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CBSE XII
MATHEMATICS
Differentail Equation
01

DIFFERENTIAL Equations

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Mathematics is both the queen and the handmaiden of the Sciences.

—E.T. Bell

Objectives

After studying the material of this chapter, you should be able to :

- Understand the definitions of Order & Degree of differential equation.
- Understand to form differential equation when the equation of family of curves is given.
- Understand to solve different types of First Order & First Degree differential equation.
- Understand to solve Homogeneous & Linear differential equations.
- Understand the applications of differential equations.

INTRODUCTION

In certain situations we notice that the relation between the rates of change of observable quantities is simpler than the relation between the quantities themselves. In such cases differential equations are taken as models for several problems in Engineering, Physical Sciences, Biological Sciences. In this chapter, we shall study some basic concepts. We shall also deal with some physical and geometrical problems which shall give rise to differential equations. Finally, we shall discuss some methods for solving differential equations. First of all, we shall deal with some definitions.

In this chapter we will study following concepts :

* Differential equations
* Solutions of differential equations.

9.1. DEFINITIONS

(a)

Definition

An equation :

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0,$$

which expresses a relation between dependent and independent variables and their derivatives of any order, is called a differential equation.

We may also define it as :

"an equation involving unknown functions and their derivatives w.r.t. one or more independent variables."

For Examples :

(*i*) $\frac{dy}{dx} = \sin x$ or $y' = \sin x$ (*ii*) $\frac{d^2y}{dx^2} + y = 0$ or $y'' + y = 0$

(*iii*) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ or $(y'')^2 + x^2(y')^3 = 0$ (*iv*) $y' + \cos y' = 0$

(*v*) $\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = 0$ (*vi*) $y = x\frac{dy}{dx} + \frac{1}{\frac{dy}{dx}}$.

Chapter at Glance

(*b*) **Order of Differential Equation.** *The order of a differential equation is the same as that of the highest derivative (or differential) it contains.*

In the above, the order of (*i*), (*iv*) and (*vi*) is 1 ; the order of (*ii*), (*iii*) and (*v*) is 2.

(*c*) **Degree of Differential Equation.** *The degree of a differential equation whose terms are polynomials in the derivatives, is defined as the highest power (positive integral index) of the highest-order derivative in it after the equation is freed from radicals and fractions in its derivatives.*

In the above, the degree of (*i*), (*ii*) is 1 ; the degree of (*iii*), (*v*) and (*vi*) is 2. The degree of (*iv*) is not defined.

(*d*) **Linear and Non-linear Differential Equations.**

A differential equation is said to be linear if the unknown function and its derivative, which occur in the equation, occur only in the first degree and are not multiplied together.

Otherwise, the differential equation is said to be *Non-linear.*

In the above, (*i*), (*ii*) are linear while (*iii*), (*v*) and (*vi*) are non-linear.

KEY POINT

A *linear differential equation is always of first degree but the converse is not true i.e.* every differential equation of the first degree may not be linear.

ILLUSTRATIVE EXAMPLES

Example 1. What is the degree of the following equation ?

(i) $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$ (*C.B.S.E. 2013*)

(ii) $\left(\frac{dy}{dx}\right)^4 - 3x\frac{d^2y}{dx^2} = 0$ (*C.B.S.E. 2013*)

(iii) $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0$

(iv) $5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x.$ (*C.B.S.E. 2010*)

Solution. (*i*) Degree = 2. $\left[\because \text{Power of } \frac{d^2y}{dx^2} = 2\right]$

(*ii*) Degree = 1. $\left[\because \text{Power of } \frac{d^2y}{dx^2} = 1\right]$

(*iii*) Degree = 3. $\left[\because \text{Power of } \frac{d^2y}{dx^2} = 3\right]$

(*iv*) Degree = 1. $\left[\because \text{Power of } \frac{d^2y}{dx^2} = 1\right]$

Example 2. Find the order and degree (if defined) of each of the following equations :

(i) $y'' + 3y' + 2y = 0$ **(ii)** $y''' + 2(y'')^2 - y' + y = 0$

(iii) $y'^2 - \sin^2 y = 0$ **(iv)** $(y'')^2 + \cos y' = 0.$

Solution. (*i*) The given equation is $y'' + 3y' + 2y = 0$.

Its order is 2 and degree is 1.

(*ii*) The given equation is $y''' + 2(y'')^2 - y' + y = 0$.

Its order is 3 and degree is 1.

(*iii*) The given equation is $y'^2 - \sin^2 y = 0$.

Its order is 1 and degree is 2.

(*iv*) The given equation is $(y'')^2 + \cos y' = 0$.

Its order is 2 and degree is not defined.

Example 3. Determine the order and degree of each of the following. Also state whether they are linear or non-linear :

(i) $t^2\frac{d^2s}{dt^2} - st\left(\frac{ds}{dt}\right)^4 = s$

(ii) $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$

(iii) $y = px + \sqrt{a^2p^2 + b^2}$, **where** $p = \frac{dy}{dx}$

(iv) $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x.$ (*N.C.E.R.T.*)

Solution. (*i*) We have : $t^2\frac{d^2s}{dt^2} - st\left(\frac{ds}{dt}\right)^4 = s.$

Here order = 2 and degree = 1.

The differential equation is **non-linear.**

(*ii*) We have : $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$

$\Rightarrow x\left(\frac{dy}{dx}\right)^2 + 3 = y^2\frac{dy}{dx}.$

Here order = 1 and degree = 2.

The differential equation is **non-linear.**

(*iii*) We have : $y = px + \sqrt{a^2p^2 + b^2}$

$\Rightarrow \quad y - px = \sqrt{a^2p^2 + b^2}.$

Squaring, $(y - px)^2 = a^2p^2 + b^2$

$\Rightarrow \quad y^2 - 2pxy + p^2x^2 = a^2p^2 + b^2$

$\Rightarrow \quad p^2(x^2-a^2)-2pxy+(y^2-b^2)=0$, where $p=\dfrac{dy}{dx}$.

Here order = 1 and degree = 2.

The differential equation is **non-linear.**

(*iv*) We have : $\dfrac{d^2y}{dx^2}=\cos 3x+\sin 3x$.

Here order = 2 and degree = 1.

The differential equation is **linear.**

Example 4. Write the order and degree of the differential equation :

$$\mathbf{y = x\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.\qquad \textit{(Tripura B. 2016)}$$

Solution. We have : $y = x\dfrac{dy}{dx}+\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

$$\Rightarrow \quad y-x\frac{dy}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Squaring,

$$y^2+x^2\left(\frac{dy}{dx}\right)^2-2xy\frac{dy}{dx}=1+\left(\frac{dy}{dx}\right)^2.$$

Here order = 1 and degree = 2.

EXERCISE 9 (a)

Fast Track Answer Type Questions

FTATQ

Indicate the order of each of the following differential equations (1 – 3) :

1. (*a*) $\log\left(\dfrac{d^2y}{dx^2}\right)=\left(\dfrac{dy}{dx}\right)^3+x$. (*C.B.S.E. Sample Paper 2019*)

(*b*) (*i*) $y'+3y=0$ (*ii*) $y'+y^2=y$.

2. (*i*) $y''+4y=0$ (*ii*) $y^v+y=0$.

3. (*i*) $y'+2y=\sin x$ (*ii*) $y^{iv}+y=\sin x$.

In each of the following differential equations, indicate its degree, wherever possible. Also give the order of each of them (4 – 8) :

4. (*i*) $y''+y^2=0$ (*ii*) $y^{iv}+y'''+y''+y'+y=0$.

5. $(y')^2+y^2-1=0$. **6.** $y'+\sin y'=0$.

7. (*i*) $y^v+y^2+e^{y'}=0$ (*ii*) $y'+e^y=0$.

8. (*i*) $(y'')^2+(y')^3+\sin y=0$ (*ii*) $y^{iv}+\sin y'''=0$

(*iii*) $y'''+y''+y'+y\sin y=0$.

Find the order and degree, if defined, of the following differential equations (9 – 10) :

9. (*i*) $\dfrac{d^2y}{dx^2}+y=0$ (*Kerala B. 2014*)

(*ii*) $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$ (*Kashmir B. 2011*)

(*iii*) $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2-y\dfrac{dy}{dx}=0$ (*Karnataka B. 2014*)

(*iv*) $y'''+y^2+e^x=0$ (*N.C.E.R.T.*)

(*v*) $\left[1+\left(\dfrac{dy}{dx}\right)^3\right]^{3/4}=\dfrac{d^2y}{dx^2}$ (*Karnataka B. 2013*)

(*vi*) $\dfrac{d^2y}{dx^2}=\dfrac{2y^3+\left(\dfrac{dy}{dx}\right)^4}{\sqrt{\dfrac{d^2y}{dx^2}}}$ (*Bihar B. 2012*)

(*vii*) $\left(\dfrac{dy}{dx}\right)^2+\dfrac{dy}{dx}-\sin^2 y=0$ (*Karnataka B. 2017*)

(*viii*) $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2-y\dfrac{dy}{dx}=0$ (*Assam B. 2017*)

(*ix*) $\dfrac{d^3y}{dx^3}+x^2\left(\dfrac{d^2y}{dx^2}\right)^3=0$. (*Assam B. 2018*)

10. $y'''+5y''+y'=0$. (*N.C.E.R.T.*)

Find the order and degree, if defined, of the following differential equations (11 – 14) and state whether they are linear or non-linear :

11. $(xy^2+x)\,dx+(y-x^2y)\,dy=0$.

12. $\sqrt{1-x^2}\,dx+\sqrt{1-y^2}\,dy=0$.

13. $\left(\dfrac{d^2y}{dx^2}\right)^2+7\left(\dfrac{dy}{dx}\right)^3+y=0$.

14. $xy\dfrac{dy}{dx}=\left(\dfrac{1+y^2}{1+x^2}\right)(1+x+x^2)$.

15. Write the order and degree of the following differential equations :

(*i*) $\left(\dfrac{ds}{dt}\right)^4+3s\dfrac{d^2s}{dt^2}=0$ (*Kashmir B. 2017*)

(*ii*) $\left(\dfrac{d^2y}{dx^2}\right)^2+\left(\dfrac{dy}{dx}\right)^3+2y=0$

(*iii*) $\left(\dfrac{dy}{dx}\right)^3+\dfrac{d^2y}{dx^2}+6=0$ (*Kashmir B. 2017*)

(*iv*) $\left(\dfrac{d^2y}{dx^2}\right)^3+\left(\dfrac{dy}{dx}\right)^4+7=0$. (*Kashmir B. 2017*)

16. Find the order of the differential equation :

$$y=\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^3}.$$

Answers

1. (*a*) 2 (*b*) (*i*) – (*ii*) 1. **2.** (*i*) 2 (*ii*) 5.
3. (*i*) 1 (*ii*) 4.
4. (*i*) Degree 1, order 2 (*ii*) Degree 1, order 4.
5. Degree 2, order 1.
6. Degree not defined, order 1.
7. (*i*) Degree not defined, order 5 (*ii*) Degree 1, order 1.
8. (*i*) Degree 2, order 2 (*ii*) Degree not defined, order 4 (*iii*) Degree 1, order 3.
9. (*i*) Order 2, degree 1 (*ii*) Order 3, degree 1 (*iii*) Order 2, degree 1 (*iv*) Order 3, degree 1. (*v*) Order 2, degree 4 (*vi*) Order 2, degree 3 (*vii*) Order 1, degree 2 (*viii*) Order 2, degree 1 (*ix*) Order 3, degree 1.
10. Order 3, degree 1.
11. **– 12.** Order 1, degree 1 ; non-linear.
13. Order 2, degree 2 ; non-linear.
14. Order 1, degree 1 ; non-linear.
15. (*i*) Order 2, degree 1 (*ii*) Order 2, degree 2 (*iii*) Order 2, degree 1 (*iv*) Order 2, degree 3.
16. 1.

Hints/Solutions

16. $\left(y-\frac{dy}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^3.$

Clearly, order = 1.

9.2. SOLUTIONS OF DIFFERENTIAL EQUATIONS

(*i*) **Solution.** A solution of a differential equation is any function $y = f(x)$, which when put in the equation, changes it into an identity.

It is a relation between the variables involved such that this relation and the differential coefficients obtained therefrom satisfy the given differential equation.

This is also called **primitive** or **integral** of the differential equation.

(*ii*) **General Solution.** *The solution of a differential equation, which contains as many arbitrary constants as the order of the differential equation, is said to be the general solution.*

This is also called **complete primitive** or **complete solution** of the differential equation.

For Example, the general solution of $\frac{d^2y}{dx^2} + y = 0$ is :

$$y = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are arbitrary constants.

(*iii*) **Particular Solution.** *The particular solution of a differential equation is that which is obtained from the general solution by giving particular values to arbitrary constants.*

For Example, the particular solution of $\frac{d^2y}{dx^2} + y = 0$ is :

$$y = \cos x.$$ [*Here $c_1 = 1$ and $c_2 = 0$ in part (ii)*]

ILLUSTRATIVE EXAMPLES

Example 1. Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation :

(i) $y = x^2 + 2x + c$; $y' - 2x - 2 = 0$ *(N.C.E.R.T.)*

(ii) $x + y = \tan^{-1} y$; $y^2 y' + y^2 + 1 = 0$. *(N.C.E.R.T.)*

Solution. (*i*) We have : $y = x^2 + 2x + c$.

Diff. w.r.t. x, $y' = 2x + 2$

$\Rightarrow \quad y' - 2x - 2 = 0.$

Hence, the verification

(*ii*) We have : $x + y = \tan^{-1} y.$

Diff. w.r.t. x, $1 + y' = \frac{1}{1+y^2} . y'$

$\Rightarrow \quad (1 + y')(1 + y^2) = y'$

$\Rightarrow 1 + y' + y^2 + y'y^2 = y'$

$\Rightarrow \quad y^2 y' + y^2 + 1 = 0.$

Hence, the verification.

Example 2. For each of the following differential equations, verify that the accompanying function is a solution (both the differential equations and the accompanying functions being defined on whole of R) :

(i) $\mathbf{y' = e^x : e^x}$

(ii) $\mathbf{(1 + x^2)\, y' = xy : \sqrt{1 + x^2}}$.

Solution. (*i*) We have : $y = e^x$.

$\therefore \quad y' = e^x$, which is the reqd. verification.

(*ii*) We have : $y=\sqrt{1+x^2}$ ...(1)

$\therefore \quad y' = \frac{1}{2\sqrt{1+x^2}}(0+2x) = \frac{x}{\sqrt{1+x^2}}$

$\Rightarrow \quad \sqrt{1+x^2}\; y' = x$

$\Rightarrow \quad (1 + x^2)\, y' = x\sqrt{1+x^2}$

$\Rightarrow \quad (1 + x^2)\, y' = xy,$ [*Using (1)*]

which is the reqd. verification.

Example 3. For each of the following differential equations, verify that the accompanying function is a solution in the domain mentioned (A, B ∈ R : parameters)

(i) $\mathbf{xy' = y\ (x \in R \setminus \{0\}) : Ax\ (x \in R \setminus \{0\})}$ (*N.C.E.R.T.*)

(ii) $\mathbf{x^3 y'' = 1 (x \in R \setminus \{0\}) : \frac{1}{2x} + Ax + B (x \in R \setminus \{0\})}$.

Solution. (*i*) We have : $y = Ax$ ...(1)

$\therefore \quad y' = A$

$\Rightarrow \quad xy' = Ax \quad \Rightarrow \quad xy' = y,$ [*Using (1)*]

which is the reqd. verification.

(*ii*) We have : $y = \frac{1}{2x} + Ax + B$.

$\therefore \quad y' = -\frac{1}{2x^2} + A$

and $y'' = \frac{1}{x^3} \quad \Rightarrow \quad x^3 y'' = 1,$

which is the reqd. verification.

Example 4. Verify that y = 3 cos (log x) + 4 sin (log x), is a solution of the differential equation :

$$\mathbf{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.}$$

Solution. We have : $y = 3 \cos(\log x) + 4 \sin(\log x)$ ....(1)

Diff. w.r.t. x, $\frac{dy}{dx} = -3\sin(\log x).\frac{1}{x} + 4\cos(\log x).\frac{1}{x}$

$\Rightarrow \quad x\frac{dy}{dx} = -3 \sin(\log x) + 4 \cos(\log x)$ ...(2)

Again diff. w.r.t. x,

$x\frac{d^2y}{dx^2} + \frac{dy}{dx}.1 = -3\cos(\log x).\frac{1}{x} - 4\sin(\log x).\frac{1}{x}$

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -3 \cos(\log x) - 4 \sin(\log x)$

$= -y$ [*Using (1)*]

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0,$

which is the reqd. verification.

Example 5. Verify that the function $\mathbf{y = e^{-3x}}$ **is a solution of the differential equation** $\mathbf{\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0}$.

(*N.C.E.R.T.*)

Solution. We have : $y = e^{-3x}$ ...(1)

$\therefore \quad \frac{dy}{dx} = -3e^{-3x}$ ...(2)

and $\frac{d^2y}{dx^2} = 9e^{-3x}$ ...(3)

Now $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y$

$= 9e^{-3x} - 3e^{-3x} - 6e^{-3x}$ [*Using (1), (2) and (3)*]

$= 0.$

Hence, (1) is a solution of $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

Example 6. Show that the function $y = e^{3x}(a + bx)$ is a solution of the differential equation :

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0. \quad (J. \& K. B. 2010)$$

Solution. We have : $y = e^{3x}(a + bx)$ ...(1)

$$\therefore \quad \frac{dy}{dx} = e^{3x}(b) + 3e^{3x}(a + bx)$$

$$= e^{3x}(b + 3a + 3bx) \quad ...(2)$$

and
$$\frac{d^2y}{dx^2} = e^{3x}(3b) + 3e^{3x}(b + 3a + 3bx)$$

$$= e^{3x}(9a + 6b + 9bx) \quad ...(3)$$

Now $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y$

$= e^{3x}(9a + 6b + 9bx) - 6e^{3x}(b + 3a + 3bx) + 9e^{3x}(a + bx)$ [*Using (1), (2) and (3)*]

$= e^{3x}(9a + 6b + 9bx - 6b - 18a - 18bx + 9a + 9bx)$

$= e^{3x}(0) = 0.$

Hence, (1) is a solution of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$

Example 7. Verify that the function :

$$y = C_1 e^{ax}\cos bx + C_2 e^{ax}\sin bx,$$

where C_1, C_2 are arbitrary constants, is a solution of the differential equation :

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0.$$

(*N.C.E.R.T.*)

Solution. The given equation is :

$y = C_1 e^{ax}\cos bx + C_2 e^{ax}\sin bx$ ...(1)

Diff. w.r.t. x, $\frac{dy}{dx} = e^{ax}[-b C_1 \sin bx + b C_2 \cos bx] + [C_1 \cos bx + C_2 \sin bx] e^{ax}. a$

$\Rightarrow \frac{dy}{dx} = e^{ax}[(b C_2 + a C_1)\cos bx + (a C_2 - b C_1)\sin bx]$ ...(2)

Again diff. w.r.t. x,

$\frac{d^2y}{dx^2} = e^{ax}[(b C_2 + a C_1)(-b \sin bx) + (a C_2 - bC_1)(b \cos bx)] + [(bC_2 + aC_1)\cos bx + (aC_2 - bC_1)\sin bx)]. ae^{ax}$

$= e^{ax}[(a^2 C_2 - 2ab C_1 - b^2 C_2)\sin bx + (a^2C_1 + 2ab C_2 - b^2 C_1)\cos bx]$ ...(3)

$\text{LHS} = \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y$

$= e^{ax}[(a^2 C_2 - 2ab C_1 - b^2 C_2)\sin bx + (a^2 C_1 + 2ab C_2 - b^2 C_1)\cos bx]$
$- 2a e^{ax}[(bC_2 + aC_1)\cos bx + (a C_2 - b C_1)\sin bx]$
$+ (a^2 + b^2)[C_1 \cos bx + C_2 \sin bx] e^{ax}$

$= e^{ax}[(a^2 C_2 - 2ab C_1 - b^2 C_2 - 2a^2 C_2 + 2ab C_1 + a^2 C_2 + b^2 C_2)\sin bx$
$+ (a^2 C_1 + 2ab C_2 - b^2C_1 - 2ab C_2 - 2a^2 C_1 + a^2 C_1 + b^2 C_1)\cos bx]$

$= e^{ax}[(0)\sin bx + (0)\cos bx]$

$= 0 = \textbf{RHS}.$

Hence, the verification.

EXERCISE 9 (b)

Fast Track Answer Type Questions

FTATQ

For each of the following differential equations (1 – 4), verify that the accompanying function (explicit or implicit) is a solution (both the differential equations and the accompanying functions being defined on the whole of R) :

1. (*i*) $y' + y = 2 : e^{-x} + 2$ (*ii*) $y' + 2y = 0 : 2e^{-2x}$

(*iii*) $y' + y = 1 + x : e^{-x} + x.$

2. (*i*) $y' = \cos x : 1 + \sin x$

(*ii*) $y'' + 4y = 0 : 3 \sin 2x.$

3. $y = x\frac{dy}{dx} + a\frac{dx}{dy}; y^2 = 4ax.$

4. $xy' - 4y = 0 ; y = cx^4.$

Very Short Answer Type Questions

VSATQ

For each of the following (5 – 10), verify that the given function is a solution of the differential equation :

5. $y''' = 6$; $y = x^3 + ax^2 + bx + c$.

6. (*i*) $\dfrac{d^2y}{dx^2} + y = 0$; $y = a\cos x + b\sin x$

(N.C.E.R.T.; Nagaland B. 2018; Kerala B.2014)

(*ii*) $y'' + y = 0$; $y = \text{A}\cos x - \text{B}\sin x$.

7. (*i*) $y'' + 4y = 0$; $y = \text{A}\cos 2x + \text{B}\sin 2x$

(Meghalaya B. 2018, 16)

(*ii*) $y'' + 4y = 0$; $y = \text{A}\cos 2x - \text{B}\sin 2x$.

8. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$; $y = \text{A}x + \dfrac{\text{B}}{x}$.

9. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$; $y = e^x(\sin x + \cos x)$.

(Bihar B. 2014)

10. $(y - x)\,dy - (y^2 - x^2)\,dx = 0$; $y = -(1 + x)$.

11. (*i*) Verify that the function $y = \text{A}\cos x + \text{B}\sin x$, where $\text{A}, \text{B} \in \mathbf{R}$, is a solution of the differential equation $\dfrac{d^2y}{dx^2} + y = 0$. *(N.C.E.R.T.)*

(*ii*) Verify that $ax^2 + by^2 = 1$ is a solution of the differential equation $x(yy_2 + y_1^2) = yy_1$.

(C.B.S.E. Sample Paper 2018)

Short Answer Type Questions

SATQ

12. Verify that each of the following functions $\phi : \mathbf{R} \to \mathbf{R}$, as defined below, is the solution of the accompanying initial value problems :

(*i*) $\phi(x) = e^x : y' = y,\ y(0) = 1$

(*ii*) $\phi(x) = x^2 + 2x + 1 : y''' = 0,\ y(0) = 1,\ y'(0) = 2,\ y''(0) = 2$

(*iii*) $\phi(x) = \cos x\ (x \in \mathbf{R}) : y'' + y = 0,\ y(0) = 1,\ y'(0) = 0$.

13. Show that the differential equation of which :

$y = 2\left(x^2 - 1\right) + ce^{-x^2}$ is a solution is :

$\dfrac{dy}{dx} + 2xy = 4x^3$.

14. Show that $y = ax^3 + bx^2 + c$ is a solution of the differential equation $\dfrac{d^3y}{dx^3} - 6a = 0$. *(J. & K. B. 2011)*

Hints to Selected Questions

8. $y = \text{A}x + \dfrac{\text{B}}{x} \Rightarrow \dfrac{dy}{dx} = \text{A} - \dfrac{\text{B}}{x^2}, \dfrac{d^2y}{dx^2} = \dfrac{2\text{B}}{x^3}$; etc.

10. The given equation can be written as :

$$(y - x)\frac{dy}{dx} - (y^2 - x^2) = 0.$$

9.3. FORMATION OF DIFFERENTIAL EQUATIONS

Let us consider the family of all straight lines, which pass through the origin.

We represent the family of curves by the equation $y = mx$ ...(1)

For different values of m, we obtain different members of the family.

To form a differential equation so as to represent the family of curves, we differentiate (1) w.r.t. x so that $\dfrac{dy}{dx} = m$.

Putting in (1), $y = x\dfrac{dy}{dx}$, which is the reqd. diff. equation.

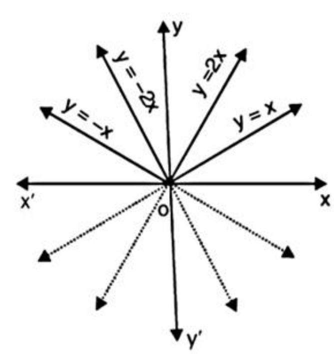

Fig.

GUIDE LINES

(*a*) When the given family 'f_1' of curves involves only one parameter ; say *a*, then it is represented by the equation of the type $f_1(x, y, a) = 0$ ...(1)

Diff. (1) w.r.t. *x*, we get $g(x, y, y', a) = 0$ ...(2)

Eliminating '*a*' from (1) and (2), we get $F(x, y, y') = 0$ ...(3),

which is the required differential equation.

(*b*) When the given family 'f_2' of curves involves two parameters ; say *a* and *b*, then it is represented by the equation of the type $f_2(x, y, a, b) = 0$ ...(4)

Diff. (4) w.r.t. *x*, we get $g(x, y, y', a, b) = 0$ ...(5)

Again diff. (5) w.r.t. *x*, we get $h(x, y, y', y'', a, b) = 0$ ...(6)

Eliminating '*a*' and '*b*' from (4), (5) and (6), we get $F(x, y, y', y'') = 0$ ...(7),

which is the required differential equation.

(*c*) When the given family 'f_n' of curves involves *n* parameters; say $a_1, a_2, ...a_n$, then it is represented by :

$f_n(x, y, a_1, a_2, ...a_n) = 0$ ...(8)

Differentiating (8) w.r.t. *x* successively *n* times, we get *n* differential equations.

Eliminating $a_1, a_2, ..., a_n$ from (8) and *n* differential equations, we get :

$F(x, y, y', y'', ...y^n) = 0,$

which is the reqd. differential equation.

KEY POINT

The order of differential equation, which represents a family of curves, is the same as the number of arbitrary constants present in the equation corresponding to the family of curves.

Frequently Asked Questions

FAQs

Example 1. For the differential equation representing the family of curves y = mx, where 'm' is an arbitrary constant.

(*H.P.B. 2016; Karnataka B. 2014; A.I.C.B.S.E. 2013*)

Or

Find the differential equation of the family of lines passing through the origin. (*A.I.C.B.S.E. 2015*)

Solution. We have : $y = mx$ ...(1),

which represents family of lines through the origin.

Diff. w.r.t. *x*, $\frac{dy}{dx} = m$ ...(2)

From (1) and (2), $y = \frac{dy}{dx}.x \Rightarrow \frac{dy}{dx} = \frac{y}{x}$,

which is the reqd. differential equation.

Example 2. Form the differential equation representing the family of curves :

y = A cos 2x + B sin 2x, where 'A' and 'B' are constants.

Solution. The given equation is :

$y = A\cos 2x + B\sin 2x$ ...(1)

Diff. w.r.t. *x*, $\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x.$

Again diff. w.r.t. *x*, $\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x$

$= -4(A\cos 2x + B\sin 2x)$

$= -4y.$ [*Using (1)*]

Hence, $\frac{d^2y}{dx^2} + 4y = 0,$

which is the reqd. differential equation.

Example 3. Find the differential equation representing the family of curves y = a e^{bx+5}, where 'a' and 'b' are arbitrary constants. (*C.B.S.E. 2018*)

Solution. We have : $y = a\,e^{bx+5}$ ...(1)

Diff w.r.t. *x*, $\frac{dy}{dx} = a\,e^{bx+5}.(b)$

$\Rightarrow \frac{dy}{dx} = by$ ...(2) [*Using (1)*]

Again, diff w.r.t. x, $\frac{d^2y}{dx^2} = b\frac{dy}{dx}$ ...(3)

Dividing (3) by (2) $\frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}} = \frac{\frac{dy}{dx}}{y}$

$\Rightarrow$ $y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

$\Rightarrow$ $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0,$

which is the reqd. differential equation.

Example 4. Find the differential equation of the family of curves :

$y = Ae^{2x} + Be^{3x}$. *(H.B. 2016)*

Solution. We have : $y = Ae^{2x} + Be^{3x}$ ...(1)

Diff. w.r.t. x, $\frac{dy}{dx} = 2Ae^{2x} + 3Be^{3x}$ ...(2)

Again diff. w.r.t. x, $\frac{d^2y}{dx^2} = 4Ae^{2x} + 9Be^{3x}$ ...(3)

(2) – 3(1) gives : $\frac{dy}{dx} - 3y = -Ae^{2x}$ ...(4)

(3) – 3(2) gives : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = -2Ae^{2x}$ ...(5)

Dividing (5) by (4), $\frac{\frac{d^2y}{dx^2} - 3\frac{dy}{dx}}{\frac{dy}{dx} - 3y} = 2$

$\Rightarrow$ $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 2\frac{dy}{dx} - 6y$

$\Rightarrow$ $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0,$

which is the reqd. differential equation.

Example 5. Find the differential equation of the family of circles $(x - a)^2 + (y - b)^2 = r^2$, where 'a' and 'b' are arbitrary constants. *(C.B.S.E. 2010 C)*

Solution. The given equation is :

$$(x - a)^2 + (y - b)^2 = r^2 \quad ...(1)$$

Differentiating (1) w.r.t. x,

$$2(x - a) + 2(y - b)y' = 0$$

$\Rightarrow$ $(x - a) + (y - b)y' = 0$ ...(2)

Differentiating (2) w.r.t. x,

$$1 + (y - b)y'' + y'^2 = 0 \quad ...(3)$$

From (3), $y - b = -\frac{y'^2 + 1}{y''}$.

Putting in (2), $x - a = -\left(-\frac{y'^2+1}{y''}\right)y' = \frac{y' + y'^3}{y''}$.

Putting in (1), $\left(\frac{y' + y'^3}{y''}\right)^2 + \left(-\frac{y'^2+1}{y''}\right)^2 = r^2$

$\Rightarrow$ $y'^2(1 + y'^2)^2 + (1 + y'^2)^2 = r^2y''^2$

$\Rightarrow$ $(1 + y'^2)^2(1 + y'^2) = r^2y''^2$

$\Rightarrow$ $(1 + y'^2)^3 = r^2y''^2,$

which is the reqd. differential equation.

Example 6. Obtain the differential equation of the family of circles, which touch the x-axis at the origin.

(N.C.E.R.T.; C.B.S.E. Sample Paper 2019 ; H.P.B. 2010; A.I.C.B.S.E. 2009 C)

Solution. Let $(0, \alpha)$ be the centre of any member of the family of circles.

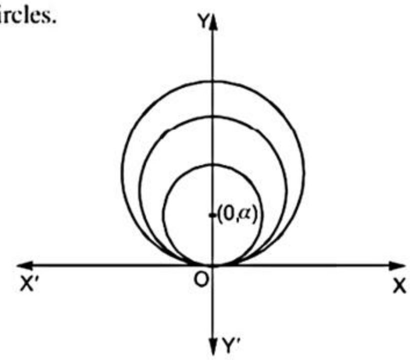

Fig.

Then the equation of the family of circles is :

$x^2 + (y - \alpha)^2 = \alpha^2 \quad \Rightarrow \quad x^2 + y^2 - 2\alpha y = 0$ ...(1)

Diff. w.r.t. x, $2x + 2y\frac{dy}{dx} - 2\alpha\frac{dy}{dx} = 0$

$\Rightarrow$ $\alpha = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}$.

Putting in (1), we get :

$$x^2 + y^2 - 2\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}y = 0$$

$\Rightarrow$ $(x^2 - y^2)\frac{dy}{dx} = 2xy,$

which is the reqd. differential equation.

Example 7. Obtain the differential equation representing the family of parabolas having vertex at the origin and axis along the positive direction of x-axis.

(N.C.E.R.T.; H.P.B. 2016, 11, 10)

Soution. Let S $(a, 0)$ be the focus of any member of the family of parabolas.

Then the equation of the family of curves is $y^2 = 4ax$ ...(1)

Diff. w.r.t. x, $2y\dfrac{dy}{dx} = 4a$ ...(2)

Using (2) in (1), we get $y^2 = \left(2y\dfrac{dy}{dx}\right)x$

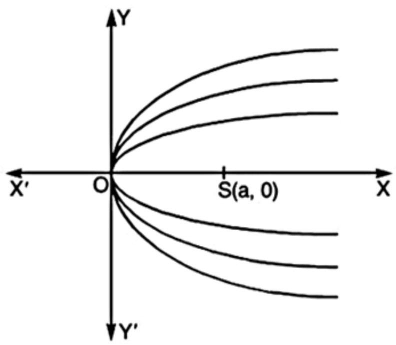

Fig.

$$\Rightarrow \quad y^2 - 2xy\frac{dy}{dx} = 0,$$

which is the reqd. differential equation.

Example 8. Obtain the differential equation of the family of ellipses having foci on y-axis and centre at the origin. *(H.P.B. 2018, 12)*

Solution. We know that the equation of the said family of ellipses is $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ ...(1),

where $a > b$

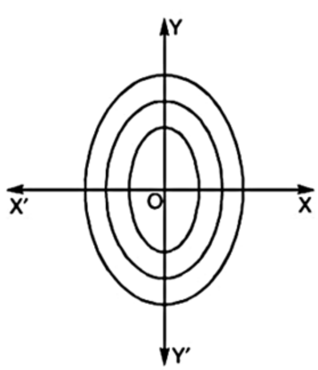

Fig.

Diff. (1) w.r.t. x, we get $\dfrac{2x}{b^2} + \dfrac{2y}{a^2}\cdot\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{y}{x}\left(\frac{dy}{dx}\right) = -\frac{a^2}{b^2} \quad ...(2)$$

Again diff. (2) w.r.t. x, we get :

$$\left(\frac{y}{x}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{x\frac{dy}{dx} - y}{x^2}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0,$$

which is the reqd. differential equation.

Example 9. A saving account pays 6% interest per year, compounded continuously. In addition, the income from another investment is credited to the account continuously at the rate of ₹4,000 per year. Form the differential equation to model this account.

Solution. Let $x\ \{= x(t)\}$ denote the amount (in ₹) in the account after t years.

Thus we have : $\dfrac{dx}{dt} = \dfrac{6}{100}x + 4000$,

which is the reqd. differential equation.

Example 10. Assume that a spherical rain drop evaporates at a rate proportional to the surface area. Form a differential equation involving the rate of change of the radius of the rain drop.

Solution. Let $r\ \{= r(t)\}$ denote the radius (in mm) of the rain drop after 't' minutes.

$\therefore \quad \dfrac{dr}{dt} = -ve \qquad$ [∵ *As t increases, r decreases*]

Also V, the volume of the rain drop $= \dfrac{4}{3}\pi r^3$ and S, the surface area of the rain drop $= 4\pi r^2$.

By the question, $\dfrac{dV}{dt} = -kS$,

where 'k' is a constant of proportionality

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k\,(4\pi r^2)$$

$$\Rightarrow \quad \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = -k\,(4\pi r^2)$$

$\Rightarrow \quad \dfrac{dr}{dt} = -k$, which is the reqd. differential equation.

EXERCISE 9 (c)

Fast Track Answer Type Questions — FTATQ

Represent the following families of curves by forming the corresponding differential equations (a, b : parameters) (1 – 6) :

1. $y = ax.$ (N.C.E.R.T.)
2. $x^2 + (y - b)^2 = 1.$
3. (i) $x^2 - y^2 = a^2$ (ii) $(x - a)^2 - y^2 = 1.$
4. (i) $y^2 = 4ax$ (ii) $y^2 = 4a(x - b)$ (iii) $(y - b)^2 = 4(x - a).$
5. (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$
6. (i) $y = ax^3$ (ii) $x^2 + y^2 = ax^3$ (iii) $y = e^{ax}.$

Very Short Answer Type Questions — VSATQ

7. Show that the differential equation of which $y = 2(x^2 - 1) + c\,e^{-x^2}$ is a solution is $\frac{dy}{dx} + 2xy = 4x^3.$

8. Show that the differential equation of which : $x^3 - y^3 = c(x^2 + y^2)^2$ is a solution is : $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy.$

9. Form the differential equation of the family of lines making equal intercepts on the co-ordinate axes. (P.B. 2018)

10. Form the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating 'a'.

11. Form the differential equation corresponding to $y^2 = a(b - x^2)$ by eliminating 'a' and 'b'.

12. Eliminate 'm' and 'a' from $y^2 = m(a^2 - x^2).$ (N.C.E.R.T.)

13. (i) Eliminate 'A' and 'B' from :

$$y = A\cos mx + B\sin mx.$$

(ii) Eliminate 'a' and 'b' from :

$$y = a\cos x + b\sin x.$$ (Nagaland B. 2015)

Short Answer Type Questions — SATQ

Form the differential equations of the family of curves (14 – 17) :

14. $v = \frac{A}{r} + B,$ where A and B are arbitrary constants. (C.B.S.E. 2015)

15. (i) $y = A\cos(x + B)$
(ii) $y = a\sin(x + b).$ (N.C.E.R.T.; H.P.B. 2017)

16. $x^2 + y^2 = 2ax.$

17. (i) $y = Ae^x + Be^{-x}$
(ii) $y = Ae^x + Be^{-x} + x^2$
(iii) $y = ae^{3x} + be^{-2x}$ (N.C.E.R.T.)
(iv) $y = Ae^{2x} + Be^{-3x}$
(v) $y = ae^{2x} + be^{-2x}.$ (Meghalaya B. 2014)

18. Find the differential equation of the family of curves $y = 2(x^2 - 1) + c\,e^{-x^2}.$

19. Obtain the differential equation by eliminating 'a' and 'b' from the equation :
(i) $y = e^x(a\cos x + b\sin x)$ (N.C.E.R.T. ; Assam B. 2016; P.B. 2013)
(ii) $y = e^x(a\cos 2x + b\sin 2x).$ (P.B. 2013)

20. Show that the differential equation of the family of circles having their centres at the origin and radius 'a' is :

$$x + y\frac{dy}{dx} = 0.$$ (Meghalaya B. 2015)

21. Find the differential equation of all circles which pass through the origin and whose centres are on the : (Nagaland B. 2016)
(i) x-axis (H. B. 2017)
(ii) y-axis. (H. B. 2017)

22. Obtain the differential equation of the family of circles :
(i) with centre at (1, 2) (Kerala B. 2015)
(ii) in the second quadrant and touching co-ordinate axes (N.C.E.R.T. ; A.I.C.B.S.E. 2012)
(iii) having radius 3.

23. Form the differential equation of the family of circles in the first quadrant, which touch the co-ordinate axes. (N.C.E.R.T.)

24. Find the differential equation for the family of all concentric circles centred at the origin and having different radii. (Meghalaya B. 2016)

25. Find the differential equation of all parabolas whose axes are parallel to y-axis.

26. Obtain the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis. (N.C.E.R.T.; H.P.B. 2018, 14, 12 ; C.B.S.E. 2011)

27. (i) Show that the differential equation that represents all parabolas each of which has a latus-rectum $4a$ and whose axes are parallel to x-axis is $2ay_2 + y_1^3 = 0.$
(ii) Show that the differential equation that represents all parabolas having their axis of symmetry coincident with the axis of x is $yy_2 + y_1^2 = 0.$

28. Obtain the differential equation of the family of ellipses having foci on x-axis and centre at the origin. (N.C.E.R.T.; H.P.B. 2016, 14, 11, 10; C.B.S.E. 2009 C)

29. Obtain the differential equation of family of hyperbolas having foci on x-axis and centre at the origin. (N.C.E.R.T.; H.P.B. 2018, 14, 12, 11)

30. A population grows at the rate of 5% per year. If $x = x(t)$ denotes the number of individuals in the population after t years, then the rate of change of x is equal to 5% of x. Form the desired differential equation.

Answers

1. $xy' - y = 0$.

2. $x^2(y'^2 + 1) = y'^2$.

3. (*i*) $x - yy' = 0$ (*ii*) $y^2y'^2 - y^2 = 1$.

4. (*i*) $y - 2xy' = 0$ (*ii*) $yy'' + y'^2 = 0$

(*iii*) $2y'' + y'^3 = 0$.

5. (*i*) $x(yy'' + y'^2) = yy'$ (*ii*) $x(yy'' + y'^2) = yy'$.

6. (*i*) $xy' = 3y$ (*ii*) $x^2 + 3y^2 = 2xyy'$ (*iii*) $xy' = y \log y$.

9. $\frac{dy}{dx} + 1 = 0$.

10. $x^2\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} - 2y^2\left(\frac{dy}{dx}\right)^2 - x^2 = 0$.

11. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$.

12. $xyy'' + xy'^2 = yy'$.

13. (*i*) $y'' + m^2y = 0$ (*ii*) $y'' + y = 0$.

14. $r\frac{d^2v}{dr^2} + 2\frac{dv}{dr} = 0$.

15. (*i*) – (*ii*) $\frac{d^2y}{dx^2} + y = 0$.

16. $2xy\frac{dy}{dx} + x^2 - y^2 = 0$.

17. (*i*) $\frac{d^2y}{dx^2} = y$ (*ii*) $\frac{d^2y}{dx^2} - y = -x^2 + 2$

(*iii*) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ (*iv*) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

(*v*) $\frac{d^2y}{dx^2} = 4y$.

18. $2xy + \frac{dy}{dx} = 4x^3$.

19. (*i*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

(*ii*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$.

21. (*i*) $y^2 = x^2 + 2xy\frac{dy}{dx}$ (*ii*) $(x^2 - y^2)\frac{dy}{dx} = 2xy$.

22. (*i*) $yy' - 2y' + x - 1 = 0$

(*ii*) $(x + y)^2 [(y')^2 + 1] = (x + yy')^2$

(*iii*) $(1 + (y')^2)^3 = 9y''$.

23. $(x - y)^2 (1 + y'^2) = (x + yy')^2$.

24. $x + yy' = 0$. **25.** $\frac{d^3y}{dx^3} = 0$.

26. $xy' - 2y = 0$. **28.** $xyy'' + xy'^2 - yy' = 0$.

29. $xy\,y'' + xy'^2 - yy' = 0$.

30. $\frac{dx}{dt} = \frac{5}{100}x$.

Hints to Selected Questions

21. (*i*) The equation of the family of circles is :

$$(x - \alpha)^2 + y^2 = \alpha^2$$

i.e. $x^2 + y^2 - 2\alpha x = 0$.

23. The equation of the family of circles is :

$$(x - a)^2 + (y - a)^2 = a^2.$$

25. The equation is $(x - h)^2 = 4a(y - k)$.

26. The equation is $x^2 = 4ay$.

27. (*i*) Take the equation as $(y - k)^2 = 4a(x - h)$

(*ii*) Take the equation as $y^2 = 4a(x - h)$.

28. The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

29. The equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

30. $\frac{dx}{dt} = 5\%$ of $x = \frac{5}{100}x$.

9.4. DIFFERENT TYPES OF SOLVING FIRST ORDER, FIRST DEGREE DIFFERENTIAL EQUATIONS

Now we shall discuss different methods of solving first order, first degree differential equations.

TYPE I. $\frac{dy}{dx} = f(x)$.

GUIDE LINES TO SOLVE

Integrating both sides, we have : $y = \int f(x)\,dx + c$, which is the reqd. solution.

ILLUSTRATIVE EXAMPLES

Example 1. Solve the following differential equation :

$$\left(1+x^2\right)\frac{dy}{dx} - x = 2\tan^{-1}x.$$

Solution. The given equation is $\left(1+x^2\right)\frac{dy}{dx} - x = 2\tan^{-1}x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{1+x^2} + \frac{2\tan^{-1}x}{1+x^2}.$$

Integrating, $y = \int \frac{x}{1+x^2}dx + 2\int \tan^{-1}x.\frac{1}{1+x^2}dx + c$

$$\Rightarrow y = \frac{1}{2}\int \frac{2x}{1+x^2}dx + 2\int (\tan^{-1}x).\frac{1}{1+x^2}dx + c$$

$$\Rightarrow y = \frac{1}{2}\log\left|1+x^2\right| + 2\frac{(\tan^{-1}x)^2}{2} + c$$

$$\left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right]$$

$$\Rightarrow y = \frac{1}{2}\log(1+x^2) + (\tan^{-1}x)^2 + c,$$

$[\because x^2 \geq 0 \Rightarrow 1 + x^2 > 0 \therefore |1 + x^2| = 1 + x^2]$

which is the reqd. solution.

Example 2. Find the general solution of the differential equation :

$$(\tan^2 x + 2\tan x + 5)\frac{dy}{dx} = 2(1+\tan x)\sec^2 x.$$

Solution. We have : $(\tan^2 x + 2\tan x + 5)\frac{dy}{dx}$

$$= 2(1+\tan x)\sec^2 x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(1+\tan x)\sec^2 x}{\tan^2 x + 2\tan x + 5}.$$

Integrating, $y = \int \frac{2(1+\tan x)\sec^2 x}{\tan^2 x + 2\tan x + 5}dx + c$ ...(1)

Let $I = \int \frac{2(1+\tan x)\sec^2 x}{\tan^2 x + 2\tan x + 5}dx$.

Put tan x = t so that $\sec^2 x\, dx = dt$.

$\therefore \quad I = \int \frac{2(1+t)}{t^2+2t+5}dt = \int \frac{2t+2}{t^2+2t+5}dt$ ...(2)

Put $t^2 + 2t + 5 = z$ so that $(2t + 2)\,dt = dz$.

$\therefore$ From (2), $I = \int \frac{dz}{z} = \log|z| = \log|t^2 + 2t + 5|$

$$= \log|\tan^2 x + 2\tan x + 5|.$$

$\therefore$ From (1), $y = \log|\tan^2 x + 2\tan x + 5| + c$,

which is the reqd. general solution.

Example 3. The marginal cost of manufacturing a certain item is given by $C'(x) = 2 + 0{\cdot}15x$.

Find the total cost function C(x), given that C(0) = 100.

Solution. We have : $C'(x) = 2 + 0{\cdot}15x$.

Integrating, $C(x) = \int (2 + 0{\cdot}15x)\,dx + A$

$\Rightarrow \quad C(x) = 2x + (0{\cdot}15)\frac{x^2}{2} + A$ ...(1)

Putting $x = 0$, $C(0) = A$.

But $C(0) = 100$. [*Given*]

$\therefore \quad A = 100.$

Putting in (1), $C(x) = 2x + (0{\cdot}15)\dfrac{x^2}{2} + 100$.

Hence, $C(x) = 2x + 0{\cdot}075x^2 + 100$,

which is the reqd. cost function.

Example 4. Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at any time.

Solution. Let 'V' and 'S' be the volume and surface area respectively of the spherical rain drop.

Then $V = \dfrac{4}{3}\pi r^3$ and $S = 4\pi r^2$,

where 'r' is the radius of the rain drop at any time t.

By the question, $\dfrac{dV}{dt} = -kS$,

where 'k' is the constant of proportionality

$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k\,(4\pi r^2)$$

$$\Rightarrow \frac{4}{3}\pi\,(3r^2)\frac{dr}{dt} = -4\pi k r^2$$

$$\Rightarrow \frac{dr}{dt} = -k.$$

Integrating, $r = -k\displaystyle\int 1 . dt + c$

$\Rightarrow \quad r = -kt + c$ ...(1)

When $t = 0, r = 3$,

$\therefore \quad 3 = 0 + c \quad \Rightarrow c = 3$.

When $t = 1, r = 2$,

$\therefore \quad 2 = -k + c \quad \Rightarrow 2 = -k + 3$

$\Rightarrow k = 1$.

Putting in (1), $r = -t + 3$ *i.e.* $r = 3 - t\,;\, 0 \le t < 3$,

which is the reqd. expression.

EXERCISE 9 (d)

Fast Track Answer Type Questions

FTATQ

Find the general solution of the following (1 – 4) :

1. $\dfrac{dy}{dx} = x^2 + \sin 3x$. *(H.B. 2010)*

2. $(x^2+1)\dfrac{dy}{dx} = 2$. *(J. & K. B. 2010)*

3. $(x+2)\dfrac{dy}{dx} = x^2 + 4x - 9$.

4. $\dfrac{dy}{dx} = x^5 + x^2 - \dfrac{2}{x}$.

5. Write the particular solution of the equation :

$\dfrac{dy}{dx} = \sin x$, given that $y(\pi) = 2$. *(Bihar B. 2012)*

Very Short Answer Type Questions

VSATQ

Find the general solution of the following (6 – 15) :

6. $\sqrt{1-x^6}\, dy = x^2\, dx$.

7. $(4 + 5\sin x)\dfrac{dy}{dx} = \cos x$.

8. $\dfrac{dy}{dx} = \cos^3 x \sin^4 x + x\sqrt{2x+1}$.

9. $\dfrac{dy}{dx} = \dfrac{1}{\sin^4 x + \cos^4 x}$.

10. $\dfrac{dy}{dx} = \sin^{-1} x$. *(N.C.E.R.T.; J. & K. B. 2011)*

11. (*i*) $(1+\cos x)\dfrac{dy}{dx} = (1 - \cos x)$

(*ii*) $(1 + \cos x)\, dy = (1 - \cos x)\, dx$.

(P.B. 2012 ; H.B. 2010)

12. $\dfrac{dy}{dx} = \log x$.

13. $\dfrac{dy}{dx} - x\sin^2 x = \dfrac{1}{x\log x}$.

14. $\dfrac{dy}{dx} + 3x = e^{-2x}$. *(J. & K. B. 2011)*

15. $\dfrac{dy}{dx} = \sin^3 x\cos^2 x + x\, e^x$.

16. Solve : $\dfrac{dy}{dx} = 2x^3 - x$, given $y = 1$, when $x = 0$.

17. Find the particular solution of :

$(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x$, $y = 1$ when $x = 0$.

(N.C.E.R.T.; Jharkhand B. 2016)

18. Find the particular solution of :

(*i*) $\cos\left(\dfrac{dy}{dx}\right) = a\ (a \in \mathbf{R})$, $y = 1$ when $x = 0$

(N.C.E.R.T.; Kashmir B. 2017)

(ii) $\sin\left(\frac{dy}{dx}\right) = a$, given that, $x = 0$, $y = 1$

(iii) $\cos\left(\frac{dy}{dx}\right) = \frac{1}{3}$, $y(0) = 2$. *(P.B. 2017)*

19. Find the particular solution of $e^{\frac{dy}{dx}} = x + 1$, given that when $x = 0$, $y = 3$.

20. Find the equation of the curve passing through the point (1, 1) whose differential equation is :
$x\,dy = (2x^2 + 1)\,dx$ $(x \neq 0)$. *(N.C.E.R.T.)*

Answers

1. $y = \frac{1}{3}x^3 - \frac{1}{3}\cos 3x + c.$ **2.** $y = 2\tan^{-1} x + c.$

3. $y = \frac{1}{2}x^2 + 2x - 13\log|x+2| + c.$

4. $\frac{1}{6}x^6 + \frac{1}{3}x^3 - 2\log|x| + c.$

5. $y = 1 - \cos x.$

6. $y = \frac{1}{3}\sin^{-1} x^3 + c.$

7. $y = \frac{1}{5}\log|4 + 5\sin x| + c.$

8. $y = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x - \frac{1}{6}(2x+1)^{3/2} + \frac{1}{10}(2x+1)^{5/2} + c.$

9. $y = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}}{\tan 2x}\right) + c.$

10. $y = x\sin^{-1} x + \sqrt{1-x^2} + c.$

11. (i) – (ii) $y = 2\tan\frac{x}{2} - x + c.$

12. $y = x(\log x - 1) + c.$

13. $y = \frac{1}{4}x^2 - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + \log|\log x| + c.$

14. $y = \frac{-3x^2}{2} - \frac{e^{-2x}}{2} + c.$

15. $y = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + (x-1)e^x + c.$

16. $y = \frac{1}{2}x^2(x^2 - 1) + 1.$

17. $y = \frac{1}{2}\log|x+1| + \frac{3}{4}(x^2+1) - \frac{1}{2}\tan^{-1} x + 1.$

18. (i) $\cos\left(\frac{y-1}{x}\right) = a$ (ii) $\sin\left(\frac{y-1}{x}\right) = a$

(iii) $\cos\left(\frac{y-2}{x}\right) = \frac{1}{3}.$

19. $y = x\log(x+1) + \log|x+1| - x + 3.$

20. $y = x^2 + \log|x|.$

Hints to Selected Questions

6. $\sqrt{1-x^6}\,dy = x^2\,dx \Rightarrow dy = \frac{x^2}{\sqrt{1-x^6}}dx.$

Integrating, $y = \int \frac{x^2}{\sqrt{1-x^6}}dx + c.$ Put $x^3 = t.$

8. $\frac{dy}{dx} = \cos^3 x\sin^4 x + x\sqrt{2x+1}.$

$\therefore \quad y = \int \sin^4 x\cos^3 x\,dx + \int x\sqrt{2x+1}\,dx.$

9. Here $y = \int \frac{1}{\sin^4 x + \cos^4 x}dx + c.$

Now $I = \int \frac{1}{\sin^4 x + \cos^4 x}dx = \int \frac{\sec^4 x}{\tan^4 x + 1}dx.$

Put $\tan x = t.$

18. (i) The given equation becomes : $\frac{dy}{dx} = \cos^{-1} a.$

19. The given equation becomes : $\frac{dy}{dx} = \log(x+1).$

20. $\frac{dy}{dx} = \frac{2x^2+1}{x}.$

$\therefore \quad y = \int \frac{2x^2+1}{x}dx + c.$

TYPE II. $\frac{dy}{dx} = f(y)$.

GUIDE-LINES TO SOLVE

Step (i) $\frac{dy}{dx} = f(y) \Rightarrow \frac{dx}{dy} = \frac{1}{f(y)}$. **Step (ii)** Integrating, $x = \int \frac{1}{f(y)} dy + c$, which is the reqd. solution.

ILLUSTRATIVE EXAMPLES

Example 1. Solve : $\frac{dy}{dx} = \frac{1+y^2}{y^3}$.

Solution. The given equation is $\frac{dy}{dx} = \frac{1+y^2}{y^3}$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{y^3}{y^2+1}.$$

Integrating, $x = \int \left(y - \frac{y}{y^2+1} \right) dy + c$

$$\Rightarrow \quad x = \frac{y^2}{2} - \frac{1}{2} \log |y^2+1| + c$$

$$\Rightarrow \quad x = \frac{y^2}{2} - \frac{1}{2} \log (y^2+1) + c,$$

$[\because y^2 \geq 0 \Rightarrow y^2 + 1 > 0$
$\therefore |y^2 + 1| = y^2 + 1]$

which is the reqd. solution.

Example 2. Solve : $\frac{dy}{dx} = \sec y$.

Solution. The given equation is $\frac{dy}{dx} = \sec y$

$$\Rightarrow \quad \frac{dx}{dy} = \cos y.$$

Integrating, $x = \int \cos y \, dy + c$

$\Rightarrow \quad x = \sin y + c$, which is the reqd. solution.

EXERCISE 9 (e)

Very Short Answer Type Questions

VSATQ

Solve the following equations :

1. $\frac{dy}{dx} + y = 1 \quad (y \neq 1)$. *(N.C.E.R.T.)*

2. $\frac{dy}{dx} = \sin^2 y$.

3. $\frac{dy}{dx} = \frac{1 - \cos 2y}{1 + \cos 2y}$.

4. $\frac{dy}{dx} = \frac{1}{y^2 + \sin y}$.

5. $\frac{dy}{dx} = \sqrt{4 - y^2} \quad (-2 < y < 2)$.

(N.C.E.R.T.; H.P.B. 2015, 10 S; Meghalaya B. 2015; Jammu B. 2012)

Answers

1. $y = 1 + ae^{-x}$.

2. $x + \cot y = c$.

3. $x + \cot y + y = c$.

4. $x = \frac{y^3}{3} - \cos y + c$.

5. $y = 2 \sin (x + c)$.

Hints to Selected Questions

4. $\frac{dy}{dx} = \frac{1}{y^2 + \sin y} \Rightarrow \frac{dx}{dy} = y^2 + \sin y$.

Integrating, $x = \frac{y^3}{3} - \cos y + c$.

TYPE III. VARIABLES SEPARABLE

GUIDE-LINES TO SOLVE : $\frac{dy}{dx} = f(x)\,\phi(y)$.

Step (i) The given equation can be written as $\frac{dy}{\phi(y)} = f(x)\,dx$.

Step (ii) Integrating, $\int \frac{1}{\phi(y)}\,dy = \int f(x)\,dx + c$, where '*c*' is a constant of integration,

which is the required solution.

Frequently Asked Questions

FAQs

Example 1. Solve the following :

$$\frac{dy}{dx} = \sqrt{4-y^2} \quad (-2 < y < 2).$$ *(Kashmir B. 2016)*

Solution. The given equation is $\frac{dy}{dx} = \sqrt{4-y^2}$

$\Rightarrow \quad \frac{dy}{\sqrt{2^2-y^2}} = dx$. | *Variables Separable*

Integrating, $\int \frac{dy}{\sqrt{2^2-y^2}} = \int 1.dx + c$

$\Rightarrow \sin^{-1}\frac{y}{2} = x + c$, which is the reqd. solution.

Example 2. Solve the following :

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}.$$ *(N.C.E.R.T.; Kashmir B. 2013)*

Solution. The given equation is :

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$\Rightarrow \quad \frac{1}{1+y^2}\,dy = \frac{1}{1+x^2}\,dx$. | *Variables Separable*

Integrating, $\int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx + c$

$\Rightarrow \quad \tan^{-1} y = \tan^{-1} x + c$,

which is the reqd. solution.

Example 3. Find the general solution of the differential equation :

$$(1 + x)(1 + y^2)\,dx + (1 + y)(1 + x^2)\,dy = 0.$$

Solution. The given equation is :

$(1 + x)(1 + y^2)\,dx + (1 + y)(1 + x^2)\,dy = 0$

$\Rightarrow \quad \frac{1+x}{1+x^2}\,dx + \frac{1+y}{1+y^2}\,dy = 0$. | *Variables Separable*

Integrating, $\int \frac{1+x}{1+x^2}\,dx + \int \frac{1+y}{1+y^2}\,dy = c$

$\Rightarrow \quad \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx + \int \frac{1}{1+y^2}\,dy + \int \frac{y}{1+y^2}\,dy = c$

$\Rightarrow \quad \int \frac{1}{1+x^2}\,dx + \frac{1}{2}\int \frac{2x}{1+x^2}\,dx + \int \frac{1}{1+y^2}\,dy + \frac{1}{2}\int \frac{2y}{1+y^2}\,dy = c$

$\Rightarrow \quad \tan^{-1} x + \frac{1}{2}\log|1+x^2| + \tan^{-1} y + \frac{1}{2}\log|1+y^2| = c$

$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y + \frac{1}{2}\log(1+x^2) + \frac{1}{2}\log(1+y^2) = c$

[$\because x^2 \ge 0 \Rightarrow 1 + x^2 > 0 \therefore |1 + x^2| = 1 + x^2$. Similarly $|1 + y^2| = 1 + y^2$]

$\Rightarrow \quad \tan^{-1}\frac{x+y}{1-xy} + \frac{1}{2}\log(1+x^2)(1+y^2) = c$,

which is the reqd. general solution.

Example 4. (*a*) **Solve : (i)** $\frac{dy}{dx} = 1 + x + y + xy$

(ii) $xyy' = 1 + x + y + xy$. *(N.C.E.R.T.)*

(*b*) **Find the particular solution of the differential equation** $\frac{dy}{dx} = 1 + x + y + xy$**, given that y = 0 when x = 1.** *(A.I.C.B.S.E. 2014)*

Solution. (*a*) (*i*) The given equation is :

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x)(1+y)$$

$$\Rightarrow \quad \frac{dy}{1+y} = (1+x)\,dx \qquad \text{| Variables Separable}$$

Integrating, $\int \frac{dy}{1+y} = \int (1+x)\,dx + c$

$$\Rightarrow \quad \log|1+y| = x + \frac{1}{2}x^2 + c,$$

which is the reqd. solution.

(*ii*) The given equation is $xyy' = 1 + x + y + xy$

$$\Rightarrow \quad xy\frac{dy}{dx} = (1+x)(1+y)$$

$$\Rightarrow \quad \frac{y}{1+y}dy = \frac{1+x}{x}dx. \qquad \text{| Variables Separable}$$

Integrating, $\int \frac{y}{1+y}dy = \int \frac{1+x}{x}dx + c$

$$\Rightarrow \quad \int \frac{(1+y)-1}{1+y}dy = \int \left(\frac{1}{x}+1\right)dx + c$$

$$\Rightarrow \quad \int 1.dy - \int \frac{1}{1+y}dy = \int \frac{1}{x}dx + \int 1.dx + c$$

$\Rightarrow \quad y \quad \log|1+y| = \log|x| + x + c$

$\Rightarrow \quad y = x + \log|x(1+y)| + c,$

which is the reqd. solution.

(*b*) From part (*a*) (*i*), $\log|1+y| = x + \frac{1}{2}x^2 + c$ ...(1)

When $x = 1$, $y = 0$, then :

$$\log|1+0| = 1 + \frac{1}{2}(1)^2 + c$$

$$\Rightarrow \quad \log 1 = 1 + \frac{1}{2} + c \quad \Rightarrow 0 = \frac{3}{2} + c$$

$$\Rightarrow \quad c = -\frac{3}{2}.$$

Putting in (1), $\log|1+y| = x + \frac{1}{2}x^2 - \frac{3}{2}$,

which is the reqd. particular solution.

Example 5. Solve :

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0.$$

(A.I.C.B.S.E. 2010)

Solution. The given equation can be written as :

$$\sqrt{(1+x^2)(1+y^2)} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\sqrt{1+x^2}}{x}dx + \frac{y}{\sqrt{1+y^2}}dy = 0. \qquad \text{| Variables Separable}$$

Integrating, $\int \frac{\sqrt{1+x^2}}{x}dx + \int \frac{y}{\sqrt{1+y^2}}dy = c$ ....(1)

Now $I_1 = \int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{\sqrt{1+x^2}}{x^2}.x\,dx.$

Put $1 + x^2 = u^2$ so that $2x dx = 2u\, du$ i.e. $x\, dx = u\, du$.

$$\therefore \quad I_1 = \int \frac{u}{u^2-1}u\,du = \int \frac{u^2}{u^2-1}du$$

$$= \int \frac{(u^2-1)+1}{u^2-1}du$$

$$= \int 1.du + \int \frac{1}{u^2-1^2}du$$

$$\text{| "Form} : \int \frac{dx}{x^2-a^2}\text{"}$$

$$= u + \frac{1}{2(1)}\log\left|\frac{u-1}{u+1}\right|$$

$$= \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|.$$

And $\quad I_2 = \int \frac{y}{\sqrt{1+y^2}}dy.$

Put $y^2 = v$ so that $2y\, dy = dv$ i.e. $y\, dy = \frac{1}{2}dv$.

$$\therefore \quad I_2 = \int \frac{\frac{1}{2}dv}{\sqrt{1+v}} = \frac{1}{2}\int (1+v)^{-1/2}dv$$

$$= \frac{1}{2}.\frac{(1+v)^{1/2}}{1/2} = \sqrt{1+v} = \sqrt{1+y^2}.$$

$\therefore$ From (1),

$$\sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + \sqrt{1+y^2} = c,$$

which is the reqd. solution.

Example 6. Solve the following initial value problems and find the corresponding solution curves :

(i) $2xy' = 5y$, $y(1) = 1$

(ii) $\sin x \cos y\, dx + \cos x \sin y\, dy = 0$, $y(0) = \frac{\pi}{4}$.

Solution. (*i*) The given equation is $2xy' = 5y$

i.e. $2x\frac{dy}{dx} = 5y$

$$\Rightarrow \quad 2\frac{dy}{y} = \frac{5}{x}dx. \qquad \text{| Variables Separable}$$

Integrating, $2\int \frac{dy}{y} = 5\int \frac{1}{x}dx + c$

$\Rightarrow$ 2 log | y | = 5 log | x | + c ...(1)

Now y (1) = 1 *i.e.* y = 1 when x = 1.

$\therefore$ 2 log | 1 | = 5 log | 1 | + c $\Rightarrow$ c = – 3 log |1|

$\Rightarrow$ c = – 3 log 1 = – 3 (0) = 0.

Putting in (1), 2 log | y | = 5 log | x | $\Rightarrow$ log y^2 = log x^5

$\Rightarrow y^2 = x^5 \Rightarrow y = |x|^{5/2}$,

which is the reqd. corresponding solution curve.

(*ii*) The given equation is :

sin x cos y dx + cos x sin y dy = 0

$$\Rightarrow -\frac{\sin x}{\cos x}dx - \frac{\sin y}{\cos y}dy = 0.$$ | *Variables Separable*

Integrating, $$\int \frac{-\sin x}{\cos x}dx + \int \frac{-\sin y}{\cos y}dy = c$$

$$\Rightarrow \log|\cos x| + \log|\cos y| = c \quad ...(1)$$

$$\left[\because \frac{d}{dx}(\cos x) = -\sin x; etc.\right]$$

Now $y(0) = \frac{\pi}{4}$ *i.e.* $y = \frac{\pi}{4}$, when x = 0.

$$\therefore \log|\cos 0| + \log\left|\cos\frac{\pi}{4}\right| = c$$

$$\Rightarrow \log|1| + \log\left|\frac{1}{\sqrt{2}}\right| = c$$

$$\Rightarrow \log 1 + \log\frac{1}{\sqrt{2}} = c$$

$$\Rightarrow c = \log 1 + \log 1 - \frac{1}{2}\log 2 = -\frac{1}{2}\log 2.$$

[$\because$ *log 1 = 0*]

Putting in (1), $\log|\cos x| + \log|\cos y| = -\frac{1}{2}\log 2$

$$\Rightarrow \log|\cos x \cos y| + \frac{1}{2}\log 2 = 0,$$

which is the reqd. corresponding solution curve.

Example 7. Find the particular solution of the following differential equation :

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1; \ y = 0 \text{ when } x = 0.$$

(C.B.S.E. 2012)

Solution. The given equation is $(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$

$$\frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$$ | *Variables Separable*

$$\frac{e^y}{2-e^y}dy = \frac{dx}{x+1}.$$

Integrating, $$\int \frac{e^y}{2-e^y}dy = \int \frac{1}{x+1}dx + c'$$

$$-\int \frac{-e^y}{2-e^y}dy = \int \frac{1}{x+1}dx + c'$$

$$\frac{1}{2-e^y} = c(x+1) \quad ...(1)$$

When x = 0, y = 0; $\frac{1}{2-e^0} = c(0+1) \Rightarrow c = \frac{1}{2-1} = \frac{1}{1} = 1.$

Putting (1), $\frac{1}{2-e^y} = 1.\ (x+1) \Rightarrow (x+1)(2-e^y) = 1,$

which is the reqd. particular solution.

Example 8. Find the particular solution of the following differential equation :

$$\mathbf{\cos y\, dx + (1 + 2 e^{-x}) \sin y\, dy = 0 ; y (0) = \frac{\pi}{4}.}$$

(C.B.S.E. Sampler Paper 2019)

Solution. The given differential equation is :

cos y dx + (1 + 2 e^{-x}) sin y dy = 0

$$\Rightarrow \frac{dx}{1+2e^{-x}} + \frac{\sin y}{\cos y}dy = 0$$ | *Variables Separable*

Integrating, $$\int \frac{e^x dx}{e^x+2} - \int \frac{-\sin y}{\cos y}dy = 0$$

$\Rightarrow$ log | e^x + 2 | = log | cos y | + constant

$\Rightarrow$ log |e^x + 2| = log | cos y | + constant

$\Rightarrow$ log |e^x + 2| = log | c cos y |

$\Rightarrow$ $e^x + 2 = c \cos y$ ...(1)

When $x = 0,\ y = \frac{\pi}{4}$,

then $1 + 2 = c\cos\frac{\pi}{4} \Rightarrow 3 = \frac{c}{\sqrt{2}} \Rightarrow c = 3\sqrt{2}$.

Putting in (1), $e^x + 2 = 3\sqrt{2}\cos y$,

which is the reqd. particular solution.

Example 9. Find the particular solution of the differential equation :

$$\mathbf{e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dx = 0, \text{ given that } y = \frac{\pi}{4}}$$

when x = 0. *(C.B.S.E. 2018)*

Solution. The given differential equation is :

$e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dx = 0$

$\Rightarrow$ $e^x \tan y\, dx = (e^x - 2) \sec^2 y\, dy$

$$\Rightarrow \frac{e^x}{e^x-2}dx = \frac{\sec^2 y}{\tan y}dy$$

| *Variables Separable*

Integrating, $$\int \frac{e^x}{e^x-2}dx = \int \frac{\sec^2 y}{\tan y}dy + \text{constant}$$ ...(1)

Now, $I_1 = \int \frac{e^x}{e^x - 2} dx$.

Put $e^x - 2 = t$ so that $e^x\, dx = dt$.

$\therefore \quad I_1 = \int \frac{dt}{t} = \log |t|$

$= \log |e^x - 2|$.

And, $I_2 = \int \frac{\sec^2 y}{\tan y} dy$.

Put $\tan y = z$ so that $\sec^2 y\, dy = dz$.

$\therefore \quad I_2 = \int \frac{dz}{z}$

$= \log |z| = \log |\tan y|$.

$\therefore$ From (1),

$\log |c| + \log |e^x - 2| = \log |\tan y|$

$\Rightarrow \quad \tan y = c(e^x - 2) \quad \ldots(2)$

When $x = 0, y = \frac{\pi}{4}$.

$\therefore \quad 1 = c(1 - 2)$

$\Rightarrow \quad c = -1$.

Putting in (2), $\log y = 2 - e^x$,

which is the reqd. particular solution.

Example 10. Solve the differential equation :

$$(1 - y^2)(1 + \log x)\, dx + 2xy\, dy = 0,$$

given that when x = 1, y = 0. *(C.B.S.E. 2016)*

Solution. The given equation is :

$(1 - y^2)(1 + \log x)\, dx + 2xy\, dy = 0$

$\Rightarrow \quad \frac{1 + \log x}{x} dx = \frac{-2y\, dy}{1 - y^2}$. | *Variables Separable*

Integrating, $\int \frac{1 + \log x}{x} dx = \int \frac{-2y}{1 - y^2} dy$

$\Rightarrow \frac{(1 + \log x)^2}{2} = \log |1 - y^2| + c \quad \ldots(1)$

When $y = 0, x = 1$.

$\therefore \frac{(1 + \log 1)^2}{2} = \log |1| + c \Rightarrow \frac{(1 + 0)^2}{2} = 0 + c \Rightarrow c = \frac{1}{2}$.

Putting in (1), $\frac{(1 + \log x)^2}{2} = \log |1 - y^2| + \frac{1}{2}$

$\Rightarrow (1 + \log x)^2 = 2 \log |1 - y^2| + 1$,

which is the reqd. solution.

Example 11. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that y = 0 when x = 0. *(N.C.E.R.T.; A.I.C.B.S.E. 2014)*

Solution. The given equation is :

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$

$\Rightarrow \quad \frac{dy}{dx} = e^{3x+4y}$

$\Rightarrow \quad e^{-4y}\, dy = e^{3x}\, dx$. | *Variables Separable*

Integrating, $\int e^{-4y}\, dy = \int e^{3x}\, dx + c$

$\Rightarrow \quad \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + c \quad \ldots(1)$

When $x = 0, y = 0$,

$\therefore \quad -\frac{1}{4} = \frac{1}{3} + c$

$\Rightarrow \quad c = -\frac{1}{4} - \frac{1}{3} = -\frac{7}{12}$.

Putting in (1), $-\frac{1}{4} \cdot \frac{1}{e^{4y}} = \frac{1}{3} e^{3x} - \frac{7}{12}$

$\Rightarrow \quad -3e^{-4y} = 4e^{3x} - 7$

$\Rightarrow \quad 4e^{3x} + 3e^{-4y} = 7$,

which is the reqd. particular solution.

EXERCISE 9 (f)

Fast Track Answer Type Questions — FTATQ

Find a one-parameter family of solutions of each of the following indicating the interval in which the solution is valid :

1. (*i*) $y' = 3y$ (*ii*) $y' = -\frac{x}{y} (y \neq 0)$ (*iii*) $\frac{dy}{dx} = \frac{y}{x}$. *(Rajasthan B. 2013)*

2. $dy + (x + 1)(y + 1)\, dx = 0$.

3. (*i*) $\frac{dr}{d\theta} = \cos\theta$ (*ii*) $y' = (\cos^2 x - \sin^2 x)\cos^2 y$.

4. (*i*) $e^y\, dx + e^x\, dy = 0$ (*ii*) $y' = e^{x+y} + e^{-x+y}$.

Very Short Answer Type Questions

VSATQ

Find the general solution of the following differential equations :

5. $\frac{dy}{dx} + y = 1\ (y \neq 1)$. (N.C.E.R.T.; Jammu B. 2017)

6. $\frac{dy}{dx} = e^{x+y}$. (Jammu B. 2017; Nagaland B. 2015)

7. (i) $(e^x + e^{-x})\frac{dy}{dx} = (e^x - e^{-x})$ (Jammu B. 2017)

(ii) $(e^x + 1)\, y\, dy = (y+1)\, e^x\, dx$

(iii) $x\,(e^{2y} - 1)\, dy + (x^2 - 1)\, e^y\, dx = 0$.

8. (i) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ (J. & K. B. 2011)

(ii) $\frac{dy}{dx} = e^{x+y} + x^2 e^y$.

9. (i) $(x^2+1)\frac{dy}{dx} = xy$. (Mizoram B. 2016)

(ii) $y(1-x^2)\frac{dy}{dx} = x(1+y^2)$. (Mizoram B. 2017)

10. (i) $\frac{dy}{dx} = (1+x^2)(1+y^2)$

(N.C.E.R.T.; H.P.B. 2017, 15, 13 S, 13; Jammu B. 2017, 14; Kashmir B. 2016; P.B. 2010)

(ii) $\frac{dy}{dx} = (4+x^2)(9+y^2)$ (H. B. 2013)

(iii) $\frac{dy}{dx} = (1+x)(1+y^2)$. (Mizoram B. 2015)

11. $(y + xy)\, dx + (x - xy^2)\, dy = 0$.

12. $\sqrt{1+x^2}\, dy + \sqrt{1+y^2}\, dx = 0$.

Short Answer Type Questions

SATQ

Solve the following (13–27) differential equations :

13. (i) $x^2 (y+1)\, dx + y^2 (x-1)\, dy = 0$ (P.B. 2012)

(ii) $x\,(x^2 - x^2y^2)\, dy + y\,(y^2 + x^2y^2)\, dx = 0$.

14. $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$.

(N.C.E.R.T. ; Rajasthan B. 2013)

15. $\frac{dy}{dx} = \frac{x+1}{2-y}\ (y \neq 2)$.

(N.C.E.R.T.; H.P.B. 2017, 10 S; Kashmir B. 2015; H.B. 2010)

16. $\frac{dy}{dx} + \frac{y(y-1)}{x(x-1)} = 0$.

17. (i) $\frac{dy}{dx} = 1 - x + y - xy$ (ii) $\frac{dy}{dx} = x - 1 + xy - y$.

18. $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$.

19. $x \sin y\, dy + (x\, e^x \log x + e^x)\, dx = 0$. (P.B. 2011)

20. $\frac{dy}{dx} = \frac{x\, e^x \log x + e^x}{x \cos y}$.

21. $\cos x \cos y\, dy + \sin x \sin y\, dx = 0$.

22. $\tan y\, dx + \sec^2 y \tan x\, dy = 0$.

23. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.

(N.C.E.R.T.; Kerala B. 2018; Assam B. 2018; H.P.B. 2015; 13 S, 13, 10 S; H.B. 2015; P.B. 2010)

24. $(1 + \cos x)\, dy = (1 - \cos y)\, dx$.

25. $\cos x\,(1 + \cos y)\, dx - \sin y\,(1 + \sin x)\, dy = 0$.

26. $\operatorname{cosec} x \log y \frac{dy}{dx} + x^2 y^2 = 0$. (C.B.S.E. 2014)

27. $y \log y\, dx - x\, dy = 0$.

(N.C.E.R.T.; H.P.B. 2017; Jammu B. 2017; Uttarakhand B. 2015; H.B. 2014)

Solve the following initial value equations (28–31) :

28. $e^x\sqrt{1-y^2}\, dx + \frac{y}{x}\, dy = 0$, given that $x = 0$ when $y = 1$.

(C.B.S.E. (F) 2012)

29. (i) $x\,(1+y^2)\, dx - y\,(1+x^2)\, dy = 0$, given that $y = 1$ when $x = 0$. (A.I.C.B.S.E. 2014)

(ii) $(1 + x^2 + y^2 + x^2y^2)\, dx + xy\, dy = 0$, given that $x = 1, y = 0$.

30. (i) $(1+y^2)(1 + \log x)\, dx + x\, dy = 0$, given that when $x = 1, y = 1$. (C.B.S.E. 2011)

(ii) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$, given that $y = 1$ when $x = 0$.

(P.B. 2018)

31. $(1 + e^{2x})\, dy + (1 + y^2)\, e^x\, dx = 0$, given that $x = 0, y = 1$.

(N.C.E.R.T. ; Assam B. 2013 ; C.B.S.E. (F) 2011)

Solve the following initial value problems and find the corresponding solution – curves (32 – 35) :

32. (i) $y' + 2y^2 = 0$, $y(1) = 1$

(ii) $xy' + 1 = 0$, $y(-1) = 0$

(iii) $y' = 2xy$, $y(0) = 1$.

33. (*i*) $x\,dy + y\,dx = xy\,dx$, $y(1) = 1$

(*ii*) $x\,(x\,dy - y\,dx) = y\,dx$, $y(1) = 1$.

34. $(x + 1)\,y' = 2\,e^{-y} - 1$, $y(0) = 0$. (*N.C.E.R.T.*)

35. $y' = y \tan x$, $y(0) = 1$. (*N.C.E.R.T.*)

Find the particular solution of the following (36 – 43) :

36. $\dfrac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$,

given that $y = 1$ when $x = 0$. (*C.B.S.E. 2012*)

37. $\dfrac{dy}{dx} = -\,4xy^2$, $y(0) = 1$.

(*N.C.E.R.T.; Jammu B. 2015*)

38. $\dfrac{dy}{dx} = y \tan x$, given that $y = 1$ when $x = 0$.

(*Kashmir B. 2017; Jammu B. 2015; C.B.S.E. 2010*)

39. $e^x\sqrt{1-y^2}\,dx + \dfrac{y}{x}\,dy = 0$, given that $y = 1$ when $x = 0$.

(*C.B.S.E. 2014*)

40. $(1 + x)\,y\,dx + (1 - y)\,x\,dy = 0$, when $y\,(1) = 1$.

(*Meghalaya B. 2013*)

41. $\sec^2 x \tan y\,dx - \sec^2 y \tan x\,dy = 0$, given that $y = \dfrac{\pi}{6}$, $x = \dfrac{\pi}{3}$. (*P.B. 2016*)

42. $\dfrac{dy}{dx} = x^3 \operatorname{cosec} y$, given that $y(0) = 0$.

43. $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$, given that $y = \dfrac{\pi}{2}$, when $x = 1$.

(*C.B.S.E. 2014*)

44. Find the particular solution of :

(*i*) $\log\left(\dfrac{dy}{dx}\right) = 2x + y$ (*ii*) $\log\left(\dfrac{dy}{dx}\right) = ax + by$,

given that $y = 0$ when $x = 0$.

45. The line normal to a given curve at each point (x, y) on the curve passes through the point (2, 0). If the curve contains the point (2, 3), find its equation.

46. (*i*) For the differential equation :

$$xy\frac{dy}{dx} = (x + 2)\,(y + 2),$$

find the solution if curve passing through the point (1, – 1).

(*ii*) Find the particular solution of the following differential equation :

$$xy\frac{dy}{dx} = (x + 2)\,(y + 2)\ ;\ y = -1 \text{ when } x = 1.$$

(*C.B.S.E. 2012*)

47. In a bank principal increases at the rate of 5% per year. In how many years ₹ 1000 double itself. (*N.C.E.R.T.*)

48. In a bank principal increases at the rate of 5% per year. An amount of ₹ 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

(*N.C.E.R.T.*)

49. A population grows at the rate of 2% per year. How long does it take for the population to double ? Use differential equation for it.

50. The surface area of a balloon, being inflated, changes at a rate proportional to time *t*.

(*i*) If initially its radius is 1 unit and after 3 seconds it is 2 units, find the radius after time *t*.

(*ii*) If initially its radius is 3 units and after 2 seconds it is 5 units, find the radius after *t* seconds.

Long Answer Type Questions

LATQ

51. A wet porous substance in the open air loses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half its moisture during the first hour, when will it have lost :

(*i*) 90%

(*ii*) 99% ;

weather conditions remaining the same ?

52. A bank pays interest by continuous compounding, that is, by treating the interest rate as the instantaneous rate of change of the principal. Suppose in an account interest accrues at 8% per year, compounded continuously. Calculate the percentage increase in such an account over one year.

[Take $e^{\cdot 08} \simeq 1\cdot0833$]

53. The slope of tangent at a point P (x, y) on a curve is $-\dfrac{x}{y}$. If the curve passes through the point (3, – 4), find the equation of the curve.

54. $\dfrac{dy}{dx} + \dfrac{y}{x} = 0$, where '$x$' denotes the percentage population living in a city and 'y' denotes the area for living healthy life of population. Find the particular solution when $x = 100$, $y = 1$.

55. Find the equation of a curve passing through the point (0, – 2), given that at any point (x, y) on the curve the product of the slope of its tangent and y-co-ordinate of the point is equal to the x-co-ordinate of the point. (*N.C.E.R.T.*)

56. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, – 3). Find the equation of the curve, given that it passes through (–2, 1). (*N.C.E.R.T.*)

Answers

1. (i) $y = c\, e^{3x}$ $(x \in \mathbf{R})$ (ii) $x^2 + y^2 = c$
(iii) $y = cx$ $(x \in \mathbf{R})$.

2. $\log|y+1| + \frac{x^2}{2} + x = c\ (x \in \mathbf{R})$.

3. (i) $r = \sin\theta + c\ (\theta \in \mathbf{R})$
(ii) $\tan y = \frac{1}{2}\sin 2x + c\ (x \in \mathbf{R})$.

4. (i) $e^{-x} + e^{-y} = c\ (x \in \mathbf{R})$ (ii) $e^{-x} - e^{-y} = e^x + c\ (x \in \mathbf{R})$.

5. $y = 1 + ce^{-x}$.

6. $e^x + e^{-y} = c$.

7. (i) $y = \log|e^x + e^{-x}| + c$
(ii) $y = \log|c(e^x + 1)(y + 1)|$
(iii) $e^y + \frac{1}{e^y} + \frac{1}{2}x^2 - \log|x| = c$.

8. (i) $e^y = e^x + \frac{x^3}{3} + c$ (ii) $e^{-y} + e^x + \frac{1}{3}x^3 + c = 0$.

9. (i) $y = c(x^2 + 1)$ (ii) $(1 + y^2)(1 - x^2) = c$.

10. (i) $\tan^{-1} y = x + \frac{1}{3}x^3 + c$
(ii) $\frac{1}{3}\tan^{-1}\frac{y}{3} = 4x + \frac{1}{3}x^3 + c$
(iii) $\tan^{-1} y = x + \frac{1}{2}x^2 + c$.

11. $\log|xy| + x - \frac{y^2}{2} = c$.

12. $(x + \sqrt{1+x^2})(y + \sqrt{1+y^2}) = c$.

13. (i) $\frac{1}{2}(x^2 + y^2) + x - y + \log|(x-1)(y+1)| = c$
(ii) $\log\left|\frac{x}{y}\right| = \frac{1}{2}\left(\frac{1}{x^2} + \frac{1}{y^2}\right) + c$.

14. $\sin^{-1} y + \sin^{-1} x = c$.

15. $x^2 + y^2 + 2x - 4y + c = 0$.

16. $(x - 1)(y - 1) = cxy$.

17. (i) $\log|1 + y| = x - \frac{1}{2}x^2 + c$
(ii) $\log|y + 1| = \frac{x^2}{2} - x + c$.

18. $x \sin\frac{y}{x} = c$.

19. $-\cos y + e^x \log x = c$.

20. $\sin y = e^x \log x + c$.

21. $\sin y = c \cos x$.

22. $\sin x \tan y = c$.

23. $\tan x \tan y = c$.

24. $-\cot\frac{y}{2} = \tan\frac{x}{2} + c$.

25. $(1 + \sin x)(1 + \cos y) = c$.

26. $-\frac{\log y}{y} - \frac{1}{y} - x^2\cos x + 2x\sin x + 2\cos x = c$.

27. $y = e^{cx}$.

28. $\sqrt{1-y^2} = (x-1)e^x + 1$.

29. (i) $2x^2 - y^2 + 1 = 0$
(ii) $x^2 + \log(x^2 + x^2y^2) = 1$.

30. (i) $\log|x| + \frac{1}{2}(\log x)^2 = -\tan^{-1} y + \frac{\pi}{4}$
(ii) $y = \frac{1+x}{1-x}$.

31. $e^x y = 1$.

32. (i) $y = \frac{1}{2x-1}, x \neq \frac{1}{2}$ (ii) $y = -\log|x|$
(iii) $y = e^{x^2}\ (x \in \mathbf{R})$.

33. (i) $y = \frac{1}{x}e^{x-1}\ (x \in \mathbf{R} \setminus \{0\})$
(ii) $y = x\, e^{1-\frac{1}{x}}\ (x > 0)$.

34. $y = \log\left(2 - \frac{1}{x+1}\right) (x \neq -1)$.

35. $y = \sec x \left(x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$.

36. $\tan^{-1} y = x + \frac{1}{3}x^3 + \frac{\pi}{4}$.

37. $y = \frac{1}{2x^2+1}$.

38. $y = \sec x$.

39. $\sqrt{1-y^2} = (x-1)e^x + 1$.

40. $\log|xy| + x - y = 0$.

41. $\tan x = 3 \tan y$.

42. $-\cos y = \frac{1}{4}x^4 - 1$.

43. $y \sin y = x^2 \log x + \frac{\pi}{2}$.

44. (i) $e^{2x} + 2e^{-y} = 3$ (ii) $be^{ax} + ae^{-by} = a + b$.

45. $(x - 2)^2 + y^2 = 9$.

46. (i) – (ii) $y = x + 2\log|x(y + 2)| - 2$.

47. $20 \log_e 2$ years. **48.** ₹ 1648.

49. 50 log 2 years.

50. (*i*) $\frac{1}{\sqrt{3}}\sqrt{t^2+3}$ (*ii*) $\sqrt{4t^2+18}$.

51. (*i*) $t = \frac{\log 10}{\log 2}$ hours (*ii*) $\frac{\log 100}{\log 2}$ hours.

52. 8·33%.

53. $x^2 + y^2 = 25$. **54.** $xy = 100$.

55. $x^2 - y^2 + 4 = 0$. **56.** $x^2 + 8x - y + 13 = 0$.

Hints to Selected Questions

8. (*i*) Given equation becomes : $e^y\, dy = (e^x + x^2)\, dx$.

11. $(y + xy)\, dx + (x - xy^2)\, dy = 0$

$\Rightarrow y(1 + x)\, dx + x(1 - y^2)\, dy = 0$

$\Rightarrow \frac{1+x}{x}dx + \frac{1-y^2}{y}dy = 0$.

17. (*i*) $\frac{dy}{dx} = 1 - x + y - xy \Rightarrow \frac{dy}{dx} = (1 - x)(1 + y)$

$\Rightarrow \frac{dy}{1+y} = (1 - x)\, dx$.

18. $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right) \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$.

Put $\frac{y}{x} = z$.

19. $x \sin y\, dy + (x e^x \log x + e^x)\, dx = 0$

$\Rightarrow \sin y\, dy + e^x\left(\log x + \frac{1}{x}\right)dx = 0$.

20. Given equation becomes :

$\cos y\, dy = e^x\left(\log x + \frac{1}{x}\right)dx$.

Integrating, $\sin y = e^x \log x + c$.

36. $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$\Rightarrow \frac{dy}{dx} = (1 + x^2)(1 + y^2)$

$\Rightarrow \frac{dy}{1+y^2} = (1 + x^2)\, dx$.

44. (*i*) The given equation becomes : $\frac{dy}{dx} = e^{2x+y}$

$\Rightarrow e^{-y}\, dy = e^{2x}\, dx$. Integrate.

45. Here $-\frac{dx}{dy} = \frac{y}{x-2} \Rightarrow (x - 2)\, dx + y\, dy = 0$.

49. Here $\frac{dx}{dt} = \frac{2x}{100} \Rightarrow \frac{dx}{x} = \frac{1}{50}dt$.

51. (*i*) – (*ii*) If '*x*' be the moisture at time *t*, then $\frac{dx}{dt} = -kx$; etc.

52. $\frac{dx}{dt} = \frac{8x}{100} \Rightarrow \frac{dx}{x} = 0{\cdot}08\, dt$.

53. Here $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\, dy = -x\, dx$.

TYPE IV. EQUATIONS REDUCIBLE TO VARIABLES SEPARALE

GUIDE-LINES TO SOLVE :

$$\frac{dy}{dx} = \phi\,(ax + by + c).$$

Step (i) Put $ax + by + c = z$. **Step (ii)** Separate the variables z and x. **Step (iii)** Integrate both sides.

ILLUSTRATIVE EXAMPLES

Example 1. Solve : $\frac{dy}{dx} = \cos(x + y)$.

Solution. We have : $\frac{dy}{dx} = \cos(x + y)$ ...(1)

Put x + y = z ...(2)

Diff. w.r.t. x, $1 + \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$.

Putting in (1), $\frac{dz}{dx} - 1 = \cos z \Rightarrow \frac{dz}{dx} = 1 + \cos z$

$\Rightarrow \frac{dz}{1+\cos z} = dx$. |*Variables Separable*

Integrating, $\int \frac{dz}{1+\cos z} = \int 1 . dx + c$,

where '*c*' is a constant of integration

$\Rightarrow \int \frac{1}{2\cos^2\frac{z}{2}}dz = x + c \Rightarrow \frac{1}{2}\int \sec^2\frac{z}{2}dz = x + c$

$\Rightarrow \frac{1}{2}.\frac{\tan\frac{z}{2}}{1/2} = x + c \Rightarrow \tan\frac{x+y}{2} = x + c$,

which is the reqd. solution.

Example 2. Find the particular solution of $\frac{dy}{dx} = \cos(x + y + 1)$, given that x = 0, y = – 1.

(P.B. 2011)

Solution. The given equation is $\frac{dy}{dx} = \cos(x + y + 1)$ ...(1)

Put x + y + 1 = z

Diff. w.r.t. x, $1 + \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$.

Putting in (1), $\frac{dz}{dx} - 1 = \cos z \Rightarrow \frac{dz}{dx} = 1 + \cos z$

$\Rightarrow \quad \frac{dz}{1+\cos z} = dx.$ | *Variables Separable*

Integrating, $\int \frac{dz}{1+\cos z} = \int 1 . dx + c$,

where 'c' is a constant of integration

$\Rightarrow \quad \int \frac{dz}{2\cos^2 \frac{z}{2}} = x + c$

$\Rightarrow \quad \frac{1}{2}\int \sec^2 \frac{z}{2} dz = x + c$

$\Rightarrow \quad \frac{1}{2} \frac{\tan \frac{z}{2}}{1/2} = x + c$

$\Rightarrow \quad \tan \frac{x+y+1}{2} = x + c$ ....(2)

When $x = 0$, $y = -1$

$\therefore \quad \tan \frac{0-1+1}{2} = 0 + c \Rightarrow \tan 0 = 0 + c$

$\Rightarrow \quad 0 = 0 + c \Rightarrow c = 0.$

Putting in (2), $\tan \frac{x+y+1}{2} = x$,

which is the reqd. particular solution.

EXERCISE 9 (g)

Short Answer Type Questions

SATQ

Solve the following differential equations :

1. $(x + y)\frac{dy}{dx} = 1.$ *(H.P.B. 2016, 10)*
2. $\frac{dy}{dx} = (4x + y + 1)^2.$
3. $(x + y + 1) \frac{dy}{dx} = 1.$ *(Nagaland B. 2016)*
4. (*i*) $\frac{dy}{dx} = (x + y)^2$ (*ii*) $(x - y)^2 \frac{dy}{dx} = a^2.$
5. $\frac{dy}{dx} = 1 + e^{x-y}.$
6. (*i*) $\cos(x + y)\, dy = dx$ (*ii*) $\frac{dy}{dx} = \sin(x + y)$ (*iii*) $\cos^{-1}\left(\frac{dy}{dx}\right) = x + y.$ *(P.B. 2014 S)*
7. $\frac{dy}{dx} = \sin(x + y) + \cos(x + y).$
8. $\frac{dy}{dx} = \cot^2(x + y).$
9. $\cos(x + y)\, dy = dx$, $y(0) = 0.$
10. $(x + y + 1)^2 dy = dx$, $y(-1) = 0.$
11. Find the particular solution of : $\frac{dy}{dx} = \cos(x + y + 2)$, given that $x = 0$, $y = -2$. *(P.B. 2011)*

Answers

1. $y = \log|x + y + 1| + c.$
2. $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c.$
3. $y - \log|x + y + 2| = c.$
4. (*i*) $x + y = \tan(x + c)$ (*ii*) $\frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right| = y + c.$
5. $-e^{-(x-y)} = -x + c.$
6. (*i*) $\tan\frac{x+y}{2} = y + c$ (*ii*) $\tan(x + y) - \sec(x + y) = x + c$ (*iii*) $\tan\frac{x+y}{2} = y + c.$
7. $\log\left|\tan\frac{x+y}{2} + 1\right| = x + c.$
8. $y = x + \frac{1}{2}\sin 2(x + y) + c.$
9. $y = \tan\left(\frac{x+y}{2}\right).$
10. $x + y + 1 = \tan y.$
11. $\tan\frac{x+y+2}{2} = x.$

Hints to Selected Questions

2. Put $4x + y + 1 = z$.
3. Put $x + y + 1 = z$.
6. (*i*) Put $x + y = z$.
7 – 9. Put $x + y = z$.
10. Put $x + y + 1 = z$.
11. Put $x + y + 2 = z$.

TYPE V. HOMOGENEOUS EQUATIONS

Definition

Homogeneous Function. *A function $f(x, y)$ is said to be homogeneous of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for any non-zero constant λ.*

For Examples : (I) $f(x, y) = y^2 + 3xy$ is homogeneous function of degree 2.

$[\because f(\lambda x, \lambda y) = (\lambda y)^2 + 3(\lambda x)(\lambda y) = \lambda^2 (y^2 + 3xy) = \lambda^2 f(x, y)]$

(II) $f(x, y) = \sin x + \cos y$ is not homogeneous function. $[\because f(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n f(x, y)$ *for any n*$]$

Observation :

$$f(x, y) = y^2 + 3xy$$

$$= x^2\left(\frac{y^2}{x^2} + 3\frac{y}{x}\right) = x^2 g\left(\frac{y}{x}\right)$$

or $$= y^2\left(1 + 3\frac{x}{y}\right) = y^2 h\left(\frac{x}{y}\right).$$

Thus $f(x, y)$ is a homogeneous function of degree n if $f(x, y) = x^n g\left(\frac{y}{x}\right)$ or $y^n h\left(\frac{x}{y}\right)$ for any n.

(*b*) **Homogeneous Differential Equation.**

Definition

A differential equation in x and y is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, where $f(x, y)$ and $g(x, y)$ are both homogeneous functions of the same degree in x and y.

GUIDE-LINES TO SOLVE

Step (i) *Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.* **Step (ii)** *Proceed as in Type III.*

To solve homogeneous differential equation of the type $\frac{dy}{dx} = f(x,y)$. ...(1)

Put y = vx *i.e.* $\frac{y}{x} = v$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ (1) becomes : $v + x\frac{dv}{dx} = g(v) \Rightarrow x\frac{dv}{dx} = g(v) - v$

$$\Rightarrow \frac{dv}{g(v) - v} = \frac{dx}{x}.$$ *|Variables Separable*

Integrating, $\int \frac{dv}{g(v) - v} = \int \frac{1}{x}dx + c$, which is the required general solution of (1) when we replace v by $\frac{y}{x}$.

Note : When the homogeneous equation is of the form $\frac{dx}{dy} = h(x, y)$.

Here we **put x = vy** *i.e.* $\frac{x}{y} = v$ and proceed as above.

Frequently Asked Questions

Example 1. Show that $\frac{dy}{dx} = \frac{x-y}{x+y}$ is homogeneous and solve it. (H.B. 2011)

Solution. The given equation is $\frac{dy}{dx} = \frac{x-y}{x+y}$ ...(1)

Here $f(x, y) = \frac{x-y}{x+y}$.

$$\therefore \quad f(\lambda x, \lambda y) = \frac{\lambda x - \lambda y}{\lambda x + \lambda y} = \frac{\lambda(x-y)}{\lambda(x+y)}$$

$$= \lambda^0 \frac{x-y}{x+y} = \lambda^0 f(x, y).$$

Thus $f(x, y)$ is homogeneous function of degree zero.

Put $y = vx$ ...(2)

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(3)

From (1), using (2) and (3), we get :

$$v + x\frac{dv}{dx} = \frac{x - vx}{x + vx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-2v-v^2}{1+v}$$

$$\Rightarrow \quad \frac{1+v}{1-2v-v^2}dv = \frac{dx}{x}. \quad |\text{Variables Separable}$$

Integrating w.r.t. x, we get :

$$\int \frac{1+v}{1-2v-v^2}dv = \int \frac{dx}{x} + \text{constant}$$

$$\Rightarrow \quad -\frac{1}{2}\int \frac{-2-2v}{1-2v-v^2}dv = \log|x| + \log|c|$$

$$\Rightarrow \quad -\frac{1}{2}\log\left|1-2v-v^2\right| = \log|x| + \log|c|$$

$$\Rightarrow \quad -\frac{1}{2}\log\left|1-\frac{2y}{x}-\frac{y^2}{x^2}\right| = \log|x| + \log|c|$$

$$\Rightarrow -\frac{1}{2}\log\left|\frac{x^2-2xy-y^2}{x^2}\right| = \log|cx|$$

$$\Rightarrow \quad \log\left|\frac{x^2-2xy-y^2}{x^2}\right| = \log\left|\frac{1}{c^2x^2}\right|$$

$$\Rightarrow \quad \frac{x^2-2xy-y^2}{x^2} = \frac{1}{c^2x^2}$$

$$\Rightarrow \ x^2 - 2xy - y^2 = \frac{1}{c^2} \ \Rightarrow x^2 - 2xy - y^2 = A,$$

where $A\left(=\frac{1}{c^2}\right)$ is an arbitrary constant,

which is the reqd. solution.

Example 2. Show that $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$ (Meghalaya B. 2016; H.P.B. 2015, 10 S; H.B. 2014)

Or

$$\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$$

is homogeneous and solve it.

Solution. The given equation is :

$$(x^2 + xy)\,dy = (x^2 + y^2)\,dx$$

i.e. $\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$ ...(1)

Here $f(x, y) = \frac{x^2+y^2}{x^2+xy}$.

$$\therefore \quad f(\lambda x, \lambda y) = \frac{\lambda^2x^2+\lambda^2y^2}{\lambda^2x^2+\lambda x.\lambda y} = \frac{\lambda^2(x^2+y^2)}{\lambda^2(x^2+xy)}$$

$$= \lambda^0\frac{x^2+y^2}{x^2+xy} = \lambda^0 f(x, y).$$

Thus $f(x, y)$ is homogeneous function of degree zero.

Put $y = vx$ ...(2)

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(3)

From (1), using (2) and (3), we get :

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{x^2+vx^2} = \frac{1+v^2}{1+v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v$$

$$= \frac{1+v^2-v-v^2}{1+v} = \frac{1-v}{1+v}$$

$$\Rightarrow \quad \frac{1+v}{1-v}dv = \frac{dx}{x}. \quad |\text{Variables Separable}$$

Integrating, $\int\frac{dx}{x} = \int\frac{1+v}{1-v}dv = \int\frac{2-(1-v)}{1-v}dv$

$$= 2\int\frac{1}{1-v}dv - \int 1.dv$$

$$= -2\int\frac{-1}{1-v}dv - \int 1.dv$$

$$\Rightarrow \quad \log|x| = -2\log|1-v| - v + c$$

$$\Rightarrow \quad \log|x| = -2\log\left|1-\frac{y}{x}\right| - \frac{y}{x} + c$$

$$\Rightarrow \log|x| = -2\log\left|\frac{x-y}{x}\right| - \frac{y}{x} + c$$

$$\Rightarrow \log|x| = -2\log|x-y| + 2\log|x| - \frac{y}{x} + c$$

$$\Rightarrow \log|x| - 2\log|x-y| - \frac{y}{x} + c = 0,$$

which is the reqd. solution.

Example 3. Solve : $\mathbf{\frac{dy}{dx} = \frac{y^3+2x^2y}{x^3+2xy^2}}$.

Solution. The given equation is $\frac{dy}{dx} = \frac{y^3+2x^2y}{x^3+2xy^2}$ ...(1)

This is homogeneous in x and y.

Put $y = vx$ ...(2)

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(3)

From (1), using (2) and (3), we get :

$$v + x\frac{dv}{dx} = \frac{v^3x^3 + 2x^2 . vx}{x^3 + 2x . v^2x^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^3+2v}{1+2v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^3+2v}{1+2v^2} - v = \frac{v-v^3}{1+2v^2}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{1+2v^2}{v-v^3}dv.$$ |*Variables Separable*

Integrating,

$$\int\frac{dx}{x} = \int\left[\frac{1}{v} + \frac{3}{2}.\frac{1}{1-v} - \frac{3}{2}.\frac{1}{1+v}\right]dv + A$$

[*Partial Fractions*]

$$\Rightarrow \log|x| = \log|v| - \frac{3}{2}\log|1-v| - \frac{3}{2}\log|1+v| + A$$

$$\Rightarrow \log|x| = \log|v| - \frac{3}{2}\log|1-v^2| + A$$

$$\Rightarrow \log|x| = \log\left|\frac{y}{x}\right| - \frac{3}{2}\log\left|1-\frac{y^2}{x^2}\right| + A$$

$$\Rightarrow \log|x| = \log|y| - \log|x| - \frac{3}{2}\log\left|\frac{x^2-y^2}{x^2}\right| + A$$

$$\Rightarrow \log|x| = \log|y| - \log|x| - \frac{3}{2}\log|x^2-y^2| + \frac{3}{2}\log x^2 + A$$

$$\Rightarrow \frac{3}{2}\log|x^2-y^2| = \log|y| - 2\log|x| + 3\log|x| + A$$

$$\left[\because \sqrt{x^2} = |x|\right]$$

$$\Rightarrow \frac{3}{2}\log|x^2-y^2| = \log|y| + \log|x| + A$$

$$\Rightarrow \frac{3}{2}\log|x^2-y^2| = \log|y| + \log|x| + \log|B|$$

[*Putting A = log | B |*]

$$\Rightarrow \log|(x^2-y^2)^{3/2}| = \log|Bxy|$$

$$\Rightarrow \quad (x^2-y^2)^{3/2} = Bxy$$

$$\Rightarrow \quad (x^2-y^2)^3 = cx^2y^2,$$ [*Putting $B^2 = c$*]

which is the reqd. solution.

Example 4. Solve : $\mathbf{x\,dy - y\,dx = \sqrt{x^2-y^2}\,dx}$. (*N.C.E.R.T.*)

Solution. The given equation is :

$$x\,dy - y\,dx = \sqrt{x^2-y^2}\,dx$$

$$\Rightarrow \quad x\,dy = \left(y + \sqrt{x^2-y^2}\right)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+\sqrt{x^2-y^2}}{x}$$ ...(1)

This is homogeneous in x and y.

Put $y = vx$ ...(2)

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(3)

From (1), using (2) and (3), we get :

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2-v^2x^2}}{x}$$

$$\Rightarrow \quad v + x\frac{dy}{dx} = v + \sqrt{1-v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1-v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}.$$ |*Variables Separable*

Integrating, $\int\frac{1}{\sqrt{1-v^2}}dv = \int\frac{dx}{x} + c$

$$\Rightarrow \quad \sin^{-1}v = \log|x| + c$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \log|x| + c,$$

which is the reqd. solution.

CAUTION

Sometimes the equation is not homogeneous but involves $\left(\frac{y}{x}\right)$, here we also solve by the substitution y = vx.

This is illustrated in the following **Example 5.**

Example 5. Solve :

$$\left(x\cos\frac{y}{x}\right)(y\,dx + x\,dy) = \left(y\sin\frac{y}{x}\right)(x\,dy - y\,dx).$$

HOTS *(N.C.E.R.T.)*

Solution. The given equation is :

$$\left(x\cos\frac{y}{x}\right)(y\,dx + x\,dy) = \left(y\sin\frac{y}{x}\right)(x\,dy - y\,dx)$$

$$\Rightarrow \left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x.\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left\{x\cos\frac{y}{x} + y\sin\frac{y}{x}\right\}y}{\left\{y\sin\frac{y}{x} - x\cos\frac{y}{x}\right\}x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[\cos\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)\right]\left(\frac{y}{x}\right)}{\left[\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)\right]}$$

[Dividing num. and denom. by x^2]

Put y = vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get :

$$v + x\frac{dv}{dx} = \frac{(\cos v + v\sin v)\,v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}.$$ *|Variables Separable*

Integrating, $\int\frac{v\sin v - \cos v}{v\cos v}dv = 2\int\frac{dx}{x} + c'$

$$\Rightarrow \int\frac{\sin v}{\cos v}dv - \int\frac{dv}{v} = 2\int\frac{dx}{x} + c'$$

$$\Rightarrow -\log|\cos v| - \log|v| = 2\log|x| - \log|c|,$$

where $c' = \log|c|$

$$\Rightarrow \log|x^2| + \log|\cos v| + \log|v| = \log|c|$$

$$\Rightarrow \log|x^2 v\cos v| = \log|c|$$

$$\Rightarrow x^2 v\cos v = c$$

$$\Rightarrow x^2\frac{y}{x}\cos\frac{y}{x} = c \Rightarrow xy\cos\frac{y}{x} = c,$$

which is the the reqd. solution.

Example 6. Find the particular solution satisfying the given condition :
(x + y) dy + (x – y) dx = 0, y = 1 when x = 1.

(N.C.E.R.T)

Solution. The given equation is :

$$(x+y)\,dy + (x-y)\,dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x} \quad ...(1)$$

This is homogeneous in x and y.

Put y = vx ...(2)

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ... (3)

From (1), using (2) and (3), we get :

$$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} \Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v-1-v^2-v}{v+1} = -\frac{v^2+1}{v+1}$$

$$\Rightarrow -\frac{v+1}{v^2+1}dv = \frac{dx}{x}.$$ *|Variables Separable*

Integrating, we get :

$$-\int\frac{v+1}{v^2+1}dv = \int\frac{dx}{x} + c$$

$$\Rightarrow -\frac{1}{2}\int\frac{2v}{v^2+1}dv - \int\frac{1}{1^2+v^2}dv = \int\frac{dx}{x} + c$$

$$\Rightarrow -\frac{1}{2}\log|v^2+1| - \frac{1}{1}\tan^{-1}v = \log|x| + c$$

$$\Rightarrow -\frac{1}{2}\log(v^2+1) - \tan^{-1}v = \log|x| + c$$

$[\because v^2 \geq 0 \Rightarrow v^2 + 1 > 0 \therefore |v^2 + 1| = v^2 + 1]$

$$\Rightarrow -\frac{1}{2}\log\left(\frac{y^2}{x^2} + 1\right) - \tan^{-1}\frac{y}{x} = \log|x| + c$$

$$\Rightarrow -\frac{1}{2}\log\left(\frac{y^2+x^2}{x^2}\right) - \tan^{-1}\frac{y}{x} = \log|x| + c$$

$$\Rightarrow -\frac{1}{2}\log(y^2+x^2) + \frac{1}{2}\log x^2 - \tan^{-1}\frac{y}{x} = \log|x| + c$$

$$\Rightarrow -\frac{1}{2}\log(x^2+y^2) + \log|x| - \tan^{-1}\frac{y}{x} = \log|x| + c$$

$\left[\because \sqrt{x^2} = |x|\right]$

$$\Rightarrow -\frac{1}{2}\log(x^2+y^2) - \tan^{-1}\frac{y}{x} = c \quad ...(4)$$

When $x = 1, y = 1$.

$$\therefore \quad -\frac{1}{2}\log(1+1) - \tan^{-1}\frac{1}{1} = c$$

$$\Rightarrow \quad -\frac{1}{2}\log 2 - \frac{\pi}{4} = c \quad \Rightarrow \quad c = -\frac{\pi}{4} - \frac{1}{2}\log 2.$$

Putting in (4),

$$-\frac{1}{2}\log(x^2+y^2) - \tan^{-1}\frac{y}{x} = -\frac{\pi}{4} - \frac{1}{2}\log 2$$

$$\Rightarrow \frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{x} = \frac{\pi}{4} + \frac{1}{2}\log 2,$$

which is the reqd. particular solution.

Example 7. Solve the following differential equation :

$$\mathbf{xy \log\left(\frac{y}{x}\right)dx + \left(y^2 - x^2\log\left(\frac{y}{x}\right)\right)dy = 0.}$$

(C.B.S.E. 2010 C)

Solution. The given equation is :

$$xy\log\left(\frac{y}{x}\right)dx + \left(y^2 - x^2\log\left(\frac{y}{x}\right)\right)dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{xy\log\left(\frac{y}{x}\right)}{y^2 - x^2\log\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{y}{x}\log\left(\frac{y}{x}\right)}{\frac{y^2}{x^2} - \log\left(\frac{y}{x}\right)} \qquad ....(1)$$

[Dividing num. and denom. by x^2]

Put $\frac{y}{x} = v$ *i.e.* $\boxed{\mathbf{y = vx}}$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ (1) becomes : $v + x\frac{dv}{dx} = -\frac{v\log v}{v^2 - \log v}$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v\log v}{v^2 - \log v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v\log v - v^3 + v\log v}{v^2 - \log v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v^3}{v^2 - \log v}$$

$$\Rightarrow \frac{v^2 - \log v}{v^3}dv = -\frac{dx}{x}. \qquad | \textit{ Variables Separable}$$

Integrating, $\int \frac{v^2 - \log v}{v^3}dv = -\int \frac{dx}{x} + c$

$$\Rightarrow \int \frac{1}{v}dv - \int \log v . v^{-3}dv = -\int \frac{dx}{x} + c$$

$$\Rightarrow \log|v| - \log v.\frac{v^{-2}}{-2} - \frac{1}{v}.\frac{v^{-2}}{-2}dv = -\log|x| + c$$

$$\Rightarrow \log|v| - -\frac{\log v}{2v^2} + \frac{1}{2} \quad v^{-3}dv = -\log|x| + c$$

$$\Rightarrow \log|v| + \frac{\log v}{2v^2} - \frac{1}{2}\frac{v^{-2}}{-2} = -\log|x| + c$$

$$\Rightarrow \log\left|\frac{y}{x}\right| + \frac{x^2}{2y^2}\log\frac{y}{x} + \frac{1}{4}\frac{x^2}{y^2} = -\log|x| + c$$

$$\Rightarrow \log|y| - \log|x| + \frac{x^2}{2y^2}\left(\log\frac{y}{x} + \frac{1}{2}\right) = -\log|x| + c$$

$$\Rightarrow \log|y| + \frac{x^2}{2y^2}\left(\log\frac{y}{x} + \frac{1}{2}\right) = c,$$

which is the reqd. solution.

Example 8. Find the particular solution of the following differential equation:

$$\mathbf{x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0,} \text{ given that when } \mathbf{x = 2, y = \pi.}$$

(H.P.B. 2016; A.I.C.B.S.E. 2012)

Solution. The given equation is $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

$$\frac{dy}{dx} - \frac{y}{x} + \sin\left(\frac{y}{x}\right) = 0$$

$$\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right) \qquad ...(1)$$

Put $\frac{y}{x} = v$ *i.e.* $\boxed{\mathbf{y = vx}}$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ (1) becomes : $v + x\frac{dv}{dx} = v - \sin v$

$$x\frac{dv}{dx} = -\sin v \qquad \text{cosec } v\, dv = -\frac{dx}{x}.$$

| *Variables Separable*

Integrating, $\int \text{cosec } v\, dv = -\int \frac{dx}{x} + \text{constant}$

$$-\log|\text{cosec } v - \cot v| = -\log|x| + \log|c|$$

$$\log|\text{cosec } v - \cot v| = \log\left|\frac{x}{c}\right|$$

$$\text{cosec } v - \cot v = \frac{x}{c}$$

$c(1 - \cos v) = x \sin v$

$$x \sin \frac{y}{x} = c\left(1 - \cos\frac{y}{x}\right) \quad \ldots(2)$$

When $x = 2$, $y = \pi$, $\therefore$ $2 \sin \frac{\pi}{2} = c\left(1 - \cos\frac{\pi}{2}\right)$

$2(1) = c(1 - 0) \Rightarrow c = 2.$

Putting in (2), $x \sin\frac{y}{x} = 2\left(1 - \cos\frac{y}{x}\right)$,

which is the reqd. particular solution.

Example 9. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of the differential equation :

$(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where c is a parameter.

(C.B.S.E. 2017)

Solution. We have : $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)dy$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \quad \text{| } \textit{Homogeneous}$$

Put y = vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x^3 - 3v^2x^3}{v^3x^3 - 3vx^3} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v} = \frac{1 - v^4}{v(v^2 - 3)}$$

$$\Rightarrow \quad \frac{v(v^2 - 3)}{1 - v^4}dv = \frac{dx}{x} \quad \text{| } \textit{Variables Separable}$$

$$\text{Integrating, } \int \frac{(v^3 - 3v)}{(1 - v^2)(1 + v^2)}dv = \int \frac{dx}{x} \quad \ldots(1)$$

$$\text{Now } \frac{v^3 - 3v}{(1 - v^2)(1 + v^2)} = \frac{v^3 - 3v}{(1 - v)(1 + v)(1 + v^2)} \equiv \frac{A}{1 - v} + \frac{B}{1 + v} + \frac{Cv + D}{1 + v^2} \quad \ldots(2)$$

$$\therefore \quad v^3 - 3v \equiv A(1 + v)(1 + v^2) + B(1 - v)(1 + v^2) + (Cv + D)(1 - v^2)$$

Putting $v = 1$, $\quad 1 - 3 = A(2)(2) \quad \Rightarrow \quad A = -\frac{1}{2}$.

Putting $v = -1$, $\quad -1 + 3 = B(2)(1 + 1) \quad \Rightarrow \quad B = \frac{1}{2}$.

Putting $v = 0$, $\quad 0 = A(1)(1) + B(1)(1) + D(1)$

$$\Rightarrow \quad 0 = -\frac{1}{2} + \frac{1}{2} + D \quad \Rightarrow \quad D = 0.$$

Comparing coeff. of v^3, $1 = A - B - C$

$$\Rightarrow \quad C = A - B - 1 = -\frac{1}{2} - \frac{1}{2} - 1 = -2.$$

$$\therefore \text{ From (2), } \frac{v^3 - 3v}{(1 - v^2)(1 + v^2)} = \frac{-1/2}{1 - v} + \frac{1/2}{1 + v} + \frac{-2v}{1 + v^2}.$$

$$\therefore \int \frac{v^3 - 3v}{(1 - v^2)(1 + v^2)}dv = \frac{1}{2}\log|1 - v| + \frac{1}{2}\log|1 + v| - \log|1 + v^2| = \frac{1}{2}\log|1 - v^2| - \log|1 + v^2|.$$

From (1), $\frac{1}{2}\log|1 - v^2| - \log|1 + v^2| = \log|x| + \log|c'|$

$$\Rightarrow \quad \log\left|\frac{\sqrt{1 - v^2}}{1 + v^2}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \quad \log\left|\frac{\sqrt{1 - \frac{y^2}{x^2}}}{1 + \frac{y^2}{x^2}}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \quad \log\left|\frac{x\sqrt{x^2 - y^2}}{x^2 + y^2}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \quad \frac{x\sqrt{x^2 - y^2}}{x^2 + y^2} = c'x$$

$$\Rightarrow \quad x\sqrt{x^2 - y^2} = c'x(x^2 + y^2)$$

$$\Rightarrow \quad \sqrt{x^2 - y^2} = c'(x^2 + y^2).$$

Squaring, $\quad x^2 - y^2 = c'^2(x^2 + y^2)^2$

$\Rightarrow \quad x^2 - y^2 = c(x^2 + y^2)^2$, where $c'^2 = c$,

which is the reqd. solution.

Example 10. Show that the differential equation :

$$2ye^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$$

is homogeneous and find the particular solution, given that x = 0 when y = 1. *(C.B.S.E. 2013)*

Solution. (*i*) The given equation is :

$2y\,e^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{2xe^{x/y} - y}{2y\,e^{x/y}} \qquad ...(1)$$

Here $\quad f(x, y) = \dfrac{2xe^{x/y} - y}{2ye^{x/y}}$.

$$\therefore \quad f(\lambda x, \lambda y) = \frac{2\lambda xe^{\frac{\lambda x}{\lambda y}} - \lambda y}{2\lambda ye^{\frac{\lambda x}{\lambda y}}}$$

$$= \frac{2xe^{x/y} - y}{2ye^{x/y}} = \lambda^0 f(x, y).$$

Thus $f(x, y)$ is homogeneous function of degree zero.

(*ii*) **Put x = vy** so that $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$.

$\therefore$ (1) becomes : $v + y\dfrac{dv}{dy} = \dfrac{2vye^v - y}{2y\,e^v}$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{1}{2e^v}$$

$$\Rightarrow \quad 2e^v\,dv = -\frac{dy}{y}. \qquad |\text{Variables Separable}$$

Integrating, $\quad 2\displaystyle\int e^v\,dv = -\int \frac{dv}{y} + c$

$\Rightarrow \quad 2\,e^v = -\log|y| + c$

$\Rightarrow \quad 2\,e^{x/y} = -\log|y| + c \qquad ....(2)$

Now $x = 0$ when $y = 1$, $\therefore 2(1) = -\log|1| + c$

$\Rightarrow \quad 2 = -\log 1 + c$

$\Rightarrow \quad 2 = -0 + c \Rightarrow c = 2.$

Putting in (2), $2e^{x/y} = -\log|y| + 2$,

which is the reqd. solution.

EXERCISE 9 (h)

Short Answer Type Questions

SATQ

Show that each of the following differential equations (1 – 11) is homogeneous and solve each of them :

1. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}$.

2. (*i*) $(x - y)\dfrac{dy}{dx} = x + 2y$

(N.C.E.R.T.; H.P.B. 2015, 11; A.I.C.B.S.E. 2010)

(*ii*) $(x - y)\,dy = (x + 2y)\,dx$. *(C.B.S.E. Sample Paper 2018)*

3. (*i*) $(x - y)\,dy - (x + y)\,dx = 0$

(N.C.E.R.T.; H.P.B. 2018, 14, 10; Uttarakhand B. 2013; P.B. 2010)

(*ii*) $\dfrac{dy}{dx} = \dfrac{y - x}{x + y}$ *(H.B. 2011)*

(*iii*) $(x + y)\,dy - (y - x)\,dx = 0$ *(P.B. 2010)*

(*iv*) $(y + x)\dfrac{dy}{dx} = (y - x)$.

4. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$.

(Meghalaya B. 2014; Bihar B. 2012)

5. (*i*) $x^2\dfrac{dy}{dx} = y(x + y)$ *(Mizoram B. 2018)*

(*ii*) $\dfrac{dy}{dx} = \dfrac{y(y + x)}{x(y - x)}$.

6. $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$.

(N.C.E.R.T. ; H.B. 2018; Assam B. 2017, 13; Meghalaya B. 2017; P.B. 2015, 13; Rajasthan B. 2013; Jammu B. 2012; Kashmir B. 2011)

7. (*i*) $(3xy + y^2)\,dx = (x^2 + xy)\,dy$ *(H.B. 2018; P.B. 2013)*

(*ii*) $(x^2 + xy)\,dy + (3xy + y^2)\,dx = 0$ *(P.B. 2011)*

(*iii*) $(y^2 - x^2)\,dy - 3xy\,dx = 0$.

8. (*i*) $2xy\,dx + (x^2 + 2y^2)\,dy = 0$ *(P.B. 2011)*

(*ii*) $2xy\,dy - (x^2 + 3y^2)\,dx = 0$. *(P.B. 2010 S)*

9. (*i*) $(x^2 - y^2)\,dx + 2xy\,dy = 0$

(N.C.E.R.T. ; H.P.B. 2018, 14, 13, 10; H.B. 2015)

(*ii*) $(x^2 + y^2)\,dx + 2xy\,dy$ *(H.B. 2018; P.B. 2011)*

(*iii*) $(x^2 + y^2)\,dx = 2xy\,dy$. *(P.B. 2014 S)*

10. (*i*) $(x^3 + y^3)\,dy - x^2y\,dx = 0$

(*ii*) $x^2y\,dx - (x^3 + y^3)\,dy = 0$

(*iii*) $\dfrac{dy}{dx} = \dfrac{x^2y}{x^3 + y^3}$

11. (*i*) $x\cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} = y\cos\left(\dfrac{y}{x}\right) + x$

(N.C.E.R.T. ; H.P.B. 2012 ; Kashmir B. 2011)

(*ii*) $ye^{\frac{x}{y}}\,dx = (xe^{\frac{x}{y}} + y^2)\,dy\ (y \neq 0)$.

(N.C.E.R.T.)

Find the particular solutions of the following problems (12 – 15) :

12. (*i*) $x^2\,dy - (x^2 + xy + y^2)dx = 0,\ y(1) = 1$ (*P. B. 2017*)

(*ii*) $(x^2 - y^2)\,dx + 2xy\,dy = 0,$ given that $y = 1$ when $x = 1$. (*Jammu B. 2016*)

13. $2x^2\dfrac{dy}{dx} - 2xy + y^2 = 0,\ y(e) = e.$ (*C.B.S.E. 2012*)

14. $\dfrac{dy}{dx} - \dfrac{y}{x} + \operatorname{cosec}\dfrac{y}{x} = 0,\ y(1) = 0.$ (*N.C.E.R.T.*)

15. (*i*) $xe^{\frac{y}{x}} - y + xy' = 0,\ y(e) = 0$ (*N.C.E.R.T.*)

(*ii*) $\left(xe^{\frac{y}{x}} + y\right)dx = x\,dy,\ y(1) = 1.$

16. Solve the following differential equations :

(*i*) $(x - y)\dfrac{dy}{dx} = (x + 2y),$

given that $y = 0$ when $x = 1$ (*A.I.C.B.S.E. 2017*)

(*ii*) $(x + y)\,dy + (x - y)\,dx = 0,$

given that $y = 1$ when $x = 1$ (*H.P.B. 2016*)

(*iii*) $x^2\,dy = (2xy + y^2)\,dx$, given that $y = 1$ when $x = 1$. (*A.I.C.B.S.E. 2015*)

17. Solve : (*i*) $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}\ ,\ x \neq 0$

(*ii*) $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx.$

(*W.Bengal B. 2017; H.P.B. 2015, 13, 11;] H.B. 2015; A.I.C.B.S.E. 2011*)

Show that the following differential equations (18-20) are homogeneous and solve them :

18. (*i*) $x\dfrac{dy}{dx} - y + x\sin\left(\dfrac{y}{x}\right) = 0$

(*N.C.E.R.T.; H.P.B. 2016, 12, 11; Jammu B. 2013*)

(*ii*) $x\cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} = y\cos\left(\dfrac{y}{x}\right) + x$ (*Kashmir B. 2012*)

(*iii*) $x\sec^2\left(\dfrac{y}{x}\right)dy = \left\{y\sec^2\left(\dfrac{y}{x}\right) + x\right\}dx.$ (*P.B. 2014*)

19. $y\,dx + x\,\log\left(\dfrac{y}{x}\right)dy - 2x\,dy = 0.$

(*N.C.E.R.T.; H.B. 2016; Jammu B. 2012; H.P.B. 2012*)

20. $(x\,dy - y\,dx)\,y\sin\left(\dfrac{y}{x}\right)$

$= \left[(y\,dx + x\,dy)\,x\cos\left(\dfrac{y}{x}\right)\right].$

(*N.C.E.R.T.; Assam B. 2018*)

21. Find the particular solution of the differential equation :

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2},$$

given that $y = 1$, when $x = 0$. (*C.B.S.E. 2015*)

22. Solve : $\left[x\sin^2\left(\dfrac{y}{x}\right) - y\right]dx + x\,dy = 0$, given that $y = \dfrac{\pi}{4}$ when $x = 1$. (*P.B. 2018; H.B. 2016*)

23. (*i*) Show that the differential equation :

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0 \text{ is homogeneous.}$$

Find the particular solution of this differential equation, given that $x = 1$ when $y = \dfrac{\pi}{2}$. (*C.B.S.E. 2013*)

(*ii*) Show that the differential equation :

$$\left(xe^{y/x} + y\right)dx = x\,dy \text{ is homogeneous.}$$

Find the particular solution of this differential equation, given that $x = 1$ when $y = 1$. (*C. B. S. E. 2013*)

24. Show that the differential equation :

$$2y\,e^{x/y}\,dx + \left(y - 2x\,e^{x/y}\right)dy = 0$$

is homogeneous and find its particular solution, given that, $x = 0$ when $y = 1$.

(*N.C.E.R.T. ; A.I.C.B.S.E. 2016; Kashmir B. 2011*)

25. Find the particular solution of the differential equation :

$$(x\,dy - y\,dx)\,y\sin\left(\frac{y}{x}\right) = \left(y\,dx + x\,dy\right)x\cos\frac{y}{x},$$

given that $y = \pi$ and $x = 3$.

26. Solve : (*i*) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$

(*ii*) $x\dfrac{dy}{dx} = y - x\tan\left(\dfrac{y}{x}\right).$

(*Assam B. 2015; Bihar B. 2014*)

27. (*i*) Show that the family of curves for which $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx.$ (*C.B.S.E. 2017*)

(*ii*) Show that the family of curves for which slope of the tangent at any point (x, y) on it is $\dfrac{x^2 + y^2}{2xy}$ is given by $x^2 - y^2 = cx.$ (*N.C.E.R.T.; Jharkhand B. 2016*)

Answers

1. $\dfrac{y^2}{2x^2} = \log|x| + c.$

2. (*i*)–(*ii*) $-\dfrac{1}{2}\log\left|x^2 + xy + y^2\right| + \sqrt{3}\tan^{-1}\dfrac{x + 2y}{\sqrt{3}x} = c.$

3. (*i*) $\tan^{-1}\dfrac{y}{x} = \dfrac{1}{2}\log(x^2 + y^2) + \log c$

(*ii*)—(*iv*) $\log(x^2 + y^2) + 2\tan^{-1}\left(\dfrac{y}{x}\right) = c.$

4. $y^2 = c\,e^{2y/x},\ (x \neq 0).$

5. (i) $-\frac{x}{y}=\log|x|+c$ (ii) $y = x\log|xy| + cx$.

6. $(x-y)^2 = cxe^{-y/x}$.

7. (i) $\log|y|+\frac{y}{x} = 3\log|x| + \log|c|$

(ii) $2x^3y + x^2y^2 = c$ (iii) $y^2(4x^2-y^2)^3 = c$.

8. (i) $3x^2y + 2y^3 = c$ (ii) $x^2+y^2 = cx^3$.

9. (i) $x^2+y^2 = cx$ (ii) $x^3+3xy^2 = c$

(iii) $\log|x|+\log\left|1-\frac{y^2}{x^2}\right|+c = 0$.

10. (i) – (iii) $y = c\,e^{\frac{x^3}{3y^3}}$.

11. (i) $\sin\left(\frac{y}{x}\right)=\log|cx|$ (ii) $e^{\frac{x}{y}} = y + c$.

12. (i) $\tan^{-1}\frac{y}{x}=\log|x|+\frac{\pi}{4}$. (ii) $x^2+y^2 = 2x$.

13. $y=\frac{2x}{1+|\log x|}$, $\left(x\neq 0,\pm\frac{1}{e}\right)$.

14. $\log|x| = \cos\frac{y}{x}-1\ (x\neq 0)$.

15. (i) $y = -x\log\ \log|x|$, $(x\neq 0)$.

(ii) $\log|x| + e^{\frac{-y}{x}} = 1\ (x\neq 0)$.

16. (i) $\frac{-1}{2}\log\left|1+\frac{y}{x}+\frac{y^2}{x^2}\right|+\sqrt{3}\tan^{-1}\frac{2y+x}{\sqrt{3}x}$

$=\log|x|+\frac{\pi}{2\sqrt{3}}$

(ii) $\frac{1}{2}\log(x^2+y^2)-\log x+\tan^{-1}\frac{y}{x}$

$=-\log|x|+\frac{\pi}{4}+\frac{1}{2}\log 2$

(iii) $x^3y=\frac{1}{4}(3x+y)$.

17. (i) – (ii) $y+\sqrt{x^2+y^2}=cx^2$.

18. (i) $x\sin\frac{y}{x}=c\left(1+\cos\frac{y}{x}\right)$; $\left(x\sin\frac{y}{x}\neq 0\right)$

(ii) $\sin\frac{y}{x}=\log|x|+c$ (iii) $\tan\frac{y}{x}=\log|x|+c$.

19. $cy=\log\frac{y}{x}-1$ 20. $\sec\left(\frac{y}{x}\right)=cxy$.

21. $\log|y|=\frac{x^2}{2y^2}$. 22. $\log|x|-\cot\frac{y}{x}+1=0$.

23. (i) $\log|x|=\cos\left(\frac{y}{x}\right)$ (ii) $-e^{-y/x}=\log|x|-e^{-1}$.

24. $2e^{\frac{x}{y}}+\log|y|=2$. 25. $2xy\cos\frac{y}{x}=3\pi$.

26. (i) $\sin\frac{y}{x}=cx$ (ii) $x\sin\frac{y}{x}=c$.

Hints to Selected Questions

17. (i) $\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}$. Put $y = vx$.

22. Put $y = vx$.

25. The given equation can be written as :

$$\left(xy\frac{dy}{dx}-y^2\right)\sin\left(\frac{y}{x}\right)=\left(xy+x^2\frac{dy}{dx}\right)\cos\left(\frac{y}{x}\right).$$

Put $\frac{y}{x}=v$ i.e. $y = vx$.

26. (i) Put $\frac{y}{x}=v$ i.e. $y = vx$.

27. (ii) Here $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$.

TYPE VI. FIRST ORDER LINEAR EQUATIONS

Definition

A differential equation of the type :

$$\frac{dy}{dx}+Py=Q,$$

where P and Q are constants or functions of x only, is called first order linear differential equation.

For Examples : (I) $\frac{dy}{dx}+y = \cos x$

(II) $\frac{dy}{dx}+xy=e^x$

(III) $\frac{dy}{dx}+\frac{1}{x}y=\log x$; etc.

are first order linear differential equations.

Another Form :

Definition

A differential equation of the type :

$$\frac{dx}{dy}+Px=Q,$$

where P and Q are constants or functions of y only, is called first order linear differential equation.

For Examples : (I) $\frac{dx}{dy}+x=\sin y$

(II) $\frac{dx}{dy}+\frac{2x}{y}=y^2 e^{-y}$

(III) $\frac{dx}{dy}+\frac{1}{y}x=\log y$; etc.

are first order linear differential equations.

To Solve : $\mathbf{\frac{dy}{dx}+Py=Q}$ ....(1),

where P and Q are constants or functions of x.

Multiplying both sides of (1) by $g(x)$, we get :

$$g(x)\frac{dy}{dx}+P.\,g(x).\,y = Q.\,g(x) \quad ....(2)$$

Select $g(x)$ in such a way that **RHS** becomes the derivative of $y.\,g(x)$

i.e. $g(x)\frac{dy}{dx}+P.\,g(x)\,y = \frac{d}{dx}[y.\,g(x)]$

$\Rightarrow$ $g(x)\frac{dy}{dx}+P.\,g(x)\,y = g(x)\frac{dy}{dx}+y.g'(x)$

$\Rightarrow$ $P.\,g(x) = g'(x)$

$\Rightarrow$ $P = \frac{g'(x)}{g(x)}.$

Integrating, $\int P\,dx = \int \frac{g'(x)}{g(x)}dx = \log|g(x)|$

$\Rightarrow$ $g(x) = e^{\int P\,dx}.$

Multiplying (1) by $g(x) = e^{\int P\,dx}$, **LHS** becomes the derivative of some function of x and y.

This function $g(x) = e^{\int P\,dx}$ is called ***Integrating Factor*** (I.F.) of (1).

Putting the value of $g(x)$ in (2), we get :

$$e^{\int P\,dx}\frac{dy}{dx}+P\,e^{\int P\,dy}.y = Q\,e^{\int P\,dx}$$

$$\Rightarrow \qquad \frac{d}{dx}\left(y e^{\int P\,dx}\right) = Q\,e^{\int P\,dx}$$

Integrating, $$y\,e^{\int P\,dx} = \int \left(Q.e^{\int P\,dx}\right) dx$$

$$\Rightarrow \qquad y = e^{-\int P\,dx} \int \left(Q.e^{\int P\,dx}\right) dx + c,$$

which is the reqd. solution.

ALGORITHMIC APPROACH TO SOLVE FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

(*a*) Write the differential equation in the type :

$$\frac{dy}{dx} + Py = Q \qquad \text{...(1),}$$

where P, Q are constants or functions of x only.

(I) Obtain I.F. $= e^{\int P\,dx}$.

(II) Multiplying (1) by I.F.

(III) Write as $$\frac{d}{dx}(y.(\text{I.F.})) = Q.(\text{I.F.})$$

(IV) Solving, $$y.(\text{I.F.}) = \int (Q.(\text{I.F.}))\,dx + c.$$

(*b*) In the case of $$\frac{dx}{dy} + Px = Q \qquad \text{...(2),}$$

where P and Q are constants or functions of y only.

$$\text{I.F.} = e^{\int P\,dy}.$$

Then, as above, the solution will be :

$$x\,(\text{I.F.}) = \int (Q.(\text{I.F.}))\,dy + c.$$

Frequently Asked Questions

FAQs

Example 1. Find the integrating factor for the following differential equation :

$$\mathbf{x \log x \frac{dy}{dx} + y = 2 \log x.}$$

(A.I. C.B.S.E. 2015)

Solution. The given differential equation is :

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

$$\Rightarrow \qquad \frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x} \qquad \text{| Linear Equation}$$

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{x\log x}dx} = e^{\int \frac{1/x}{\log x}dx}$$

$$= e^{\log|\log x|}$$

$$= |\log x|$$

$$= \log x.$$

Example 2. Solve the differential equation :

$$x\frac{dy}{dx}+3y = \frac{\log x}{x^3}. \qquad \textit{(P.B. 2014)}$$

Solution. The given linear equation is :

$$x\frac{dy}{dx}+3y = \frac{\log x}{x^3}$$

i.e. $$\frac{dy}{dx}+\frac{3y}{x} = \frac{\log x}{x^4} \quad ...(1) \text{ | Linear Equation}$$

Comparing with $\frac{dy}{dx}$ + Py = Q, we have :

$$\text{'P'} = \frac{3}{x} \text{ and 'Q'} = \frac{\log x}{x^4}.$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{3}{x}dx} = e^{3\int \frac{1}{x}dx}$$

$$= e^{3\log|x|} = e^{\log x^3} = x^3.$$

Multiplying (1) by x^3, we get :

$$x^3\frac{dy}{dx}+3yx^2 = \frac{\log x}{x}$$

$$\Rightarrow \quad \frac{d}{dx}(y.x^3) = \log x\cdot\frac{1}{x}.$$

Integrating, $$y\,.\,x^3 = \int \log x.\frac{1}{x}dx+c$$

$$\Rightarrow \quad yx^3 = \frac{1}{2}(\log x)^2+c$$

$\Rightarrow y = \frac{x^{-3}}{2}(\log x)^2+cx^{-3}$, which is the reqd. solution.

Example 3. Find the general solution of the following :

$$x\frac{dy}{dx}+y-x+xy\cot x = 0\ (x \neq 0). \qquad \textit{(N.C.E.R.T.)}$$

Solution. The given equation is :

$$x\frac{dy}{dx}+y-x+xy\cot x = 0$$

$$\Rightarrow \quad x\frac{dy}{dx}+y(1+x\cot x) = x$$

$$\Rightarrow \quad \frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y = 1 \quad ...(1)$$

|*Linear Equation*

Comparing with $\frac{dy}{dx}$ + Py = Q, we have :

'P' = $\frac{1}{x}$ + cot x and 'Q' = 1.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int\left(\frac{1}{x}+\cot x\right)dx}$$

$$= e^{\int\frac{1}{x}dx}.e^{\int\cot x\,dx} = e^{\log|x|}\,e^{\log|\sin x|}$$

$$= e^{\log|x\sin x|} = x\sin x.$$

Multiplying (1) by $x \sin x$, we get :

$$x\sin x.\frac{dy}{dx}+x\sin x\left(\frac{1}{x}+\cot x\right)y = x\sin x$$

$$\Rightarrow \quad x\sin x.\frac{dy}{dx}+(\sin x+x\cos x)\,y = x\sin x$$

$$\Rightarrow \quad \frac{d}{dx}(y.\,x\sin x) = x\sin x.$$

Integrating, $y.\,x\sin x = \int x\sin x\,dx+c$

$$= x(-\cos x)-\int(1)(-\cos x)\,dx+c$$

[*Integrating by parts*]

$$= -x\cos x+\int\cos x\,dx+c$$

$$= -x\cos x+\sin x+c$$

$\Rightarrow y = -\cot x+\frac{1}{x}+c\,(x\sin x)^{-1}$,

which is the reqd. general solution.

Example 4. Solve the differential equation :

$x\,dy + (y - x^3)\,dx = 0$. *(A.I.C.B.S.E. 2011)*

Solution. The given equation is $x\,dy + (y - x^3)\,dx = 0$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y-x^3}{x} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x} = x^2 \quad ....(1) \text{ | Linear Equation}$$

Comparing with $\frac{dy}{dx}$ + Py = Q, we have :

$$\text{'P'} = \frac{1}{x} \text{ and 'Q'} = x^2.$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int\frac{1}{x}dx}$$

$$= e^{\log|x|} = x.$$

Multiplying (1) by x, we get :

$$x\frac{dy}{dx}+y = x^3$$

$$\Rightarrow \quad \frac{d}{dx}(yx) = x^3.$$

Integrating, $yx = \int x^3\, dx + c$

$\Rightarrow$ $xy = \frac{1}{4}x^4 + c$

$\Rightarrow$ $y = \frac{1}{4}x^3 + cx^{-1},\ x \neq 0,$

which is the reqd. solution.

Example 5. Solve the differential equation :

$\sec x \frac{dy}{dx} - y = \sin x.$ *(A.I.C.B.S.E. 2009 C)*

Solution. The given equation is :

$$\sec x \frac{dy}{dx} - y = \sin x$$

$\Rightarrow$ $\frac{dy}{dx} - \cos x \,.\, y = \sin x \cos x$ [*Dividing by sec x*]

$\Rightarrow$ $\frac{dy}{dx} - \cos x \,.\, y = \frac{1}{2}\sin 2x$ ....(1)

| *Linear Equation*

Comparing with $\frac{dy}{dx} + Py = Q$, we have :

$$\text{'P'} = -\cos x \text{ and 'Q'} = \frac{1}{2}\sin 2x.$$

$\therefore$ $\text{I.F.} = e^{\int P\,dx} = e^{-\int \cos x\,dx}$

$= e^{-\sin x}.$

Multiplying (1) by $e^{-\sin x}$, we get :

$$e^{-\sin x}.\frac{dy}{dx} - \cos x\, e^{-\sin x} y = \frac{1}{2}e^{-\sin x}\sin 2x$$

$\Rightarrow$ $\frac{d}{dx}\left(y.e^{-\sin x}\right) = \frac{1}{2}e^{-\sin x}\sin 2x.$

Integrating, $y \,.\, e^{-\sin x} = \frac{1}{2}\int e^{-\sin x}\sin 2x\,.\,dx + c$ ....(2)

Now $I = \int e^{-\sin x}\sin 2x\,dx$

$= 2\int e^{-\sin x}\sin x \cos x\,dx.$

Put sin x = t so that $\cos x\,dx = dt.$

$\therefore$ $I = 2\int e^{-t}.t\,dt = 2\int t\,e^{-t}\,dt$

$= 2\left[t.\frac{e^{-t}}{-1} - \int (1)\frac{e^{-t}}{-1}dt\right]$

[*Integrating by parts*]

$= -2t\,e^{-t} + 2\int e^{-t}\,dt$

$= -2te^{-t} + 2\frac{e^{-t}}{-1} = -2e^{-t}(t+1)$

$= -2e^{-\sin x}(\sin x + 1).$

$\therefore$ From (2), $y \,.\, e^{-\sin x} = -\frac{1}{2}.2\,e^{-\sin x}(\sin x + 1) + c$

$\Rightarrow$ $y = -\sin x - 1 + ce^{\sin x},$

which is the reqd. solution.

Example 6. Solve : $(2x - 10y^3)\frac{dy}{dx} + y = 0.$

(W. Bangal B. 2018)

Solution. The given equation can be written as :

$$\frac{dx}{dy} + \frac{2}{y}x = 10y^2 \quad ...(1)$$

| *Linear Equation*

Comparing with $\frac{dx}{dy} + Px = Q$, we have :

$\text{'P'} = \frac{2}{y}$ and $\text{'Q'} = 10y^2.$

$\therefore$ $\text{I.F.} = e^{\int P dy} = e^{\int \frac{2}{y}dy} = e^{2\log|y|}$

$= e^{\log y^2} = y^2.$

Multiplying (1) by y^2, we get :

$$y^2.\frac{dx}{dy} + 2yx = 10y^4$$

$\Rightarrow$ $\frac{d}{dy}(x\,.\,y^2) = 10y^4.$

Integrating, $x\,.\,y^2 = 10\int y^4\,dy + c$

$\Rightarrow xy^2 = \frac{10y^5}{5} + c$

$\Rightarrow xy^2 = 2y^5 + c$, which is the reqd. solution.

Example 7. Find the general solution of the following differential equation :

$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx} = 0.$ *(C.B.S.E. 2016)*

Solution. The given equation is :

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x - e^{\tan^{-1} y})\frac{dy}{dx} = -(1+y^2)$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x - e^{\tan^{-1} y}}{-(1+y^2)}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1} y}}{1+y^2} \quad ...(1)$$

| *Linear Equation*

Comparing with $\frac{dx}{dy} + Px = Q$, we have :

$$\text{'P'} = \frac{1}{1+y^2} \text{ and 'Q'} = \frac{e^{\tan^{-1} y}}{1+y^2}.$$

$$\therefore \text{I.F.} = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}.$$

Multiplying (1) with $e^{\tan^{-1} y}$, we get :

$$e^{\tan^{-1} y}.\frac{dx}{dy} + \frac{x}{1+y^2} e^{\tan^{-1} y} = \frac{(e^{\tan^{-1} y})^2}{1+y^2}$$

$$\Rightarrow \frac{d}{dy}\left(x e^{\tan^{-1} y}\right) = \frac{(e^{\tan^{-1} y})^2}{1+y^2}.$$

$$\text{Integrating, } x e^{\tan^{-1} y} = \int \frac{(e^{\tan^{-1} y})^2}{1+y^2} dy + c \quad ...(2)$$

$$\text{Now I} = \int \frac{(e^{\tan^{-1} y})^2}{1+y^2} dy.$$

Put $\tan^{-1} y = t$ so that $\frac{1}{1+y^2} dy = dt$.

$$\therefore \text{I} = \int (e^t)^2 dt = \int e^{2t} dt = \frac{e^{2t}}{2} = \frac{e^{2\tan^{-1} y}}{2}.$$

Putting in (2), $x e^{\tan^{-1} y} = \frac{1}{2} e^{2\tan^{-1} y} + c$

$$\Rightarrow x = \frac{1}{2} e^{\tan^{-1} y} + c\, e^{-\tan^{-1} y} + c,$$

which is the reqd. solution.

Example 8. **(*i*) Solve the differential equation :**
$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$.

(N.C.E.R.T.; C.B.S.E. 2015; J. & K.B. 2011)

(*ii*) Find the particular solution when x = 0, y = 0.

(A.I.C.B.S.E. 2013)

Solution. (*i*) The given equation can be written as :

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \quad ....(1) \mid \textit{Linear Equation}$$

Comparing with $\frac{dx}{dy} + Px = Q$, we have :

$$\text{'P'} = \frac{1}{1+y^2} \text{ and 'Q'} = \frac{\tan^{-1} y}{1+y^2}.$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}.$$

Multiplying (1) by $e^{\tan^{-1} y}$, we get :

$$e^{\tan^{-1} y}.\frac{dx}{dy} + \frac{x}{1+y^2} e^{\tan^{-1} y} = \frac{\tan^{-1} y}{1+y^2}.e^{\tan^{-1} y}$$

$$\Rightarrow \frac{d}{dy}\left(x.e^{\tan^{-1} y}\right) = \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y}.$$

$$\text{Integrating, } x e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y} dy + c \quad ...(2)$$

$$\text{Now} \quad \text{I} = \int \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y} dy.$$

Put $\tan^{-1} y = t$ so that $\left(\frac{1}{1+y^2}\right) dy = dt$.

$$\therefore \quad \text{I} = \int t\, e^t\, dt$$

$$= t\,e^t - \int (1)\, e^t\, dt$$

[*Integrating by parts*]

$$= te^t - e^t = e^t (t-1)$$

$$= e^{\tan^{-1} y} (\tan^{-1} y - 1).$$

$$\therefore \quad \text{From (2), } x\, e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + c$$

$$\Rightarrow \quad x = (\tan^{-1} y - 1) + c\, e^{-\tan^{-1} y} \quad ...(3),$$

which is the reqd. solution.

(*ii*) When $x = 0, y = 0$.

$$\therefore \quad 0 = (\tan^{-1} 0 - 1) + ce^{-\tan^{-1} 0}$$

$$\Rightarrow \quad 0 = (0 - 1) + ce^{-0}$$

$$\Rightarrow \quad 0 = -1 + c \Rightarrow c = 1.$$

Putting in (3), $x = (\tan^{-1} y - 1) + e^{-\tan^{-1} y}$,

which is the reqd. particular solution.

Example 9. Find the general solution of the differential equation :

$$\cos^2 x \frac{dy}{dx} + y = \tan x. \qquad \textit{(Kashmir B. 2017)}$$

Find the particular solution which satisfies y = 0 at x = 0. *(Tripura B. 2016)*

Solution. The given equation is :

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \quad \frac{dy}{dx} + y \sec^2 x = \tan x \sec^2 x \qquad ....(1)$$

| *Linear Equation*

Comparing with $\frac{dy}{dx} + Py = Q$, we have :

'P' $= \sec^2 x$ and 'Q' $= \tan x \,.\, \sec^2 x$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \sec^2 x\,dx} = e^{\tan x}.$$

Multiplying (1) by $e^{\tan x}$, we get :

$$e^{\tan x} \cdot \frac{dy}{dx} + e^{\tan x} \cdot y \sec^2 x = \tan x \sec^2 x\, e^{\tan x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(y \,.\, e^{\tan x}\right) = \tan x \sec^2 x\, e^{\tan x}.$$

Integrating, $y \,.\, e^{\tan x} = \int \tan x \sec^2 x\; e^{\tan x} dx + c$ ...(2)

Now $\quad I = \int \tan x \sec^2 x\; e^{\tan x} dx.$

Put tan x = t so that $\sec^2 x\, dx = dt$.

$$\therefore \quad I = \int t \,.\, e^t\, dt$$

$$= t\,e^t - \int (1)\, e^t\, dt$$

[Integrating by parts]

$$= t\,e^t - e^t = e^t\,(t - 1)$$

$$= e^{\tan x}\,(\tan x - 1).$$

$\therefore$ From (2), $y \,.\, e^{\tan x} = e^{\tan x}(\tan x - 1) + c$

$$\Rightarrow \quad y = (\tan x - 1) + c\,e^{-\tan x} \qquad ....(3)$$

When $x = 0$, $y = 0$, $\therefore$ $0 = (-1) + c \Rightarrow c = 1$.

Putting in (3), $y = (\tan x - 1) + e^{-\tan x}$,

which is the reqd. solution.

Example 10. Find the particular solution of differential equation :

$$\tan x \frac{dy}{dx} + y = 2x \tan x + x^2, x \neq 0,$$

given that y = 0 when $\mathbf{x = \frac{\pi}{2}}$**.** *(P.B. 2018)*

Solution. The given equation is :

$$\tan x \frac{dy}{dx} + y = 2x \tan x + x^2$$

$$\Rightarrow \quad \frac{dy}{dx} + y \cot x = 2x + x^2 \cot x \qquad ....(1)$$

| *Linear Equation*

Comparing with $\frac{dy}{dx} + Py = Q$, we have :

'P' $= \cot x$ and 'Q' $= 2x + x^2 \cot x$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \cot x\,dx}$$

$$= e^{\log|\sin x|} = \sin x.$$

Multiplying (1) by $\sin x$, we get :

$$\sin x . \frac{dy}{dx} + y \cos x = 2x \sin x + x^2 \cos x$$

$$\Rightarrow \quad \frac{d}{dx}(y . \sin x) = 2x \sin x + x^2 \cos x.$$

Integrating,

$$y \sin x = \int (2x \sin x + x^2 \cos x)\, dx + c$$

$$\Rightarrow \quad y \sin x = \int 2x \sin x\, dx + \int x^2 \cos x\, dx + c$$

$$\Rightarrow \quad y \sin x = \int x^2 \cos x\, dx + \int 2x \sin x\, dx + c$$

$$\Rightarrow \quad y \sin x = x^2 \sin x - \int 2x \sin x\, dx + \int 2x \sin x\, dx + c$$

$$\Rightarrow \quad y \sin x = x^2 \sin x + c \qquad \text{....(2)}$$

When $y = 0$, $x = \frac{\pi}{2}$, $\therefore\ 0 = \frac{\pi^2}{4}\sin\frac{\pi}{2} + c$

$$\Rightarrow \quad 0 = \frac{\pi^2}{4}(1) + c \Rightarrow c = -\frac{\pi^2}{4}.$$

Putting in (2), $y \sin x = x^2 \sin x - \frac{\pi^2}{4}$

$$\Rightarrow \quad y = x^2 - \frac{\pi^2}{4}\operatorname{cosec} x\ (x \neq 0),$$

which is the reqd. particular solution.

Example 11. Solve the differential equation :

$(x + 2y^2)\frac{dy}{dx} = y$; given that when $x = 2$, $y = 1$.

Solution. The given equation is $(x + 2y^2)\frac{dy}{dx} = y$

$$\Rightarrow x + 2y^2 = y\frac{dx}{dy} \qquad \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y \qquad \text{...(1)}$$

|*Linear Equation*

Comparing with $\frac{dx}{dy} + Px = Q$, we have :

'P' $= -\frac{1}{y}$ and 'Q' $= 2y$.

$$\therefore \quad \text{I.F.} = e^{\int P dy} = e^{\int -\frac{1}{y} dy}$$

$$= e^{-\int \frac{1}{y} dy} = e^{-\log|y|} = \frac{1}{y}.$$

Multiplying (1) by $\frac{1}{y}$, $\frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = 2$

$$\Rightarrow \frac{d}{dy}\left(x.\frac{1}{y}\right) = 2.$$

Example 12. Find the general solution of the differential equation :

$$\mathbf{\frac{dy}{dx} = \frac{y\tan y - x\tan y - xy}{y\tan y}}.$$

(C.B.S.E. Sample Paper 2019)

Solution. The given differential equation is :

$$\frac{dy}{dx} = \frac{y\tan y - x\tan y - xy}{y\tan y}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{y} + \frac{1}{\tan y}\right)x = 1$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{y} + \cot y\right)x = 1 \qquad \text{...(1)}$$ | *Linear Equation*

Comparing with $\frac{dx}{dy} + Px = Q$, we have :

'P' $= \frac{1}{y} + \cot y$ and 'Q' $= 1$.

$$\therefore \quad \text{I.F.} = e^{\int P\, dy} = e^{\int \left(\frac{1}{y} + \cot y\right) dx}$$

$$= e^{\log y + \log \sin y} = e^{\log (y \sin y)} = y \sin y.$$

Multiplying (1) by $y \sin y$, we get :

$$\Rightarrow (y\sin y)\frac{dx}{dy} + \left(\frac{1}{y} + \cot y\right)x\, y\sin y = y \sin y$$

$$\Rightarrow \frac{d}{dx}(x\,(y\sin y)) = y \sin y.$$

Integrating, $xy \sin y = \int y\sin y\, dy + c$

$$\Rightarrow xy \sin y = y(-\cos y) - \int (1)(-\cos y)\, dy + c$$

$$\Rightarrow xy \sin y = -y \cos y + \sin y + c$$

$$\Rightarrow \quad x = \frac{\sin y - y\cos y + c}{y\sin y},$$

which is the reqd. solution.

EXERCISE 9 (i)

Very Short Answer Type Questions

VSATQ

1. Find the integrating factor of the differential equation :

(*i*) $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$ *(C.B.S.E. 2015)*

(*ii*) $\cos x\frac{dy}{dx}+y=\sin x;\ 0\le x<\frac{\pi}{2}$ *(Kerala B. 2015)*

(*iii*) $\cos x\frac{dy}{dx}+y=2x+x^2$ *(P.B. 2017)*

(*iv*) $x\frac{dy}{dx}-y=x^2$. *(Meghalaya B. 2017)*

Find the general solution of the following (2–8) differential equations :

2. $\frac{dy}{dx}+2y=3$. *(P.B. 2015)*

3. (*i*) $\frac{dy}{dx}+y=1,\ (y\neq 1)$ *(Jammu B. 2015W, 12)*

(*ii*) $\frac{dy}{dx}+y=x$. *(Assam B. 2016)*

4. (*i*) $\frac{dy}{dx}-y=3x^3$ (*ii*) $\frac{dy}{dx}-y=xe^x$.

5. (*i*) $y'+2y=e^{2x}$

(*ii*) $xy'-y=(x+1)e^{-x}$

(*iii*) $\frac{dy}{dx}+\frac{y}{x}=e^x,\ (x>0)$.

6. (*i*) $\frac{dy}{dx}+3y=e^{-2x}$ *(N.C.E.R.T.; H.P.B. 2017, 12, 10)*

(*ii*) $\frac{dy}{dx}+2y=6e^x$.

7. $x\frac{dy}{dx}-y=(x-1)e^x$.

8. $\frac{dy}{dx}+3y=2x$. *(P.B. 2015)*

9. Assume that the rise in the price $p=p(t)$ of a product is proportional to the difference between the demand $w(t)$ and the supply $s(t)$ and that the demand depends on the price as a first degree polynomial. Set up a differential equation for the price.

Short Answer Type Questions

SATQ

Solve the following (10 – 33) differential equations :

10. (*i*) $x\frac{dy}{dx}=y-x$ *(P.B. 2010)*

(*ii*) $\frac{dy}{dx}+\frac{y}{x}=x^2$ *(N.C.E.R.T.; H.P.B. 2017, 12)*

Or

$xdy+(y-x^2y)\,dx=0$. *(Mizoram B. 2016)*

11. (*i*) $x\frac{dy}{dx}+2y=x^2\ (x\neq 0)$

(N.C.E.R.T.; H.P.B. 2016; Kerala B. 2014; P.B. 2013, 10 ; Kashmir B. 2011)

(*ii*) $x\frac{dy}{dx}-y=2x^3$. *(P.B. 2013)*

12. $\frac{dy}{dx}+\frac{y}{2x}=3x^2$. *(H.B. 2010)*

13. $\frac{dy}{dx}+2y=\sin 5x$.

14. (*i*) $\frac{dy}{dx}+y=\cos x$ *(N.C.E.R.T.)*

(*ii*) $\frac{dy}{dx}-y=\cos x$ *(N.C.E.R.T. ; Kashmir B. 2012)*

(*iii*) $\frac{dy}{dx}+2y=\cos 3x$.

15. (*i*) $\frac{dy}{dx}-y=\sin x$ *(A.I.C.B.S.E. 2017)*

(*ii*) $\frac{dy}{dx}=y-2\sin x$.

16. $\frac{dy}{dx} - 2y = \cos 3x.$ (J. & K.B. 2011)

17. $\frac{dy}{dx} + \sec x \cdot y = \tan x \left(0 \le x < \frac{\pi}{2}\right).$

(N.C.E.R.T. ; H.P.B. 2017; Jammu B. 2012)

18. (i) $\frac{dy}{dx} + 2y \tan x = \sin x$ (Mizoram B. 2017)

(ii) $\tan x \frac{dy}{dx} + 2y = \sec x.$

19. $\cos x \frac{dy}{dx} + y = \sin x.$ (H.B. 2013)

20. $(y + 3x^2)\frac{dx}{dy} = x.$ (A.I.C.B.S.E. 2011)

21. $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}.$ (A.I.C.B.S.E. 2014)

22. (i) $\frac{dy}{dx} + y = \cos x - \sin x$ (C.B.S.E. 2009)

(ii) $\frac{dy}{dx} + y = \sin x + \cos x.$

23. (i) $\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x$

(ii) $\frac{dy}{dx} + y \cot x = 2 \cos x.$

24. (i) $x\frac{dy}{dx} + 2y = x \cos x$ (Mizoram B. 2016)

(ii) $\frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}.$ (J. & K. B. 2011)

25. $\frac{dy}{dx} + y \sec x = \tan x.$

26. (i) $x \frac{dy}{dx} + y = x \log x$ (P.B. 2010 S)

(ii) $x\frac{dy}{dx} - y = \log x$ (Mizoram B. 2015)

(iii) $x\frac{dy}{dx} + y - x + xy \cot x = 0.$ (N.C.E.R.T.)

27. (i) $x \frac{dy}{dx} + 2y = x^2 \log x$

(N.C.E.R.T. ; Kerala B. 2018; Jammu B. 2015; H.P.B. 2016, 14, 13; H.B. 2013, 10; J. & K. B. 2010)

(ii) $x \log x \frac{dy}{dx} + y = 2 \log x.$

(Bihar B. 2012 ; C.B.S.E. 2009 C)

28. $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x.$

(N.C.E.R.T.; H.P.B. 2013, 10, 09 ; Jammu B. 2013 ; C.B.S.E. 2010, 09)

29. (i) $\sin x \frac{dy}{dx} + \cos x \cdot y = \cos x \cdot \sin^2 x$

(ii) $(x^2 - 1) \frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1}$

(C.B.S.E. 2014; Meghalaya B. 2014)

(iii) $\frac{dy}{dx} + \frac{2x}{x^2+1} y = x^2 + 2.$ (Kerala B. 2013)

30. (i) $(1 + x^2) \frac{dy}{dx} + y = \tan^{-1} x$

(Assam B. 2015; Meghalaya B. 2013 ; P.B. 2010 S ; C.B.S.E. 2009)

(ii) $(1+x^2)\frac{dy}{dx} + 2xy = \cos x$ (Meghalaya B. 2018, 13)

(iii) $(1-x^2)\frac{dy}{dx} - xy = 1.$

(W. Bengal B. 2018; Assam B. 2017)

31. (i) $y\, dx + (x - y^2)\, dy = 0$

(N.C.E.R.T.; H.P.B. 2018; Nagaland B. 2018)

(ii) $y\, dx - (x + 2y^2)\, dy = 0.$

(N.C.E.R.T.; A.I.C.B.S.E. 2017; (W. Bengal B. 2017; Uttarakhand B. 2013)

32. $(\tan^{-1} x - y)\, dx = (1 + x^2)\, dy.$ (A.I.C.B.S.E. 2017)

33. $\frac{dy}{dx} - \frac{y}{x} = \left(\frac{x-1}{x}\right) e^x.$

Solve the following initial value problems (34 – 39) :

34. $\frac{dy}{dx} = 2x + y$, given that $x = 0, y = 0.$ (P.B. 2011)

35. (i) $x \frac{dy}{dx} + y = x^3, y(2) = 1$ (P.B. 2012)

(ii) $x \frac{dy}{dx} + 2y = x^2, y(1) = \frac{1}{4}$ (Uttarakhand B. 2015)

(iii) $x\frac{dy}{dx} + 2y = x^2\ (x \ne 0)$, given that $y = 0$ when $x = 1.$

(Kerala B. 2016)

36. $x\frac{dy}{dx} + y = x \cos x + \sin x$, given that $y = 1$, when $x = \frac{\pi}{2}.$ (C.B.S.E. 2017)

37. (*i*) $\frac{dy}{dx} + 2y \tan x = \sin x$, $y = 0$ when $x = \frac{\pi}{3}$

(N.C.E.R.T.; C.B.S.E. 2018; H.P.B. 2013 S, 10 S ; C.B.S.E. (F) 2011)

(*ii*) $\frac{dy}{dx} + y \tan x = \sec x$, given that $x = 0$ and $y = 0$.

(Meghalaya B. 2016)

(*iii*) $\frac{dy}{dx} - 3y \cot x = \sin 2x$, satisfying the condition $y = 2$, when $x = \frac{\pi}{2}$.

(H.P.B. 2015)

38. $\cos^3 x \frac{dy}{dx} - y \sin x \cot x = \cos x$; $y\left(\frac{\pi}{4}\right) = 1$.

(J. & K. B. 2010)

39. $ye^y\, dx = (y^3 + 2x\, e^y)\, dy$, $y(0) = 1$.

(C.B.S.E. Sample Paper 2018)

40. Find the particular solution of the differential equation :

$$\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x}, \text{ given that } y = 1 \text{ when } x = 0.$$

(A.I.C.B.S.E. 2016)

Long Answer Type Questions

LATQ

41. Solve : (*i*) $\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x$, given that $y(0) = 0$

(*ii*) $\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x$, $x \neq 0$,

given that $y = 0$, when $x = \frac{\pi}{2}$

(N.C.E.R.T.)

(*iii*) $\frac{dy}{dx} - 3y \cot x = \sin 2x$,

given that $y = 2$ when $x = \frac{\pi}{2}$

(N.C.E.R.T. ; Meghalaya B. 2015; H.P.B. 2013 S, 10 S)

42. Find the particular solution of the differential equation: $(1+x^2)\frac{dy}{dx} = e^{m\tan^{-1}x} - y$, given that $y = 1$ when $x = 0$.

(A.I.C.B.S.E. 2015)

43. Find the particular solution of the differential equation:

(*i*) $\frac{dx}{dy} + x \cot y = 2y + y^2 \cot y$, $(y \neq 0)$, given that $x = 0$ when $x = \frac{\pi}{2}$.

(A. I.C.B.S.E. 2013)

(*ii*) $\tan x \frac{dy}{dx} = 2x \tan x + x^2 - y$; $(\tan x \neq 0)$, given that $y = 0$ when $x = \frac{\pi}{2}$.

(C.B.S.E. 2017)

44. Solve the differential equation :

$$(x + 2y^2)\frac{dy}{dx} = y.$$

Given that when $x = 2$, $y = 1$.

If 'x' denotes the percentage of people who are polite and 'y' denotes the percentage of people who are intelligent. Find x when $y = 2\%$.

Answers

1. (*i*) $e^{2\sqrt{x}}$ (*ii*)–(*iii*) $\sec x + \tan x$ (*iv*) $\frac{1}{x}$.

2. $-\frac{1}{2}\log(3 - 2y) = x + c$.

3. (*i*) $y = 1 + ce^{-x}$ (*ii*) $y = (x - 1) + ce^{-x}$.

4. (*i*) $y + 3(x^3 + 3x^2 + 6x + 6) = ce^x$

(*ii*) $y = e^x\left(\frac{x^2}{2} + c\right)$.

5. (*i*) $y = \frac{1}{4}e^{2x} + ce^{-2x}$

(*ii*) $y = -e^{-x} + cx$

(*iii*) $y = \frac{1}{x}(x - 1)e^x + c$.

6. (*i*) $y = e^{-2x} + ce^{-3x}$

(*ii*) $y = c\, e^{-2x} + 2\, e^x$.

7. $y = e^x + cx$.

8. $y = \frac{2}{3}x - \frac{2}{9} + ce^{-3x}$.

9. $\frac{dp}{dt} = k(w - s)$, where $w = ap + b$.

10. (*i*) $y = -x \log|x| + cx$ (*ii*) $y = \frac{1}{4}x^3 + \frac{c}{x}$.

11. (*i*) $y = \frac{1}{4}x^2 + cx^{-2}$ (*ii*) $y = x^3 + cx$.

12. $y = \frac{6}{7}x^3 + \frac{c}{\sqrt{x}}$.

13. $y = \frac{1}{29}(2\sin 5x - 5\cos 5x) + ce^{-2x}$.

14. (i) $y = ce^{-x} + \frac{1}{2}(\cos x + \sin x)$

(ii) $y = \frac{1}{2}(\sin x - \cos x) + ce^{x}$

(iii) $y = \frac{3}{13}\sin 3x + \frac{2}{13}\cos 3x + ce^{-2x}$.

15. (i) $y = -\frac{1}{2}(\sin x + \cos x) + ce^{x}$

(ii) $y = \sin x + \cos x + ce^{x}$.

16. $y = \frac{3}{13}\sin 3x - \frac{2}{13}\cos 3x + ce^{2x}$.

17. $y(\sec x + \tan x) = \sec x + \tan x - x + c$.

18. (i) $y\sec^2 x = \sec x + c$

(ii) $y\sin^2 x + \cos x = c$.

19. $y(\sec x + \tan x) = \sec x + \tan x - x + c$.

20. $y = 3x^2 + cx$.

21. $y = \frac{1}{2}e^{\tan^{-1}x} + ce^{-\tan^{-1}x}$.

22. (i) $y = \cos x + ce^{-x}$

(ii) $y = \sin x + c\,e^{-x}$.

23. (i) $y = x^2 + c\cos x$ (ii) $y\sin x = -\frac{\cos 2x}{2} + c$.

24. (i) $y = \sin x + \frac{2}{x}\cos x - \frac{2}{x^2}\sin x + c$

(ii) $y = \sin x + \frac{c}{x}$.

25. $y = 1 + (c - x)(\sec x + \tan x)^{-1}$.

26. (i) $xy = \frac{x^2}{4}(2\log x - 1) + c$

(ii) $y + \log x + 1 = cx$

(iii) $xy\sin x = \sin x - x\cos x + c$.

27. (i) $y = \frac{1}{4}x^2\log x - \frac{1}{16}x^2 + cx^{-2}$

(ii) $y\log x = (\log x)^2 + c\ (x > 0)$.

28. $y\log x = -\frac{2\log x}{x} - \frac{2}{x} + c$.

29. (i) $y\sin x = \frac{1}{3}\sin^3 x + c$

(ii) $y(x^2 - 1) = \log\left|\frac{x-1}{x+1}\right| + c$

(iii) $y = (1 + x^2)x + (1 + x^2)\tan^{-1}x + c(1 + x^2)$.

30. (i) $y = (\tan^{-1}x - 1) + c\,e^{-\tan^{-1}x}$

(ii) $y(1 + x^2) = \sin x + c$

(iii) $y\sqrt{1-x^2} = \sin^{-1}x + c$.

31. (i) $xy = \frac{1}{3}y^3 + c$ (ii) $x = 2y^2 + cy$.

32. $y = (\tan^{-1}x - 1) + c\,e^{-\tan^{-1}x}$.

33. $y = e^{x} + cx$.

34. $y = -2x - 2 + 2e^{x}$.

35. (i) $xy = \frac{1}{4}x^4 - 2$ (ii) $x^2(4y - x^2) = 0$

(iii) $x^2 y = \frac{1}{4}x^4 - \frac{1}{4}$.

36. $y = \sin x$.

37. (i) $y = \cos x - 2\cos^2 x$

(ii) $y = \sin x$

(iii) $y + 2\sin^2 x = 4\sin^3 x$.

38. $y = -1 + \frac{2}{e}e^{\tan x}$.

39. $x = -\frac{y^2}{e^{y}} + \frac{y^2}{e}$.

40. $y(1 + \sin x) = -\frac{x^2}{2} + 1$.

41. (i) $y = x^2$

(ii) $y\sin x = x^2\sin x - \frac{\pi^2}{4}$

(iii) $y = 4\sin^3 x - 2\sin^2 x$.

42. $y = \frac{e^{m\tan^{-1}x}}{m+1} + (m+1)\,e^{-\tan^{-1}x}$.

43. (i) $x = y^2 - \frac{\pi^2}{4}\operatorname{cosec} y\,(y \neq 0)$.

(ii) $y\sin x = x^2\sin x - \frac{\pi^2}{4}$.

44. $x = 2y^2$, 8.

Hints to Selected Questions

9. Here $\frac{dp}{dt} = k(w - s)$, where $w = ap + b$.

27. (*i*) The given equation is $x\frac{dy}{dx} + 2y = x^2 \log x$

$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = x \log x.$

28. The given equation is $x \log x \frac{dy}{dx} + y = \frac{2}{x}\log x$

$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x^2}.$

30. (*i*) – (*ii*) Divide by $(1 + x^2)$

(*iii*) Divide by $(1 - x^2)$.

31. (*i*) $\frac{dx}{dy} + \frac{x}{y} = y$, which is linear in x.

38. Divide $\cos^3 x$.