x^2-2x+1) \geq \frac{1}{8}$$

(D) $\int \frac{x^4+1}{x(x^2+1)^2}\,dx = \int \frac{(x^2+1)^2-2x^2}{x(x^2+1)^2}dx$

$$= \int\left(\frac{1}{x} - \frac{2x}{(x^2+1)^2}\right)dx = \ln|x| + \frac{1}{x^2+1} + C$$

$\therefore a = b = 1$

6. **A→ p, q, t ; B→ p ; C → q ; D → p, s**

(A) Let $3x = \cos\theta \Rightarrow 3dx = -\sin\theta\,d\theta$, then integral is

$$-\frac{1}{3}\int \frac{\frac{\cos\theta}{3}+\theta^2}{\sin\theta}\sin\theta\,d\theta = -\frac{1}{3}\int\left(\frac{1}{3}\cos\theta+\theta^2\right)d\theta$$

$$= -\frac{1}{9}\sin\theta - \frac{1}{9}\theta^3 + c$$

$$= -\frac{1}{9}\sqrt{1-9x^2} - \frac{1}{9}(\cos^{-1}3x)^3 + c$$

Hence $A = -\frac{1}{9}$, $B = -\frac{1}{9}$.

(B) Given integral is

$$\int \frac{2x+1}{(x(x+1))^2-1}dx = \int \frac{dt}{t^2-1} = -\frac{1}{2}\ln\left|\frac{x^2+x+1}{x^2+x-1}\right|+c$$

(C) $\int x^5(x^{10}+x^5+1)(2x^{10}+3x^5+6)^{1/5}dx$

$$=\int (x^{14}+x^9+1)(2x^{15}+3x^{10}+6x^5)^{1/5}dx$$

Putting $2x^{15}+3x^{10}+6x^5=t^5$

$$\Rightarrow (x^{14}+x^9+x^4)dx = \frac{5t^4}{30}dt$$

$$=\frac{1}{6}\int t^5 dt = \frac{1}{36}t^6$$

$$=\frac{1}{36}(2x^{15}+3x^{10}+6x^5)^{6/5}+c.$$

(D) $I=\int \frac{x^3-x}{\sqrt{1-x^2}}\cdot\frac{1}{(1+\sqrt{1-x^2})}dx$

let $1+\sqrt{1-x^2}=z$ $,\Rightarrow \frac{-2x}{2\sqrt{1-x^2}}dx = dz$

$$I=\int \frac{-x(1-x^2)}{\sqrt{1-x^2}}\cdot\frac{1}{(1+\sqrt{1-x^2})}dx$$

$$=\int (z-1)^2\frac{dz}{z}$$

$$=\int \frac{z^2-2z+1}{z}dz = \int z dz - 2\int dz + \int \frac{1}{z}dz$$

$$=\frac{z^2}{2}-2z+\ell n|z|+c$$

$$=\frac{(1+\sqrt{1-x})^2}{2} -2(1+\sqrt{1-x})^2+\ell n\left|1+\sqrt{1-x^2}\right|+c$$

$$=-\frac{(2+x^2+2\sqrt{1-x^2})}{2} + \ell n 1+\sqrt{1-x^2}+c$$

$$=-(1+\sqrt{1-x^2})-\frac{x^2}{2}+\ell n 1+\sqrt{1-x^2}+c$$

F NUMERIC/INTEGER ANSWER TYPE

1. Ans. : 3

$$I=\int \frac{\sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta$$

$$=\int \frac{\left(2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}\right).\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta$$

$$=\int \frac{2\sin^2\frac{\theta}{2}\sin\theta d\theta}{2\cos^2\frac{\theta}{2}\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}$$

Put $\cos\theta = t \Rightarrow -\sin\theta d\theta = dt$

Also $\cos\theta = 2\cos^2\frac{\theta}{2}-1 = 1-2\sin^2\frac{\theta}{2}=t$

$$\therefore \quad I=\int \frac{\frac{1-t}{2}(-dt)}{(1+t)\sqrt{t^3+t^2+t}} = \frac{1}{2}\int \frac{(t^2-1)dt}{(t+1)^2\sqrt{t^3+t^2+t}}$$

$$=\frac{1}{2}\int \frac{\left(1-\frac{1}{t^2}\right)(dt)}{\left(t+\frac{1}{t}+2\right)\sqrt{t+\frac{1}{t}+1}}$$

Put $t+\frac{1}{t}+1=u^2 \Rightarrow \left(1-\frac{1}{t^2}\right)dt = 2u\,du$

$$I=\frac{1}{2}\int \frac{2udu}{\left(1+u^2\right)u} = \tan^{-1}u = \tan^{-1}\sqrt{1+\frac{1}{t}+1}+c$$

$$=\tan^{-1}(\cos\theta+\sec\theta+1)^{1/2}+c$$

So, $f(\theta)=\cos\theta+\sec\theta+1 \geq 2+1=3$

2. Ans. : 1

Let $P=\int e^{ax}\cos bx dx$, $Q=\int e^{ax}\sin bx dx$

$$P+iQ=\int e^{ax}(\cos bx+i\sin bx)dx$$

[We may apply integration by parts twice also]

$$\therefore \quad \int e^{ax}.e^{ibx}dx = \int e^{(a+ib)x}dx$$

$$=\frac{e^{(a+ib)x}}{(a+ib)}+c = \frac{e^{ax}(\cos bx+i\sin bx)(a-ib)}{a^2+b^2}+c$$

$$= \frac{e^{ax}\{(a\cos bx + b\sin bx)\} + ie^{ax}\{(a\sin bx - b\cos bx)\}}{(a^2+b^2)}$$

$$= \frac{e^{ax}(a\cos bx + b\sin bx)}{(a^2+b^2)} + i\frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

Equating real and imaginary parts on both sides, we get

$$P = \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{(a^2+b^2)} + c$$

$$= \frac{1}{r}e^{ax}\cos(bx-\phi) + c \text{ and}$$

$$Q = \int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{(a^2+b^2)} + c$$

$$= \frac{1}{r}e^{ax}\sin(bx+\phi) + c$$

where $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

$\therefore\ (P^2+Q^2)r^2 = e^{2ax}$

(neglecting constant of integration)

$\therefore\ (P^2+Q^2)(a^2+b^2) = e^{2ax}$

3. Ans. : 2

$$I = \int \sin 4x\, e^{\tan^2 x}dx = \int 2\sin 2x\cos 2x e^{\tan^2 x}dx$$

$$= 4\int \sin x\cos x\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)e^{\tan^2 x}dx$$

$$= 4\int \tan x.\sec^2 x.\cos^6 x(1-\tan^2 x)e^{\tan^2 x}dx$$

Put $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\, dx = dt$

$$\therefore\ I = 2\int\frac{(1-t)e^t}{(1+t)^3}dt = -2\int\left[\frac{t+1-2}{(t+1)^3}\right]e^t dt$$

$$= -2\int\left[\frac{1}{(t+1)^2} + \frac{-2}{(t+1)^3}\right]e^t dt$$

$$= -2\frac{e^t}{(t+1)^2} + c = -2\cos^4 x\,.\,e^{\tan^2 x} + c$$

4. Ans. : 1

$$I = \int \operatorname{cosec}^2 x\, ln\left(\cos x + \sqrt{\cos 2x}\right)dx$$

$$= -\cot x.\log_e\left(\cos x + \sqrt{\cos 2x}\right)$$

$$-\int(-\cot x), \frac{1}{\cos x + \sqrt{\cos 2x}}$$

$$\left\{-\sin x + \frac{1}{2}(\cos 2x)^{-1/2}(-\sin 2x).2\right\}dx$$

$$= -\cot x\, ln\left(\cos x + \sqrt{\cos 2x}\right)$$

$$-\int \cot x\frac{\sin x\sqrt{\cos 2x} + \sin 2x}{\sqrt{\cos 2x}\left(\cos x + \sqrt{\cos 2x}\right)}dx$$

$$= -\cot x\, ln\left(\cos x + \sqrt{\cos 2x}\right)$$

$$-\int\frac{\cos x\sqrt{\cos 2x} - \cos^2 x\cos 2x}{\cos 2x\sin^2 x}dx$$

$$= -\cot x\, ln\left(\cos x + \sqrt{\cos 2x}\right)$$

$$-\int\frac{\cos x}{\sqrt{\cos 2x}\sin^2 x}dx + \int\cot^2 x dx$$

Now, $I_1 = \int\frac{\cos x dx}{\sqrt{\cos 2x}\sin^2 x}$

$$= \int\frac{\cos x\, dx}{\sin^2 x\sqrt{1-2\sin^2 x}} = \int\frac{dt}{t^2\sqrt{1-2t^2}}$$

Put $t = \frac{1}{u} \Rightarrow dt = -\frac{1}{u^2}du$

$$\therefore\ I_1 = -\int\frac{udu}{\sqrt{u^2-2}} = -\sqrt{u^2-2} = -\sqrt{\cos ec^2 x - 2}$$

Thus $I = -\cot x\, ln\left(\cos x + \sqrt{\cos 2x}\right)$

$$+\sqrt{\cos ec^2 x - 2} - \cot x - x + c$$

$\therefore\ f(x) = -\cot x$ *and* $g(x) = \sqrt{\cos ec^2 x - 2}$

5. Ans: 2

$$I=\sqrt{2}\int\frac{(\cos x-\sin x)}{\sqrt{\sin 2x}(4+3\sin 2x)}dx$$

Put $\cos x+\sin x=z$

$(\cos x-\sin x)\,dx=dz$

$$\therefore I=\sqrt{2}\int\frac{dz}{\sqrt{(z^2-1)}(4+3)(z^2-1))}$$

Put $z=\sec\theta$ $\qquad dz=\sec\theta\tan\theta d\theta$

$$I=\sqrt{2}\int\frac{\sec\theta\tan\theta d\theta}{\tan\theta(3\sec^2\theta+1)}$$

$$\therefore I=\sqrt{2}\int\frac{\frac{\sin\theta}{\cos^2\theta}}{\frac{\sin\theta}{\cos\theta}\left(\frac{3+\cos^2\theta}{\cos^2\theta}\right)}d\theta$$

$$=\sqrt{2}\int\frac{\cos\theta}{4-\sin^2\theta}d\theta$$

Let $\sin\theta=t\Rightarrow\cos\theta\,d\theta=dt$

$$\therefore I=\sqrt{2}\int\frac{dt}{4-t^2}\Rightarrow\frac{1}{2\sqrt{2}}\ell n\left|\frac{t+2}{t-2}\right|+c$$

$$=\frac{1}{2\sqrt{2}}\ell n\left|\frac{\sin\theta+2}{\sin\theta-2}\right|+c,$$

where $\sin\theta=\frac{\sqrt{z^2-1}}{z}\quad=\frac{\sqrt{\sin 2x}}{\sqrt{1+\sin 2x}}$

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