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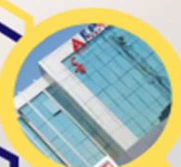
02

MATHEMATICS
REVISION MODULE

DIFFERENTIAL EQUATIONS

DIFFERENTIAL Equations



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DIFFERENTIAL EQUATIONS

POINTS TO REMEMBER

1. **Definition:** An equation involving the independent variable x (say), dependent variable y (say) and the differential coefficients of dependent variable with respect to independent variable *i.e.,* $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$, *etc.* is called a **differential equation**.

   e.g., $\frac{dy}{dx} + 4y = x, \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = x^2$ are **differential equations.**

2. **Order and Degree of a Differential Equation:** The order of a differential equation is the order of the highest derivative occurring in the differential equation.

   The degree of a differential equation is the degree of the highest order derivative occurring in the equation, when the ***differential coefficients*** *are made* ***free from radicals, fractions*** *and it is written as a polynomial in differential co-efficient.*

   Example: Consider three differential equations:

   (*i*) $\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right) - \frac{dy}{dx} + y = 0$ (*ii*) $\frac{d^2y}{dx^2} = 1 + \sqrt{\frac{dy}{dx}}$ (*iii*) $\left(\frac{d^2y}{dx^2}\right)^3 + \sin\left(\frac{dy}{dx}\right) = 0$

   Solution:

   (*i*) In this equation, the highest order derivative is 3 and its power is 1. Therefore, its order is 3 and degree 1.

   (*ii*) In this equation, the differential co-efficient is not free from radical. Therefore, it is made free from radical as

   $$\frac{d^2y}{dx^2} - 1 = \sqrt{\frac{dy}{dx}} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 + 1 - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \qquad \text{[Squaring both sides]}$$

   Hence, order is 2 and degree is 2.

   (*iii*) In this equation order of highest order derivative is 2 therefore, its order is 2, but this differential equation cannot be written in the form of polynomial in differential co-efficient.

   Hence, its degree is not defined.

   [**Note :** The order and degree of differential equations are always positive integers.]

3. **Classification of Differential Equations:**

   (*A*) *Differential equations are classified according to their order:*

   (*i*) **First order differential equations:** First order differential equations are those in which only the first order derivative of the dependent variable occurs.

POINTS TO REMEMBER

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(*ii*) **Higher order differential equations:** Differential equations of order two or more are referred as higher order differential equations.

(B) *Another classification of differential equations refers to its linearity means linear and non linear differential equations:*

Linear and non-linear differential equations: *A differential equation, in which the dependent variable and its derivatives occur only in the 1st degree and are not multiplied together, is called a linear differential equation otherwise it is non linear.*

Note: *Every linear differential equation is always of the 1st degree but every differential equation of the 1st degree need not be the linear differential equation.*

4. **Solution of a differential equation:** The solution of a differential equation is a relation between dependent and the independent variables which satisfies the given differential equation *i.e.*, when this relation is substituted in given differential equation, makes left hand and right hand sides identically equal.

Note: *If any relation contains n arbitrary constants, then the differential equation of nth order will be obtained after eliminating all the arbitrary constants.*

5. **General and particular solutions of differential equations:** The general solution of a differential equation of *n*th order is a relation between dependent and independent variables having *n* arbitrary constants.

The solution obtained from the general solution by giving the particular values to these arbitrary constants is called the particular solution.

6. **Forms of the solution of differential equations:** The general solution may have more than one forms but the arbitrary constants must be same in the number.

7. **Solution of differential equations:** In this chapter, we shall only find the solutions of differential equations *viz.* differential equations with variables separable form, homogeneous and linear differential equations.

8. **Type 1:**

(A) **Variables Separable Form:** If in the given equation, it is possible to get all the terms containing x and dx to one side and all the terms containing y and dy to the other, the variables are said to be separable.

Procedure to solve the differential equations with variables separable form:

Consider the equation $\frac{dy}{dx} = X.Y$ where X is a function of x only and Y is a function of y only.

(*i*) Put the equation in the form $\frac{1}{Y}.dy = X.\,dx$

(*ii*) Integrating both the sides, we get

$$\int \frac{dy}{Y} = \int X\,dx + C, \text{ where C is an arbitrary constant.}$$

Thus, the required solution is obtained.

(B) **Equations Reducible to Variables Separable Form:** Equations of the form $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to form in which the variables are separable form.

Procedure to solve an equation reducible to variables separable form:

(*i*) Write the given equation in form $\frac{dy}{dx} = f(ax+by+c)$.

(*ii*) Put $ax + by + c = z$, so that $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$.

(*iii*) Putting this $\frac{dy}{dx}$ in the given equation, we get $\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$. This equation is reduced in the form : $\frac{dz}{a + b\,f(z)} = dx$. After integrating, we get the required result.

Type 2 : Homogeneous Function and Homogeneous Differential Equation

Homogeneous function: A function $F(x, y)$ is called homogeneous function of degree n if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y), \text{ where } \lambda \text{ is non-zero real number.}$$

Homogeneous differential equation: A differential equation of the form $\frac{dy}{dx} = F(x,y)$ is called homogeneous differential equation, if $F(x, y)$ is a homogeneous function of degree zero, *i.e.*, $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$.

Example: $(x^2 + xy)dy = (x^2 + y^2)dx$

$\Rightarrow$ $\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$ is homogeneous differential equation because

Here, $F(x,y) = \frac{x^2+y^2}{x^2+xy}$

$$\therefore \quad F(\lambda x,\lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\lambda^2 x^2 + \lambda x.\lambda y} = \frac{\lambda^2(x^2+y^2)}{\lambda^2(x^2+xy)} = \lambda^0 F(x,y)$$

Hence, $F(x, y)$ is homogeneous function of degree zero.

Therefore, $\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$ is a homogeneous differential equation.

To solve this type of equation we proceed as follows:

(*i*) Suppose $y = vx$ and so $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

(*ii*) The value $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ is substituted in given equation. The equation reduces to variable separable form, which can be solved by integrating both sides.

(*iii*) Finally, v is replaced by $\frac{y}{x}$ to get the required solution.

[**Note :** *If the homogeneous differential equation is in the form* $\frac{dx}{dy} = F(x,y)$ *then we substitute* $x = vy$ *and so* $\frac{dx}{dy} = v + y\frac{dv}{dy}$ *and proceed as above.*]

Type 3: Linear Differential Equations Form: A linear differential equation is that in which the dependent variable and its differential co-efficient occur in the first degree and not multiplied together.

Thus, the standard form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q, \text{ where } P \text{ and } Q \text{ are functions of } x \text{ or constants.}$$

Now, we find a function F of x, by which we can multiply both sides of the given equation so that the LHS becomes a complete differential. Such a function F is called the integrating factor (IF)

In this case $\text{IF} = e^{\int P dx}$ and solution is given by $y\, e^{\int P dx} = \int (Q\, e^{\int P dx})dx + C$

9. Sometimes the Equation can be Made Linear Differential as Follows:

$\frac{dx}{dy} + Px = Q$ in which x is treated as dependent variable while y is treated as independent variable and P, Q are function of y or constant.

In this case $\text{IF} = e^{\int P dy}$ and solution is given by,

$$x\, e^{\int P\, dy} = \int Q\,(e^{\int P dy})dy + C$$

COMPETENCY-FOCUSED QUESTIONS AS PER NEP

Multiple Choice Questions

Choose and write the correct option in the following questions.

1. The degree of the differential equation $x^2 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^3$ is [CBSE 2020 (65/3/1)]

(a) 1 (b) 2 (c) 3 (d) 6

2. The degree of the differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$ is [NCERT Exemplar]

(a) 1 (b) 2 (c) 3 (d) Not defined

3. The order and degree of differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = \frac{d^2y}{dx^2}$ respectively, are

(a) 1, 2 (b) 2, 2 (c) 2, 1 (d) 4, 2

4. The solution of the differential equation $2x.\frac{dy}{dx} - y = 3$ represents a family of

(a) straight lines (b) circles (c) parabolas (d) ellipses

5. The integrating factor of the differential equation $\frac{dy}{dx}(x\log x) + y = 2\log x$ is [NCERT Exemplar]

(a) e^x (b) $\log x$ (c) $\log(\log x)$ (d) x

6. A solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is [NCERT Exemplar]

(a) $y = 2$ (b) $y = 2x$ (c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

7. Which of the following is not a homogeneous function of x and y?

(a) $x^2 + 2xy$ (b) $2x - y$ (c) $\cos^2\left(\frac{y}{x}\right) + \frac{y}{x}$ (d) $\sin x - \cos y$

8. The solution of the differential equation $\frac{dx}{x} + \frac{dy}{y} = 0$ is [CBSE 2023 (65/3/2)]

(a) $\frac{1}{x} + \frac{1}{y} = C$ (b) $\log x - \log y = C$ (c) $xy = C$ (d) $x + y = C$

9. The solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is

(a) $y = \frac{x^2 + C}{4x^2}$ (b) $y = \frac{x^2}{4} + C$ (c) $y = \frac{x^4 + C}{x^2}$ (d) $y = \frac{x^4 + C}{4x^2}$

10. Degree of the differential equation $\sin x + \cos\left(\frac{dy}{dx}\right) = y^2$ is [CBSE 2023 (65/2/1)]

(a) 2 (b) 1 (c) not defined (d) 0

11. The degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$ is

(a) 4 (b) $\frac{3}{2}$ (c) Not defined (d) 2

12. The order and degree of a differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}+x^{\frac{1}{5}}=0$, respectively, are

(a) 2 and not defined
(b) 2 and 2
(c) 2 and 3
(d) 3 and 3

13. The solution of the differential equation $\frac{dy}{dx}+\frac{2y}{x}=0$ with $y(1)=1$ is given by

(a) $y=\frac{1}{x^2}$
(b) $x=\frac{1}{y^2}$
(c) $x=\frac{1}{y}$
(d) $y=\frac{1}{x}$

14. The general solution of $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$ is

(a) $y=e^{x-y}-x^2e^{-y}+C$
(b) $e^y-e^x=\frac{x^3}{3}+C$
(c) $e^x+e^y=\frac{x^3}{3}+C$
(d) $e^x-e^y=\frac{x^3}{3}+C$

15. The integrating factor of the differential equation $(1-y^2)\frac{dx}{dy}+yx=ay,\ (-1<y<1)$ is [CBSE 2023 (65/2/1]

(a) $\frac{1}{y^2-1}$
(b) $\frac{1}{\sqrt{y^2-1}}$
(c) $\frac{1}{1-y^2}$
(d) $\frac{1}{\sqrt{1-y^2}}$

16. Solution of $\frac{dy}{dx}-y=1,\ y(0)=1$ is given by

(a) $xy=-e^x$
(b) $xy=-e^{-x}$
(c) $xy=-1$
(d) $y=2e^x-1$

17. The number of solution of $\frac{dy}{dx}=\frac{y+1}{x-1}$ when $y(1)=2$ is

(a) none
(b) one
(c) two
(d) infinite

18. Integrating factor of the differential equation $(1-x^2)\frac{dy}{dx}-xy=1$ is

(a) $-x$
(b) $\frac{x}{1+x^2}$
(c) $\sqrt{1-x^2}$
(d) $\frac{1}{2}\log(1-x^2)$

19. What is the product of the order and degree of the differential equation $\frac{d^2y}{dx^2}\sin y+\left(\frac{dy}{dx}\right)^3\cos y=\sqrt{y}$? [CBSE 2023 (65/3/2)]

(a) 3
(b) 2
(c) 6
(d) not defined

20. The sum of the order and the degree of the differential equation $\frac{d^2y}{dx^2}+\left[\frac{dy}{dx}\right]^3=\sin y$ is: [CBSE 2023 (65/1/1)]

(a) 5
(b) 2
(c) 3
(d) 4

21. The general solution of the differential equation $x\,dy-(1+x^2)\,dx=dx$ is: [CBSE 2023 (65/1/1)]

(a) $y=2x+\frac{x^3}{3}+C$
(b) $y=2\log x+\frac{x^3}{3}+C$
(c) $y=\frac{x^2}{2}+C$
(d) $y=2\log x+\frac{x^2}{2}+C$

Answers

1. (*a*)	**2.** (*d*)	**3.** (*c*)	**4.** (*c*)	**5.** (*b*)	**6.** (*c*)
7. (*d*)	**8.** (*c*)	**9.** (*d*)	**10.** (*c*)	**11.** (*d*)	**12.** (*a*)
13. (*a*)	**14.** (*b*)	**15.** (*d*)	**16.** (*d*)	**17.** (*b*)	**18.** (*c*)
19. (*b*)	**20.** (*c*)	**21.** (*d*)			

Solutions of Selected Multiple Choice Questions

2. The given differential equation is not a polynomial equation in terms of its derivatives, so its degree is not defined.

$\therefore$ Option (*d*) is correct.

8. Given differential equation be

$$\frac{dx}{x}+\frac{dy}{y}=0 \Rightarrow \frac{dx}{x}=-\frac{dy}{y}$$

$\Rightarrow \quad \int\frac{dx}{x}=-\int\frac{dy}{y}$ (on integrating both sides)

$\Rightarrow \quad \log x=-\log y+\log C$

$\Rightarrow \quad \log x+\log y=\log C \quad \Rightarrow \quad \log xy=\log C$

$\Rightarrow \quad xy=C$

$\therefore$ Option (*c*) is correct.

9. Given differential equation is

$$x\frac{dy}{dx}+2y=x^2 \Rightarrow \frac{dy}{dx}+\frac{2}{x}y=x$$

It is of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{2}{x}, Q=x$.

$\therefore \quad \text{IF}=e^{\int Pdx}=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$

$\therefore$ Solution is $y\times x^2=\int x\times x^2\,dx+C_1=\int x^3dx+C_1$

$\Rightarrow \quad yx^2=\frac{x^4}{4}+C_1$

$\Rightarrow \quad yx^2=\frac{x^4+4C_1}{4}=\frac{x^4+C}{4}$ where $C=4C_1$

$\Rightarrow \quad y=\frac{x^4+C}{4x^2}$

$\therefore$ Option (*d*) is correct.

10. Given differential equation be

$$\sin x+\cos\left(\frac{dy}{dx}\right)=y^2, \text{ which is not a polynomial in } \frac{dy}{dx}.$$

Thus, degree is not defined.

$\therefore$ Option (*c*) is correct.

14. Given, $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$

$\Rightarrow \quad \frac{dy}{dx}=e^{-y}(e^x+x^2) \quad \Rightarrow \quad \frac{dy}{e^{-y}}=(e^x+x^2)dx$

$\Rightarrow \quad e^y dy=(e^x+x^2)dx=e^x dx+x^2dx$

Integrating, we get

$$\int e^y dy = \int e^x dx + \int x^2 dx$$

$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C \qquad \Rightarrow \qquad e^y - e^x = \frac{x^3}{3} + C$

$\therefore$ Option (*b*) is correct.

15. Given differential equation be

$$(1-y^2)\frac{dx}{dy} + yx = ay, \quad -1 < y < 1$$

$\Rightarrow \quad \frac{dx}{dy} + \frac{y}{1-y^2}.x = \frac{ay}{1-y^2}$

It is a linear differential equation of the form

$\frac{dx}{dy} + P.x = Q$, where P, Q be the function of y or constant.

$\therefore \quad P = \frac{y}{1-y^2}$ and $Q = \frac{ay}{1-y^2}$

$\therefore \quad IF = e^{\int P\,dy} = e^{\int \frac{y}{1-y^2}dy} = e^{-\frac{1}{2}\int \frac{2y}{1-y^2}dy}$

$$= e^{-\frac{1}{2}\log|1-y^2|} = e^{\log|1-y^2|^{-\frac{1}{2}}}$$

$$= (1-y^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-y^2}}$$

$\therefore$ Option (*d*) is correct.

16. Given that, $\frac{dy}{dx} - y = 1$

$\Rightarrow \quad \frac{dy}{dx} = 1 + y \qquad \Rightarrow \qquad \frac{dy}{1+y} = dx$

On integrating both sides, we get $\log(1+y) = x + C$

When $x = 0$ and $y = 1$, then

$$\log 2 = 0 + C \Rightarrow \quad C = \log 2$$

The required solution is $\log(1+y) = x + \log 2$

$\Rightarrow \quad \log\left(\frac{1+y}{2}\right) = x \qquad \Rightarrow \qquad \frac{1+y}{2} = e^x$

$\Rightarrow \quad 1 + y = 2e^x \qquad \Rightarrow \qquad y = 2e^x - 1$

$\therefore$ Option (*d*) is correct.

18. Given that, $(1-x^2)\frac{dy}{dx} - xy = 1$

$\Rightarrow \quad \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$, which is a linear differential equation.

$\therefore \quad \text{IF} = e^{\int \frac{-x}{1-x^2}dx}$, Let $1 - x^2 = t \Rightarrow -2xdx = dt \Rightarrow -xdx = \frac{dt}{2}$

$\Rightarrow \quad e^{\frac{1}{2}\int \frac{dt}{t}} = e^{\frac{1}{2}\log t} = e^{\frac{1}{2}\log(1-x^2)} = \sqrt{1-x^2}$

$\therefore$ Option (*c*) is correct.

19. Given differential equation be

$$\frac{d^2y}{dx^2}\sin y + \left(\frac{dy}{dx}\right)^3 \cos y = \sqrt{y}$$

Order = 2, degree = 1

$\Rightarrow$ Product of order and degree = 2 × 1 = 2

$\therefore$ Option (*b*) is correct.

20. Given differential equation be

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = \sin y$$

Its order = 2 and degree = 1

$\Rightarrow$ Sum of the order and the degree = 2 + 1 = 3

$\therefore$ Option (*c*) is correct.

21. Given differential equation,

$$x\,dy - (1 + x^2)dx = dx \quad \Rightarrow \quad x\,dy = (1 + x^2 + 1)dx = (2 + x^2)dx$$

$$\Rightarrow \quad dy = \frac{2 + x^2}{x}\,dx$$

On integrating both sides, we have

$$\Rightarrow \quad \int dy = \int \frac{2 + x^2}{x}\,dx = 2\int \frac{1}{x}\,dx + \int x\,dx \quad \Rightarrow \quad y = 2\log x + \frac{x^2}{2} + C$$

$\therefore$ Option (*d*) is correct.

COMPETENCY-FOCUSED QUESTIONS AS PER NEP

Assertion-Reason Questions

The following questions consist of two statements—Assertion(A) and Reason(R). Answer these questions selecting the appropriate option given below:

(a) Both A and R are true and R is the correct explanation for A.

(b) Both A and R are true but R is not the correct explanation for A.

(c) A is true but R is false.

(d) A is false but R is true.

1. **Assertion (A):** The degree of the differential equation $\frac{d^2y}{dx^2} = 1 + \sqrt{\frac{dy}{dx}}$ is 2.

   Reason (R): The degree of a differential equation is the degree of the highest order derivative occurring in the equation, when differential co-efficients are made free from radicals, fractions and it is written as a polynomial in differential coefficient.

2. **Assertion (A):** Solution of the differential equation $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1}x$ is

   $ye^{\tan^{-1}x} = (\tan^{-1}x - 1)e^{\tan^{-1}x} + C$

   Reason (R): The differential equation of the form $\frac{dy}{dx} + Py = Q$, where P, Q be the functions of x or constant, is a linear type differential equation.

3. **Assertion (A):** The integrating factor of differential equation $\frac{dx}{dy} + (\tan y).x = \sec^2 y$ is $\sec y$.

Reason (R): Linear differential equation of the form $\frac{dx}{dy} + Px = Q$, where P, $Q = f(y)$ or constant has integrating factor, IF $= e^{\int P\,dy}$

4. **Assertion (A):** General solution of differential equation $\frac{dy}{dx} = \frac{y}{x}$ is $y = Cx$.

Reason (R): The differential equation $\frac{d^2y}{dx^2} + y = 0$ has order 2.

5. **Assertion (A):** Solution of the differential equation $e^{dy/dx} = x^2$ is $y = 2(x\log x - x) + C$.

Reason (R): The integrating factor of the differential equation $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1}x$ is $e^{\tan^{-1}x}$.

Answers

1. (*a*) 2. (*b*) 3. (*a*) 4. (*b*) 5. (*b*)

Solutions of Assertion-Reason Questions

1. We have, $$\frac{d^2y}{dx^2} = 1 + \sqrt{\frac{dy}{dx}}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2} - 1\right)^2 = \left(\sqrt{\frac{dy}{dx}}\right)^2 \quad \text{[Squaring both sides]}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - 2\frac{d^2y}{dx^2} + 1 = \frac{dy}{dx} \quad \Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^2 - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 1 = 0$$

$\therefore$ Degree = 2

Clearly, both Assertion (*A*) and Reason (*R*) are true and Reason (*R*) is the correct explanation of Assertion (*A*).

$\therefore$ Option (*a*) is correct.

2. $$(1 + x^2)\frac{dy}{dx} + y = \tan^{-1}x \Rightarrow \frac{dy}{dx} + \frac{1}{1+x^2}.y = \frac{\tan^{-1}x}{1+x^2}$$

$$\therefore \quad \text{IF} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$$

Solution will be $$y \times e^{\tan^{-1}x} = \int \frac{\tan^{-1}x}{1+x^2} \times e^{\tan^{-1}x}\,dx \quad \ldots(i)$$

Let $e^{\tan^{-1}x} = t \Rightarrow \frac{e^{\tan^{-1}x}}{1+x^2}dx = dt$ and $\log(e^{\tan^{-1}x}) = \log t \Rightarrow \tan^{-1}x = \log t$

From equation (*i*), $yx\,e^{\tan^{-1}x} = \int \log t.\,dt = t\log t - t + C = t(\log t - 1) + C$

$$y\,e^{\tan^{-1}x} = e^{\tan^{-1}x}(\log(\tan^{-1}x) - 1) + C$$

Clearly, both Assertion (*A*) and Reason (*R*) are true but Reason (*R*) is not the correct explanation of Assertion (*A*).

$\therefore$ Option (*b*) is correct.

3. $\dfrac{dx}{dy} + (\tan y) . x = \sec^2 y$

Here, $\quad \text{IF} = e^{\int \tan y\, dy} \quad = e^{\log \sec y} \quad = \sec y$

Clearly, both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

$\therefore$ Option (a) is correct.

4. We have

$$\frac{dy}{dx} = \frac{y}{x} \quad \Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

Integrating, we get

$$\log y = \log x + \log C$$

$$\Rightarrow \quad \log y = \log (Cx) \quad \Rightarrow \quad y = Cx$$

So statement A is true.

Also statement R is true but R is not correct explanation of A.

$\therefore$ Option (b) is correct.

5. We have, $\quad e^{dy/dx} = x^2$

Taking logarith both sides, we get

$$\log(e^{dy/dx}) = \log x^2 \;\Rightarrow\; \frac{dy}{dx} = \log x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\log x \quad \Rightarrow \; dy = 2\log x\, dx$$

Integrating, we get

$$y = 2[\int \log x \times 1\, dx]$$

$$= 2\left[\log x \int dx - \int\left\{\frac{d}{dx}(\log x)\int dx\right\} dx\right] = 2\left[\log x \times x - \int \frac{1}{x} \times x\, dx\right]$$

$$\Rightarrow \quad y = 2[x \log x - x] + C$$

$$\Rightarrow \quad y = 2[x \log x - x] + C$$

$$\Rightarrow \quad y = 2x(\log x - 1) + C$$

So statement A is correct.

For statement R,

$$(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{1 + x^2} y = \frac{1}{1 + x^2}\tan^{-1} x$$

It is of the form

$$\frac{dy}{dx} + Py = Q$$

Where, $P = \dfrac{1}{1 + x^2}, Q = \dfrac{1}{1 + x^2}\tan^{-1} x$

$$\therefore \quad \text{IF} = e^{\int P dy} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$$

So statement R is also correct, but R is not correct explanation of statement A.

$\therefore$ Option (b) is correct.

$\Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{v^2-1}{2v} - v = \dfrac{v^2-1-2v^2}{2v} = -\dfrac{(1+v^2)}{2v}$

$\Rightarrow \quad \dfrac{2v}{1+v^2}dv = -\dfrac{dx}{x}$

Integrating, we get

$\int \dfrac{2v}{1+v^2}dv = -\int \dfrac{dx}{x}$

$\Rightarrow \quad \log|1+v^2| = -\log|x| + \log C$

$\Rightarrow \quad \log|1+v^2| + \log|x| = \log C$

$\Rightarrow \quad \log|(1+v^2)x| = \log C$

$\Rightarrow \quad (1+v^2)x = C$

$\Rightarrow \quad \left\{1+\left(\dfrac{y}{x}\right)^2\right\}x = C \quad \Rightarrow \quad \left(\dfrac{x^2+y^2}{x^2}\right)x = C$

$\Rightarrow \quad x^2+y^2 = Cx$

2. Read the following passage and answer the following questions.

If an equation is of the form

$$\frac{dy}{dx} + Py = Q$$

Where P, Q are functions of x then such equation is known as linear differential equation. Its solution is given by

$$y \times \text{IF} = \int Q \times \text{IF}\, dx + C$$

Where $\text{IF} = e^{\int P dx}$

Now suppose we have equation. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

(i) Write the value of *P*.

(ii) Write the value of *Q*.

(iii) (a) Find the general solution of given differential equation.

OR

(iii) (b) If the value of *Q* replace by sin *x* find the solution.

Sol. Given differential equation is $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

It is of the form $\dfrac{dy}{dx} + Py = Q$

(i) Here $P = \dfrac{1}{x}$

(ii) Here $Q = x^2$

(iii) (a) $\text{IF} = e^{\int P dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Solution is given by

$y \times x = \int x^2 \times x\, dx + C \quad \Rightarrow \quad yx = \int x^3 dx + C$

$\Rightarrow \quad yx = \dfrac{x^4}{4} + C \quad \Rightarrow \quad y = \dfrac{x^3}{4} + \dfrac{C}{x}$

OR

(*iii*) (*b*) If $Q = \sin x$

From (*iii*) above IF = x

Solution is given by

$$y \times x = \int x \sin x \, dx + C$$

$$yx = x\int \sin x \, dx - \int\left\{\frac{d}{dx}(x)\int \sin x \, dx\right\}dx + C$$

$$yx = -x\cos x - \int(-\cos x)dx + C$$

$$yx = -x\cos x + \int \cos x \, dx + C$$

$$\Rightarrow \quad yx = -x\cos x + \sin x + C$$

$$\Rightarrow \quad y = -\cos x + \frac{\sin x}{x} + \frac{C}{x}$$

3. Read the following passage and answer the following questions.

Polio drops are delivered to 50*K* children in a district. The rate at which polio drops are given is directly proportional to the number of children who have not been administered the drops. By the end of 2nd week half the children have been given the polio drops. How many will have been given the drops by the end of 3rd week can be estimated using the solution to the differential equation $\frac{dy}{dx} = k(50 - y)$ where x denotes the number of weeks and y the number of children who have been given the drops.

(*i*) (*a*) Find the solution of the differential equation $\frac{dy}{dx} = k(50 - y)$.

(*b*) Find the value of C in the particular solution given that $y(0) = 0$ and $k = 0.049$.

(*ii*) Find the solution that may be used to find the number of children who have been given the polio drops.

Sol. (*i*) (*a*) We have,

$$\frac{dy}{dx} = k(50 - y)$$

$$\Rightarrow \quad \int \frac{dy}{50 - y} = \int k\,dx \quad \Rightarrow \quad -\log|50 - y| = kx + C$$

(*b*) Given $y(0) = 0$ and $k = 0.049$

$$\therefore \quad -\log|50 - y| = kx + C$$

$$\Rightarrow \quad -\log|50 - 0| = 0.049 \times 0 + C$$

$$\Rightarrow \quad -\log 50 = C \quad \Rightarrow \quad C = \log\frac{1}{50}$$

(*ii*) We have,

$$-\log|50 - y| = kx + \log\frac{1}{50} \qquad \text{[from (i) (a), (b)]}$$

$$\Rightarrow \quad -kx = \log|50 - y| + \log\frac{1}{50} \quad \Rightarrow -kx = \log\frac{50 - y}{50}$$

$$\Rightarrow \quad e^{-kx} = \frac{50 - y}{50} = 1 - \frac{y}{50} \quad \Rightarrow \frac{y}{50} = 1 - e^{-kx} \Rightarrow y = 50(1 - e^{-kx})$$

This is the required solution to find the number of children who have been given the polio drops.

CONCEPTUAL QUESTIONS

1. **How many arbitrary constants are there in the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2; y(0) = 1$?** *[CBSE Sample Paper 2021]*

Sol. 0

2. **For what value of n is the following a homogeneous differential equation?**

$$\frac{dy}{dx} = \frac{x^3 - y^n}{x^2y + xy^2}$$ *[CBSE Sample Paper 2021]*

Sol. 3

3. **Find the general solution of the differential equation $e^{y-x}\frac{dy}{dx} = 1$.** *[CBSE 2020 (65/2/1)]*

Sol. Given differential equation is $e^y dy = e^x\, dx$ ½

Integrating to get $e^y = e^x + C$ ½

[CBSE Marking Scheme 2020 (65/2/1)]

4. **Find the integrating factor of the differential equation**

$$x\frac{dy}{dx} = 2x^2 + y$$ *[CBSE 2020 (65/2/2)]*

Sol. Integrating factor is $e^{\int \frac{-1}{x}dx}$ or $\left\{\text{writing given equation as } \frac{dy}{dx} - \frac{y}{x} = 2x\right.$ ½

$= \frac{1}{x}$ ½

[CBSE Marking Scheme 2020 (65/2/2)]

5. **Find the order and degree of differential equation:**

$$\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$$ *[NCERT Exemplar]*

Sol. Order is 4 but degree is not defined because given differential equation cannot be written in the form of polynomial in differential co-efficient.

6. **Find the general solution of the differential equation $e^{y-x}\frac{dy}{dx} = 1$.**

Sol. $e^{y-x}\frac{dy}{dx} = 1 \Rightarrow \frac{e^y}{e^x}\frac{dy}{dx} = 1$

$\Rightarrow e^y\, dy = e^x\, dx$

On integrating we have

$\int e^y dx = \int e^x dx$

$\Rightarrow e^y = e^x + C \Rightarrow y = \log(e^x + C)$

7. **Write the sum of the order and degree of the following differential equation:**

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$$ *[CBSE Allahabad 2015]*

Sol. Given differential equation is

$$\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^3\right] = 0 \Rightarrow 3\left(\frac{dy}{dx}\right)^2 . \frac{d^2y}{dx^2} = 0$$

i.e., order = 2, degree = 1

$\therefore$ Required sum = 2 + 1 = 3.

8. Solve the differential equation $(y+3x^2)\dfrac{dx}{dy}=x$.

Sol. $(y + 3x^2)dx = xdy$

$\Rightarrow \quad ydx + 3x^2dx = xdy$

$\Rightarrow \quad 3x^2dx = xdy - ydx$

$\Rightarrow \quad 3dx = \dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

Integrating, we get

$\Rightarrow \quad 3x = \dfrac{y}{x} + C \quad \Rightarrow \quad 3x^2 = y + Cx$

$\Rightarrow \quad y - 3x^2 + Cx = 0.$

9. Solve the following differential equation:

$$\frac{dy}{dx} + y = \cos x - \sin x$$

Sol. Given differential equation,

$$\frac{dy}{dx} + y = \cos x - \sin x$$

It is a linear differential equation of the type

$\dfrac{dy}{dx} + Py = Q$, where P, Q be the function of x or constants.

$\therefore \quad P = 1, Q = \cos x - \sin x$

Now, integrating factor, I F = $e^{\int P dx} = e^{\int 1 dx} = e^x$

$\therefore$ Solution be $y \times \text{I F} = \int Q \times \text{I F}\, dx$

$\Rightarrow \quad y\, e^x = \int e^x(\cos x - \sin x)\, dx$

$\Rightarrow \quad y\, e^x = e^x \cos x + C$

$\Rightarrow \quad y = \cos x + C\, e^{-x}$

CONSTRUCTED RESPONSE QUESTIONS AS PER NEP

Very Short Answer Questions

1. Find the general solution of $y^2dx + (x^2 - xy + y^2)\,dy = 0$. [*NCERT Exemplar*]

Sol. Given, differential equation is $y^2dx + (x^2 - xy + y^2)dy = 0$.

$\Rightarrow \quad y^2dx = -(x^2 - xy + y^2)dy$

$\Rightarrow \quad y^2\dfrac{dx}{dy} = -(x^2 - xy + y^2)$

$\Rightarrow \quad \dfrac{dx}{dy} = -\left(\dfrac{x^2}{y^2} - \dfrac{x}{y} + 1\right) \qquad \ldots(i)$

Which is a homogeneous differential equation.

Put $\frac{x}{y} = v$ or $x = vy$

$\Rightarrow \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

On substituting these values in equation (*i*), we get

$$v + y\frac{dv}{dy} = -[v^2 - v + 1]$$

$\Rightarrow \quad y\frac{dv}{dy} = -v^2 + v - 1 - v$

$\Rightarrow \quad y\frac{dv}{dy} = -v^2 - 1 \Longrightarrow \frac{dv}{v^2+1} = -\frac{dy}{y}$

On integrating both sides, we get

$$\tan^{-1}(v) = -\log y + C$$

$\Rightarrow \quad \tan^{-1}\left(\frac{x}{y}\right) + \log y = C \qquad \left[\because v = \frac{x}{y}\right]$

2. Solve the differential equation $(y + 3x^2)\frac{dx}{dy} = x$. **[*CBSE 2019 (65/5/2)*]**

Sol. $(y + 3x^2)dx = xdy \quad \Rightarrow \quad ydx + 3x^2dx = xdy$

$\Rightarrow \quad 3x^2dx = xdy - ydx$

$\Rightarrow \quad 3dx = \frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

Integrating, we get

$\Rightarrow \quad 3x = \frac{y}{x} + C \qquad \Rightarrow \qquad 3x^2 = y + Cx$

$\Rightarrow \quad y - 3x^2 + Cx = 0.$

3. Write the integrating factor of the following differential equation:

$$(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0$$ **[*CBSE Allahabad 2015*]**

Sol. $(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0$

$\Rightarrow \quad (2xy - \cot y)\frac{dy}{dx} = -(1 + y^2)$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{1 + y^2}{2xy - \cot y}$

$\Rightarrow \quad \frac{dx}{dy} = -\frac{(2xy - \cot y)}{1 + y^2}$

$\Rightarrow \quad \frac{dx}{dy} + \frac{2y}{1 + y^2}.x = \frac{\cot y}{1 + y^2}$

It is in the form $\frac{dx}{dy} + Px = Q$, where P and Q are function of y.

$\Rightarrow \quad \text{IF} = e^{\int P\,dy} = e^{\int \frac{2y}{1+y^2}dy} = e^{\log|1+y^2|} = 1 + y^2$

Solve the following differential equations Q(4 – 6).

4. $\frac{dy}{dx} = y\tan x;\ y = 1$ **when** $x = 0.$

Sol. The given equation is $\frac{dy}{dx} = y \tan x$

$\Rightarrow \quad \frac{1}{y}dy = \tan x\, dx$

$\Rightarrow \quad \int \frac{1}{y}dy = \int \tan x dx$

$\Rightarrow \quad \log|y| = -\log|\cos x| + \log C$

$\Rightarrow \quad \log|y| + \log|\cos x| = \log C$

$\Rightarrow \quad \log|y\cos x| = \log C \quad \Rightarrow \quad y\cos x = C$

Putting $y = 1$ and $x = 0$

We have, $\quad 1 . \cos(0) = C \quad \Rightarrow \quad C = 1$

$\therefore \quad y\cos x = 1 \quad \Rightarrow \quad y = \frac{1}{\cos x} \quad \Rightarrow \quad y = \sec x$

5. $\cos\left(\frac{dy}{dx}\right) = a\ (-1 \le a \le 1);\ y = 1$ **when** $x = 0.$

Sol. The given equation is $\cos\left(\frac{dy}{dx}\right) = a.$

$\Rightarrow \quad \frac{dy}{dx} = \cos^{-1}a \quad \Rightarrow \quad dy = \cos^{-1}a\, dx$

$\Rightarrow \quad \int dy = \int \cos^{-1}a\, dx$

$\Rightarrow \quad \int dy = \cos^{-1}a \int dx \quad \Rightarrow \quad y = (\cos^{-1}a)\,x + C$

Putting $y = 1$ and $x = 0$

We have,

$1 = \cos^{-1}(a) \times 0 + C \quad \Rightarrow \quad C = 1$

$\therefore \quad y = (\cos^{-1}a)\,x + 1 \quad \Rightarrow \quad \cos^{-1}a = \frac{y-1}{x}$

$\Rightarrow \quad a = \cos\left(\frac{y-1}{x}\right).$

6. $\frac{dy}{dx} + \frac{1+y^2}{2y} = 0.$

Sol. We have,

$\frac{dy}{dx} + \frac{1+y^2}{2y} = 0 \Rightarrow \frac{dy}{dx} = -\frac{1+y^2}{2y} \Rightarrow \frac{2y}{1+y^2}dy = -dx$

$\Rightarrow \quad \int \frac{2y}{1+y^2}dy = -\int dx$ putting $1 + y^2 = t \quad \Rightarrow \quad 2y\,dy = dt$

$\Rightarrow \quad \int \frac{1}{t}dt = -\int dx + C$

$\Rightarrow \quad \log t = -x + C$

$\Rightarrow \quad \log|1 + y^2| + x = C.$

7. Solve the differential equation $\cos x\dfrac{dy}{dx} = \cos 3x - \cos 2x$.

Sol. $\cos x\dfrac{dy}{dx} = \cos 3x - \cos 2x$

$\Rightarrow \quad \cos x\dfrac{dy}{dx} = (4\cos^3 x - 3\cos x) - (2\cos^2 x - 1)$

$\Rightarrow \quad \dfrac{dy}{dx} = 4\cos^2 x - 3 - 2\cos x + \dfrac{1}{\cos x}$

$\Rightarrow \quad dy = \left(\dfrac{4(1+\cos 2x)}{2} - 3 - 2\cos x + \sec x\right)dx$

$\Rightarrow \quad \int dy = \int(2 + 2\cos 2x - 3 - 2\cos x + \sec x)dx = \int(2\cos 2x - 1 - 2\cos x - \sec x)dx$

$\Rightarrow \quad y = \dfrac{2\sin 2x}{2} - x - 2\sin x - \log|\sec x + \tan x| + C$

$\Rightarrow \quad y = \sin 2x - x - 2\sin x - \log|\sec x + \tan x| + C$

8. Find the general solution of the differential equation

$$\log\left(\frac{dy}{dx}\right) = ax + by$$ **[CBSE 2021-22 (Term-2)]**

Sol.

$\log\left(\dfrac{dy}{dx}\right) = ax + by$

$\Rightarrow \dfrac{dy}{dx} = e^{ax+by}$

$\Rightarrow \dfrac{dy}{dx} = e^{ax} \cdot e^{by}$ $\quad [e^{a+b} = e^a \cdot e^b]$

$\Rightarrow \dfrac{dy}{e^{by}} = e^{ax}dx$

$\Rightarrow e^{-by}dy = e^{ax}dx$

On integrating both sides.

$\int e^{-by}dy = \int e^{ax}dx$

$\Rightarrow \dfrac{-e^{-by}}{b} = \dfrac{e^{ax}}{a} + c$

$\dfrac{e^{ax}}{a} + \dfrac{e^{-by}}{b} = c'$ $\quad [c' = -c]$

where c & c' are constants.

Answer: $\dfrac{e^{ax}}{a} + \dfrac{e^{-by}}{b} = c'$

[Topper's Answer 2022]

9. **Find the sum of the order the degree of the differential equation:**

$$\left(x+\frac{dy}{dx}\right)^2=\left(\frac{dy}{dx}\right)^2+1$$

[CBSE 2021-22 (Term-2) (65/1/1)]

Sol. Given differential equation can be written as

$$x^2+\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}=\left(\frac{dy}{dx}\right)^2+1$$

i.e., $x^2+2x\frac{dy}{dx}=1$; Order = 1, degree = 1 ½+½

Sum of order and degree = 1 + 1 = 2 1

[CBSE Marking Scheme 2022 (65/1/1)]

AS PER NEP CONSTRUCTED RESPONSE QUESTIONS

Short Answer Questions

1. **Find the general solution of the differential equation** $x\frac{dy}{dx}=y(\log y-\log x+1)$.

[CBSE 2021-22 (Term-2) (65/3/2)]

Sol.

$x\frac{dy}{dx}=y(\log y-\log x+1)$

$\Rightarrow \frac{dy}{dx}=\frac{y}{x}\left(\log\left(\frac{y}{x}\right)+1\right)$ $\left[\log a-\log b=\log\left(\frac{a}{b}\right)\right]$

on putting $x=\lambda x$, $y=\lambda y$.

~~$f(\lambda x,\lambda y)=\frac{\lambda y}{\lambda x}\left(\log\left(\frac{\lambda y}{\lambda x}\right)+1\right)$~~

$f(\lambda x,\lambda y)=\frac{y}{x}\left(\log\left(\frac{y}{x}\right)+1\right)$

$=f(x,y)$

Thus, this equation is homogenous equation.

Let $\frac{y}{x}=t$ or $y=tx$

on differentiating with respect to x

$\frac{dy}{dx}=t+x\frac{dt}{dx}$

$\frac{dy}{dx}=\frac{y}{x}\left(\log\left(\frac{y}{x}\right)+1\right)$

$$t + x\frac{dt}{dx} = t(\log t + 1) \quad \left[\because \frac{y}{x} = t\right]$$

$$t + x\frac{dt}{dx} = t\log t + t$$

$$x\frac{dt}{dx} = t\log t$$

$$\frac{dt}{t\log t} = \frac{dx}{x}$$

on integrating both sides

$$\int \frac{dt}{t\log t} = \int \frac{dx}{x}$$

$$\int \frac{dt}{t\log t} = \log_e x + c$$

let $\log t = u$

on differentiating,

$$\frac{1}{t}dt = du$$

$$\int \frac{du}{u} = \ln x + c \quad [\log_e x = \ln x]$$

$$\ln u = \ln x + c \quad [c \text{ is integration constant}]$$

$$\ln(\ln t) = \ln x + c \quad [u = \log_e t = \ln t].$$

$$\ln(\ln(y/x)) = \ln x + c$$

$$\ln(\ln(y/x)) - \ln x = c$$

$$\ln\left(\frac{\ln(y/x)}{x}\right) = c$$

$$[\log a - \log b = \log(a/b)]$$

Answer: $\ln\left(\frac{\ln(y/x)}{x}\right) = c$

[where $\ln x = \log_e x$]

[Topper's Answer 2022]

2. Find the particular solution of the differential equation $x\frac{dy}{dx} + x\cos^2\left(\frac{y}{x}\right) = y$; given that when $x = 1,\ y = \frac{\pi}{4}$. [CBSE 2021-22 (65/1/1) (Term-2)]

Sol. Given differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x} \qquad ...(1) \qquad 1$$

Let $y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ ½

Equation (1), Becomes $v + x\frac{dv}{dx} = v - \cos^2 v$

$$\sec^2 v\, dv = -\frac{dx}{x}$$

Integrating both sides we get ½

$$\tan v = -\log|x| + c$$

$$\tan\frac{y}{x} = -\log|x| + c$$

$$x = 1,\ y = \frac{\pi}{4} \Rightarrow c = 1 \qquad 1$$

$\therefore$ Particular solution is $\tan\frac{y}{x} = -\log|x| + 1$ ½

[CBSE Marking Scheme 2022]

3. Find the particular solution of the differential equation

$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$, given that $y = \frac{x}{4}$ at $x = 1$. [CBSE 2020 (65/1/1)]

Sol.

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) = f(y/x) \quad -(i)$$

$\therefore$ It is a homogeneous function.

let $\frac{y}{x} = V \Rightarrow y = Vx$

Differentiating with respect to x

$$\Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}.$$

$\therefore$ equation (i) can be written as

$$V + x\frac{dV}{dx} = V - \tan V$$

$$\Rightarrow x\frac{dV}{dx} = -\tan V$$

$$\Rightarrow \int -\cot V\, dV = \int\frac{dx}{x}$$

$$\Rightarrow -\log\sin V = \log x \mp \log c$$, $-\log C$ is integration constant.

$\Rightarrow \log \sin V + \log x = \log c$

$\Rightarrow x \sin V = c$

$\Rightarrow x \sin\left(\frac{y}{x}\right) = c$

At $x = 1,\ y = \pi/4$

$\sin\left(\frac{\pi}{4}\right) = c = \frac{1}{\sqrt{2}}$

$\Rightarrow x \sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$ Answer

[Topper's Answer 2020]

4. Solve the differential equation $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$. [CBSE 2019 (65/1/1)]

Sol.

$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$

It is linear DE of form $\frac{dy}{dx} + Py = Q$

$P = \frac{2x}{1+x^2}$, $Q = \frac{4x^2}{1+x^2}$

I.F $= e^{\int P\,dx} = e^{\ln|1+x^2|} = 1+x^2$

Sol^n of DE :

$y(1+x^2) = \int \frac{4x^2(1+x^2)}{1+x^2}dx + C$

$= \int 4x^2\,dx + C$

$y(1+x^2) = \frac{4x^3}{3} + C$

$x = 0, y = 0$

$\therefore C = 0$

$\Rightarrow 3y(1+x^2) = 4x^3$

[Topper's Answer 2019]

5. Find the general solution of the differential equation : $\frac{d}{dx}(xy^2) = 2y(1+x^2)$ [CBSE 2023 (65/3/2)]

Sol. Given differential equation be $\frac{d(xy^2)}{dx} = 2y(1+x^2)$

$\Rightarrow \quad x\frac{dy^2}{dx} + y^2\frac{dx}{dx} = 2y\,(1+x^2)$

$\Rightarrow \quad 2xy\frac{dy}{dx} + y^2 = 2y\,(1 + x^2)$

$\Rightarrow \quad 2x\frac{dy}{dx} + y = 2\,(1 + x^2)$

$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{2x}\cdot y = \frac{2(1+x^2)}{2x}$

$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{2x}y = \frac{1+x^2}{x}$

It is a linear differential equation of the type

$\frac{dy}{dx} + Py = Q$, where P, Q be the function of a constant

$\therefore \quad P = \frac{1}{2x},\; Q = \frac{1+x^2}{x}$

$\therefore \quad IF = e^{\int P\,dx} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\log x} = e^{\log\sqrt{x}} = \sqrt{x}$

Its solution be

$y \times IF = \int Q \times IF\,dx$

$\Rightarrow \quad y \times \sqrt{x} = \int \frac{1+x^2}{x} \times \sqrt{x}\,dx = \int \frac{1+x^2}{\sqrt{x}}dx = \int\left(x^{-\frac{1}{2}} + x^{3/2}\right)dx$

$\Rightarrow \quad y\sqrt{x} = 2x^{1/2} + \frac{2}{5}x^{5/2} + C$

$\Rightarrow \quad y = 2 + \frac{2}{5}x^2 + C$

6. Solve the following differential equation : $xe^{y/x} - y + x\frac{dy}{dx} = 0$ [CBSE 2023 (65/3/2)]

Sol. Given differential equation be

$x\,e^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0$

$\Rightarrow x\frac{dy}{dx} = y - x\,e^{y/x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - e^{y/x}$

It is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we have

$v + x\frac{dv}{dx} = v - e^v \quad \Rightarrow \quad x\frac{dv}{dx} = -e^v$

$\Rightarrow \quad \int\frac{dv}{e^v} = -\int\frac{dx}{x}$ (on integrating)

$\Rightarrow \quad \int e^{-v}dv = -\int\frac{dx}{x}$

$\Rightarrow \quad -e^{-v} = -\log x + C$

$\Rightarrow \quad \log x - e^{-\frac{y}{x}} = C$

7. Solve the differential equation given by $x\,dy - y\,dx - \sqrt{x^2+y^2}\,dx = 0$. [CBSE 2023 (65/1/1)]

Sol. Given differential equation be

$x\,dy - y\,dx - \sqrt{x^2+y^2}\,dx = 0$

$\Rightarrow \; x\,dy = \left(y + \sqrt{x^2+y^2}\right)dx$

$\Rightarrow \; \frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$

It is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x} = v + \sqrt{1+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2} \quad \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

On integrating both sides, we have

$$\int\frac{dv}{\sqrt{1+v^2}} = \int\frac{dx}{x} \quad \Rightarrow \log\left|v + \sqrt{1+v^2}\right| = \log x + \log C$$

$$\Rightarrow \quad \log\left|\frac{v+\sqrt{1+v^2}}{x}\right| = \log C$$

$$\Rightarrow \quad \frac{\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}}{x} = C \quad \Rightarrow y + \sqrt{x^2+y^2} = Cx^2$$

8. Find the particular solution of the differential equation $\frac{dy}{dx} + \sec^2 x.\, y = \tan x.\sec^2 x$, given that $y(0) = 0$. [*CBSE 2023 (65/1/1)*]

Sol. Given differential equation be

$$\frac{dy}{dx} + \sec^2 x.y = \tan x.\sec^2 x, \quad \text{given } y(0) = 0$$

It is a linear differential equation of type

$\frac{dy}{dx} + Py = Q$, where P, Q be the function of x or constant

$\therefore \; P = \sec^2 x,\; Q = \tan x \sec^2 x$

$\therefore \; IF = e^{\int P dx} = e^{\int \sec^2 x dx} = e^{\tan x}$

Its solution be

$$y \times IF = \int Q \times IF\, dx$$

$$\Rightarrow \quad y \times e^{\tan x} = \int \tan x.\sec^2 x.\, e^{\tan x} dx$$

Let $\tan x = t \quad \Rightarrow \sec^2 x\, dx = dt$

$$= \int t.e^t dt = t\int e^t dt - \int\left(\frac{dt}{dt}.\int e^t dt\right)dt$$

$$= t.e^t - \int e^t dt = te^t - e^t + C = e^t(t-1) + C$$

$$\Rightarrow \quad y\, e^{\tan x} = e^{\tan x}(\tan x - 1) + C$$

$$\Rightarrow \quad y = \tan x - 1 + C\, e^{-\tan x}$$

Given, $y(0) = 0 \;\Rightarrow\; 0 = \tan 0 - 1 + Ce^{\tan 0}$

$\Rightarrow 0 = C - 1 \quad \Rightarrow \quad C = 1$

Particular solution is given by

$$y = \tan x - 1 + e^{-\tan x}$$

9. Find the general solution of the differential equation:
$(xy - x^2)\, dy = y^2 dx$. [*CBSE 2023 (65/2/1)*]

Sol. Given differential equation be

$$(xy - x^2)\, dy = y^2 dx$$

$\Rightarrow \quad \frac{dy}{dx}=\frac{y^2}{xy-x^2}$

$\Rightarrow \quad \frac{dx}{dy}=\frac{xy-x^2}{y^2}$

It is a homogeneous differential equation.

Put $x = vy \quad \Rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$

$\Rightarrow \quad v+y\frac{dv}{dy}=\frac{vy^2-v^2y^2}{y^2}=\frac{v-v^2}{1}=v-v^2$

$\Rightarrow \quad v+y\frac{dv}{dy}=v-v^2$

$\Rightarrow \quad y\frac{dv}{dy}=-v^2 \qquad \Rightarrow \frac{dv}{v^2}=-\frac{dy}{y}$

On integrating both sides, we have

$\int\frac{dv}{v^2}=-\int\frac{dy}{y} \qquad \Rightarrow \int v^{-2}dv=-\log y+C$

$\Rightarrow \quad \frac{v^{-1}}{-1}=-\log y+C \qquad \Rightarrow \log y-\frac{1}{v}=C$

$\Rightarrow \quad \log y-\frac{y}{x}=C$

10. Find the general solution of the differential equation:

$$(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$$ *[CBSE 2023 (65/2/1)]*

Sol. Given differential equation be

$(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$

$\Rightarrow \quad \frac{dy}{dx}+\frac{2x}{x^2+1}.y=\frac{\sqrt{x^2+4}}{x^2+1}$

It is a linear differential equation of the type

$\frac{dy}{dx}+Py=Q$

Hence, $P=\frac{2x}{x^2+1}, Q=\frac{\sqrt{x^2+4}}{x^2+1}$

$\therefore \quad IF=e^{\int P\,dx}=e^{\int\frac{2x}{x^2+1}dx}=e^{\log(x^2+1)}=(x^2+1)$

Thus, its solution be

$y\times IF=\int Q\times IF\,dx$

$\Rightarrow \quad y\times(x^2+1)=\int\frac{\sqrt{x^2+4}}{x^2+1}\times(x^2+1)\,dx=\int\sqrt{x^2+4}\,dx$

$\Rightarrow \quad y\,(x^2+1)=\frac{x}{2}\sqrt{x^2+4}+\frac{4}{2}\log\left|x+\sqrt{x^2+4}\right|+C$

$=\frac{x}{2}\sqrt{x^2+4}+2\log\left|x+\sqrt{x^2+4}\right|+C$

11. **Find the particular solution of the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$, given that $y = 0$ when $x = \frac{\pi}{3}$.** *[CBSE 2018]*

Sol.

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

On comparing the above equation with the standard linear equation

$$\frac{dy}{dx} + Py = Q$$

we get, $P = 2\tan x$, $Q = \sin x$

Therefore, $I.F. = e^{\int P dx}$

$$I.F. = e^{2\int \tan x \, dx}$$

$$= e^{2(\log \sec x)}$$

$$= e^{\log(\sec x)^2} = \sec^2 x$$

Now,

$$y \cdot I.F. = \int Q \times I.F. \, dx.$$

$$y \cdot \sec^2 x = \int \sin x \times \sec^2 x \, dx$$

$$y \cdot \sec^2 x = \int \sin x \times \frac{1}{\cos^2 x} dx$$

$$y \cdot \sec^2 x = \int \tan x \cdot \sec x \, dx$$

Put $\sec x = t$

$(\sec x \tan x) dx = dt$

$$y \cdot \sec^2 x = \int dt$$

$$y \cdot \sec^2 x = t + C$$

$$y \cdot \sec^2 x = \sec x + C$$

Now when $y = 0$, $x = \pi/3$.

$$0 = \sec \frac{\pi}{3} + C, \quad 0 = +2 + C$$

$$C = -2$$

Therefore,

Solution ⇒ $y \cdot \sec^2 x = \sec x - 2$.

or. $y = \frac{1}{\sec x} - \frac{2}{\sec^2 x}$

or. $y = \sec x^{-1} - 2(\sec^2 x)^{-1}$ Ans.

[Topper's Answer 2018]

12. **Solve the following differential equation: $(1 + e^{y/x})dy + e^{y/x}\left(1 - \frac{y}{x}\right)dx = 0, (x \neq 0)$.** *[CBSE 2020 (65/2/1)]*

Sol. Given differential equation

$$(1 + e^{y/x})dy + e^{y/x}\left(1 - \frac{y}{x}\right)dx = 0$$

$\Rightarrow \quad (1+e^{y/x})dy = \left(\frac{y}{x}-1\right)e^{y/x}dx$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(\frac{y}{x}-1\right)e^{y/x}}{(1+e^{y/x})}$

It is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

We have,

$$v + x\frac{dv}{dx} = \frac{(v-1)}{1+e^v}e^v = \frac{ve^v - e^v}{1+e^v}$$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{ve^v - e^v}{1+e^v} - v = \frac{ve^v - e^v - v - ve^v}{1+e^v}$

$\Rightarrow \quad x\frac{dv}{dx} = -\frac{(v+e^v)}{1+e^v}$

$\Rightarrow \quad \frac{1+e^v}{v+e^v}dv = -\frac{dx}{x}$

On integrating both sides, we have

$$\int \frac{1+e^v}{v+e^v}dv = -\int \frac{dx}{x}$$

$\Rightarrow \quad \log|v+e^v| = -\log|x| + \log|C|$

$\Rightarrow \quad \log|v+e^v| + \log|x| = \log|C|$

$\Rightarrow \quad \log|x(v+e^v)| = \log|C|$

$\Rightarrow \quad x(v+e^v) = C \quad \Rightarrow \quad x\left(\frac{y}{x}+e^{y/x}\right) = C$

$\Rightarrow \quad y + x\,e^{y/x} = C$

13. Solve the differential equation $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$. **[*CBSE (AI) 2014*]**

Sol. Given differential equation is

$$(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$

$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{1+x^2}y = \frac{e^{\tan^{-1}x}}{1+x^2} \quad \ldots(i)$

Equation (*i*) is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{1}{1+x^2}, Q = \frac{e^{\tan^{-1}x}}{1+x^2}$$

$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

Therefore, general solution of required differential equation is

$$y.e^{\tan^{-1}x} = \int e^{\tan^{-1}x}.\frac{e^{\tan^{-1}x}}{1+x^2}dx + C$$

$\Rightarrow \quad y.e^{\tan^{-1}x} = \int \frac{e^{2\tan^{-1}x}}{1+x^2}dx + C \quad \ldots(ii)$

Let $\tan^{-1} x = z$ $\Rightarrow$ $\frac{1}{1+x^2}dx = dz$

(*ii*) becomes

$$y.e^{\tan^{-1}x} = \int e^{2z}dz + C \Rightarrow y.e^{\tan^{-1}x} = \frac{e^{2z}}{2} + C$$

$\Rightarrow$ $y.e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + C$ [Putting $z = \tan^{-1} x$]

$\Rightarrow$ $y = \frac{e^{\tan^{-1}x}}{2} + C.e^{-\tan^{-1}x}$ [Dividing both sides by $e^{\tan^{-1}x}$]

It is the required solution.

14. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$ given that $y = 1$ when $x = 0$. [*CBSE Delhi 2014*]

Sol. We have, $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$

$\Rightarrow$ $e^x\sqrt{1-y^2}\,dx = -\frac{y}{x}dy$ $\Rightarrow$ $xe^x dx = -\frac{y}{\sqrt{1-y^2}}dy$

$\Rightarrow$ $\int \underset{I\ II}{xe^x} dx = -\int \frac{y}{\sqrt{1-y^2}}dy$

$\Rightarrow$ $xe^x - \int e^x dx = \frac{1}{2}\int\frac{dt}{\sqrt{t}}$, where $t = 1-y^2$ $\Rightarrow$ $\frac{dt}{2} = -ydy$ (Using ILATE on LHS)

$\Rightarrow$ $xe^x - e^x = \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$ $\Rightarrow$ $xe^x - e^x = \sqrt{t} + C$

$\Rightarrow$ $xe^x - e^x = \sqrt{1-y^2} + C$, is the general solution.

Putting $y = 1$ and $x = 0$, we get

$$0e^0 - e^0 = \sqrt{1-1^2} + C \Rightarrow C = -1$$

Therefore, required particular solution is $xe^x - e^x = \sqrt{1-y^2} - 1$.

15. Solve the differential equation: [*CBSE (AI) 2013; Delhi 2015*]

$$(\tan^{-1} y - x)\,dy = (1+y^2)\,dx$$

Sol. The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \quad ...(i)$$

Now, (*i*) is of the form $\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1}y}{1+y^2}$

Therefore, IF $= e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Thus, the solution of the given differential equation is

$$xe^{\tan^{-1}y} = \int\left(\frac{\tan^{-1}y}{1+y^2}\right)e^{\tan^{-1}y}dy + C \quad ...(ii)$$

Let $I = \int\left(\frac{\tan^{-1}y}{1+y^2}\right)e^{\tan^{-1}y}dy$

Substituting $\tan^{-1} y = t$ so that $\left(\frac{1}{1+y^2}\right)dy = dt$, we get

$$I = \int te^t dt = te^t - \int 1.e^t dt = te^t - e^t \equiv e^t(t-1)$$

or $I = e^{\tan^{-1}y}(\tan^{-1}y - 1)$

Substituting the value of I in equation (*ii*), we get

$$x.e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C$$

or $x = (\tan^{-1}y - 1) + C\,e^{-\tan^{-1}y}$ is the required solution.

16. Solve the differential equation $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$, where $x \in (-\infty, -1) \cup (1, \infty)$.

[*CBSE Delhi 2014; (AI) 2010; (F) 2009, 2011*]

Sol. The given differential equation is $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$.

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{2}{(x^2-1)^2} \quad ...(i)$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

$$\therefore \text{IF} = e^{\int P\,dx} = e^{\int 2x/(x^2-1)dx} = e^{\log|x^2-1|} = x^2-1$$

Multiplying both sides of (*i*) by IF $= x^2-1$, we get $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

$$\Rightarrow d\left(y(x^2-1)\right) = \frac{2}{x^2-1}$$

Integrating both sides, we get

$$y(x^2-1) = \int \frac{2}{x^2-1}dx + C \qquad [\text{Using}: y\,(\text{IF}) = \int Q.(\text{IF})dx + C]$$

$$\Rightarrow y(x^2-1) = \frac{2}{2}\log\left|\frac{x-1}{x+1}\right| + C \quad \Rightarrow \quad y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$$

This is the required solution.

17. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$ given that $y = 0$ when $x = 1$.

[*CBSE (AI) 2014*]

Sol. Given differential equation is $\frac{dy}{dx} = 1 + x + y + xy$.

$$\Rightarrow \frac{dy}{dx} = (1+x) + y(1+x) \quad \Rightarrow \quad \frac{dy}{dx} = (1+x)(1+y) \quad \Rightarrow \quad \frac{dy}{1+y} = (1+x)dx$$

Integrating both sides, we get $\log|1+y| = \int(1+x)dx$

$$\Rightarrow \log|1+y| = x + \frac{x^2}{2} + C \text{ is the general solution.}$$

Putting $x = 1, y = 0$, we get

$$\log 1 = 1 + \frac{1}{2} + C \quad \Rightarrow \quad 0 = \frac{3}{2} + C \quad \Rightarrow \quad C = \frac{-3}{2}$$

Hence, particular solution is $\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$.

18. Solve the differential equation $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$. [*CBSE (F) 2014*]

Sol. Given differential equation is $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$

$\Rightarrow \dfrac{dy}{dx}+\left(\dfrac{1}{x.\log x}\right).y=\dfrac{2}{x^2}$ (Divide each term by $x \log x$)

It is in the form $\dfrac{dy}{dx}+Py=Q$, where $P=\dfrac{1}{x.\log x}$ and $Q=\dfrac{2}{x^2}$.

$\therefore$ IF $=e^{\int Pdx}=e^{\int \frac{dx}{x\log x}}$

Put $\log x = z \Rightarrow \dfrac{dx}{x}=dz$

IF $e^{\int \frac{1}{z}dz}=e^{\log z}=z=\log x$

$\therefore$ General solution is

$$y.\log x=\int \log x.\frac{2}{x^2}dx+C \quad \Rightarrow \quad y\log x = 2\int \frac{\log x}{x^2}dx+C$$

Let $\log x = z \Rightarrow \dfrac{1}{x}dx=dz$ also $\log x = z \Rightarrow x=e^z$

$$\therefore\ y\log x=2\int \frac{z}{e^z}dz+C \quad \Rightarrow \quad y\log x=2\int z.e^{-z}dz+C$$

$$\Rightarrow y\log x=2\left[z.\frac{e^{-z}}{-1}-\int \frac{e^{-z}}{-1}dz\right]+C \quad \Rightarrow \quad y\log x=2[-ze^{-z}+\int e^{-z}dz]+C$$

$$\Rightarrow\ y\log x=-2ze^{-z}-2e^{-z}+C \quad \Rightarrow \quad y\log x=-2\log x\, e^{-\log x}-2e^{-\log x}+C$$

$$\Rightarrow\ y\log x=-2\log x.\frac{1}{x}-\frac{2}{x}+C \qquad \left[\because e^{-\log x}=e^{\log\frac{1}{x}}=\frac{1}{x}\right]$$

$$\Rightarrow\ y\log x=-\frac{2}{x}(1+\log x)+C$$

19. Show that the differential equation $x\dfrac{dy}{dx}\sin\left(\dfrac{y}{x}\right)+x-y\sin\left(\dfrac{y}{x}\right)=0$ is homogeneous. Find the particular solution of this differential equation, given that $x=1$ when $y=\dfrac{\pi}{2}$. [*CBSE Delhi 2013*]

Sol. Given differential equation is $x\dfrac{dy}{dx}\sin\dfrac{y}{x}+x-y\sin\dfrac{y}{x}=0$.

Dividing both sides by $x\sin\dfrac{y}{x}$, we get

$$\frac{dy}{dx}+\operatorname{cosec}\frac{y}{x}-\frac{y}{x}=0 \quad \Rightarrow \quad \frac{dy}{dx}=\frac{y}{x}-\operatorname{cosec}\frac{y}{x} \qquad \ldots(i)$$

Let $F(x,y)=\dfrac{y}{x}-\operatorname{cosec}\dfrac{y}{x}$

$$\therefore\ F(\lambda x,\lambda y)=\frac{\lambda y}{\lambda x}-\operatorname{cosec}\frac{\lambda y}{\lambda x}=\lambda^0\left[\frac{y}{x}-\operatorname{cosec}\frac{y}{x}\right]=\lambda^0F(x,y)$$

Hence, differential equation (*i*) is homogeneous.

Let $y=vx \Rightarrow \dfrac{dy}{dx}=v+x.\dfrac{dv}{dx}$

Now, equation (*i*) becomes

$$v+x.\frac{dv}{dx}=\frac{vx}{x}-\operatorname{cosec}\frac{vx}{x}$$

$$v+x.\frac{dv}{dx}=v-\operatorname{cosec}v \quad \Rightarrow \quad x.\frac{dv}{dx}=-\operatorname{cosec}v$$

$\Rightarrow \ -\sin v\,dv = \dfrac{dx}{x} \qquad \Rightarrow \qquad -\int \sin v\,dv = \int \dfrac{dx}{x}$

$\Rightarrow \ \cos v = \log|x| + C \qquad \Rightarrow \qquad \cos\dfrac{y}{x} = \log|x| + C$ ...(*ii*)

Putting $y = \dfrac{\pi}{2}, x = 1$ in (*ii*), we get

$\therefore \quad \cos\dfrac{\pi}{2} = \log 1 + C \quad \Rightarrow \ 0 = 0 + C \ \Rightarrow \ C = 0$

Hence, particular solution is

$\cos\dfrac{y}{x} = \log|x| + 0 \qquad i.e., \qquad \cos\dfrac{y}{x} = \log|x|$

20. Solve the differential equation:

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$ **[*CBSE (AI) 2010; (F) 2015*]**

Sol. Given $\sqrt{1+x^2+y^2+x^2y^2} + xy\dfrac{dy}{dx} = 0$

By simplifying the equation, we get

$xy\dfrac{dy}{dx} = -\sqrt{1+x^2+y^2+x^2y^2} = -\sqrt{1+x^2+y^2(1+x^2)}$

$\Rightarrow \quad xy\dfrac{dy}{dx} = -\sqrt{(1+x^2)(1+y^2)} = -\sqrt{(1+x^2)}\sqrt{(1+y^2)}$

$\Rightarrow \quad \dfrac{y}{\sqrt{(1+y^2)}}dy = -\dfrac{\sqrt{(1+x^2)}}{x}dx$

Integrating both sides, we get

$\int \dfrac{y}{\sqrt{(1+y^2)}}dy = -\int\dfrac{\sqrt{(1+x^2)}}{x}dx = -\int\dfrac{\sqrt{1+x^2}}{x^2}\times\ x dx$ ...(*i*)

Let $1+y^2 = t \Rightarrow 2y\,dy = dt$ and $1+x^2 = m^2 \ \Rightarrow \ 2x\,dx = 2m\,dm \ \Rightarrow \ x\,dx = m\,dm$

$\therefore \ (i) \ \Rightarrow \ \dfrac{1}{2}\int\dfrac{1}{\sqrt{t}}dt = -\int\dfrac{m}{m^2-1}.m\,dm$

$\Rightarrow \quad \dfrac{1}{2}\dfrac{t^{1/2}}{1/2} + \int\dfrac{m^2}{m^2-1}dm = 0 \quad \Rightarrow \quad \sqrt{t} + \int\dfrac{m^2+1-1}{m^2-1}dm = 0$

$\Rightarrow \quad \sqrt{t} + \int\left(1+\dfrac{1}{m^2-1}\right)dm = 0 \quad \Rightarrow \quad \sqrt{t} + m + \dfrac{1}{2}\log\left|\dfrac{m-1}{m+1}\right| = 0$

Now, substituting the value of t and m, we get

$$\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C = 0$$

21. $(x^2+y^2)dy = xy\,dx$. If $y(1) = 1$ and $y(x_0)e$, then find the value of x_0. **[*CBSE Bhubneshwar 2015*]**

Sol. Given differential equation is $(x^2+y^2)dy = xy\,dx$

It is also written as

$\dfrac{dy}{dx} = \dfrac{xy}{x^2+y^2}$ ...(*i*)

Now, to solve let $y = vx$. [$\because$ (*i*) is a homogeneous equation]

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = v + \frac{xdv}{dx}$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (*i*), we get

$$v + x\frac{dv}{dx} = \frac{x.vx}{x^2 + (vx)^2}$$

$$v + x\frac{dv}{dx} = \frac{vx^2}{x^2 + v^2x^2} \Rightarrow v + x\frac{dv}{dx} = \frac{vx^2}{x^2(1+v^2)} = \frac{v}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{(1+v^2)} - v \Rightarrow x\frac{dv}{dx} = \frac{v - v - v^3}{(1+v^2)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^3}{(1+v^2)} \Rightarrow \frac{(1+v^2)dv}{v^3} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{(1+v^2)dv}{v^3} = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v^3} + \int \frac{dv}{v} = -\log|x| + C \Rightarrow -\frac{1}{2v^2} + \log|v| = -\log|x| + C$$

$$\Rightarrow -\frac{x^2}{2y^2} + \log\left|\frac{y}{x}\right| = -\log|x| + C \Rightarrow -\frac{x^2}{2y^2} + \log|y| - \log|x| = -\log|x| + C$$

$$\Rightarrow -\frac{x^2}{2y^2} + \log|y| = C \qquad \ldots(ii)$$

Given, $x = 1, y = 1$

$$\Rightarrow -\frac{1}{2\times 1} + \log|1| = C \Rightarrow -\frac{1}{2} = C \qquad [\because \log 1 = 0]$$

Now (*ii*) becomes

$$-\frac{x^2}{2y^2} + \log|y| = -\frac{1}{2} \Rightarrow \log|y| = \frac{x^2}{2y^2} - \frac{1}{2} \Rightarrow \log|y| = \frac{x^2 - y^2}{2y^2} \qquad \ldots(iii)$$

Putting $x = x_0$ and $y = e$ in (*iii*), we get

$$\log|e| = \frac{x_0^2 - e^2}{2e^2} \Rightarrow 1 = \frac{x_0^2 - e^2}{2e^2} \Rightarrow x_0^2 - e^2 = 2e^2$$

$$\Rightarrow x_0^2 = 3e^2 \Rightarrow x_0 = \sqrt{3}e$$

22. Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

[CBSE (AI) 2010, 2017; (F) 2013; Ajmer 2015]

Sol. Given, $(x - y)\frac{dy}{dx} = x + 2y$

By simplifying the above equation, we get

$$\frac{dy}{dx} = \frac{x + 2y}{x - y} \qquad \ldots(i)$$

Let $F(x,y) = \dfrac{x+2y}{x-y}$

then $F(\lambda x,\lambda y) = \dfrac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \dfrac{\lambda(x+2y)}{\lambda(x-y)} = \lambda^0 F(x,y)$

$F(x, y)$ is homogeneous function and hence given differential equation is homogeneous.

Now, let $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting these values in equation (*i*), we get

$$v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v}{1-v} - v = \frac{1+2v-v+v^2}{1-v} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1+v+v^2}dv = \frac{dx}{x}$$

By integrating both sides, we get

$$\int\frac{1-v}{1+v+v^2}dv = \int\frac{dx}{x} \qquad \ldots(ii)$$

$$\text{LHS} = \int\frac{1-v}{v^2+v+1}dv$$

Let $1 - v = A(2v+1) + B = 2Av + (A+B)$

Comparing coefficients of both sides, we get

$2A = -1,\ A + B = 1$ or $A = -\dfrac{1}{2},\ B = \dfrac{3}{2}$

$$\therefore \quad \int\frac{1-v}{v^2+v+1}dv = \int\frac{-\frac{1}{2}(2v+1)+\frac{3}{2}}{v^2+v+1}dv$$

$$= -\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv + \frac{3}{2}\int\frac{dv}{v^2+v+1}$$

$$= -\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv + \frac{3}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= -\frac{1}{2}\log|v^2+v+1| + \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

Now, substituting it in equation (*ii*), we get

$$-\frac{1}{2}\log|v^2+v+1| + \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = \log x + C$$

$$\Rightarrow \quad -\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right| + \sqrt{3}\tan^{-1}\left(\frac{\frac{2y}{x}+1}{\sqrt{3}}\right) = \log x + C$$

$$\Rightarrow \quad -\frac{1}{2}\log|x^2+xy+y^2| + \frac{1}{2}\log x^2 + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = \log x + C$$

$$\Rightarrow \quad -\frac{1}{2}\log|x^2+xy+y^2| + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = C$$

23. Solve $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$. *[NCERT Exemplar]*

Sol. Given, $$\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$$

Put $x + y = z \Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

On substituting these values in equation (*i*), we get

$$\left(\frac{dz}{dx} - 1\right) = \cos z + \sin z \Rightarrow \frac{dz}{dx} = (\cos z + \sin z + 1) \Rightarrow \frac{dz}{\cos z + \sin z + 1} = dx$$

On integrating both sides, we get

$$\int \frac{dz}{\cos z + \sin z + 1} = \int 1 dx$$

$$\Rightarrow \int \frac{dz}{\frac{1-\tan^2 z/2}{1+\tan^2 z/2} + \frac{2\tan z/2}{1+\tan^2 z/2} + 1} = \int dx$$

$$\Rightarrow \int \frac{dz}{\frac{1-\tan^2 z/2 + 2\tan z/2 + 1 + \tan^2 z/2}{(1+\tan^2 z/2)}} = \int dx$$

$$\Rightarrow \int \frac{(1+\tan^2 z/2)dz}{2+2\tan z/2} = \int dx \Rightarrow \int \frac{\sec^2 z/2\, dz}{2(1+\tan z/2)} = \int dx$$

Put $1 + \tan z/2 = t \Rightarrow \left(\frac{1}{2}\sec^2 z/2\right)dz = dt$

$$\Rightarrow \int \frac{dt}{t} = \int dx \Rightarrow \log|t| = x + C$$

$$\Rightarrow \log|1+\tan z/2| = x + C \Rightarrow \log\left|1 + \tan\frac{(x+y)}{2}\right| = x + C$$

24. Find the particular solution of the differential equation:

$(1 - y^2)(1 + \log x)\, dx + 2xy\, dy = 0$ given that $y = 0$ when $x = 1$ *[CBSE Delhi 2016]*

Sol. We have $(1 - y^2)(1 + \log x)\, dx + 2xy\, dy = 0$

$$\Rightarrow 2xy\, dy = -(1 - y^2)(1 + \log x)\, dx \Rightarrow \frac{2y\, dy}{1-y^2} = -\frac{(1+\log x)dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \int \frac{2y}{1-y^2}dy = -\int \frac{(1+\log x)}{x}dx \Rightarrow -\log|1-y^2| = -\int \frac{(1+\log x)}{x}dx$$

$$\Rightarrow -\log|1-y^2| = -\int z\,dz \left[\text{Let } 1 + \log x = z \Rightarrow \frac{1}{x}dx = dz\right]$$

$$\Rightarrow \log|1-y^2| = \frac{z^2}{2} + C \Rightarrow \log|1-y^2| = \frac{(1+\log x)^2}{2} + C$$

Putting $x = 1$ and $y = 0$, we get

$$\Rightarrow \log 1 = \frac{(1+\log 1)^2}{2} + C \Rightarrow 0 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$$

Hence, particular solution is $\log|1-y^2| = \frac{(1+\log x)^2}{2} - \frac{1}{2}$.

25. **Find the general solution of the following differential equation:**

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$ [CBSE Delhi 2016]

Sol. We have $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

$$\Rightarrow \quad (x-e^{\tan^{-1}y})\frac{dy}{dx}=-(1+y^2)$$

$$\Rightarrow \quad \frac{dy}{dx}=-\left(\frac{1+y^2}{x-e^{\tan^{-1}y}}\right) \quad \Rightarrow \quad \frac{dx}{dy}=-\left(\frac{x-e^{\tan^{-1}y}}{1+y^2}\right)$$

$$\Rightarrow \quad \frac{dx}{dy}=-\frac{x}{1+y^2}+\frac{e^{\tan^{-1}y}}{1+y^2} \quad \Rightarrow \quad \frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{e^{\tan^{-1}y}}{1+y^2}$$

It is in the form $\frac{dx}{dy}+Px=Q$, where $P=\frac{1}{1+y^2}$ and $Q=\frac{e^{\tan^{-1}y}}{1+y^2}$.

$$\therefore \quad \text{IF}=e^{\int P.dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Therefore, general solution is $x.e^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}.e^{\tan^{-1}y}dy+C$.

$$\Rightarrow \quad x.e^{\tan^{-1}y}=\int e^z.e^z dz+C \qquad \left[\text{Let } \tan^{-1}y=z \Rightarrow \frac{1}{1+y^2}dy=dz\right]$$

$$\Rightarrow \quad x.e^{\tan^{-1}y}=\int e^{2z}dz+C$$

$$\Rightarrow \quad x.e^{\tan^{-1}y}=\frac{e^{2z}}{2}+C \quad \Rightarrow \quad x.e^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+C$$

$$\Rightarrow \quad x=\frac{1}{2}e^{\tan^{-1}y}+C.e^{-\tan^{-1}y}$$

26. **Find the particular solution of differential equation: $\frac{dy}{dx}=-\frac{x+y\cos x}{1+\sin x}$ given that $y=1$ when $x=0$.** [CBSE (North) 2016]

Sol. We have

$$\frac{dy}{dx}=-\frac{x+y\cos x}{1+\sin x}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{x}{1+\sin x}-\frac{y\cos x}{1+\sin x} \quad \Rightarrow \quad \frac{dy}{dx}+\frac{\cos x}{1+\sin x}y=-\frac{x}{1+\sin x}$$

It is in the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{\cos x}{1+\sin x}, Q=-\frac{x}{1+\sin x}$.

Now $\text{IF}=e^{\int\frac{\cos x}{1+\sin x}dx}=e^{\log|1+\sin x|}=1+\sin x$

Therefore, general solution is

$$y(1+\sin x)=\int-\frac{x}{1+\sin x}(1+\sin x)dx+C \qquad =-\int x\,dx+C$$

$$\Rightarrow y(1+\sin x)=-\frac{x^2}{2}+C$$

$$1(1+\sin 0)=0+C \qquad [\text{Given } y=1 \text{ and } x=0] \qquad \Rightarrow \qquad C=1$$

Hence, particular solution is

$$y(1+\sin x) = -\frac{x^2}{2}+1$$

$$\Rightarrow \quad y = \frac{2-x^2}{2(1+\sin x)}$$

CONSTRUCTED RESPONSE QUESTIONS AS PER NEP

Long Answer Questions

1. Solve the following differential equation:

$3e^x \tan y\, dx + (2-e^x)\sec^2 y\, dy = 0$, given that when $x=0, y=\frac{\pi}{4}$ *[CBSE (F) 2012]*

Sol. Given, $3e^x \tan y\, dx + (2-e^x)\sec^2 y\, dy = 0$

$\Rightarrow \quad (2-e^x)\sec^2 y\, dy = -3e^x \tan y\, dx$

$\Rightarrow \quad \frac{\sec^2 y}{\tan y} dy = \frac{-3e^x}{2-e^x} dx$

$\Rightarrow \quad \int \frac{\sec^2 y\, dy}{\tan y} = 3\int \frac{-e^x dx}{2-e^x}$

$\Rightarrow \quad \log|\tan y| = 3\log|2-e^x| + \log C$

$\Rightarrow \quad \log|\tan y| = \log|C.(2-e^x)^3|$

$\Rightarrow \quad \tan y = C(2-e^x)^3$

Putting $x=0, y=\frac{\pi}{4}$, we get

$\Rightarrow \quad \tan\frac{\pi}{4} = C(2-e^0)^3 \quad \Rightarrow \quad 1 = C(2-1)^3 \quad \Rightarrow \quad 1 = C$

Therefore, particular solution is $\tan y = (2-e^x)^3$.

2. Solve: $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$ *[CBSE (AI) 2011]*

Sol. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x}, x \neq 0$$

Clearly, it is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in it, we get

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2+v^2x^2}+vx}{x} \quad \Rightarrow \quad v + x\frac{dv}{dx} = \sqrt{1+v^2}+v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2} \quad \Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{1}{\sqrt{1+v^2}} dv = \int \frac{1}{x} dx \quad \Rightarrow \quad \log|v+\sqrt{1+v^2}| = \log|x| + \log C$$

$\Rightarrow \quad |v+\sqrt{1+v^2}| = |Cx| \quad \Rightarrow \quad \left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = |Cx|$ $\quad [\because v = y/x]$

$\Rightarrow \quad \{y+\sqrt{x^2+y^2}\}^2 = C^2x^2$ [Squaring both sides]

Hence, $\{y+\sqrt{x^2+y^2}\}^2 = C^2x^2$ gives the required solution.

3. Show that the differential equation $(xe^{y/x} + y)\,dx = x\,dy$ is homogeneous. Find the particular solution of this differential equation, given that $x = 1$ when $y = 1$. **[*CBSE Delhi 2013*]**

Sol. Given differential equation is $\left(x.e^{\frac{y}{x}}+y\right)dx = xdy \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x.e^{\frac{y}{x}}+y}{x}$ ...(*i*)

Let $F(x,y) = \dfrac{x.e^{\frac{y}{x}}+y}{x}$

$\Rightarrow \quad F(\lambda x,\lambda y) = \dfrac{\lambda x.e^{\frac{\lambda y}{\lambda x}}+\lambda y}{\lambda x} = \lambda^0 \dfrac{x.e^{\frac{y}{x}}+y}{x} = \lambda^0 F(x,y)$

Hence, given differential equation (*i*) is homogeneous.

Let $y = vx \quad \Rightarrow \quad \dfrac{dy}{dx} = v + x.\dfrac{dv}{dx}$

Now, given differential equation (*i*) would become

$v + x\dfrac{dv}{dx} = \dfrac{x.e^{\frac{vx}{x}}+vx}{x} \quad \Rightarrow \quad v + x.\dfrac{dv}{dx} = e^v + v \quad \Rightarrow \quad x.\dfrac{dv}{dx} = e^v$

$\Rightarrow \dfrac{dv}{e^v} = \dfrac{dx}{x} \quad \Rightarrow \quad \int e^{-v}dv = \int \dfrac{dv}{v} \quad \Rightarrow \quad \dfrac{e^{-v}}{-1} = \log x + C$

$\Rightarrow -e^{-\frac{y}{x}} = \log x + C \quad \Rightarrow \quad -\dfrac{1}{e^{\frac{y}{x}}} = \log x + C \quad \Rightarrow \quad e^{\frac{y}{x}}.\log x + Ce^{\frac{y}{x}} + 1 = 0$

Putting $x = 1, y = 1$, we get

$\therefore \quad e\log 1 + Ce + 1 = 0 \quad \Rightarrow \quad C = -\dfrac{1}{e}$

$\therefore$ The required particular solution is

$$e^{\frac{y}{x}}.\log x - \frac{1}{e}e^{\frac{y}{x}} + 1 = 0 \quad \text{or} \quad e^{\frac{y}{x}}\log x - e^{\frac{y}{x}-1} + 1 = 0$$

4. Show that the differential equation $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$. **[*CBSE (AI) 2013*]**

Sol. Given differential equation is $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y - x\sin^2\left(\frac{y}{x}\right)}{x}$ ...(*i*)

Let $F(x,y) = \dfrac{y - x\sin^2\left(\frac{y}{x}\right)}{x}$

Then $F(\lambda x,\lambda y) = \dfrac{\lambda y - \lambda x\sin^2\frac{\lambda y}{\lambda x}}{\lambda x} = \lambda^0 \dfrac{y - x\sin^2\frac{y}{x}}{x} = \lambda^0 F(x,y)$

Hence, differential equation (*i*) is homogeneous.

Now, let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting these value in (*i*), we get

$$v + x\frac{dv}{dx} = \frac{vx - x\sin^2\frac{vx}{x}}{x} \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{x\{v - \sin^2 v\}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin^2 v \quad \Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v \quad \Rightarrow \quad \frac{dv}{\sin^2 v} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \operatorname{cosec}^2 v\, dv = -\int \frac{1}{x}dx \quad \Rightarrow \quad -\cot v = -\log x + C \quad \Rightarrow \quad \log x - \cot\left(\frac{y}{x}\right) = C \quad \ldots(ii)$$

Putting $y = \frac{\pi}{4}$ and $x = 1$ in (*ii*), we get

$$\log 1 - \cot\frac{\pi}{4} = C \quad \Rightarrow \quad 0 - 1 = C \quad \Rightarrow \quad C = -1$$

Hence, particular solution is

$$\log x - \cot\left(\frac{y}{x}\right) = -1 \quad \Rightarrow \quad \log x - \cot\left(\frac{y}{x}\right) + 1 = 0$$

5. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$ given that $y = \frac{\pi}{2}$ when $x = 1$. **[*CBSE Delhi* 2014] [*HOTS*]**

Sol. Given differential equation is $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

$$\Rightarrow \quad (\sin y + y\cos y)\, dy = x(2\log x + 1)dx$$

$$\Rightarrow \quad \int \sin y\, dy + \int y\cos y\, dy = 2\int x\log x\, dx + \int x\, dx$$

$$\Rightarrow \quad \int \sin y\, dy + \left[y\sin y - \int \sin y\, dy\right] = 2\left[\log x\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx\right] + \int x\, dx$$

$$\Rightarrow \quad \int \sin y\, dy + y\sin y - \int \sin y\, dy = x^2\log x - \int x\, dx + \int x\, dx + C$$

$$\Rightarrow \quad y\sin y = x^2\log x + C, \text{ is general solution.} \quad \ldots(i)$$

For particular solution, we put $y = \frac{\pi}{2}$ when $x = 1$

(*i*) becomes $\frac{\pi}{2}\sin\frac{\pi}{2} = 1.\log 1 + C \quad \Rightarrow \quad \frac{\pi}{2} = C \quad [\because \log 1 = 0]$

Putting the value of C in (*i*), we get the required particular solution

$$y\sin y = x^2\log x + \frac{\pi}{2}$$

6. If a curve $y = f(x)$, passing through the point (1, 2) is the solution of the differential equation

$$2x^2\, dy = (2xy + y^2)dx.$$

Find the value of $f(1/2)$.

Sol. We have

$$2x^2\, dy = (2xy + y^2)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy + y^2}{2x^2} \quad \ldots(i)$$

$$f(x, y) = \frac{2xy + y^2}{2x^2}$$

$$\Rightarrow \quad f(\lambda x, \lambda y) = \frac{2\lambda^2 xy + \lambda^2 y^2}{2\lambda^2 x^2} = \frac{2xy + y^2}{2x^2} = \lambda 0 f(x,y)$$

$\Rightarrow$ Given differential equation is a homogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Using these values in (*i*), we get

$$v + x\frac{dv}{dx} = \frac{2x\,vx + v^2x^2}{2x^2} = \frac{2x^2v + v^2x^2}{2x^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{2v + v^2}{2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v + v^2}{2} - v = \frac{2v + v^2 - 2v}{2} = \frac{v^2}{2}$$

$$\Rightarrow \quad \frac{dv}{v^2} = \frac{1}{2}\frac{dx}{x} \Rightarrow 2\frac{dv}{v^2} = \frac{dx}{x}$$

Integrating, we get

$$\Rightarrow \quad \frac{-2}{v} = \log|x| + C$$

$$\Rightarrow \quad \frac{-2x}{y} = \log|x| + C \qquad ...(ii)$$

Since (*ii*) passes through (1, 2).

$$\therefore \quad \frac{-2}{2} = \log|1| + C \Rightarrow C = -1$$

From (*ii*)

$$\frac{-2x}{y} = \log x - 1$$

$$\Rightarrow \quad \frac{2x}{y} = 1 - \log x \Rightarrow y = \frac{2x}{1 - \log x}$$

$$\therefore \quad y\left(\frac{1}{2}\right) = \frac{2 \times \frac{1}{2}}{1 - \log\frac{1}{2}} = \frac{1}{1 + \log 2}$$

i.e., $\quad y\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{1}{1 + \log 2}$.

7. If $\cos x\frac{dy}{dx} - y\sin x = 6x, 0 < x < \frac{\pi}{2}$ and $y\left(\frac{\pi}{3}\right) = 0$. Find the value of $y\left(\frac{\pi}{6}\right)$.

Sol. We have

$$\cos x\frac{dy}{dx} - y\sin x = 6x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y\sin x}{\cos x} = \frac{6x}{\cos x} \Rightarrow \frac{dy}{dx} + \left(\frac{-\sin x}{\cos x}\right)y = \frac{6x}{\cos x}$$

It is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-\sin x}{\cos x}, Q = \frac{6x}{\cos x}$.

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int \frac{-\sin x}{\cos x}dx} = e^{\log|\cos x|} = \cos x$$

$\therefore$ Solution is $y\cos x = \int \frac{6x}{\cos x}\cos x\,dx + C$

$\Rightarrow \quad y\cos x = \int 6x\,dx + C = \frac{6x^2}{2} + C = 3x^2 + C$

$\because \quad y\left(\frac{\pi}{3}\right) = 0$

$\Rightarrow \quad 0 = 3\times\frac{\pi^2}{9} + C = \frac{\pi^2}{3} + C \Rightarrow C = -\frac{\pi^2}{3}$

$\therefore \quad y\cos x = 3x^2 - \frac{\pi^2}{3}$

$\Rightarrow \quad y = \left(3x^2 - \frac{\pi^2}{3}\right)\sec x$

$\therefore \quad y\left(\frac{\pi}{6}\right) = \left(\frac{3\pi^2}{36} - \frac{\pi^2}{3}\right)\sec\frac{\pi}{6}$

$= \left(\frac{\pi^2}{12} - \frac{\pi^2}{3}\right)\times\frac{2}{\sqrt{3}} = \frac{-3\pi^2}{12}\times\frac{2}{\sqrt{3}} = \frac{-\pi^2}{2\sqrt{3}}$

8. Solve: $x\frac{dy}{dx} + y - x + xy\cot x = 0 (x \neq 0)$ **[*NCERT*][*CBSE (East) 2016*]**

Sol. The given differential equation $x\frac{dy}{dx} + y - x + xy\cot x = 0 (x \neq 0)$

$\Rightarrow \quad \frac{dy}{dx} + \left(\cot x + \frac{1}{x}\right)y = 1$ (Dividing both sides by x) ...(*i*)

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \cot x + \frac{1}{x}$ and $Q = 1$.

So, $IF = e^{\int\left(\cot x + \frac{1}{x}\right)dx} = e^{\log|\sin x| + \log|x|}$

$= e^{\log|x\sin x|} = x\sin x$

Multiplying both sides by IF in equation (*i*), we get

$x\sin x\frac{dy}{dx} + x\sin x\left(\cot x + \frac{1}{x}\right)y = x\sin x$

$\Rightarrow x\sin x\frac{dy}{dx} + (x\cos x + \sin x)y = x\sin x \quad \Rightarrow \quad \frac{d}{dx}(yx\sin x) = x\sin x$ [By product rule]

On integrating both sides, we get

$\Rightarrow \quad d(yx\sin x) = x\sin x\,dx$

$yx\sin x = \int x\sin x\,dx + C$ ...(*ii*)

Let $I = \int x\sin x\,dx = x\times(-\cos x) - \int 1.(-\cos x)dx$ (Using by parts)

$I = -x\cos x + \sin x$

Putting the value of I in (*ii*), we get

$y\,x\sin x = -x\cos x + \sin x + C$

$\Rightarrow \quad y\,x\sin x = \sin x - x\cos x + C$

Hence, $y = \frac{1}{x} - \cot x + \frac{C}{x\sin x}$ is the required solution.

9. Find the particular solution of the differential equation
$(1 + e^{2x})dy + (1 + y^2)\, e^x\, dx = 0$ given that $y = 1$ when $x = 0$. [NCERT] [CBSE (F) 2011]

Sol. We have, $(1 + e^{2x})\, dy + (1 + y^2)\, e^x\, dx = 0$ and given that $y = 1$, when $x = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{-(1+y^2)e^x}{1+e^{2x}} \quad \Rightarrow \quad \frac{dy}{-(1+y^2)} = \frac{e^x dx}{1+e^{2x}}$$

Integrating both sides, we get

$$-\int \frac{dy}{1+y^2} = \int \frac{e^x dx}{1+e^{2x}} \quad \Rightarrow \quad -\tan^{-1} y = \int \frac{e^x dx}{1+(e^x)^2}$$

$$\Rightarrow \quad -\tan^{-1} y = \int \frac{dt}{1+t^2} \qquad [\text{Putting } e^x = t \Rightarrow e^x dx = dt]$$

$$\Rightarrow \quad -\tan^{-1} y = \tan^{-1}(t) + C$$

$$\Rightarrow \quad -\tan^{-1} y = \tan^{-1}(e^x) + C \qquad ...(i)$$

Put $x = 0$, $y = 1$ in (*i*), we get

$$-\tan^{-1} 1 = \tan^{-1}(e^0) + C \quad \Rightarrow \quad -\frac{\pi}{4} = \frac{\pi}{4} + C \quad \Rightarrow \quad C = -\frac{\pi}{2}$$

Putting the value of C in (*i*), we get

$$-\tan^{-1} y = \tan^{-1}(e^x) - \frac{\pi}{2} \quad \Rightarrow \quad \frac{\pi}{2} = \tan^{-1}(e^x) + \tan^{-1} y$$

Hence, $\tan^{-1}(e^x) + \tan^{-1} y = \frac{\pi}{2}$ is the required solution.

Questions for Practice

Objective Type Questions

1. Choose and write the correct option in each of the following questions.

(*i*) The degree of the differential equation is $\left(1 + \frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$ [NCERT Exemplar]

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

(*ii*) The order and degree of the differential equation $\frac{d^4y}{dx^4} = y + \left(\frac{dy}{dx}\right)^4$ are respectively

(*a*) 4, 1 (*b*) 4, 2 (*c*) 2, 2 (*d*) 2, 4

(*iii*) The integrating factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$ is

(*a*) e^{-x} (*b*) e^{-y} (*c*) $\frac{1}{x}$ (*d*) x

(*iv*) Solution of the differential equation $x\frac{dy}{dx} + y = x\, e^x$ is

(*a*) $xy = e^x(1 - x) + C$ (*b*) $xy = e^x(x + 1) + C$
(*c*) $xy = e^y(y - 1) + C$ (*d*) $xy = e^x(x - 1) + C$

(*v*) The general solution of the differential equation $e^x\, dy + (y\, e^x + 2x)\, dx = 0$ is

(*a*) $x\, e^y + x^2 = C$ (*b*) $x\, e^y + y^2 = C$ (*c*) $y\, e^x + x^2 = C$ (*d*) $y\, e^y + x^2 = C$

■ **Conceptual Questions**

2. What is the degree of the following differential equation:

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x\ ?$$ [CBSE Delhi 2010]

3. Write the degree of the following differential equation:

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$ [CBSE Delhi 2013]

4. Write the sum of the order and degree of the following differential equation:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + x^4 = 0$$ [CBSE (F) 2015]

5. Find the product of the order and degree $x\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 + y^2 = 0$. [CBSE Chennai 2015]

6. Write the integrating factor of $\frac{dy}{dx}(x\log x) + y = 2\log x$. [CBSE Punchkula 2015]

7. Solve: $e^{dy/dx} = x^2$

8. State whether $y = e^{-x}(x + a)$ is the solution of differential equation:

$$\frac{dy}{dx} + y = e^{-x}$$

9. Solve : $\frac{dy}{dx} - \frac{y(x+1)}{x} = 0$

■ **Very Short Answer Questions**

10. Write the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$.

11. Write the integrating factor of $\frac{dy}{dx} + y = \frac{1+y}{x}$.

12. Given that $\frac{dy}{dx} = e^{-2y}$ and $y = 0$ when $x = 5$. Find the value of x when $y = 3$.

13. Find the general solution of the differential equation $\frac{dy}{dx} = 2^{y-x}$.

■ **Short Answer Questions**

14. Solve the following differential equation:

$$\cos^2 x\frac{dy}{dx} + y = \tan x$$ [CBSE Delhi 2008, 2011; (AI) 2009]

15. Solve the differential equation:

$$(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$$ [CBSE (AI) 2010]

16. Solve the differential equation:

$(x^2 + 3xy + y^2)dx - x^2dy = 0$ given that $y = 0$, when $x = 1$ [CBSE (East) 2016]

17. Solve the differential equation : $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^3$ [CBSE (South) 2016]

18. Find the particular solution of this differential equation $x^2\frac{dy}{dx} - xy = 1 + \cos\left(\frac{y}{x}\right), x \neq 0$. Find the particular solution of this differential equation, given that when $x = 1, y = \frac{\pi}{2}$. [CBSE (F) 2013]

19. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$ given that $y = 1$, when $x = 0$. **[CBSE Delhi 2015]**

20. Solve the differential equation $\frac{dy}{dx}+y\cot x=2\cos x$, given that $y = 0$, when $x=\frac{\pi}{2}$. **[CBSE (F) 2014]**

21. Solve the differential equation $(x^2 - yx^2)\,dy + (y^2 + x^2y^2)\,dx = 0$, given that $y = 1$, when $x = 1$. **[CBSE (F) 2014]**

22. Solve the differential equation:
$$\frac{dy}{dx}=\frac{x+y}{x-y}$$ **[CBSE 2019 (65/3/1)]**

23. Solve the differential equation:
$$(1+x^2)\,dy+2xy\,dx=\cot x\,dx$$ **[CBSE 2019 (65/3/1)]**

24. Find the general solution of the differential equation $x^2y\,dx-(x^3+y^3)\,dy=0$. **[CBSE 2020 (65/3/1)]**

25. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right)=3x+4y$, given that $y = 0$ when $x = 0$. **[CBSE (AI) 2014]**

Long Answer Questions

26. Solve the following differential equation, given that $y = 0$, when $x=\frac{\pi}{4}$:
$$\sin 2x\frac{dy}{dx}-y=\tan x$$ **[CBSE Chennai 2015]**

27. Solve the following differential equation:
$$\left[y-x\cos\left(\frac{y}{x}\right)\right]dy+\left[y\cos\left(\frac{y}{x}\right)-2x\sin\left(\frac{y}{x}\right)\right]dx=0$$ **[CBSE (F) 2015]**

28. Find the particular solution of the differential equation $(1+x^2)\frac{dy}{dx}=\left(e^{m\tan^{-1}x}-y\right)$ given that $y = 1$, when $x = 0$. **[CBSE Panchkula 2015]**

29. Find the particular solution of the following differential equation:
$$xy\frac{dy}{dx}=(x+2)(y+2);\ y=-1 \text{ when } x=1$$ **[CBSE Delhi 2012]**

30. Find the particular solution of the differential equation
$$\frac{dx}{dy}+x\cot y=2y+y^2\cot y,\ (y\neq 0) \text{ given that } x=0 \text{ when } y=\frac{\pi}{2}.$$ **[CBSE (AI) 2013]**

31. Find the particular solution of the differential equation $x(1 + y^2)dx - y(1 + x^2)dy = 0$ given that $y = 1$ when $x = 0$. **[CBSE (AI) 2014]**

32. Find the particular solution of the differential equation satisfying the given conditions
$$x^2\,dy + (xy + y^2)\,dx = 0;\ y = 1 \text{ when } x = 1.$$ **[CBSE Delhi 2010]**

33. $(x^2 + y^2)\,dy = xy\,dx$. If $y(1) = 1$ and $y(x_0) = e$, then find the value of x_0. **[CBSE Bhubaneswar 2015]**

34. Find the particular solution of the differential equation $(y-\sin x)dx+(\tan x)dy=0$ satisfying the condition that $y = 0$ when $x = 0$. **[CBSE Guwahati 2015]**

35. If $\frac{ydx-xdy}{y}=0$, $x, y>0$ and $y(1)=x^2$ then find the value of $y(5)$

Answers

1. (i) (b) (ii) (a) (iii) (c) (iv) (d) (v) (c)
2. 1
3. 2
4. 4
5. 4
6. $\log x$
7. $y = 2(x\log x - x) + C$
8. Yes
9. $y = x e^{x+C}$
10. $y = Cx$
11. $\frac{e^x}{x}$
12. $\frac{e^6+9}{2}$
13. $2^{-x} - 2^{-y} = C$
14. $y = \tan x - 1 + C e^{-\tan x}$
15. $(x^2+1)y = \frac{x}{2}\sqrt{x^2+4} + 2\log\left|x + \sqrt{x^2+4}\right|$
16. $y = \frac{x\log|x|}{1-\log|x|}$
17. $\frac{y}{x+1} = (x+1)\frac{e^{3x}}{3} - \frac{e^{3x}}{9} + C$
18. $\tan\left(\frac{y}{2x}\right) = -\frac{1}{2x^2} + \frac{3}{2}$
19. $-\frac{x^2}{2y^2} + \log|y| = 0$
20. $2y\sin x = -(1+\cos 2x)$
21. $\log|y| + \frac{1}{y} = -\frac{1}{x} + x + 1$
22. $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2+y^2) + C$
23. $y = \frac{1}{1+x^2}\log|\sin x| + \frac{C}{1+x^2}$
24. $\log|y| = \frac{x^3}{3y^3} + C$
25. $4e^{3x} + 3e^{-4y} = 7$
26. $y = \tan x - \sqrt{\tan x}$
27. $y^2 - 2x^2\cos\left(\frac{y}{x}\right) = C$
28. $y\, e^{\tan^{-1}x} = \frac{e^{(m+1)\tan^{-1}x}}{m+1} + \frac{m}{m+1}$
29. $x + 2\log|x| - 2$
30. $x\sin y = y^2\sin y - \frac{\pi^2}{4}$
31. $y^2 = 2x^2 + 1$
32. $3x^2y = y + 2x$
33. $x_0 = \sqrt{3}\,e$
34. $y = \frac{1}{2}\sin x$
35. 25