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CIRCLES

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Welcome to the Circles Study Module tailored for IIT-JEE Mathematics aspirants. This module offers a comprehensive review of the properties, theorems, and applications of circles, essential for success in the IIT-JEE examination.

- **Comprehensive Coverage:** This module covers all essential concepts of circles, ensuring thorough preparation for the IIT-JEE Mathematics examination.
- **Clear Explanation:** Concepts are explained clearly and concisely, supported by illustrative examples and diagrams to facilitate learning.
- **Practice-oriented Approach:** Ample practice questions, problem-solving exercises, and mock tests enable students to apply their knowledge and reinforce their understanding of circles.

KEY FEATURES

- **Basic Concepts:** Explore the fundamental properties of circles, including definitions of radius, diameter, chord, secant, tangent, and arc. Understand the central angle, inscribed angle, and their relationships with arc measures.

- **Equations of Circles:** Study the standard and general forms of the equation of a circle in the coordinate plane. Learn how to derive equations of circles given various conditions, such as center and radius, diameter endpoints, and points on the circle.

- **Tangent and Secant Properties:** Delve into the properties of tangents and secants to circles, including theorems related to tangent-secant angles, secant-secant angles, and the power of a point. Understand how to apply these properties to solve problems involving circles.

- **Chord Properties and Intersecting Circles:** Investigate properties of chords, secants, and tangents intersecting circles, including theorems related to intersecting chords, the angle between chords, and cyclic quadrilaterals. Learn how to prove and apply these properties in geometric proofs.

- **Circle Geometry and Coordinate Geometry:** Explore applications of circle geometry in coordinate geometry, including finding the equation of circles, determining the position of points concerning circles, and solving problems involving tangents and secants.

- **Constructions and Loci:** Learn geometric constructions related to circles, such as constructing tangents, chords, and inscribed/circumscribed polygons. Understand how to locate points satisfying specific conditions (loci) related to circles.

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CIRCLES

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DEFINITION OF A CIRCLE

A circle is the locus of a point which moves in a plane, so that its distance from a *fixed point* in the plane is always *constant*.

The fixed point is called the *centre* of the circle and the constant distance is called its *radius*.

EQUATIONS OF A CIRCLE

1. An equation of a circle with centre (h, k) and radius r is

$$(x-h)^2+(y-k)^2=r^2$$

Illustration 1

A circle of area 9π square units has two of its diameters along the lines $x + y = 5$ and $x - y = 1$. Find the equation of the circle.

Solution: The centre of the circle is the point of intersection of the diameters $x + y = 5$ and $x - y = 1$, which is $(3, 2)$. If r is the radius of the circle, then $\pi r^2 = 9\pi \Rightarrow r = 3$ and the equation of the circle is $(x-3)^2 + (y-2)^2 = 3^2$

$$\Rightarrow \quad x^2+y^2-6x-4y+4=0$$

2. An equation of a circle with centre $(0, 0)$ and radius r is

$$x^2+y^2=r^2$$

3. An equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

Illustration 2

The line $3x + 4y - 12 = 0$ meets the coordinate axes at A and B. Find the equation of the circle drawn on AB as diameter.

Solution: $3x + 4y - 12 = 0$ meets x-axis at $A(4, 0)$ and y-axis at $B(0, 3)$. Equation of the circle on the line joining A and B as diameter is

$$(x-0)(x-4)+(y-0)(y-3)=0$$

$$\Rightarrow \quad x^2+y^2-4x-3y=0$$

4. General equation of a circle is

$$x^2+y^2+2gx+2fy+c=0$$

where g, f and c are constants.

(i) Centre of this circle is $(-g, -f)$.

(ii) Its radius is $\sqrt{g^2+f^2-c}$, $(g^2+f^2 \ge c)$.

(iii) Length of the intercept made by this circle on the x-axis is $2\sqrt{g^2-c}$ if $g^2 - c \ge 0$, and that on the y-axis is $2\sqrt{f^2-c}$ if $f^2 - c \ge 0$.

Illustration 3

Show that $x + y - 1 = 0$ is the equation of a diameter of the circle $x^2 + y^2 - 6x + 4y + c = 0$ for all values of c.

Solution: Centre of the circle is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

i.e $(3, -2)$ which lies on the line $x + y - 1 = 0$ and any line passing through the centre of the circle is a diameter of the circle. Hence $x + y - 1 = 0$ is a diameter of the circle.

5. General equation of second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

in x and y, represents a circle if and only if

(i) coefficient of x^2 equals coefficient of y^2, i.e., $a = b \ne 0$.

(ii) coefficient of xy is zero, i.e., $h = 0$.

(iii) $g^2 + f^2 - ac \ge 0$

Illustration 4

Find the centre and radius of the circle

$3x^2 + (a+1)y^2 + 6x - 9y + a + 4 = 0$

Solution: Since the given equation represents a circle, co-efficient of x^2 = co-efficient of y^2

$\Rightarrow 3 = a + 1 \Rightarrow a = 2$ and the equation of the circle becomes

$$3x^2+3y^2+6x-9y+6=0 \text{ or } x^2+y^2+2x-3y+2=0$$

whose centre is $(-1, 3/2)$ and

$$\text{radius} = \sqrt{1+(3/2)^2-2} = \frac{\sqrt{5}}{2}.$$

SOME RESULTS REGARDING CIRCLES

1. *Position of a point with respect to a circle.* Point $P(x_1, y_1)$ lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =$ or < 0.

Illustration 5

Find the values of a for which the point (a, a), $a > 0$, lies outside the circle $x^2 + y^2 - 2x + 6y - 6 = 0$

Solution: The point (a, a) lies outside the given circle if $a^2 + a^2 - 2a + 6a - 6 > 0$

$\Rightarrow 2(a^2 + 2a - 3) > 0 \quad \Rightarrow (a + 3)(a - 1) > 0$

$\Rightarrow a > 1$ as $a > 0$

2. *Parametric coordinates* of any point on the circle $(x - h)^2 + (y - k)^2 = r^2$ are given by $(h + r\cos\theta, k + r\sin\theta)$, with $0 \le \theta < 2\pi$. In particular, parametric coordinates of any point on the circle $x^2 + y^2 = r^2$ are $(r\cos\theta, r\sin\theta)$ with $0 \le \theta < 2\pi$.

3. An equation of the *tangent* to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) on the circle is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

4. An equation of the *normal* to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at point (x_1, y_1) on the circle is

$$\frac{y - y_1}{y_1 + f} = \frac{x - x_1}{x_1 + g}$$

5. Equations of the *tangent* and *normal* to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) on the circle are, respectively,

$$xx_1 + yy_1 = r^2 \quad \text{and} \quad \frac{x}{x_1} = \frac{y}{y_1}$$

Illustration 6

Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point in the first quadrant where the diameter $4x - 3y = 0$ meets the circle.

Solution: The diameter meets the given circle at the point (3, 4) in the first quadrant.

Equation of the tangent to the circle at this point is $x(3) + y(4) = 25 \Rightarrow 3x + 4y = 25$

6. The line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$ if and only if $c^2 = r^2(1 + m^2)$.

7. The lines $y = mx \pm r\sqrt{1 + m^2}$ are tangents to the circle $x^2 + y^2 = r^2$, for all finite values of m. If m is infinite, the tangents are $x \pm r = 0$.

Illustration 7

The line $y = \sqrt{3x} + 4$ touches a circle with centre at the origin. Find the radius of the circle.

Solution: Let the radius of the circle be r, so its equation is $x^2 + y^2 = r^2$. As the line $y = \sqrt{3}x + 4$ touches this circle,

$$(4)^2 = r^2(1 + (\sqrt{3})^2) \quad \Rightarrow r^2 = 4 \Rightarrow r = 2$$

so the required radius is 2.

8. An equation of the *chord* of the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, whose mid-point is (x_1, y_1), is $T = S_1$, where

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

and $\qquad S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

In particular, an equation of the chord of the circle $x^2 + y^2 = r^2$, whose mid-point is (x_1, y_1), is $xx_1 + yy_1 = x_1^2 + y_1^2$.

9. An equation of the *chord of contact* of the tangents drawn from a point (x_1, y_1) outside the circle $S = 0$, is $T = 0$. (S and T are as defined in (8) above.)

10. *Length of the tangent* drawn from a point (x_1, y_1) outside the circle $S = 0$, to the circle, is $\sqrt{S_1}$. (S and S_1 are as defined in (8) above.)

Illustration 8

Find the length of a tangent drawn from the point (3, 4) to the circle $x^2 + y^2 - 4x + 6y - 3 = 0$.

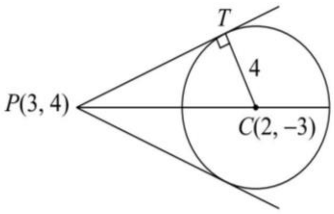

Fig. 17.1

Solution: Centre of the circle is $(2, -3)$ and radius $= \sqrt{2^2 + (-3)^2 + 3} = 4$. So if PT is the tangent from $P(3, 4)$ to the circle, then

$$(PT)^2 = (PC)^2 - (CT)^2$$
$$= (3 - 2)^2 + (4 + 3)^2 - 4^2 = 34.$$

The required length of the tangent is $\sqrt{34}$.

Note

If $S \equiv x^2 + y^2 - 4x + 6y - 3$ then length of the tangent from $P(3,4)$ to this circle is $S = \sqrt{(3)^2 + (4)^2 - 4(3) + 6(4) - 3} = \sqrt{34}$.

11. Two circles with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$, and radii r_1, r_2 respectively,

(i) *touch each other externally* if $|C_1C_2| = r_1 + r_2$. The point of contact is

$$\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

Illustration 9

Show that the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 - 8x + 12 = 0$ touch each other. Find the point of contact.

Solution: Centre of the first circle is (1, 0) and its radius is 1. Centre of the second circle is (4, 0) and its radius in 2. Since the distance between the centres = 3, which is equal to the sum of the radii. The two circles touch each other externally and the point of contact is the point (2, 0) which divides the join of (1, 0) and (4, 0) in the ratio 1 : 2.

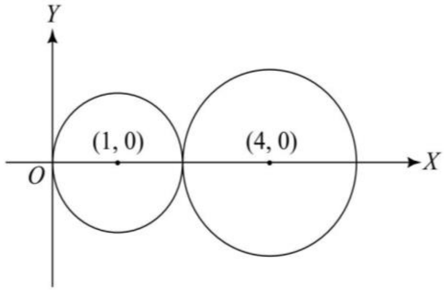

Fig. 17.2

(ii) *touch each other internally* if $|C_1 C_2| = |r_1 - r_2|$, $r_1 \neq r_2$. The point of contact is

$$\left(\frac{r_1 x_2 - r_2 x_1}{r_1 - r_2}, \frac{r_1 y_2 - r_2 y_1}{r_1 - r_2}\right)$$

12. An equation of the *family of circles* passing through the points (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

13. An equation of the family of circles which touch the line $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite value of m, is

$$(x - x_1)^2 + (y - y_1)^2 + \lambda[(y - y_1) - m(x - x_1)] = 0$$

If m is infinite, equation becomes $(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$.

14. Let QR be a chord of a circle passing through the point $P(x_1, y_1)$, and let the tangents at the extremities Q and R of this chord intersect at the point $L(h, k)$ (Fig. 17.3). Then locus of L is called as *polar* of P with respect to the circle, and P is called the *pole* of its polar.

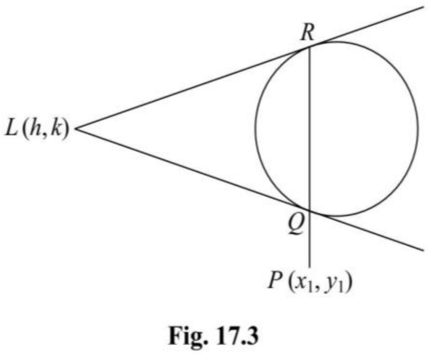

Fig. 17.3

(i) *Equation of the polar* of P (x_1, y_1) with respect to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, is $T = 0$, where T is as defined in (8) above.

(ii) If the polar of P with respect to a circle passes through Q, then the polar of Q with respect to the same circle passes through P. Two such points P and Q are called *conjugate points* of the circle.

15. If lengths of the tangents drawn from a point P to the two circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

are equal, then the locus of P is called the *radical axis* of the two circles $S_1 = 0$ and $S_2 = 0$, and its equation is $S_1 - S_2 = 0$, i.e.,

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

(i) Radical axes of two circles is perpendicular to the line joining their centres.

(ii) Radical axes of three circles, taken in pairs, pass through a fixed point called the *radical centre* of the three circles, if the centres of these circles are non-collinear.

SPECIAL FORMS OF EQUATION OF A CIRCLE

1. An equation of a circle with centre (r, r), radius $|r|$ is $(x - r)^2 + (y - r)^2 = r^2$. This touches the coordinate axes at the points $(r, 0)$ and $(0, r)$.

Illustration 10

A circle of radius 3 units touches both the axes. Find its equation.

Solution: As the circle touches both the axes, the distance of the centre from both the axes in 3 units. So the centre of the circle can be $(\pm 3, \pm 3)$ and hence there are four circles with radius 3, touching both the axes and their equations are

$$x^2 + y^2 \pm 6x \pm 6y + 9 = 0.$$

2. An equation of a circle with centre (x_1, r), radius $|r|$ is $(x - x_1)^2 + (y - r)^2 = r^2$. This touches the x-axis at $(x_1, 0)$.

3. An equation of a circle with centre (r, y_1), radius $|r|$ is $(x - r)^2 + (y - y_1)^2 = r^2$. This touches the y-axis at $(0, y_1)$.

4. An equation of a circle with centre $(a/2, b/2)$ and radius $\sqrt{(a^2 + b^2)/4}$ is

$$x^2 + y^2 - ax - by = 0$$

This circle passes through the origin (0, 0), and has intercepts a and b on the x and y axes, respectively.

SYSTEMS OF CIRCLES

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c$, $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c'$ and $L \equiv ax + by + k'$.

1. If two circles $S = 0$ and $S' = 0$ intersect at real and distinct points, then $S + \lambda S' = 0$ $(\lambda \neq -1)$ represents a *family of circles* passing through these points (λ being a parameter),

and $S - S' = 0$ (for $\lambda = -1$) represents the *common chord* of the circles.

2. If two circles $S = 0$ and $S' = 0$ touch each other, then $S - S' = 0$ represents equation of the *common tangent* to the two circles at their point of contact.

3. If two circles $S = 0$ and $S' = 0$ intersect each other *orthogonally* (the tangents at the point of intersection of the two circles are at right angles), then

$$2gg' + 2ff' = c + c'.$$

4. If the circle $S = 0$ intersects the line $L = 0$ at two real and distinct points, then $S + \lambda L = 0$ represents a family of circles passing through these points.

Illustration 11

Find the equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter.

Solution: Equation of any circle passing through the points of intersection of the chord and the circle is

$$x^2 + y^2 - 16 + \lambda(3x + y + 5) = 0$$

The chord $3x + y + 5 = 0$ is a diameter of this circle if the centre $\left(\frac{-3\lambda}{2}, \frac{-\lambda}{2}\right)$ of the circle lies on the chord.

$\Rightarrow \quad 3\left(\frac{-3\lambda}{2}\right) - \frac{\lambda}{2} + 5 = 0$

$\Rightarrow \quad \lambda = 1$ and the required equation of the circle is $x^2 + y^2 + 3x + y - 11 = 0$.

5. If $L = 0$ is a tangent to the circle $S = 0$ at P, then $S + \lambda L = 0$ represents a family of circles touching $S = 0$ at P, and having $L = 0$ as the common tangent at P.

6. *Coaxial Circles* A system of circle is said to be coaxial if every pair of circles of the system have the same radical axis.

The simplest form of the equation of a coaxial system of circles is $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is constant, the common radical axis of the system being y-axis and the line of centres being x-axis.

The Limiting points of the coaxial system of circles are the members of the system which are of zero radius. Thus the limiting points of the coaxial system of circles

$$x^2 + y^2 + 2gx + c = 0 \text{ are } (\pm\sqrt{c}, 0) \text{ if } c \geq 0.$$

The equation $S + \lambda S' = 0$ ($\lambda \neq -1$) represents a family of coaxial circles, two of whose members are given to be $S = 0$ and $S' = 0$.

Conjugate systems (or orthogonal systems) of circles Two system of circles such that every circle of one system cuts every circle of the other system orthogonally are said to be conjugate system of circles. For instance,

$$x^2 + y^2 + 2gx + c = 0 \quad \text{and} \quad x^2 + y^2 + 2fy - c = 0,$$

where g and f are variables and c is constant, represent two systems of coaxial circles which are conjugate.

COMMON TANGENTS TO TWO CIRCLES

If $(x - g_1)^2 + (y - f_1)^2 = a_1^2$ and $(x - g_2)^2 + (y - f_2)^2 = a_2^2$ are two circles with centres C_1 (g_1, f_1) and C_2 (g_2, f_2) and radii a_1 and a_2 respectively, then we have the following results regarding their common tangents.

1. When $C_1 C_2 > a_1 + a_2$ i.e., distance between the centres is greater than the sum of their radii, the two circles do not intersect with each other and *four* common tangents can be drawn to two circles. Two of them are *direct common tangents* and the other two are *transverse common tangents*. The points T_1, T_2 of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles and divide it externally and internally respectively in the ratio of their radii.

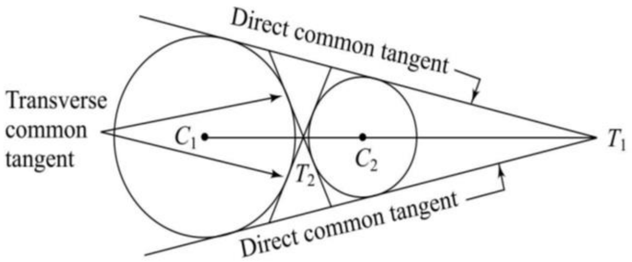

Fig. 17.4

$$\frac{C_1T_1}{T_1C_2} = \frac{a_1}{a_2} \text{ (externally)} \quad \frac{C_1T_2}{T_2C_2} = \frac{a_1}{a_2} \text{ internally}$$

2. When $C_1 C_2 = a_1 + a_2$ i.e., the distance between the centres is equal to the sum of the radii, the two circles touch each other externally, two direct common tangents are real and distinct and the transverse common tangents coincide.

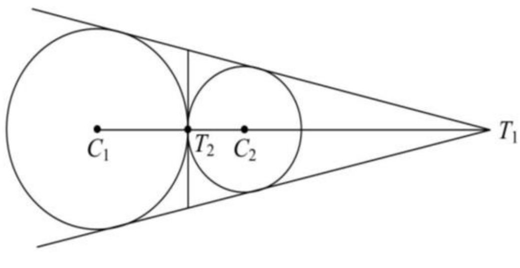

Fig. 17.5

3. When $C_1 C_2 < a_1 + a_2$ i.e., the distance between the centres is less than the sum of the radii, the circles intersect at two real and distinct points, the two direct common tangents are real and distinct while the transverse common tangents are imaginary.

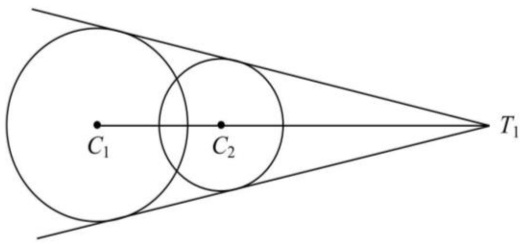

Fig. 17.6

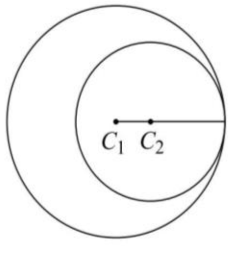

Fig. 17.7

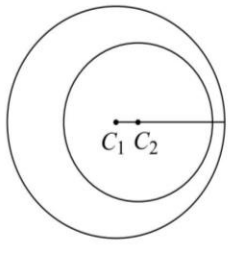

Fig. 17.8

4. When $C_1 C_2 = |a_1 - a_2|$ $(a_1 \neq a_2)$ i.e., the distance between the centres is equal to the difference of their radii, the circles touch each other internally, two direct common tangents are real and coincident while the transverse common tangents are imaginary. (Fig 16.7)
5. When $C_1 C_2 < |a_1 - a_2|$ $a_1 \neq a_2$ i.e., the distance between the centres is less than the difference of the radii, one circle with smaller radius lies inside the other and the four common tangents are all imaginary. (Fig 16.8)

Illustration 12

Find the number of common tangents to the circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$.

Solution: Centre of the circle $x^2 + y^2 = 16$ is (0, 0) and its radius is 4. Centre of the circle $x^2 + y^2 - 2y = 0$ is (0, 1) and its radius is 1.

Distance between the centre = 1 which is less than the difference between the radii.

Second circle lies inside the first circle so there is no real common tangent to the given circles.

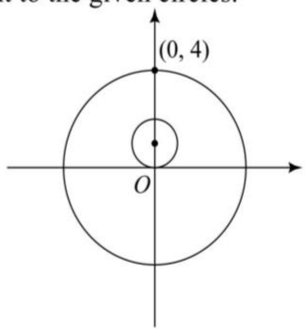

Fig. 17.9

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: If each of the lines $5x + 8y = 13$ and $4x - y = 3$ contains a diameter of the circle $x^2 + y^2 - 2(a^2 - 7a + 11)x - 2(a^2 - 6a + 6)y + b^3 + 1 = 0$, then

(a) $a = 5$ and $b \notin (-1, 1)$
(b) $a = 1$ and $b \notin (-1, 1)$
(c) $a = 2$ and $b \in (-\infty, 1)$
(d) $a = 5$ and $b \in (-\infty, 1)$

Ans. (d)

Solution: The point of intersection (1, 1) of the given lines is the centre of the circle.

$\Rightarrow \quad a^2 - 7a + 11 = 1, a^2 - 6a + 6 = 1$

$\Rightarrow \quad a = 5$ and the equation of the circle is $x^2 + y^2 - 2x - 2y + b^3 + 1 = 0$

so radius of the circle = $\sqrt{1 + 1 - (b^3 + 1)} = \sqrt{1 - b^3}$

For radius to be real $b^3 < 1 \Rightarrow b \in (-\infty, 1)$

Example 2: If a circle touches the axis of x at (5, 0) and passes through the point (4, – 1), it also passes through the point

(a) (–1, 6) (b) (6, –1)
(c) (1, 6) (d) (6, 1)

Ans. (b)

Solution: Centre of the circle is on the line $x = 5$ at a distance equal to the radius of the circle from the axis of x.

Let the coordinates of the centre be $(5, k)$, so the equation of the circle is

$$(x - 5)^2 + (y - k)^2 = k^2$$

As it passes through the point (4, – 1)

$$(4 - 5)^2 + (-1 - k)^2 = k^2 \quad \Rightarrow k = -1$$

and the equation of the circle is

$$(x-5)^2+(y+1)^2=1$$

which passes through the point (6, – 1)

Note

(6, – 1) is the other end of the diameters through (4, –1) of the circle.

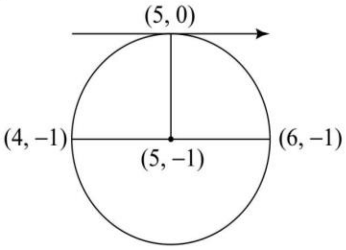

Fig. 17.10

Example 3: For all values of a, equation of a diameter of the circle $x^2+y^2-2ax+2ay+a^2=0$ is

(a) $x+y=a$ (b) $x-y=a$
(c) $x+y=2a$ (d) $x-y=2a$.

Ans. (d)

Solution: Centre of the circle is $(a, -a)$ and the line $x-y=2a$ passes through it. So it is a diameter of the given circle.

Example 4: $S: x^2+y^2-6x+4y-3=0$ is a circle and $L: 4x+3y+19=0$ is a straight line.

(a) L is a chord of S. (b) L is a diameter of S
(c) L is a tangent to S (d) none of these

Ans. (d)

Solution: Centre of S is $(3, -2)$ and its radius is $\sqrt{3^2+(-2)^2+3}=4$

Length of the perpendicular of (3, – 2) from the line L is

$\left|\frac{4(3)+3(-2)+19}{\sqrt{(4)^2+(3)^2}}\right|=5$ which is greater than the radius of S. So the line L lies outside the circle S.

Example 5: Circles $x^2+y^2+2x-8y-8=0$ and $x^2+y^2+2x-6y-6=0$

(a) touch each other internally at (–1, – 1)
(b) touch each other externally at (–1, –1)
(c) intersect each other at (–1, –1)
(d) none of these.

Ans. (a)

Solution: Centre of the first circle is (–1, 4) and its radius is $\sqrt{1+16+8}=5$. Centre of the second circle is (– 1, 3) and its radius is $\sqrt{1+9+6}=4$.

Distance between the centres = 1 = difference between the radii, so the two circles touch each other internally at a common point (– 1, –1)

Example 6: Vertices of an isosceles triangle of area a^2 are $(-a, 0)$ and $(a, 0)$. Equation of the circumcircle of the triangle is

(a) $x^2+y^2+2ax-2ay+a^2=0$
(b) $x^2+y^2-2ax+2ay+a^2=0$
(c) $x^2+y^2=a^2$
(d) none of these

Ans. (c)

Solution: Since the triangle is isosceles, third vertex lies on the line $x=0$, perpendicular to the base and passing through the mid-point (0, 0) of the base. As the area is a^2, distance of the vertex from the base is a as the length of the base is $2a$. So vertex of the triangle is $(0, \pm a)$ and let the equation of the circle passing through the vertices of the triangle $x^2+y^2+2gx+2fy+c=0$, then $a^2+2ga+c=0$, $a^2-2ga+c=0$

and $a^2\pm 2fa+c=0$.

$\Rightarrow$ $c=-a^2$, $g=f=0$ and the equation of the required circle is $x^2+y^2=a^2$.

Example 7: If the circle $x^2+y^2-6x-8y+(25-a^2)=0$ touches the axis of y, then a equals.

(a) 0 (b) ±4
(c) ±2 (d) ±3

Ans. (d)

Solution: If the circle touches the axis of y, distance of its centre (3, 4) from the axis of y is equal to the radius $\sqrt{(3)^2+(4)^2-(25-a^2)}$ of the circle.

$\Rightarrow$ $3=\sqrt{a^2}$ $\Rightarrow$ $a=\pm 3$

Example 8: A circle is described on the line joining the points (2, – 3) and (– 4, 7) as a diameter. The circle

(a) passes through the origin
(b) touches the axis of x.
(c) touches the axis of y
(d) origin lies inside the circle

Ans. (d)

Solution: Equation of the circle is

$(x-2)(x+4)+(y+3)(y-7)=0$

$\Rightarrow$ $x^2+y^2+2x-4y-29=0$.

Centre is (1, – 2), radius = $\sqrt{34}$

Distance of the centre from the origin is $\sqrt{5}<\sqrt{34}$.

So the origin lies inside the circle.

Example 9: A circle with centre at the centroid of the triangle with vertices (2, 3), (6, 7) and (7, 5) passes through the origin, radius of the circle in units is

(a) 5 (b) $5\sqrt{2}$
(c) 4 (d) none of these

Ans. (b)

◎ **Solution:** Centroid of the triangle is $\left(\frac{2+6+7}{3}, \frac{3+7+5}{3}\right)$ = (5, 5). Let the equation of the circle be $(x-5)^2 + (y-5)^2 = r^2$, r being the radius.

$\Rightarrow$ $x^2 + y^2 - 10x - 10y + 50 - r^2 = 0$ which passes through the origin if $r^2 = 50 \Rightarrow r = 5\sqrt{2}$.

◉ **Example 10:** $S: x^2 + y^2 - 8x + 10y = 0$ and $L: x - y - 9 = 0$ are the equations of a circle and a line.

(a) L is a normal to the circle S.

(b) S is the only circle having radius $\sqrt{41}$ and a diameter along L.

(c) L is a tangent to the circle S.

(d) L does not intersect the circle S.

Ans. (a)

◎ **Solution:** L passes through the centre (4, –5) of the circle S and hence is a diameter and every diameter is a normal to the circle.

◉ **Example 11:** A circle has its centre on the y-axis and passes through the origin, touches another circle with centre (2, 2) and radius 2, then the radius of the circle is

(a) 1 (b) 1/2

(c) 1/3 (d) 1/4.

Ans. (b)

◎ **Solution:** Let the centre of the circle be (0, k). As it passes through the origin, its radius is k.
Since it touches the second circle

$$\sqrt{(0-2)^2 + (k-2)^2} = k + 2$$

$$\Rightarrow (k+2)^2 - (k-2)^2 = 4 \Rightarrow k = \frac{1}{2}$$

◉ **Example 12:** If the point (3, 4) lies inside and the point (– 3, – 4) lies outside the circle $x^2 + y^2 - 7x + 5y - p = 0$, then the set of all possible values of p is

(a) (24, 25) (b) (25, 26)

(c) (24, 26) (d) (0, 24)

Ans. (c)

◎ **Solution:** (3, 4) lies inside the circle

$\Rightarrow (3)^2 + (4)^2 - 7(3) + 5(4) - p < 0 \Rightarrow 24 < p$

(–3, – 4) lies outside the circle

$\Rightarrow (-3)^2 + (-4)^2 - 7(-3) + 5(-4) - p > 0.$

$\Rightarrow p < 26.$

So $p \in (24, 26)$

◉ **Example 13:** The mid point of chord by the circle $x^2 + y^2 = 16$ on the line $x + y + 1 = 0$ is

(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

(c) $\left(\frac{1}{2}, \frac{-3}{2}\right)$ (d) $\left(\frac{3}{4}, -\frac{7}{4}\right)$

Ans. (b)

◎ **Solution:** The line meets the circle at points whose x-coordinates are the roots of the equation

$$x^2 + (x+1)^2 = 16 \Rightarrow 2x^2 + 2x - 15 = 0$$

$$\Rightarrow x_1 + x_2 = -1$$

So if (h, k) is the required point.

$$h = \frac{x_1 + x_2}{2} = -\frac{1}{2} \text{ and } h + k + 1 = 0 \Rightarrow k = -\frac{1}{2}$$

and the required point is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

◉ **Example 14:** If a chord of a circle $x^2 + y^2 = 25$ with one extremity at (4, 3) subtends a right angle at the centre of this circle, then the coordinates of the other extremity of this chord can be.

(a) (– 3, –4) (b) (4, –3)

(c) (3, 4) (d) (–3, 4)

Ans. (d)

◎ **Solution:** Let P be the point (4, 3) and Q (h, k) be the other extremity of the chord through P. Centre of the circle is the origin 0. Then OP is perpendicular to OQ.

$$\Rightarrow \frac{k}{h} \times \frac{3}{4} = -1 \Rightarrow \frac{k}{h} = -\frac{4}{3}$$

which is satisfied by (h, k) = (–3, 4)

◉ **Example 15:** Equation of a common chord of the circles $x^2 + y^2 + 6x - 10y + 9 = 0$ and $x^2 + y^2 - 10x + 6y + 25 = 0$ is

(a) $x + y + 4 = 0$ (b) $x - y + 4 = 0$

(c) $x + y + 1 = 0$ (d) $x - y - 1 = 0$

Ans. (d)

◎ **Solution:** Equation of the common chord is

$x^2 + y^2 + 6x - 10y + 9 - (x^2 + y^2 - 10x + 6y + 25) = 0$

$\Rightarrow x - y - 1 = 0$

◉ **Example 16:** $x + ay = a^2 + 1$ is a tangent to the circle $x^2 + y^2 = 10$ for

(a) any values of a (b) only one value of a

(c) two values of a (d) no value of a.

Ans. (c)

◎ **Solution:** If $x + ay = a^2 + 1$ touches the circle, length of the perpendicular of the centre (0, 0) of the circle from the line is $\sqrt{10}$, the radius of the circle.

$$\Rightarrow \left|\frac{0 + 0.9 - (a^2+1)}{\sqrt{1+a^2}}\right| = \sqrt{10}$$

$$\Rightarrow \sqrt{1+a^2} = \sqrt{10} \Rightarrow a^2 = 9 \Rightarrow a = \pm 3$$

◉ **Example 17:** If the circles $x^2 + y^2 + 5x - 6y - 1 = 0$ and $x^2 + y^2 + ax - y + 1 = 0$ intersect orthogonally (the tangents at the point of intersection of the circles are at right angles), the value of a is

(a) 6/5 (b) 5/6
(c) – 6/5 (d) – 5/6

Ans. (c)

◎ **Solution:** Let $C_1\left(-\frac{5}{2},3\right)$ be the centre and $r_1 = \sqrt{\left(-\frac{5}{2}\right)^2+(3)^2+1}$ be the radius of the first circle and $C_2\left(-\frac{a}{2},\frac{1}{2}\right)$ be the centre and $r_2 = \sqrt{\left(-\frac{a}{2}\right)^2+\left(\frac{1}{2}\right)^2-1}$ be the radius of the second circle. If P is a point of intersection of the two circles then $(C_1P)^2+(C_2P)^2 = (C_1C_2)^2$

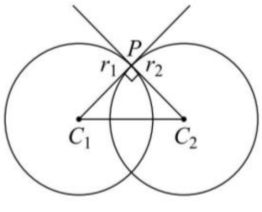

Fig. 17.11

$$\Rightarrow \left(-\frac{5}{2}\right)^2+(3)^2+1+\left(-\frac{a}{2}\right)^2+\left(\frac{1}{2}\right)^2-1=\left(\frac{-5}{2}+\frac{a}{2}\right)^2+\left(3-\frac{1}{2}\right)^2$$

$$\Rightarrow \quad a=-\frac{6}{5}$$

Note

We could also use $2gg' + 2ff' = c + c'$

◉ **Example 18:** Length of a tangent drawn from the origin to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$ in units is

(a) 4 (b) 6
(c) 8 (d) $2\sqrt{2}$

Ans. (d)

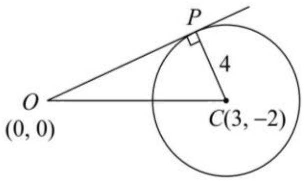

Fig. 17.12

◎ **Solution:** If P is the point of contact of the tangent from O and C is the centre of the circle, then

$$(OP)^2 = (OC)^2 - (CP)^2$$
$$= 3^2 + (-2)^2 - (3^2 + (-2)^2 - 8) = 8$$

◉ **Example 19:** A circle passing through the intersection of the circles $x^2 + y^2 + 5x + 4 = 0$ and $x^2 + y^2 + 5y - 4 = 0$ also passes through the origin. The centre of the circle is

(a) $\left(\frac{5}{2},\frac{5}{2}\right)$ (b) $\left(\frac{5}{4},\frac{5}{4}\right)$
(c) $\left(-\frac{5}{4},-\frac{5}{4}\right)$ (d) $\left(-\frac{5}{2},-\frac{5}{2}\right)$

Ans. (c)

◎ **Solution:** Let the equation of the circle be
$x^2 + y^2 + 5x + 4 + \lambda(x^2 + y^2 + 5y - 4) = 0$
It passes through the origin if $4 - 4\lambda = 0$
$\Rightarrow \quad \lambda = 1$ and the equation of the circle is
$2x^2 + 2y^2 + 5x + 5y = 0$
Centre of the circle is $\left(-\frac{5}{4},-\frac{5}{4}\right)$

◉ **Example 20:** The locus of the point from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = 36$ is

(a) $x^2 + y^2 = 42$ (b) $x^2 + y^2 = 48$
(c) $x^2 + y^2 = 60$ (d) $x^2 + y^2 = 72$.

Ans. (d)

◎ **Solution:** $y = mx \pm 6\sqrt{1+m^2}$ is the equation of any tangent to the circle. If it passes through (h, k), then
$k = mh \pm 6\sqrt{1+m^2}$
$\Rightarrow \quad (k - mh)^2 = 36(1 + m^2)$
$\Rightarrow \quad (36 - h^2)m^2 + 2mhk + (36 - k^2) = 0$
which gives two values of m say m_1 and m_2, slopes of two tangents passing thought (h, k). These tangents are perpendicular

If $\quad m_1 m_2 = -1 \quad \Rightarrow \quad \frac{36-k^2}{36-h^2} = -1$

$\Rightarrow \quad h^2 + k^2 = 72 \quad \Rightarrow$ Locus of (h, k) is $x^2 + y^2 = 72$.

LEVEL 1

Straight Objective Type Questions

◉ **Example 21:** Four distinct points (2, 3), (0, 2), (4, 5) and (0, t) are concyclic if the value of t is

(a) – 2 (b) 2
(c) 17 (d) – 17

Ans. (c)

◎ **Solution:** An equation of a circle through (2, 3) and (4, 5) is

$$(x-2)(x-4)+(y-3)(y-5)+\lambda\begin{vmatrix} x & y & 1\\ 2 & 3 & 1\\ 4 & 5 & 1\end{vmatrix}=0$$

It will pass through (0, 2) if

$$(-2)(-4) + (-1)(-3) + \lambda \begin{vmatrix} 0 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 11 + 2\lambda = 0$

$\Rightarrow \quad \lambda = -11/2$ and it will pass through (0, t) if

$$(-2)(-4) + (t-3)(t-5) - \frac{11}{2} \begin{vmatrix} 0 & t & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad t^2 - 19t + 34 = 0$

$\Rightarrow \quad t = 2, 17$

Example 22: The locus of the middle points of the chords of the circle $x^2 + y^2 = 4a^2$ which subtend a right angle at the centre of the circle is

(a) $x + y = 2a$ (b) $x^2 + y^2 = a^2$
(c) $x^2 + y^2 = 2a^2$ (d) $x^2 + y^2 = x + y$

Ans. (c)

Solution: Let $M(h, k)$ be the middle point of the chord AB of the given circle $x^2 + y^2 = 4a^2$, with centre at O (0, 0) and radius equal to $2a$.
Then OM is perpendicular to AB
Since AOB is a right angled triangle

$$4(AM)^2 = (AB)^2 = (OA)^2 + (OB)^2$$
$$= (2a)^2 + (2a)^2 = 8a^2$$

$\Rightarrow \quad AM = \sqrt{2}\, a$

Also $\quad (OA)^2 = (OM)^2 + (AM)^2$

$\Rightarrow \quad (2a)^2 = h^2 + k^2 + 2a^2$

$\Rightarrow \quad h^2 + k^2 = 2a^2$

$\Rightarrow \quad$ Locus of (h, k) is $x^2 + y^2 = 2a^2$.

Example 23: Two tangents are drawn from the origin to a circle with centre at (2, – 1). If the equation of one of the tangents is $3x + y = 0$, the equation of the other tangent is

(a) $3x - y = 0$ (b) $x + 3y = 0$
(c) $x - 3y = 0$ (d) $x + 2y = 0$

Ans. (c)

Solution: Let the equation of the other tangent from the origin be $y = mx$, then length of the perpendiculars from the centre (2, –1) on the two tangents is same.

$$\Rightarrow \quad \left|\frac{2m+1}{\sqrt{1+m^2}}\right| = \left|\frac{6-1}{\sqrt{9+1}}\right| = \frac{5}{\sqrt{10}}$$

$\Rightarrow \quad 10(2m + 1)^2 = 25(1 + m^2)$

$\Rightarrow \quad 3m^2 + 8m - 3 = 0$

$\Rightarrow \quad (3m - 1)(m + 3) = 0$

$\Rightarrow \quad m = -3 \quad$ or $\quad 1/3$.

$m = -3$ represents the given tangent hence the slope of the required tangent is 1/3 and its equation is $y = (1/3)\, x$

$\Rightarrow \quad x - 3y = 0.$

Example 24: A line meets the coordinate axes in A and B. A circle is circumscribed about the triangle OAB. If the distances from A and B of the tangent to the circle at the origin O be m and n, then the diameter of the circle is

(a) $m(m + n)$ (b) $m + n$
(c) $n(m + n)$ (d) $m^2 + n^2$

Ans. (b)

Solution: Let the coordinates of A be $(a, 0)$ and of B be $(0, b)$, then AOB being a right angled triangle AB is a diameter of the circle, so equation of the circle is $(x - a)(x - 0) + (y - b)(y - 0) = 0$

$$\Rightarrow x^2 + y^2 - ax - by = 0$$

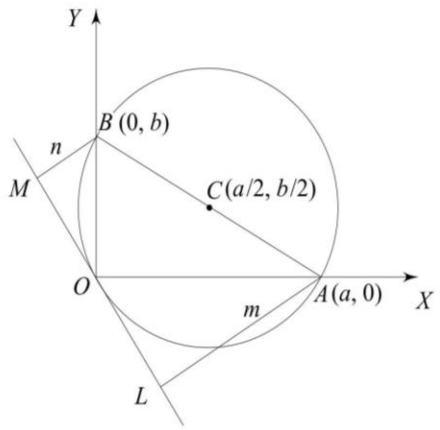

Fig. 17.13

Equation of the tangent at the origin is $ax + by = 0$ (i)
Let AL and BM be the perpendicular from A and B on (i)

then $\quad AL = \left|\frac{a^2}{\sqrt{a^2+b^2}}\right| = m$ and $BM = \left|\frac{b^2}{\sqrt{a^2+b^2}}\right| = n$

$\Rightarrow \quad m + n = \sqrt{a^2 + b^2}$ = diameter of the circle.

Example 25: A variable chord of the circle $x^2 + y^2 - 2ax = 0$ is drawn through the origin. Locus of the centre of the circle drawn on this chord as diameter is

(a) $x^2 + y^2 + ax = 0$ (b) $x^2 + y^2 + ay = 0$
(c) $x^2 + y^2 - ax = 0$ (d) $x^2 + y^2 - ay = 0$

Ans. (c)

Solution: Let (h, k) be the centre of the required circle then (h, k) is the middle point of the chord say OA of the given circle $x^2 + y^2 - 2ax = 0$, O being the origin.

$\Rightarrow \quad A\,(2h, 2k)$ lies on the circle

$\Rightarrow \quad h^2 + k^2 - ah = 0$

$\Rightarrow \quad$ Locus of (h, k) is $x^2 + y^2 - ax = 0$.

Example 26: If a circle passes through the point (3, 4) and cuts the circle $x^2 + y^2 = a^2$ orthogonally, the equation of the locus of its centre is

(a) $3x + 4y - a^2 = 0$
(b) $6x + 8y = a^2 + 25$
(c) $6x + 8y + a^2 + 25 = 0$
(d) $3x + 4y = a^2 + 25$

Ans. (b)

Solution: Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$

Since it passes through (3, 4)

$$9 + 16 + 6g + 8f + c = 0$$

$$\Rightarrow \quad 6g + 8f + c = -25 \quad \text{(ii)}$$

As (i) cuts the circle $x^2 + y^2 = a^2$ orthogonally

$$2g \times 0 + 2f \times 0 = c - a^2$$

$$\Rightarrow \quad c = a^2.$$

So from (ii) we get

$$6g + 8f + a^2 + 25 = 0$$

Hence locus of the centre $(-g, -f)$ is

$$6x + 8y - (a^2 + 25) = 0.$$

Example 27: An equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and cutting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally is

(a) $x^2 + y^2 - 2x - 6y = 0$
(b) $x^2 + y^2 - 6x - 2y = 0$
(c) $x^2 + y^2 - 4x - 4y = 0$
(d) $x^2 + y^2 - 8x = 0$

Ans. (c)

Solution: Let the centre of the circle lying on the line $x + y = 4$ be $(g, 4 - g)$. Since the required circle passes through the origin, equation of the circle is

$$x^2 + y^2 - 2gx - 2(4 - g)y = 0$$

Since it cuts the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally,

$$-2g(-2) - 2(4 - g)(1) = 4$$

$$\Rightarrow \quad 6g = 12 \quad \Rightarrow \quad g = 2$$

Hence an equation of the required circle is

$$x^2 + y^2 - 4x - 4y = 0.$$

Example 28: If O is the origin and OP, OQ are the tangents from the origin to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$, the circumcentre of the triangle OPQ is

(a) $(3, -2)$ (b) $(3/2, -1)$
(c) $(3/4, -1/2)$ (d) $(-3/2, 1)$

Ans. (b)

Solution: We note that PQ is the chord of contact of the tangents from the origin to the circle

$$x^2 + y^2 - 6x + 4y + 8 = 0 \quad \text{(i)}$$

Equation of PQ is $3x - 2y - 8 = 0$ (ii)

Equation of a circle passing through the intersection of (i) and (ii) is

$$x^2 + y^2 - 6x + 4y + 8 + \lambda(3x - 2y - 8) = 0 \quad \text{(iii)}$$

If this represents the circumcircle of the triangle OAB, it passess through O (0, 0), so $\lambda = 1$ and the equation (iii) becomes $x^2 + y^2 - 3x + 2y = 0$.

So that the required coordinates of the centre are $(3/2, -1)$

Example 29: The circle passing through three distinct points $(1, t)$, $(t, 1)$ and (t, t) passes through the point

(a) (1, 1) (b) (–1, –1)
(c) (–1, 1) (d) (1, –1)

for all values of t

Ans. (a)

Solution: Equation of a circle passing through $(1, t)$ and $(t, 1)$ is

$$(x - 1)(x - t) + (y - t)(y - 1) + \lambda \begin{vmatrix} x & y & 1 \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} = 0$$

It passes through (t, t) if $0 + 0 + \lambda \begin{vmatrix} t & t & 1 \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} = 0$

$$\Rightarrow \quad \lambda = 0$$

$\Rightarrow$ The circle through the given points is $(x - 1)(x - t) + (y - t)(y - 1) = 0$, which clearly passes through (1,1)

Example 30: If OA and OB are tangents from the origin O, to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, $c > 0$ and C is the centre of the circle, then area of the quadrilateral $OACB$ is

(a) $\frac{1}{2}\sqrt{c(g^2 + f^2 - c)}$ (b) $\sqrt{c(g^2 + f^2 - c)}$

(c) $c\sqrt{g^2 + f^2 - c}$ (d) $\sqrt{\frac{g^2 + f^2 - c}{c}}$

Ans. (b)

Solution: Since $OA = OB = \sqrt{c}$ (length of the tangent from the origin to the circle) and $CA = CB = \sqrt{g^2 + f^2 - c}$

Area of the quadrilateal $OACB$

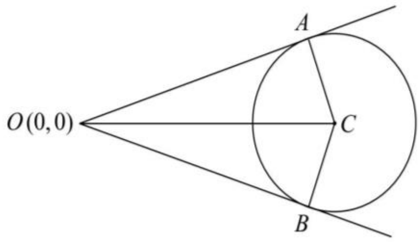

Fig. 17.14

$$= 2 \text{ Area of the triangle } OAC$$

$$= 2 \times (1/2)\, OA \times CA$$

$$= \sqrt{c}\sqrt{g^2 + f^2 - c}$$

Example 31: If the line $y = x + 3$ meets the circle $x^2 + y^2 = a^2$ at A and B, then equation of the circle on AB as diameter is

(a) $x^2 + y^2 + 3x - 3y - a^2 + 9 = 0$
(b) $x^2 + y^2 - 3x + 3y - a^2 - 9 = 0$
(c) $x^2 + y^2 + 3x + 3y + a^2 - 9 = 0$
(d) $x^2 + y^2 - 3x + 3y - a^2 + 9 = 0$

Ans. (a)

◎ **Solution:** Equation of any circle passing through the intersection of the given circle and the given line is
$x^2 + y^2 - a^2 + \lambda(y - x - 3) = 0$ (i)

If AB is a diameter of the circle, the centre $\left(\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$ of (i) lies on the given line $y - x - 3 = 0$

$\Rightarrow \quad -\frac{\lambda}{2} - \frac{\lambda}{2} - 3 = 0 \quad \Rightarrow \quad \lambda = -3.$

So that equation of the required circle is

$$x^2 + y^2 + 3x - 3y - a^2 + 9 = 0.$$

◉ **Example 32:** The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$, and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. The radius of the circle with AB as diameter is

(a) $\sqrt{a^2 + b^2 + p^2 + q^2}$

(b) $\sqrt{a^2 + p^2}$

(c) $\sqrt{b^2 + q^2}$

(d) none of these

Ans. (a)

◎ **Solution:** Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2), respectively. From the given conditions, we then have $x_1 + x_2 = -2a$, $x_1x_2 = -b^2$, $y_1 + y_2 = -2p$ and $y_1y_2 = -q^2$. Therefore the required radius is

$$(1/2)\,AB = (1/2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= (1/2)\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 - 4(x_1x_2 + y_1y_2)}$$

$$= (1/2)\sqrt{4a^2 + 4p^2 + 4(b^2 + q^2)}$$

$$= \sqrt{a^2 + b^2 + p^2 + q^2}.$$

◉ **Example 33:** If two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g_1x + 2f_1y = 0$ touch each other, then

(a) $f_1g = fg_1$ (b) $ff_1 = gg_1$

(c) $f^2 + g^2 = f_1^2 + g_1^2$ (d) none of these

Ans. (a)

◎ **Solution:** Both the circles clearly pass through the origin (0, 0). They will therefore touch each other if they have a common tangent at the origin. Now the tangent to the first circle at (0, 0) is $gx + fy = 0$, and that of the second circle is $g_1x + f_1y = 0$. If these two equations are to represent the same line, we must have $g/g_1 = f/f_1$, i.e., $f_1g = fg_1$.

◉ **Example 34:** If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes at concyclic points, then

(a) $a_1a_2 + b_1b_2 = 0$ (b) $a_1a_2 - b_1b_2 = 0$

(c) $a_1b_1 + a_2b_2 = 0$ (d) $a_1b_1 - a_2b_2 = 0$

Ans. (b)

◎ **Solution:** The given lines meet x-axis at $(-c_1/a_1, 0)$ and $(-c_2/a_2, 0)$, y-axis at $(0, -c_1/b_1)$ and $(0, -c_2/b_2)$. If an equation of the circle passing through these points is $x^2 + y^2 + 2gx + 2fy + c = 0$, then $-c_1/a_1, -c_2/a_2$ are the roots of $x^2 + 2gx + c = 0 \Rightarrow c = c_1c_2/a_1a_2$. Similarly $c = c_1c_2/b_1b_2$ $\Rightarrow a_1a_2 - b_1b_2 = 0$.

◉ **Example 35:** The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + \alpha = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + \beta = 0$ is

(a) $\sqrt{\beta - \alpha}$ (b) $\sqrt{\alpha - \beta}$

(c) $\sqrt{\alpha\beta}$ (d) $\sqrt{\alpha/\beta}$

Ans. (a)

◎ **Solution:** Let (h, k) be any point on the first circle, then

$$h^2 + k^2 + 2gh + 2fk + \alpha = 0 \quad (1)$$

length of the tangent from (h, k) to the second circle is

$$\sqrt{h^2 + k^2 + 2gh + 2fk + \beta} = \sqrt{\beta - \alpha}.$$

◉ **Example 36:** The coordinates of the point on the circle $x^2 + y^2 - 2x - 4y - 11 = 0$ farthest from the origin are

(a) $\left(2 + 8/\sqrt{5}, 1 + 4/\sqrt{5}\right)$

(b) $\left(1 + 4/\sqrt{5}, 2 + 8/\sqrt{5}\right)$

(c) $\left(1 + 8/\sqrt{5}, 2 + 4/\sqrt{5}\right)$

(d) none of these

Ans. (b)

◎ **Solution:** The required point lies on the normal to the circle through the origin, i.e on the line $2x = y$ which gives

$$x^2 + 4x^2 - 2x - 8x - 11 = 0$$

$\Rightarrow \quad 5x^2 - 10x - 11 = 0$

$\Rightarrow \quad x = 1 \pm 4/\sqrt{5}$ and $y = 2(1 \pm 4/\sqrt{5})$

and the coordinates of the required point farthest from the origin are $(1 + 4/\sqrt{5}, 2 + 8/\sqrt{5})$.

◉ **Example 37:** C_1 is a circle with centre at the origin and radius equal to r and C_2 is a circle with centre at $(3r, 0)$ and radius equal to $2r$. The number of common tangents that can be drawn to the two circles are

(a) 1 (b) 2

(c) 3 (d) 4

Ans. (c)

◎ **Solution:** The given circles C_1 and C_2 touch each other externally as the distance between the centres of the two circles is equal to the sum of their radii, hence the number of common tangents to these circles is 3.

Example 38: An equilateral triangle is inscribed in the circle $x^2 + y^2 = a^2$ with the vertex at $(a, 0)$. The equation of the side opposite to this vertex is

(a) $2x - a = 0$ (b) $x + a = 0$
(c) $2x + a = 0$ (d) $3x - 2a = 0$

Ans. (c)

Solution: $A\ (a, 0)$ be the vertex of the equilateral triangles ABC inscribed in the circle $x^2 + y^2 = a^2$

Let M be the middle point of the side BC, then MOA is perpendicular to BC and O being the centroid of the triangle $OA = 2\,(OM)$

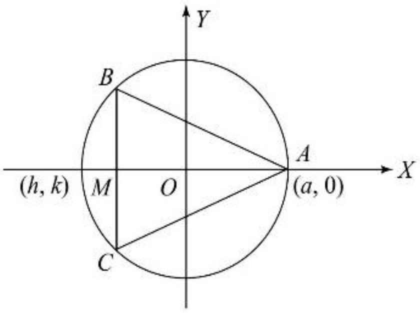

Fig. 17.15

(Circumcentre and Centroid of an equilateral triangle are same.)

So if (h, k) be the coordinates of M,

then $\dfrac{2h+a}{3} = 0$ and $\dfrac{2k+0}{3} = 0$

$\Rightarrow \quad h = -(a/2)$ and $k = 0$

and hence the equation of BC is $x = -a/2$ or $2x + a = 0$.

Example 39: Length of the common chord of the circles

$(x-1)^2 + (y+1)^2 = c^2$ and $(x+1)^2 + (y-1)^2 = c^2$ is

(a) $\frac{1}{2}\sqrt{c^2-2}$ (b) $\sqrt{c^2-2}$

(c) $2\sqrt{c^2-2}$ (d) $c + 2$

Ans. (c)

Solution: Equation of the common chord AB of the given circles is

$$(x-1)^2 + (y+1)^2 - (x+1)^2 - (y-1)^2 = 0$$

$\Rightarrow \quad y = x$

Let $C_1\ (1, -1)$ be the centre of the first circle and M be the mid-point of AB, then

$$C_1A = c,\ C_1M = \left|\frac{1+1}{\sqrt{2}}\right| = \sqrt{2}$$

and $AB = 2AM = 2\sqrt{(C_1A)^2 - (C_1M)^2} = 2\sqrt{c^2-2}$.

Example 40: The lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units. An equation of this circle is $(\pi = 22/7)$

(a) $x^2 + y^2 + 2x - 2y = 62$
(b) $x^2 + y^2 + 2x - 2y = 47$
(c) $x^2 + y^2 - 2x + 2y = 47$
(d) $x^2 + y^2 - 2x + 2y = 62$

Ans. (c)

Solution: The centre of the circle is the point of intersection of the given diameters $2x - 3y = 5$ and $3x - 4y = 7$, which is $(1, -1)$ and if r is the radius of the circle, then

$\pi r^2 = 154 \quad \Rightarrow \quad r^2 = 154 \times \dfrac{7}{22}$

$\Rightarrow \quad r = 7.$

Hence an equation of the required circle is

$$(x-1)^2 + (y+1)^2 = 7^2 \quad \Rightarrow \quad x^2 + y^2 - 2x + 2y = 47.$$

Example 41: The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is

(a) $x^2 + y^2 = 9a^2$ (b) $x^2 + y^2 = 16a^2$
(c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = a^2$

Ans. (c)

Solution: As the centre of the circumcircle of an equilateral triangle is its centroid and the distance of the centroid from the vertex of the triangle is 2/3 of its median through that vertex. So the distance of the centre from a vertex of the triangle is $(2/3) \times 3a = 2a$ and hence the equation of the circumcircle is

$$x^2 + y^2 = 4a^2.$$

Example 42: The circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ intersect each other in two distinct points if

(a) $r < 2$ (b) $r > 8$
(c) $2 < r < 8$ (d) $2 \le r \le 8$

Ans. (c)

Solution: Centres of the given circles are (5, 0) and (0, 0) and their radii are 3 and r respectively. The two circles will intersect in two distinct points if the distance between their centres is greater than the difference and less than the sum of their radii

$\Rightarrow \quad |3 - r| < 5 < 3 + r \quad \Rightarrow \quad 2 < r < 8.$

Example 43: A line is drawn through the point $P(3, 11)$ to cut the circle $x^2 + y^2 = 9$ at A and B. Then $PA \cdot PB$ is equal to

(a) 9 (b) 121
(c) 205 (d) 139

Ans. (b)

Solution: From geometry we know $PA\ .\ PB = (PT)^2$ where PT is the length of the tangent from P to the circle.

Hence $\quad PA\ .\ PB = (3)^2 + (11)^2 - 9 = 11^2 = 121.$

Alternate Solution

Equation of any line through (3, 11) is

$$\frac{x-3}{\cos\theta} = \frac{y-11}{\sin\theta} = r \text{ (say)}$$

Then the coordinates of a point on this line at a distance r from (3, 11) are $(3 + r\cos\theta,\ 11 + r\sin\theta)$ and if this lies on the given circle $x^2 + y^2 = 9$

then $(3 + r\cos\theta)^2 + (11 + r\sin\theta)^2 = 9$

$\Rightarrow\quad 9 + 121 + 2r(3\cos\theta + 11\sin\theta) + r^2 = 9$

$\Rightarrow\quad r^2 + 2r(3\cos\theta + 11\sin\theta) + 121 = 0$

which is quadratic in r, gives two values of r say r_1 and r_2 and hence the distances of the points A and B from P.

Thus, $PA\cdot PB = r_1 r_2 = 121$.

Example 44: If the line $x\cos\alpha + y\sin\alpha = p$ represents the common chord $APQB$ of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ $(a > b)$ as shown in the Fig. 17.16, then AP is equal to

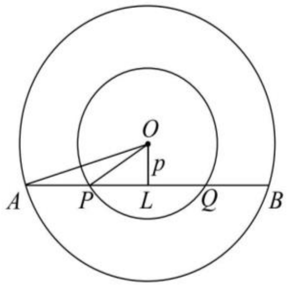

Fig. 17.16

(a) $\sqrt{a^2+p^2}+\sqrt{b^2+p^2}$

(b) $\sqrt{a^2-p^2}+\sqrt{b^2-p^2}$

(c) $\sqrt{a^2-p^2}-\sqrt{b^2-p^2}$

(d) $\sqrt{a^2+p^2}-\sqrt{b^2+p^2}$

Ans. (c)

Solution: The given circles are concentric with centre at (0, 0) and the length of the perpendicular from (0, 0) on the given line is p. Let $OL = p$

then $\quad AL = \sqrt{(OA)^2-(OL)^2} = \sqrt{a^2-p^2}$

and $\quad PL = \sqrt{(OP)^2-(OL)^2} = \sqrt{b^2-p^2}$

$\Rightarrow\quad AP = \sqrt{a^2-p^2} - \sqrt{b^2-p^2}$

Example 45: A circle touches both the coordinate axes and the line $x - y = a\sqrt{2}$ $(a > 0)$, the coordinates of the centre of the circle can be

(a) (a, a) (b) $(a, -a)$

(c) $(-a, a)$ (d) none of these

Ans. (a)

Solution: Let r be the radius of the circle. Since it touches the coordinate axes and the line $x - y = a\sqrt{2}$, the coordinates of the centre of the circle can be (r, r), $(-r, -r)$ or $(r, -r)$ (As $r > 0$ and the line $x - y = a\sqrt{2}$ meets the co-ordinates axes at $(a\sqrt{2}, 0)$ and $(0, -a\sqrt{2})$).

If the centre is (r, r) or $(-r, -r)$, then

$$\left|\frac{-a\sqrt{2}}{\sqrt{1+1}}\right| = r \qquad \Rightarrow \qquad r = a$$

So (a, a) can be the coordinates of the centre of the circle,

If the centre is $(r, -r)$ we have

$$\left|\frac{r+r-a\sqrt{2}}{\sqrt{2}}\right| = r \qquad \Rightarrow \qquad r = \left(\sqrt{2}\pm 1\right)a.$$

Example 46: If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is

(a) 4 (b) $2\sqrt{5}$

(c) 5 (d) $3\sqrt{5}$

Ans. (c)

Solution: The line $5x - 2y + 6 = 0$ meets y-axis at (0, 3). So the coordinates of Q are (0, 3) and PQ is the length of the tangent from Q (0, 3) to the circle $x^2 + y^2 + 6x + 6y - 2 = 0$. Hence

$$PQ = \sqrt{0+3^2+6\times 0+6\times 3-2} = \sqrt{25} = 5.$$

Example 47: A triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (– 4, 3) respectively, then $\angle QPR$ is equal to

(a) $\pi/2$ (b) $\pi/3$

(c) $\pi/4$ (d) $\pi/6$

Ans. (c)

Solution: We know that $\angle QPR = \frac{1}{2}\angle QOR$; O being the centre (0, 0) of the given circle $x^2 + y^2 = 25$. Let m_1 = slope of $OQ = 4/3$ and m_2 = slope of $OR = -3/4$

As $\quad m_1 m_2 = -1,\ \angle QOR = \pi/2$

$\Rightarrow\quad \angle QPR = \pi/4$

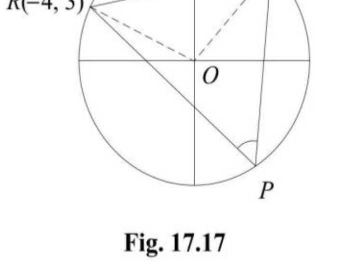

Fig. 17.17

Example 48: Two vertices of an equilateral triangle are (– 1, 0) and (1, 0) and its third vertex lies above the x-axis, an equation of the circumcircle is

(a) $3x^2 + 3y^2 - 2\sqrt{3}y = 3$

(b) $2x^2 + 2y^2 - 3\sqrt{2}y = 2$

(c) $x^2 + y^2 - 2y = 1$

(d) none of these

Ans. (a)

Solution: Let A (– 1, 0), B (1, 0) and C (0, b) be the vertices of the triangle, as C lies on the locus of points equidistance from A(–1, 0) and B(1, 0) i.e., y-axis.

Then $\quad AB = AC \quad \Rightarrow \quad \sqrt{1+b^2} = 2$

$\Rightarrow \quad b^2 = 3 \quad \Rightarrow \quad b = \sqrt{3} \qquad [\because b > 0]$

Since the triangle is equilateral, the centre of the circumcircle is at the centroid of the triangle which is (0, $\sqrt{3}/3$). Thus the equation of the circumcircle is

$$(x - 0)^2 + (y - 1/\sqrt{3})^2 = (1 - 0)^2 + (0 - 1/\sqrt{3})^2$$

$$\Rightarrow \quad x^2 + y^2 - (2/\sqrt{3})\,y + 1/3 = 1 + 1/3$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 2\sqrt{3}\,y = 3.$$

Example 49: An isosceles triangle is inscribed in the circle $x^2 + y^2 - 6x - 8y = 0$ with vertex at the origin and one of the equal sides along the axis of x. Equation of the other side through the origin is

(a) $7x - 24y = 0$ (b) $24x - 7y = 0$

(c) $7x + 24y = 0$ (d) $24x + 7y = 0$

Ans. (d)

Solution: Centre of the circle is (3, 4) and it passes through the origin. If $y = mx$ is the equation of the required line, then length of the perpendicular from the centre on this line is equal to the length of the perpendicular from the centre on the axis of x.

$$\Rightarrow \quad \frac{3m-4}{\sqrt{1+m^2}} = \pm 4$$

$$\Rightarrow \quad 9m^2 - 24m + 16 = 16(1 + m^2)$$

$$\Rightarrow \quad m = -24/7$$

($\because m = 0$ corresponds to x-axis)

and hence the required equation is $24x + 7y = 0$.

Example 50: Equations of the common tangents of the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$ are

(a) $x = 0$ (b) $y = 4$

(c) $3x + 4y = 10$ (d) all of these

Ans. (d)

Solution: Equations of the given circles can be written as C_1: $(x - 1)^2 + (y - 3)^2 = 1$ having centre at (1, 3) and radius 1 and C_2: $(x + 3)^2 + (y - 1)^2 = 9$ having centre at (–3, 1) and radius 3.

Both the circles clearly touch $x = 0$ and $y = 4$

Now $\quad \left|\dfrac{3\times 1 + 4\times 3 - 10}{\sqrt{3^2+4^2}}\right| = 1$

and $\quad \left|\dfrac{3(-3) + 4(1) - 10}{\sqrt{3^2+4^2}}\right| = 3$

Shows that both the circles touch $3x + 4y = 10$ also.

Example 51: A circle whose radius is 5 and which touches externally the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5) intersects in real distinct points the line

(a) $x = 0$ (b) $y = 0$

(c) $y = x$ (d) none of these

Ans. (c)

Solution: Centre of the given circle is A (1, 2) and its radius is $\sqrt{1 + (2)^2 + 20} = 5$. Point of contact P is (5, 5). Let B (h, k) be the centre of the required circle of radius 5, then P is the mid-point of AB, so that

$$\frac{h+1}{2} = 5 \text{ and } \frac{k+2}{2} = 5 \quad \Rightarrow \quad h = 9, k = 8$$

and an equation of the required circle is

$$x^2 + y^2 - 18x - 16y + 120 = 0$$

If $x = 0$, $y^2 - 16y + 120 = 0$ does not give real values of y

If $y = 0$, $x^2 - 18x + 120 = 0$ does not give real values of x

If $\quad y = x,\ 2x^2 - 34x + 120 = 0$

or $\quad x^2 - 17x + 60 = 0 \quad \Rightarrow \quad x = 5, 12$

This shows that the circle intersects the line $y = x$ at two real distinct points.

Example 52: A point moves such that the sum of the squares of its distances from the sides of a square of side unity is equal to 9. The locus of such a point is a circle

(a) inscribed in the square

(b) circumscribing the square

(c) inside the square

(d) containing the square

Ans. (d)

Solution: Let P (x, y) be any point on the locus and $OABC$ be the square of side unity.

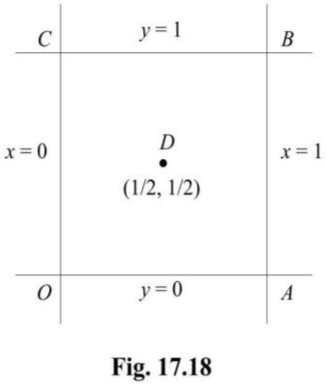

Fig. 17.18

Then, according to the given condition

$$x^2 + y^2 + (x - 1)^2 + (y - 1)^2 = 9$$

$\Rightarrow \quad 2(x^2 + y^2) - 2x - 2y - 7 = 0$

$\Rightarrow \quad x^2 + y^2 - x - y - 7/2 = 0$

which is a circle with centre (1/2, 1/2) coinciding with the centre D of the square and the radius of the circle is $\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{7}{2}} = 2$, showing that the circle contains the square.

Example 53: Equation of a circle with centre (–4, 3) touching internally and containing the circle $x^2 + y^2 = 1$ is

(a) $x^2 + y^2 + 8x - 6y + 9 = 0$
(b) $x^2 + y^2 - 8x + 6y + 9 = 0$
(c) $x^2 + y^2 + 8x - 6y - 11 = 0$
(d) $x^2 + y^2 - 8x + 6y - 11 = 0$

Ans. (c)

Solution: Let the equation of the required circle be

$$(x + 4)^2 + (y - 3)^2 = r^2 \quad \text{(i)}$$

If (i) touches the circle $x^2 + y^2 = 1$ internally (ii)
the distance between the centres (– 4, 3) and (0, 0) of these circles is equal to the difference of their radii

$\Rightarrow \quad \sqrt{4^2 + 3^2} = r - 1 \Rightarrow r = 5 + 1 \Rightarrow r = 6$

So that an equation of the required circle is

$$x^2 + y^2 + 8x - 6y - 11 = 0.$$

Example 54: The tangents drawn from the origin to the circle $x^2 + y^2 - 2px - 2qy + q^2 = 0$ are perpendicular if

(a) $p^2 + q^2 = 1$ (b) $p^2 - q^2 = 1$
(c) $p^2 - q^2 = 0$ (d) none of these

Ans. (c)

Solution: Equation of the given circle can be written as

$$(x - p)^2 + (y - q)^2 = p^2 \quad \text{(i)}$$

This has (p, q) as the centre and p as the radius showing that it touches y-axis. So one of the tangents from the origin to the circle is y-axis.

$\Rightarrow$ Other tangent from the origin to the circle must be x-axis. which is possible if $q = \pm\ p$.

$\Rightarrow \quad p^2 - q^2 = 0.$

Example 55: A square is inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ with its sides parallel to the axes of coordinates. The coordinates of the vertex farthest from the origin are

(a) (9, – 8) (b) (8, – 9)
(c) (8, 5) (d) (– 6, 9)

Ans. (b)

Solution: The centre of the given circle is (1, –2) and its radius is $\sqrt{98}$, so the coordinates of the vertices of the square inscribed in the given circle are $(1 + \sqrt{98}\cos\theta, - 2 + \sqrt{98}\sin\theta)$ where $\theta = \pm\pi/4, \pm 3\pi/4$.

S^2 = square of the distance of a vertex from the origin.

$= 5 + 98 + 2\sqrt{98}(\cos\theta - 2\sin\theta)$ which is maximum when

$\theta = -\pi/4$, so the required point is $(1 + 7, - 2 - 7)$
$= (8, - 9)$

Example 56: An equation of a circle touching the axes of coordinates and the line $x\cos\alpha + y\sin\alpha = 2$ is $x^2 + y^2 - 2gx + 2gy + g^2 = 0$ where $g =$

(a) $2(\cos\alpha + \sin\alpha + 1)^{-1}$
(b) $2(\cos\alpha - \sin\alpha + 1)^{-1}$
(c) $2(\cos\alpha + \sin\alpha - 1)^{-1}$
(d) $- 2(\cos\alpha - \sin\alpha - 1)^{-1}$

Ans. (b)

Solution: Centre of the circle is $(g, -g)$ and radius is $|g|$. If it touches the line $x\cos\alpha + y\sin\alpha = 2$, the

$g\cos\alpha - g\sin\alpha - 2 = \pm g$

$\Rightarrow \quad g(\cos\alpha - \sin\alpha \pm 1) = 2$

$\Rightarrow \quad g = \dfrac{2}{\cos\alpha - \sin\alpha \pm 1}$

which is satisfied by (b).

Example 57: If common chord of the circle C with centre at (2, 1) and radius r and the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a diameter of the second circle, then the value of r is

(a) 3 (b) 2
(c) 3/2 (d) 1

Ans. (a)

Solution: Equation of C is $(x - 2)^2 + (y - 1)^2 = r^2$

or $\quad x^2 + y^2 - 4x - 2y = r^2 - 5$

Equation of the common chord is

$$x^2 + y^2 - 4x - 2y - (r^2 - 5) - [x^2 + y^2 - 2x - 6y + 6] = 0$$

$\Rightarrow \quad 2x - 4y + r^2 + 1 = 0$

Since it is a diameters of the second circle, the centre (1, 3) lies on it.

$\Rightarrow \quad 2 - 12 + r^2 + 1 = 0 \quad \Rightarrow \quad r = 3.$

Example 58: The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is

(a) $4 \le x^2 + y^2 \le 64$ (b) $x^2 + y^2 \le 25$
(c) $x^2 + y^2 \ge 25$ (d) $3 \le x^2 + y^2 \le 9$.

Ans. (a)

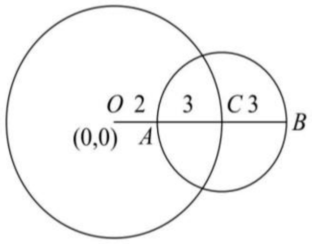

Fig. 17.19

◎ **Solution:** Distance of any point (x, y) on a circle with centre C lying on the circle $x^2 + y^2 = 25$ and radius 3 from the origin lies between OA (= 2) and OB (= 8)
Hence $4 \le x^2 + y^2 \le 64$.

◉ **Example 59:** If a circle passes through (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is

(a) $2ax - 2by + (a^2 + b^2 + 4) = 0$
(b) $2ax + 2by - (a^2 + b^2 + 4) = 0$
(c) $2ax + 2by + (a^2 + b^2 + 4) = 0$
(d) $2ax - 2by - (a^2 + b^2 + 4) = 0$

Ans. (b)

◎ **Solution:** Let the equation of the circle be
$x^2 + y^2 - 2gx - 2fy + c = 0$. As it passes through (a, b)
$$a^2 + b^2 - 2ga - 2fb + c = 0$$
Since it intersects the circle $x^2 + y^2 = 4$ orthogonally
$$2g \times 0 + 2f \times 0 = c - 4 \Rightarrow c = 4$$
and the locus of (g, f), the centre is
$$2ax + 2by - (a^2 + b^2 + 4) = 0$$

◉ **Example 60:** Consider a family of circles which are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) are the coordinates of the centre of the circles, then the set of values of k is given by the interval

(a) $0 < k < 1/2$ (b) $k \ge 1/2$
(c) $-1/2 \le k \le 1/2$ (d) $k \le 1/2$.

Ans. (b)

◎ **Solution:** Equation of the circle with centre (h, k) and touching x-axis is
$$(x - h)^2 + (y - k)^2 = k^2$$
which passes through $(-1, 1)$
if $\quad h^2 + 2h - 2k + 2 = 0$
or if $\quad (h + 1)^2 = 2k - 1$
For real values of h, $2k - 1 \ge 0$.
$\Rightarrow \quad k \ge 1/2$

◉ **Example 61:** If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q then the line $5x + by - a = 0$ passes through P and Q for

(a) Infinitely many values of a
(b) Exactly two values of a
(c) Exactly one value of a
(d) No value of a

Ans. (d)

◎ **Solution:** Equation of the line passing through the points P and Q is
$$x^2 + y^2 + 2ax + cy + a - (x^2 + y^2 - 3ax + dy - 1) = 0$$
$\Rightarrow \quad 5ax + (c - d)y + a + 1 = 0$
which coincides with $5x + by - a = 0$
if $\quad \dfrac{5a}{5} = \dfrac{c - d}{b} = \dfrac{a + 1}{-a}$.
or if $a^2 + a + 1 = 0$ which does not give any real value of a.

◉ **Example 62:** A circle touches the x-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is

(a) a hyperbola (b) a parabola
(c) an ellipse (d) a circle

Ans. (b)

◎ **Solution:** Let (h, k) be the centre of the circle. Since it touches x-axis, its radius is k. As it also touches the circle with centre (0, 3) and radius 2 (externally as x-axis lies outside the given circle) distance between the centres is equals to the sum of their radii.
So $\quad h^2 + (k - 3)^2 = (k + 2)^2$
$\Rightarrow \quad h^2 = 5(2k - 1)$
Locus of the centre (h, k) is
$$x^2 = 5(2y - 1)$$
which is a parabola.

◉ **Example 63:** Three distinct points A, B, C, are given in the two dimensional coordinate plane such that the ratio of the distance of anyone of them from the point (1, 0) to the distance from the point $(-1, 0)$ is equal to 1/3. Then the circumcentre of the triangle ABC is at the point

(a) (5/2, 0) (b) (5/3, 0)
(c) (0, 0) (d) (5/4,0)

Ans. (d)

◎ **Solution:** Let $Q = (1, 0)$ and $R = (-1, 0)$
Let $P(x, y)$ be any point such that
$$\frac{PQ}{PR} = \frac{1}{3} \Rightarrow 9(PQ)^2 = (PR)^2$$
$\Rightarrow \quad 9[(x - 1)^2 + y^2] = (x + 1)^2 + y^2$
$\Rightarrow \quad 8x^2 + 8y^2 - 20x + 8 = 0$
A, B, C lie on this locus of P which is a circle circumscribing the triangle ABC.
So the circumcentre of the triangle ABC is the centre (5/4, 0) of this circle.

◉ **Example 64:** Let $A(1,2)$ $B(3, 4)$ be two points and $C(x, y)$ be a point such that area of the triangle ABC is 3 sq. units and $(x - 1)(x - 3) + (y - 2)(y - 4) = 0$. Then maximum number of positions of C, in the xy plane is

(a) 2 (b) 4
(c) 8 (d) none of these

Ans. (d)

Solution: C lies on the circle on AB as diameter of length $2\sqrt{2}$.
So radius of the circle is $\sqrt{2}$.
Area of the triangle $ABC = 3 = (1/2)\, AB \times$ altitude from C on AB

$$\Rightarrow \quad \text{altitude} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}} > \sqrt{2}.$$

Maximum altitude from C on AB is equal to the radius of the circle.
So no such C exists.

Example 65: A circle touches y-axis at (0, 3) and makes an intercept of 2 units on the positive x-axis. Intercept made by the circle on the line $\sqrt{10}\, x - 3y = 1$ in units is

(a) 3 (b) 6
(c) $2\sqrt{10}$ (d) 10.

Ans. (c)

Solution:

$$AB = 2,\ AD = 1$$
$$CD = 3 = OE$$
$$AC = \sqrt{(AD)^2 + (CD)^2}$$
$$= \sqrt{1 + 3^2} = \sqrt{10} = CE$$

So the coordinates of C are $\left(\sqrt{10}, 3\right)$

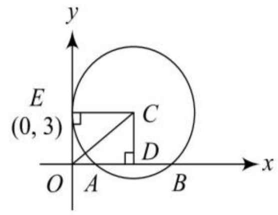

Fig. 17.20

The line $\sqrt{10}\,x - 3y = 1$ passes through the centre $C\left(\sqrt{10}, 3\right)$ of the circle and hence is a diameter and the required intercept is twice the radius of the circle.

Example 66: Equation of the circle which circumscribes the square formed by the lines $xy - 9x - 2y + 18 = 0$ and $xy - 5x - 6y + 30 = 0$ is

(a) $x^2 + y^2 - 4x - 10y + 21 = 0$
(b) $x^2 + y^2 - 8x - 14y + 57 = 0$
(c) $x^2 + y^2 - 12x - 10y + 53 = 0$
(d) none of these.

Ans. (b)

Solution: Given equations represent the lines $x = 2$, $y = 9$, $x = 6$, $y = 5$ which forms a square of each side equal to 4, vertex $A(2, 5)$, $C(6, 9)$. Centre of the circle is the mid point of AC and the radius is (1/2) AC. Hence the equation of the circle is

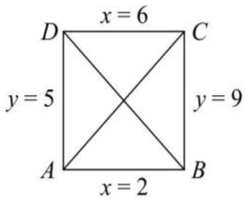

Fig. 17.21

$$(x - 4)^2 + (y - 7)^2 = \left(2\sqrt{2}\right)^2$$
$$\Rightarrow \quad x^2 + y^2 - 8x - 14y + 16 + 49 - 8 = 0$$
$$\Rightarrow \quad x^2 + y^2 - 8x - 14y + 57 = 0.$$

Example 67: If the centre of the circle passing through the origin and the points of intersection of the pair of straight lines $xy - 7x + 3y - 21 = 0$ with the coordinate axes lies on the $x + y = k$, then k is equal to.

(a) 0 (b) 1
(c) 2 (d) 4

Ans. (c)

Solution: Equation the circle passing through the points (– 3, 0) and (0, 7) where the lines meet the axes, and the origin is $x^2 + y^2 + 3x - 7y = 0$. Whose centre (– 3/2, 7/2) lies on the line $x + y = k$

$$\Rightarrow \quad k = 2$$

Example 68: If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for

(a) all except two values of p
(b) exactly one value of p
(c) all values of p
(d) all except one value of p.

Ans. (d)

Solution: Equation of a circle passing through P and Q is $x^2 + y^2 + 3x + 7y + 2p - 5 + \lambda\,(x^2 + y^2 + 2x + 2y - p^2) = 0$
which passes through (1, 1)

if $\quad (7 + 2p) - \lambda\,(p^2 - 6) = 0$

$$\Rightarrow \quad \lambda = \frac{7 + 2p}{p^2 - 6} \neq -1$$
$$\Rightarrow \quad p^2 - 6 + 7 + 2p \neq 0 \quad \Rightarrow \quad (p + 1)^2 \neq 0$$
$$\Rightarrow \quad p \neq -1$$

Example 69: Equation of circle passing through (1, 5) and (4, 1) and touching y-axis is $x^2 + y^2 - 5x - 6y + 9 + \lambda(4x + 3y - 19) = 0$ where λ is equal to

(a) 0, – 40/9 (b) 0

(c) 40/9 (d) – 40/9

Ans. (a)

Solution: Equation of any circle through the given points is $(x - 1)(x - 4) + (y - 5)(y - 1) + \lambda(4x + 3y - 19) = 0$ where $4x + 3y - 19 = 0$ is the equation of the line joining (1, 5) and (4, 1)

$$\Rightarrow \quad x^2 + y^2 - 5x - 6y + 9 + \lambda(4x + 3y - 19) = 0$$

Which touches y-axis if the roots of the equation

$y^2 - 6y + 9 + \lambda(3y - 19) = 0$ are equal. (Taking the intersection of the circle with $x = 0$)

$$\Rightarrow \quad (3\lambda - 6)^2 = 4(9 - 19\lambda)$$

$$\Rightarrow \quad 9\lambda^2 + 40\lambda = 0 \Rightarrow \lambda = 0, -40/9.$$

Example 70: The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

(a) $15 < m < 65$

(b) $35 < m < 85$

(c) $-85 < m < -35$

(d) $-35 < m < 15$

Ans. (d)

Solution: The line intersects the circle at two distinct points if length of the perpendicular from (2, 4), the centre of the circle, to the line is less than the radius.

$$\Rightarrow \quad \frac{|(3)(2) - 4(4) - m|}{\sqrt{(3)^2 + (4)^2}} < \sqrt{4 + 16 + 5}$$

$$\Rightarrow \quad |m + 10| < 25$$

$$\Rightarrow \quad -25 < m + 10 < 25$$

$$\Rightarrow \quad -35 < m < 15.$$

Example 71: The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is

(a) $x^2 + y^2 - 2x - 2y + 1 = 0$

(b) $x^2 + y^2 - x - y = 0$

(c) $x^2 + y^2 + 2x + 2y - 7 = 0$

(d) $x^2 + y^2 + x + y - 2 = 0$

Ans. (b)

Solution: Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

As it passes through (1, 0) and (0, 1), we get

$$1 + 2g + c = 0,\ 1 + 2f + c = 0$$

If r is the radius of the circle, then

$$r^2 = g^2 + f^2 - c = \frac{1}{4}(1 + c)^2 + \frac{1}{4}(1 + c)^2 - c$$

$$= \frac{1}{2}(1 + c^2)$$

Now r^2 will be least if $c = 0$ and the required equation is $x^2 + y^2 - x - y = 0$

Example 72: The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ ($c > 0$) touch each other

(a) $|a| = 2c$ (b) $2|a| = c$

(c) $|a| = c$ (d) $a = 2c$

Ans. (c)

Solution: Given circles can touch each other internally see the figures. This is possible if

$$c = a \quad \text{or} \quad c = -a$$

$$\Rightarrow \quad c = |a|.$$

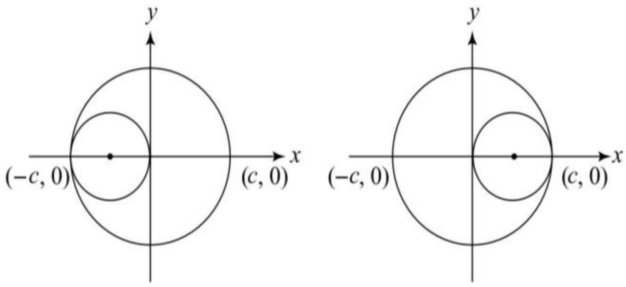

Fig. 17.22

Example 73: The circle passing through the point (–1, 0) and touching y-axis at (0, 2) also passes through the point

(a) (–3/2, 0) (b) (–5/2, 2)

(c) (–3/2, 5/2) (d) (–4, 0)

Ans. (d)

Solution: Let he equation of the circle be

$$(x - k)^2 + (y - 2)^2 = k^2$$

which passes through the point (–1, 0)

if $\quad (-1 - k)^2 + 4 = k^2 \Rightarrow k = -5/2$

and the equation of the circle is

$$x^2 + y^2 + 5x - 4y + 4 = 0$$

which passes through (–4, 0)

Example 74: The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is

(a) $20(x^2 + y^2) - 36x + 45y = 0$

(b) $20(x^2 + y^2) + 36x - 45y = 0$

(c) $36(x^2 + y^2) - 20x + 45y = 0$

(d) $36(x^2 + y^2) + 20x - 45y = 0$

Ans. (a)

Solution: Equation of the chord contact in terms of the mid-point (α, β) is $x\alpha + y\beta = \alpha^2 + \beta^2$

Let $P(5t, 4t - 4)$ be a point on $4x - 5y = 20$

Equation of the chord of contact of P is

$$x(5t) + y(4t - 4) = 9$$

comparing the equations we get

$$\frac{5t}{\alpha} = \frac{4t - 4}{\beta} = \frac{9}{\alpha^2 + \beta^2} = \frac{4(5t) - 5(4t - 4)}{4\alpha - 5\beta}$$

$\Rightarrow \quad 20(\alpha^2+\beta^2)-36\alpha+45\beta=0$

Locus of (α, β) is $20(x^2+y^2)-36x+45y=0$

Example 75: The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is

(a) 6/5 (b) 5/3

(c) 10/3 (d) 3/5

Ans. (c)

Solution: Let the equation of the circle be $(x-1)^2+(y-k)^2=k^2$. Since it passes through (2, 3)

$1+(3-k)^2=k^2 \Rightarrow k=5/3$

$\Rightarrow$ diameter $=2k=10/3$.

Assertion-Reason Type Questions

Example 76: Statement-1: Limiting points of a family of coaxial circles are (1, 2) and (2, 1). No circle of this family passes through the origin.

Statement-2 : Equation of a circle passing through (1, 2) and (2, 1), the centre of which does not lie on the join of these points is $x^2+y^2-3x-3y+4=0$

Ans. (c)

Solution: In statement 1, two members of the family are $(x-1)^2+(y-2)^2=0$ and $(x-2)^2+(y-1)^2=0$. Equation of any member of this family is $x^2+y^2-2x-4y+5+\lambda(x^2+y^2-4x-2y+5)=0$ which passes through the origin if $\lambda=-1$, in which case the equation does not represent a circle. So the statement is true.

In statement-2, Equation represents the circle on the join of the given points as diameter. So the statement is false.

Example 77: Statement-1: Equation of a tangent to the circle $x^2+y^2=50$ at a point which has positive integral coordinates (α, β) $(\alpha \neq \beta)$ is $x+7y=50$ or $7x+y=50$.

Statement-2 : There are 12 points on the circle $x^2+y^2=50$ with integral coordinates.

Ans. (a)

Solution: Statement-2 is true as the required points are, $(\pm 1, \pm 7)$, $(\pm 7, \pm 1)$ $(\pm 5, \pm 5)$ out of which the points satisfying the conditions in statement are (1, 7) and (7, 1) only and the tangents at these points are respectively $x+7y=50$ and $7x+y=50$ so statement–1 is also true.

Example 78: Statement-1: The centre of the circle passing through the points (3, 8) and (5, 4) and having smallest radius is (4, 6)

Statement-2 : The centre of a circle passing through two given points lies at the mid-point of the line joining the points.

Ans. (c)

Solution: In statement-1, the radius is smallest when the line joining the given points is a diameter of the circle and hence its centre is the mid-point $\left(\frac{3+5}{2}, \frac{4+8}{2}\right)$ i.e (4, 6) of the line joining the given points. So statement-1 is True.

Statement-2 is false as the line joining the given points may not be a diameter for each circle passing through these points.

Example 79: Statement-1: The circle $x^2+y^2-8x-4y+16=0$ touches x-axis at the point (4, 0)

Statement-2 : The circle $(x-a)^2+(y-r)^2=r^2$ touches x-axis at the point $(a, 0)$

Ans. (a)

Solution: Statement-2 is true as the centre of the circle is (a, r) and its radius is r, centre is at a distance r from x-axis. In statement-2, the circle is $(x-4)^2+(y-2)^2=4=2^2$, using statement-2, statement-1 is also true.

Example 80: Statement-1 : Point (3, – 1) lies outside the circle $2x^2+2y^2-3x+5y-7=0$

Statement-2 : A point (α, β) lies outside the circle $x^2+y^2+2gx+2fy+c=0$ if $\alpha^2+\beta^2+2g\alpha+2f\beta+c>0$

Ans. (d)

Solution: Statement-2 is true because the distance between the point (α, β) and the centre $(-g, -f)$ of the circle is greater than its radius.

$\Rightarrow \quad (\alpha+g)^2+(\beta+f)^2>g^2+f^2-c$

$\Rightarrow \quad \alpha^2+\beta^2+2g\alpha+2f\beta+c>0.$

Using it in statement-1, $2\times 9+2\times 1-3\times 3+5(-1)-7=-1<0$ so statement-1 is false.

Example 81: Tangents are drawn from the point (17, 7) to the circle $x^2+y^2=169$

Statement-1: Tangents are mutually perpendicular.

Statement-2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2+y^2=338$

Ans. (a)

Solution: Statement-2 is true, because equation of any tangent to the circle is $y=mx\pm 13\sqrt{1+m^2}$. If it passes through (h, k), then $k=mh\pm 13\sqrt{1+m^2}$

$\Rightarrow \quad (169-h^2)m^2+2mhk+(169-k^2)=0$

which gives two values of m say m_1, m_2, the slopes of the tangents drawn from (h, k) to the circle

$$m_1 m_2 = -1 \quad \Rightarrow \quad \frac{169 - k^2}{169 - h^2} = -1 \quad \Rightarrow \quad h^2 + k^2 = 338$$

$\Rightarrow$ Locus of (h, k) is $x^2 + y^2 = 338$

Since the point (17, 7) in statement-1 satisfies the above equation, using statement-2, statement-1 is also true.

Example 82: Statement-1: Limiting points of a family of co-axial system of circles are (1, 1) and (3, 3). The member of this family passing through the origin is $2x^2 + 2y^2 - 3x - 3y = 0$

Statement-2: Equation of the tangent to the circle $2x^2 + 2y^2 - 3x - 3y = 0$ at the origin is $x + y = 0$

Ans. (b)

Solution: In statement-1, two members of the co-axial system are circles with centres at (1, 1), (3, 3) and radius zero. So the equation of the system of circles is

$$(x - 1)^2 + (y - 1)^2 + \lambda[(x - 3)^2 + (y - 3)^2] = 0$$

If it passes through the origin.

$1 + 1 + \lambda[9 + 9] = 0 \Rightarrow \lambda = -1/9$ and the required circle is $2x^2 + 2y^2 - 3x - 3y = 0$

So statement-1 is true.

Statement-2 is also true but does not lead the statement-1.

Example 83: Statement 1: $x^2 + y^2 - 6x - 10y - 2 = 0$ is the only circle of radius 6 units having a diameter along the line $5x - 2y - 5 = 0$.

Statement-2: $5x - 2y - 5 = 0$ is a normal to the circle $x^2 + y^2 - 6x - 10y - 2 = 0$.

Ans. (d)

Solution: Given circle is $(x - 3)^2 + (y - 5)^2 = 36$

$\Rightarrow$ centre of the circle is (3, 5) and radius is 6. Since the line $5x - 2y - 5 = 0$ passes through (3, 5) it is a diameter of the circle. So this is a circle of radius 6 units having $5x - 2y - 5 = 0$ as a diameter. But any other point on this can also be taken as the centre, giving us another circle of radius 6 with the same diameter. So statement-1 is false. Since every diameter is normal to the circle, statement-2 is true.

Example 84: Statement-1: The circle $x^2 + y^2 - 8x + 10y + 32 = 0$ does not touch or intersect any axis.

Statement-2: A circle with centre (a, b) and radius r does not touch or intersect any axis if $r < \min [|a|, |b|]$

Ans. (a)

Solution: We find by fig. $r < |a|$ and $r < |b|$

$\Rightarrow r < \min (|a|, |b|)$

So statement-2 is true, using it statement-1 is also true, as the centre of the circle is $(4, -5)$ and $r = 3$.

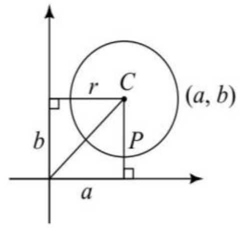

Fig. 17.23

Example 85: $C_1: x^2 + y^2 + 2x - 2y + 2 - a^2 = 0$

$C_2: x^2 + y^2 - 10x - 14y + 74 - a^2 = 0$

$L: x + y - 6 = 0$

Statement 1: L is a common tangent to both the circles C_1 and C_2 if $a = 3\sqrt{2}$.

Statement 2: L is a common chord of both the circles C_1 and C_2 for all values of a.

Ans. (c)

Solution: $C_1 : (x + 1)^2 + (y - 1)^2 = a^2$

$C_2 : (x - 5)^2 + (y - 7)^2 = a^2$

Centre of C_1 is $(-1, 1)$ and of C_2 is (5, 7) and both have same radius equal to a. Distance between the centre $= 6\sqrt{2} = 2a$ (when $a = 3\sqrt{2}$) = sum of the radii, so the circle touch each other externally and the equation of a common tangent is $(x^2 + y^2 + 2x - 2y + 2 - a^2) - (x^2 + y^2 - 10x - 14y + 74 - a^2) = 0$

$\Rightarrow \quad 12x + 12y - 72 = 0$ or $x + y - 6 = 0$

So statement-1 is true and statement-2 is false as for $a < 3\sqrt{2}$, circles C_1 and C_2 will not touch or intersect each other, L will lie outside both the circles.

LEVEL 2

Straight Objective Type Questions

Example 86: Let $ABCD$ be a quadrilateral with area 18, side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(a) 3 (b) 2

(c) 3/2 (d) 1

Ans. (b)

Solution: Let $CD = a$, then $AB = 2a$ and r be the radius of the circle, then $AD = 2r$. Let A be the origin and AB and AD as x-axis and y-axis respectively.

The coordinates of A, B, C, D are respectively

(0, 0), (2*a*, 0), (*a*, 2*r*), (0, 2*r*)

Area $(ABCD) = (1/2)(a + 2a)(2r) = 18$

$\Rightarrow \quad ar = 6.$

Equation of *BC* is

$2rx + ay - 4ar = 0$ and the coordinates of the centre of the circle are (r, r)

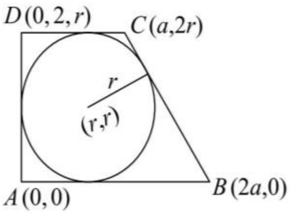

Fig. 17.24

Since the circle touches *BC*,

$$\frac{2r^2 + ar - 4ar}{\sqrt{4r^2 + a^2}} = r$$

$\Rightarrow \quad 4r^4 - 72r^2 + 324 = 4r^4 + 36$

$\Rightarrow \quad r = 2.$

Example 87: An equation of the chord of the circle $x^2 + y^2 = a^2$ passing through the point (2, 3) farthest from the centre is

(a) $2x + 3y = 13$ (b) $3x - y = 3$
(c) $x - 2y + 4 = 0$ (d) $x - y + 1 = 0$

Ans. (a)

Solution: Let *P* (2, 3) be the given point, *M* be the middle point of a chord of the circle $x^2 + y^2 = a^2$ through *P*. Then the distance of the centre *O* of the circle from the chord is *OM*.

and $(OM)^2 = (OP)^2 - (PM)^2$ which is maximum when *PM* is minimum, i.e., *M* coincides with *P* i.e., *P* is the middle point of the chord. Hence the equation of the chord is

$2 \cdot x + 3 \cdot y - a^2$
$= (2)^2 + (3)^2 - a^2$
$\Rightarrow \quad 2x + 3y = 13.$

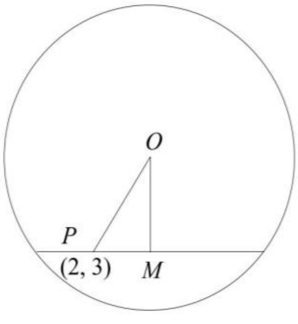

Fig. 17.25

Example 88: An isosceles right angled triangle is inscribed in the circle $x^2 + y^2 = r^2$. If the coordinates of an end of the hypotenuse are (*a b*), the coordinates of the vertex are

(a) $(-a, -b)$ (b) $(b, -a)$
(c) (b, a) (d) $(-b, -a)$

Ans. (b)

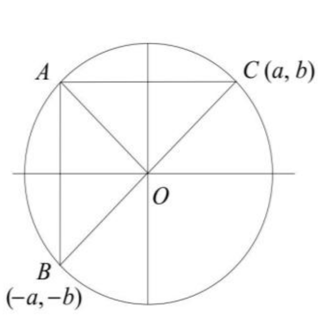

Fig. 17.26

Solution: Since the hypotenuse of a right angled triangle inscribed in a circle is a diameter of the circle, if the coordinates of the end *C* of the hypotenuse *BC* are (*a*, *b*), the coordinates of *B* are (– *a*, – *b*).

Equation of *BC* is $\frac{y}{x} = \frac{b}{a}$. If *A* is the vertex of the isosceles triangle then *OA* is perpendicular to *BC* and the equation of *AO* is $\frac{y}{x} = \frac{-a}{b}$ which meets the circle $x^2 + y^2 = r^2$ at points for which

$$\left(\frac{a^2}{b^2} + 1\right) x^2 = r^2 = a^2 + b^2$$

$[\because (a, b)$ lies on $x^2 + y^2 = r^2]$

$\Rightarrow \quad x^2 = b^2 \Rightarrow x = \pm b \Rightarrow y = \mp a.$

coordinates of *A* are (– *b*, *a*) or (*b*, – *a*).

Example 89: Two rods of lengths *a* and *b* slide along the *x*-axis and *y*-axis respectively in such a manner that their ends are concyclic. The locus of the centre of the circle passing through the end points is

(a) $4(x^2 + y^2) = a^2 + b^2$ (b) $x^2 + y^2 = a^2 + b^2$
(c) $4(x^2 - y^2) = a^2 - b^2$ (d) $x^2 - y^2 = a^2 - b^2$

Ans. (c)

Solution: Let *C* (*h*, *k*) be the centre of the circle passing through the end points of the rod *AB* and *PQ* of lengths *a* and *b* respectively, *CL* and *CM* be perpendiculars from *C* on *AB* and *PQ* respectively. (Fig. 17.27)

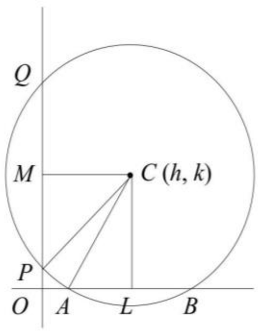

Fig. 17.27

then $AL = (1/2)AB = a/2$
$PM = (1/2)PQ = b/2$

and $CA = CP$ (radii of the same circle)

$$\Rightarrow \quad k^2 + \frac{a^2}{4} = h^2 + \frac{b^2}{4}$$

$\Rightarrow \quad 4(h^2 - k^2) = a^2 - b^2$

so that locus of (*h*, *k*) is $4(x^2 - y^2) = a^2 - b^2$

Example 90: If the point (1, 4) lies inside the circle $x^2 + y^2 - 6x - 10y + p = 0$ and the circle does not touch or intersect the coordinate axes, then

(a) $0 < p < 34$ (b) $25 < p < 29$
(c) $9 < p < 25$ (d) $9 < p < 29$

Ans. (b)

◎ **Solution:** Since the circle does not touch or intersect the coordinates axes, the absolute values of x and y coordinates of the centre are greater than the radius of the circle. Coordinates of the centre of the circle are (3, 5) and the radius is $\sqrt{9+25-p}$

so that $3 > \sqrt{9+25-p} \Rightarrow p > 25$ (1)

$5 > \sqrt{9+25-p} \Rightarrow p > 9$ (2)

and the point (1, 4) lies inside the circle

$\Rightarrow \quad 1 + 16 - 6 - 10 \times 4 + p < 0$

$\Rightarrow \quad p < 29$ (3)

From (1), (2), (3) we get

$$25 < p < 29.$$

Example 91: The lengths of the intercepts made by any circle on the coordinates axes are equal if the centre lies on the line (s) represented by

(a) $x + y + 1 = 0$ (b) $x + y = 1$
(c) $x^2 - y^2 = 0$ (d) $x - y = 1$

Ans. (c)

◎ **Solution:** Let the equation of any circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$

For intercept made by the circle on x-axis, put $y = 0$ in (i)

$\Rightarrow \quad x^2 + 2gx + c = 0$

If x_1, x_2 are roots of (*ii*), then length of the intercept on x-axis is

$$|x_1 - x_2| = \sqrt{(x_1+x_2)^2 - 4x_1x_2} = 2\sqrt{g^2 - c}$$

Similarly length of the intercept of the y-axis is $2\sqrt{f^2 - c}$

Since the lengths of these intercepts are equal

$\sqrt{g^2 - c} = \sqrt{f^2 - c}$

$\Rightarrow \quad g^2 = f^2 = (-g)^2 = (-f)^2$

Therefore, centre lies on $x^2 - y^2 = 0$

Example 92: If $a > 2b > 0$ then the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is

(a) $\dfrac{2b}{\sqrt{a^2 - 4b^2}}$ (b) $\dfrac{\sqrt{a^2 - 4b^2}}{2b}$

(c) $\dfrac{2b}{a - 2b}$ (d) $\dfrac{b}{a - 2b}$

Ans. (a)

◎ **Solution:** $y = mx - b\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = b^2$ for all values of m. If it also touches the circle $(x - a)^2 + y^2 = b^2$, then the length of the perpendicular from its centre $(a, 0)$ on this line is equal to the radius b of the circle, which gives

$$\frac{ma - b\sqrt{1+m^2}}{\sqrt{1+m^2}} = \pm b$$

Taking negative value on R.H.S. we get $m = 0$, so we neglect it. Taking the positive value on R.H.S. we get

$$ma = 2b\sqrt{1+m^2}$$

$\Rightarrow \quad m^2(a^2 - 4b^2) = 4b^2$

$\Rightarrow \quad m = \dfrac{2b}{\sqrt{a^2 - 4b^2}}.$

Example 93: Let PQ and RS be tangents at the extremities of a diameter PR of a circle of radius r. Such that PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals.

(a) $\sqrt{PQ.RS}$ (b) $\dfrac{PQ + RS}{2}$

(c) $\dfrac{2PQ.RS}{PQ + RS}$ (d) $\sqrt{\dfrac{PQ^2 + RS^2}{2}}$

Ans. (a)

◎ **Solution:** From Fig. 17.28, we have

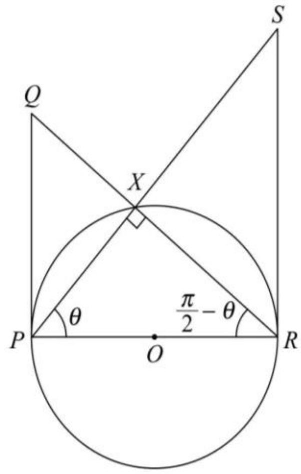

Fig. 17.28

$$\frac{PQ}{PR} = \tan(\pi/2 - \theta) = \cot\theta.$$

and $\dfrac{RS}{PR} = \tan\theta$

$\Rightarrow \quad \dfrac{PQ}{PR}.\dfrac{RS}{PR} = 1$

$\Rightarrow \quad (PR)^2 = PQ\,.\,PS$

$\Rightarrow \quad (2r)^2 = PQ\,.\,PS$

$\Rightarrow \quad 2r = \sqrt{PQ.PS}$

Example 94: If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the x-axis, then

(a) $p^2 = q^2$ (b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

Ans. (d)

Solution: Let PQ be a chord of the given circle passing through P (p, q) and the coordinates of Q be (x, y). Since PQ is bisected by the x-axis, the mid-point of PQ lies on the x-axis which gives $y = -q$.

Now Q lies on the circle $x^2 + y^2 - px - qy = 0$

so $x^2 + q^2 - px + q^2 = 0$

$\Rightarrow$ $x^2 - px + 2q^2 = 0$ (i)

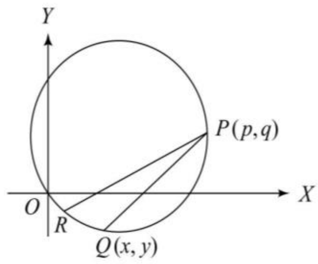

Fig. 17.29

which gives two values of x and hence the coordinates of two points Q and R (say), so that the chords PQ and PR are bisected by x-axis. If the chords PQ and PR are distinct, the roots of (i) are real distinct.

$\Rightarrow$ the discriminant $p^2 - 8q^2 > 0$

$\Rightarrow$ $p^2 > 8q^2$.

Example 95: Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then L_1 can be represented by

(a) $x + y = 0$ (b) $x - y = 0$
(c) $7x + y = 0$ (d) $x - 7y = 0$

Ans. (b)

Solution: Let the equation of L_1 be $y = mx$. Since the intercepts made by the circle on L_1 and L_2 are equal, their distances from the centre of the circle are also equal. Centre of the given circle is $(1/2, -3/2)$, so that we have

$$\left|\frac{\frac{1}{2}-\frac{3}{2}-1}{\sqrt{1+1}}\right| = \left|\frac{m\times\frac{1}{2}+\frac{3}{2}}{\sqrt{m^2+1}}\right|$$

$$\Rightarrow \quad \frac{2}{\sqrt{2}} = \frac{|m+3|}{2\sqrt{m^2+1}}$$

$$\Rightarrow \quad 8(m^2 + 1) = (m + 3)^2$$

$$\Rightarrow \quad 7m^2 - 6m - 1 = 0$$

$$\Rightarrow \quad (m - 1)(7m + 1) = 0$$

$\Rightarrow$ $m = 1$ or $m = -1/7$

So the equations representing L_1 are

$$y = x$$

or $$y = (-1/7)\,x$$

$\Rightarrow$ $x - y = 0$ or $x + 7y = 0$

So (b) is the correct answer

Example 96: Let $A_0 A_1 A_2 A_3 A_4 A_5$ be a regular hexagon inscribed in a unit circle with centre at the origin. Then the product of the lengths of the line segments $A_0 A_1$, $A_0 A_2$ and $A_0 A_4$ is

(a) 3/4 (b) $3\sqrt{3}$
(c) 3 (d) $3\sqrt{3}/2$

Ans. (c)

Solution:

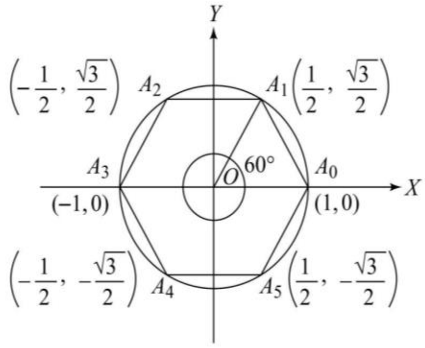

Fig. 17.30

Let O be the centre of the circle of unit radius and the coordinates of A_0 be $(1, 0)$.

Since each side of the regular hexagon makes an angle of 60º at the centre O.

Coordinates of A_1 are $(\cos 60º, \sin 60º) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

A_2 are $(\cos 120º, \sin 120º) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

A_3 are $(-1, 0)$

A_4 are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and A_5 are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Now $$A_0 A_1 = \sqrt{\left(1-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$$A_0 A_2 = \sqrt{\left(1+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{3}{4}}$$

$$= \sqrt{3} = A_0 A_4$$

So that $$(A_0 A_1)(A_0 A_2)(A_0 A_4) = 3$$

Example 97: For each natural number k, let C_k denote the circle with radius k centimeters and centre at the origin O. On the circle C_k a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at $(1, 0)$. If the particle crosses the positive direction of x-axis for the first time on the circle C_n, then $n =$

(a) 4 (b) 5
(c) 6 (d) 7

Ans. (d)

◎ **Solution:** The motion of the particle on the first four circles is shown with bold line in the figure. Note that on every circle the particle travels just one radian. The particle crosses the positive direction of x-axis first time on C_n, where n is the least positive integer such that $n \geq 2\pi \Rightarrow n = 7$.

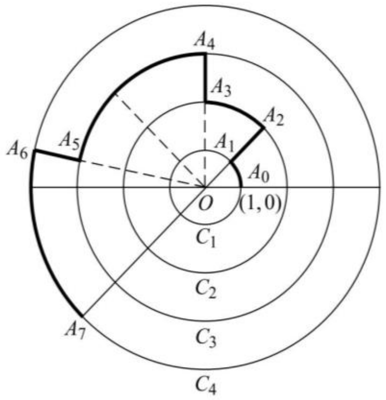

Fig. 17.31

Example 98: C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0) respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Equation of the common tangent to C_1 and C_3 which does not pass through C_2 is

(a) $x - \sqrt{3}\,y + 2 = 0$ (b) $\sqrt{3}\,x - y + 2 = 0$
(c) $\sqrt{3}\,x - y - 2 = 0$ (d) $x + \sqrt{3}\,y + 2 = 0$

Ans. (b)

◎ **Solution:** Equation of any circle through (0, 0) and (1, 0) is

$$(x-0)(x-1) + (y-0)(y-0) + \lambda \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x^2 + y^2 - x + \lambda y = 0$$

If it represents C_3, its radius = 1

$$\Rightarrow \quad 1 = (1/4) + (\lambda^2/4) \Rightarrow \lambda = \pm\sqrt{3}$$

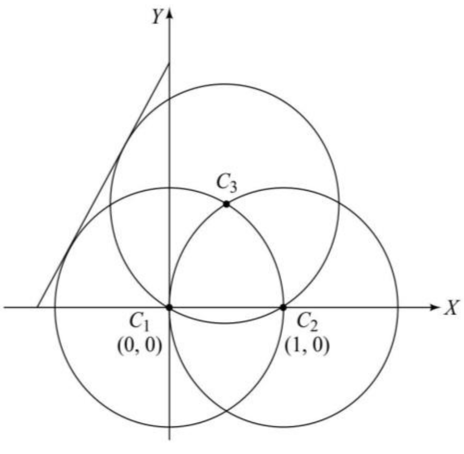

Fig. 17.32

As the centre of C_3, lies above the x-axis, we take $\lambda = -\sqrt{3}$ and thus an equation of C_3 is $x^2 + y^2 - x - \sqrt{3}y = 0$

Since C_1 and C_3 intersect and are of unit radius, their common tangents are parallel to the line joining their centres (0, 0) and (1/2, $\sqrt{3}/2$).

So, let the equation of a common tangent be

$$\sqrt{3}\,x - y + k = 0$$

It will touch C_1, if

$$\left|\frac{k}{\sqrt{3+1}}\right| = 1 \Rightarrow k = \pm 2$$

From the figure (16.32), we observe that the required tangent makes positive intercept on the y-axis and negative on the x-axis and hence its equation is $\sqrt{3}\,x - y + 2 = 0$.

Example 99: A chord of the circle $x^2 + y^2 - 4x - 6y = 0$ passing through the origin subtends an angle $\tan^{-1}(7/4)$ at the point where the circle meets positive y-axis. Equation of the chord is

(a) $2x + 3y = 0$ (b) $x + 2y = 0$
(c) $x - 2y = 0$ (d) $2x - 3y = 0$

Ans. (c)

◎ **Solution:** The given circle passes through the origin O and meets the positive Y-axis at B (0, 6). Let OP be the chord of the circle passing through the origin subtending an angle θ at B, where $\tan\theta = 7/4$

$$\Rightarrow \quad \angle OBP = \theta$$

Equation of the tangent OT at O to the given circle is

$$2x + 3y = 0$$

$\Rightarrow$ slope of the tangent = – 2/3

So that, if $\angle XOT = \alpha$, $\tan\alpha = 2/3$

From geometry, $\angle POT = \angle OBP = \theta$

$\Rightarrow \angle POX = \theta - \alpha$

and
$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

$$= \frac{\frac{7}{4} - \frac{2}{3}}{1 + \frac{7}{4} \times \frac{2}{3}} = \frac{13}{26} = \frac{1}{2}$$

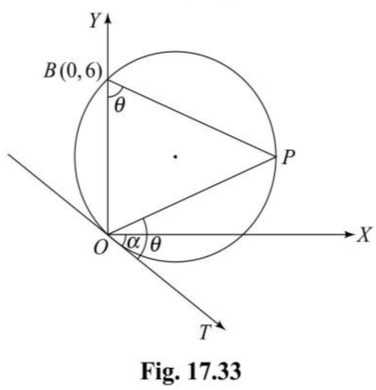

Fig. 17.33

Hence the equation of OP is $y = x \tan(\theta - \alpha)$

$\Rightarrow \quad x - 2y = 0$

Example 100: On the line joining the points A (0, 4) and B (3, 0), a square $ABCD$ is constructed on the side of the line away from the origin. Equation of the circle having centre at C and touching the axis of x is

(a) $x^2 + y^2 - 14x - 6y + 49 = 0$
(b) $x^2 + y^2 - 14x - 6y + 9 = 0$
(c) $x^2 + y^2 - 6x - 14y + 49 = 0$
(d) $x^2 + y^2 - 6x - 14y + 9 = 0$

Ans. (a)

Solution: Let $\angle ABO = \theta$ then $\angle CBL = 90° - \theta$, CL being perpendicular to x-axis. The coordinates of C are (OL, LC)

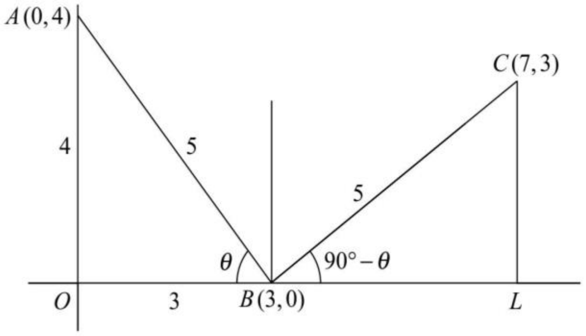

Fig. 17.34

$$OL = OB + BL = 3 + 5 \sin\theta$$
$$= 3 + 5 \times (4/5) = 7$$
$$CL = 5\cos\theta = 5 \times (3/5) = 3$$

So the coordinate of C are (7, 3) and the equation of the circle having C as centre and touching x-axis is

$(x - 7)^2 + (y - 3)^2 = (CL)^2 = 9$

$\Rightarrow \quad x^2 + y^2 - 14x - 6y + 49 = 0.$

Example 101: A circle with centre at the origin and radius equal to a meets the axis of x at A and B. $P(\alpha)$ and $Q(\beta)$ are two points on this circle so that $\alpha - \beta = 2\gamma$, where γ is a constant. The locus of the point of intersection of AP and BQ is

(a) $x^2 - y^2 - 2ay \tan\gamma = a^2$
(b) $x^2 + y^2 - 2ay \tan\gamma = a^2$
(c) $x^2 + y^2 + 2ay \tan\gamma = a^2$
(d) $x^2 - y^2 + 2ay \tan\gamma = a^2$

Ans. (b)

Solution: Coordinates of A are $(-a, 0)$ and of P are $(a\cos\alpha, a\sin\alpha)$

$\therefore$ Equation of AP is

$$y = \frac{a\sin\alpha}{a(\cos\alpha + 1)}(x + a)$$

or $\quad y = \tan(\alpha/2)(x + a) \qquad$ (i)

Similarly equation of BQ is

$$y = \frac{a\sin\beta}{a(\cos\beta - 1)}(x - a)$$

or $\quad y = -\cot(\beta/2)(x - a) \qquad$ (ii)

We now eliminate α, β from (i) and (ii)

From (i) and (ii) $\tan(\alpha/2) = \dfrac{y}{a + x}$, $\tan(\beta/2) = \dfrac{a - x}{y}$

Now $\quad \alpha - \beta = 2\gamma$

$$\Rightarrow \quad \tan\gamma = \frac{\tan(\alpha/2) - \tan(\beta/2)}{1 + \tan(\alpha/2)\tan(\beta/2)} = \frac{\frac{y}{a+x} - \frac{a-x}{y}}{1 + \frac{y}{a+x}\cdot\frac{a-x}{y}}$$

$$\Rightarrow \quad \tan\gamma = \frac{y^2 - (a^2 - x^2)}{(a+x)y + (a-x)y} = \frac{x^2 + y^2 - a^2}{2ay}$$

$$\Rightarrow \quad x^2 + y^2 - 2ay\tan\gamma = a^2$$

which is the required locus.

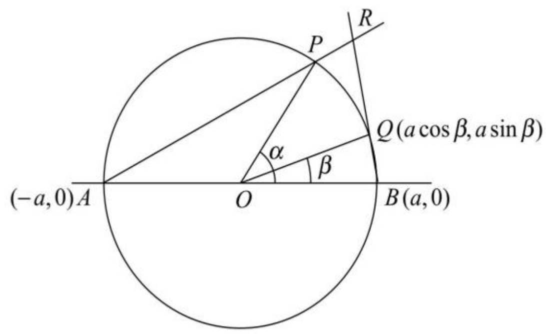

Fig. 17.35

Example 102: Equation of a circle having radius equal to twice the radius of the circle $x^2 + y^2 + (2p + 3)x + (3 - 2p)y + p - 3 = 0$ and touching it at the origin is

(a) $x^2 + y^2 + 9x - 3y = 0$
(b) $x^2 + y^2 - 9x + 3y = 0$
(c) $x^2 + y^2 + 18x + 6y = 0$
(d) $x^2 + y^2 + 18x - 6y = 0$

Ans. (d)

Solution: Since the given circle passes through the origin $p - 3 = 0 \Rightarrow p = 3$ and the equation of the given circle is

$$x^2 + y^2 + 9x - 3y = 0$$

Equation of the tangent at the origin to this circle is

$$9x - 3y = 0 \qquad \text{(i)}$$

Let the equation of the required circle which also passes through the origin be

$$x^2 + y^2 + 2gx + 2fy = 0.$$

Equation of the tangent at the origin to this circle is

$$gx + fy = 0 \qquad \text{(ii)}$$

If (i) and (ii) represent the same line, then

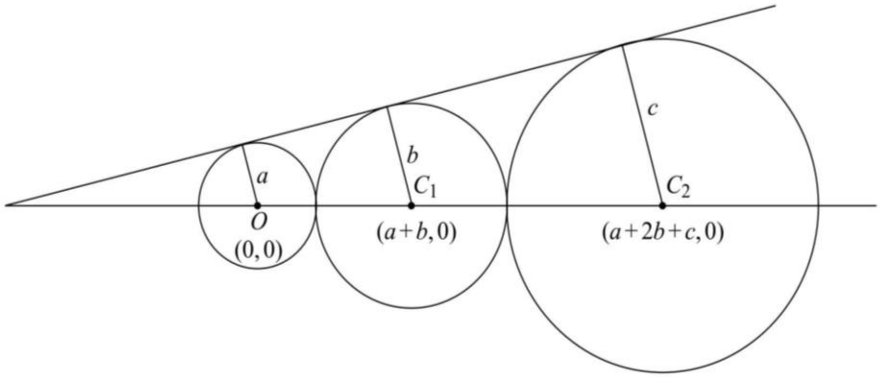

Fig. 17.36

$$\frac{g}{9} = \frac{f}{-3} = k \text{ (say)} \qquad \text{(iii)}$$

We are given that $\sqrt{g^2 + f^2} = 2\sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{-3}{2}\right)^2} = \sqrt{81+9}$

From (iii) we get $|k|\sqrt{9^2 + 3^2} = \sqrt{90} \Rightarrow k = \pm 1$

For $k = 1$,

$g = 9, f = -3$ and the equation of the required circle is $x^2 + y^2 + 18x - 6y = 0$.

Example 103: A circle C_1 of radius b touches the circle $x^2 + y^2 = a^2$ externally and has its centre on the positive x-axis; another circle C_2 of radius c touches the circle C_1 externally and has its centre on the positive x-axis. Given $a < b < c$, then the three circles have a common tangent if a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Ans. (b)

Solution: Refer Fig. 17.36. The centre of C_1 is $(a + b, 0)$ and the centre of C_2 is $(a + 2b + c, 0)$

Let $y = mx + k$ be a tangent common to the three circles. Since it touches $x^2 + y^2 = a^2$, C_1 and C_2

$$\frac{k}{\sqrt{1+m^2}} = \pm a, \quad \frac{m(a+b)+k}{\sqrt{1+m^2}} = \pm b$$

$$\text{and } \frac{m(a+2b+c)+k}{\sqrt{1+m^2}} = \pm c$$

As the centre of the three circles lie on the same side of the line $y = mx + k$, taking the same sign, say positive, in the three relations we get,

$$\frac{k}{\sqrt{1+m^2}} = a = b - \frac{m+1}{1-m} = c - m\left(\frac{1}{5}, \frac{9}{5}\right)$$

$\Rightarrow \quad \dfrac{a+b}{b-a} = \dfrac{a+2b+c}{c-a}$ (eliminating m)

$\Rightarrow \quad (a + b)(c - a) = (b - a)(a + 2b + c)$

$\Rightarrow \quad ac - a^2 + bc - ba = ba - a^2 + 2b^2 - 2ab + bc - ac$

$\Rightarrow \quad 2ac = 2b^2 \Rightarrow ac = b^2$

$\Rightarrow \quad a, b, c$ are in G.P.

Example 104: If two circles, each of radius 5 units, touch each other at (1, 2) and the equation of their common tangent is $4x + 3y = 10$, then equation of the circle, a portion of which lies in all the quadrants is

(a) $x^2 + y^2 - 10x - 10y + 25 = 0$
(b) $x^2 + y^2 + 6x + 2y - 15 = 0$
(c) $x^2 + y^2 + 2x + 6y - 15 = 0$
(d) $x^2 + y^2 + 10x + 10y + 25 = 0$

Ans. (b)

Solution: The centres of the two circles will lie on the line through P (1, 2) perpendicular to the common tangent $4x + 3y = 10$. If C_1 and C_2 are the centres of these circles then $PC_1 = 5 = r_1$, $PC_2 = -5 = r_2$. Also C_1, C_2 lie on the line $\dfrac{x-1}{\cos\theta} = \dfrac{y-2}{\sin\theta} = r$ where $\tan\theta = 3/4$. When $r = r_1$ the coordinates of C_1 are $(5\cos\theta + 1, 5\sin\theta + 2)$ or (5, 5) as $\cos\theta = 4/5$, $\sin\theta = 3/5$.

When $r = r_2$, the coordinates of C_2 are $(-3, -1)$

The circle with centre C_1 (5, 5) and radius 5 touches both the coordinate axes and hence lies completely in the first quadrant.

Therefore the required circle is with centre $(-3, -1)$ and radius 5, so its equation is

$$(x + 3)^2 + (y + 1)^2 = 5^2$$

or $\quad x^2 + y^2 + 6x + 2y - 15 = 0$

Since the origin lies inside the circle, a portion of the circle lies in all the quadrants.

Example 105: The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2 + y^2 = a^2$ which touches the circle $x^2 + y^2 - 2ax = 0$ is

(a) $y^2 = a(a - 2x)$
(b) $x^2 = a(a - 2y)$
(c) $x^2 + y^2 = (x + a)^2$
(d) $x^2 + y^2 = (y + a)^2$

Ans. (a)

◎ **Solution:** Let $P(h, k)$ be the point of intersection of the tangents at the extrimities of the chord AB of the circle $x^2 + y^2 = a^2$. Then AB is the chord of contact of the tangents from P to the circle and so its equation is

$$hx + ky = a^2 \qquad \text{(i)}$$

If (i) touches the circle $x^2 + y^2 - 2ax = 0$

then $$\frac{(h)(a) + (k)(0) - a^2}{\sqrt{h^2 + k^2}} = \pm a$$

$\Rightarrow$ $(h - a)^2 = h^2 + k^2$

$\Rightarrow$ locus of (h, k) is $(x - a)^2 = x^2 + y^2$

$\Rightarrow$ $y^2 = a(a - 2x)$.

◉ **Example 106:** If θ is the angle subtended by the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at a point $P(x_1, y_1)$ outside the circle and $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, then $\cos\theta$ is equal to

(a) $\dfrac{S_1 + c - g^2 - f^2}{S_1 - c + g^2 + f^2}$ (b) $\dfrac{S_1 - c + g^2 + f^2}{S_1 + c - g^2 + f^2}$

(c) $\dfrac{S_1 + c + g^2 - f^2}{S_1 - c + g^2 - f^2}$ (d) $\dfrac{S_1 - c + g^2 + f^2}{S_1 + c + g^2 - f^2}$

Ans. (a)

◎ **Solution:** Let PA and PB be the tangents from $P(x_1, y_1)$ to the given circle with centre $C(-g, -f)$ such that $\angle APB = \theta$ then $\angle APC = \angle CPB = \theta/2$

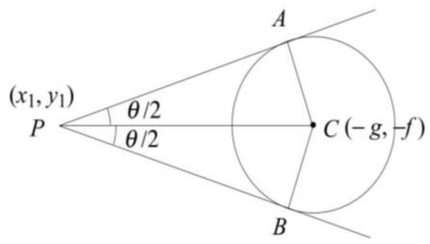

Fig. 17.37

From right angled triangle PAC

$$\tan\frac{\theta}{2} = \frac{CA}{PA} = \sqrt{\frac{g^2 + f^2 - c}{S_1}}$$

Now $$\cos\theta = \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} = \frac{S_1 - (g^2 + f^2 - c)}{S_1 + (g^2 + f^2 - c)}$$

$$= \frac{S_1 + c - g^2 - f^2}{S_1 - c + g^2 + f^2}$$

◉ **Example 107:** If the area of the quadrilateral formed by the tangent from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the pair of radii at the points of contact of these tangents to the circle is 8 square units, then c is a root of the equation

(a) $c^2 - 32c + 64 = 0$ (b) $c^2 - 34c + 64 = 0$
(c) $c^2 + 2c - 64 = 0$ (d) $c^2 + 34c - 64 = 0$

Ans. (b)

◎ **Solution:** Let OA, OB be the tangents from the origin to the given circle with centre $C(-3, 5)$ and radius $\sqrt{9 + 25 - c}$ $= \sqrt{34 - c}$.

Then area of the quadrilateral $OACB$

$= 2 \times$ area of the triangle OAC

$= 2 \times (1/2) \times OA \times AC$

Now OA = length of the tangent from the origin to the given circle $= \sqrt{c}$

and AC = radius of the circle $= \sqrt{34 - c}$

so that $\sqrt{c}\,\sqrt{34 - c} = 8$ (given) $\Rightarrow$ $c(34 - c) = 64$

$\Rightarrow$ $c^2 - 34c + 64 = 0$

◉ **Example 108:** An equation of the circle in which the chord joining the points (1, 2) and (2, – 1) subtends an angle of $\pi/4$ at any point on the circumference is

(a) $x^2 + y^2 - 15 = 0$
(b) $x^2 + y^2 - 6x - 2y + 5 = 0$
(c) $x^2 + y^2 + 6x + 2y - 15 = 0$
(d) $x^2 + y^2 - 2x - 4y + 4 = 0$

Ans. (b)

◎ **Solution:** Let $P(x, y)$ be any point on the circle passing through A (1, 2) and B (2, –1) the slope of $AP = \dfrac{y - 2}{x - 1}$ and slope of $BP = \dfrac{y + 1}{x - 2}$

Since they include an angle of $\pi/4$ so

$$\frac{\dfrac{y+1}{x-2} - \dfrac{y-2}{x-1}}{1 + \dfrac{y-2}{x-1} \times \dfrac{y+1}{x-2}} = \pm\tan\frac{\pi}{4} = \pm 1$$

$\Rightarrow$ $(y + 1)(x - 1) - (x - 2)(y - 2) = \pm[(x - 1)(x - 2) + (y - 2)(y + 1)]$

$\Rightarrow$ $3x + y - 5 = \pm[x^2 + y^2 - 3x - y]$

$\Rightarrow$ $x^2 + y^2 - 5 = 0$ or $x^2 + y^2 - 6x - 2y + 5 = 0$.

◉ **Example 109:** An equation of a common tangent to the circles $x^2 + y^2 + 14x - 4y + 28 = 0$ and

$x^2 + y^2 - 14x + 4y - 28 = 0$ is

(a) $x - 7 = 0$ (b) $y + 7 = 0$
(c) $28x + 45y + 371 = 0$ (d) None of these

Ans. (c)

◎ **Solution:** By subtracting the equation of one of the circles from that of the other, we get the equation of their common chord as $7x - 2y + 14 = 0$. This chord intersects the circles at two real, distinct points, so the given circles have two common tangents. Let PA_1A_2 be a common tangent to these circle, whose centres are $O_1(-7, 2)$ and $O_2(7, -2)$, and radii 5 and 9, respectively. Then triangles PO_1A_1 and PO_2A_2 are similar, so

$$\frac{PO_1}{O_1A_1} = \frac{PO_2}{O_2A_2} \Rightarrow \frac{PO_1}{5} = \frac{PO_2}{9}.$$

$\Rightarrow$ P divides O_1O_2 externally in the ratio 5 : 9, so its coordinates are (– 49/2, 7). Thus any line through P is given by

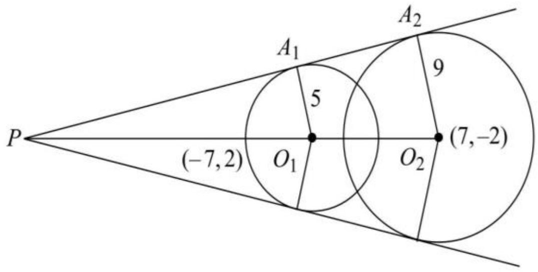

Fig. 17.38

$$y - 7 = m\left(x + \frac{49}{2}\right). \qquad (1)$$

This line will touch the circle of radius 5 centred at O_1 if

$$\frac{m\left(-7+\frac{49}{2}\right)-2+7}{\sqrt{1+m^2}} = \pm 5$$

$$\Rightarrow \quad \frac{35m}{2} + 5 = \pm 5\sqrt{1+m^2}$$

$$\Rightarrow \quad \left(\frac{7m}{2}+1\right)^2 = 1 + m^2$$

$$\Rightarrow \quad \frac{45m^2}{4} + 7m = 0,$$

$\Rightarrow$ $m = 0$ or $m = -28/45$.

Substituting $m = -28/45$ in (1) we get the equation given in (c).

Example 110: The locus of the point, the sum of the squares of whose distances from n fixed points A_i (x_i, y_i), $i = 1, 2, \ldots, n$ is equal to k^2 is a circle

(a) passing through the origin
(b) with centre at the origin
(c) with centre at the point of mean position of the given points
(d) none of these

Ans. (c)

Solution: Let (x, y) be any point on the locus, then

$$\sum_{i=1}^{n} [(x - x_i)^2 + (y - y_i)^2] = k^2$$

$$\Rightarrow \quad n(x^2 + y^2) - 2x\sum_{i=1}^{n} x_i - 2y\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - k^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)x - 2\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)y + \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - k^2\right) = 0$$

which is a circle with centre $\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\right)$ the point of mean position of the given points.

Example 111: An isosceles triangle ABC is inscribed in a circle $x^2 + y^2 = a^2$ with the vertex A at $(a, 0)$ and the base angles B and C each equal to 75°, then length of the base BC is

(a) $a/2$ (b) a
(c) $2a/\sqrt{3}$ (d) $\sqrt{3}\,a/2$

Ans. (b)

Solution: $\angle B = \angle C = 75°$

$\Rightarrow$ $\angle BAC = 30°$

$\Rightarrow$ $\angle BOC = 60°$

$\Rightarrow$ BOC is an equilateral triangle

$\Rightarrow$ $BC = OB$ = the radius of the circle

$\Rightarrow$ $BC = a$.

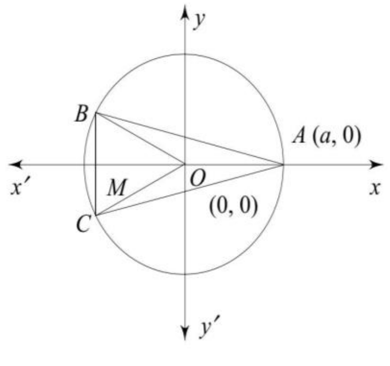

Fig. 17.39

Example 112: If the locus of a point which moves so that the line joining the points of contact of the tangents drawn from it to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = a^2$, is the circle $x^2 + y^2 = c^2$, then a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Ans. (b)

Solution: Let $P(h, k)$ be any point on the locus. Equation of the chord of contact of P with respect to the circle $x^2 + y^2 = b^2$ is $hx + ky = b^2$.

If it touches the circle $x^2 + y^2 = a^2$, then

$$\left|\frac{-b^2}{\sqrt{h^2+k^2}}\right| = a \Rightarrow a^2(h^2+k^2) = b^4$$

So that the locus of $P(h, k)$ is $x^2 + y^2 = (b^2/a)^2$

$$\therefore \quad c^2 = \left(\frac{b^2}{a}\right)^2 \Rightarrow ac = b^2$$

$\Rightarrow a, b, c$ are in G.P.

Example 113: If the line $3x - 4y - k = 0$, $(k > 0)$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b), then $k + a + b$ is equal to

(a) 20 (b) 22
(c) –30 (d) –28

Ans. (a)

Solution: Since the given line touches the given circle, the length of the perpendicular from the centre (2, 4) of the circle to the line $3x - 4y - k = 0$ is equal to the radius $\sqrt{4+16+5} = 5$ of the circle.

$$\Rightarrow \frac{3\times 2 - 4\times 4 - k}{\sqrt{9+16}} = \pm 5$$

$\Rightarrow \quad k = 15 \qquad [\because k > 0]$

Now equation of the tangent at (a, b) to the given circle is

$$xa + yb - 2(x + a) - 4(y + b) - 5 = 0$$

$\Rightarrow \quad (a - 2)x + (b - 4)y - (2a + 4b + 5) = 0.$

If it represents the given line $3x - 4y - k = 0$

then $\dfrac{a-2}{3} = \dfrac{b-4}{-4} = \dfrac{2a+4b+5}{k} = l$ (say)

then $a = 3l + 2, b = 4 - 4l$ and $2a + 4b + 5 = kl \qquad (1)$

$\Rightarrow \quad 2(3l + 2) + 4(4 - 4l) + 5 = 15l \qquad (\because k = 15)$

$\Rightarrow \quad l = 1 \Rightarrow a = 5, b = 0$ and $k + a + b = 20$.

Example 114: The circles $x^2 + y^2 + 2g_1x - a^2 = 0$ and $x^2 + y^2 + 2g_2x - a^2 = 0$ cut each other orthogonally. If p_1, p_2 are perpendiculars from $(0, a)$ and $(0, -a)$ on a common tangent of these circles then $p_1 p_2 =$

(a) $a^2/2$ (b) a^2
(c) $2a^2$ (d) $a^2 + 2$

Ans. (b)

Solution: Since the given circles cut each other orthogonally

$$g_1 g_2 + a^2 = 0 \qquad (1)$$

If $lx + my = 1$ is a common tangent of these circles, then

$$\frac{-lg_1 - 1}{\sqrt{l^2+m^2}} = \pm\sqrt{g_1^2 + a^2}$$

$\Rightarrow \quad (lg_1 + 1)^2 = (l^2 + m^2)(g_1^2 + a^2)$

$\Rightarrow \quad m^2 g_1^2 - 2lg_1 + a^2(l^2 + m^2) - 1 = 0$

Similarly $m^2 g_2^2 - 2lg_2 + a^2(l^2 + m^2) - 1 = 0$

So that g_1, g_2 are the roots of the equation

$$m^2g^2 - 2lg + a^2(l^2 + m^2) - 1 = 0$$

$$\Rightarrow \quad g_1g_2 = \frac{a^2(l^2+m^2)-1}{m^2} = -a^2 \text{ by (1)}$$

$$\Rightarrow \quad a^2(l^2 + m^2) = 1 - a^2m^2 \qquad (2)$$

Now $$p_1p_2 = \frac{|ma-1|}{\sqrt{l^2+m^2}}\cdot\frac{|-ma-1|}{\sqrt{l^2+m^2}}$$

$$= \frac{|1-m^2a^2|}{l^2+m^2} = a^2 \text{ by (2)}$$

Example 115: A circle C touches the x-axis and the circle $x^2 + (y - 1)^2 = 1$ externally, then locus of the centre of the circle C is given by

(a) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$.
(b) $\{(x, y) : y = x^2\} \cup \{(0, y) : y \le 0\}$
(c) $\{(x, y) : x^2 + (y - 1)^2 = 0\} \cup \{(0, y) : y \le 0\}$
(d) $\{(x, y) : x^2 + 4y = 0\} \cup \{(0, y) : y \le 0\}$

Ans. (a)

Solution: Let the centre of C be $(h, \pm r)$ and radius r because it touches x-axis.

Since it touches the given circle with radius 1 and centre (0, 1), externally

$$h^2 + (1 \pm r)^2 = (r + 1)^2.$$

$\Rightarrow \quad h^2 = (r + 1)^2 - (r \pm 1)^2$

$\Rightarrow \quad h^2 = 0$ or $h^2 = 4r$.

$\therefore$ Locus of (h, r) is

$\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$.

EXERCISE

Concept-based Straight Objective Type Questions

1. Equation of the circle passing through the origin and having its centre on the line $y = 3x$ at a distance $\sqrt{10}$ from the origin is

(a) $x^2 + y^2 - 2x + 6y = 0$ (b) $x^2 + y^2 + 2x - 6y = 0$
(c) $x^2 + y^2 - 2x - 6y = 0$ (d) none of these

2. The radius of the circle
$3x^2 + by^2 + 4bx - 6by + b^2 = 0$ is
(a) 1 (b) 3
(c) $\sqrt{10}$ (d) $\sqrt{11}$

3. Equation of the circle described on the line segment of $3x + 4y = 12$ intercepted between the axes as a diameter is
(a) $x^2 + y^2 - 4x - 3y = 0$
(b) $x^2 + y^2 + 4x - 3y = 0$
(c) $x^2 + y^2 - 4x + 3y = 0$
(d) $x^2 + y^2 + 4x + 3y = 0$

4. The point (1, 2) lies inside and (3, 4) outside the circle $x^2 + y^2 - 7x + 15y - c = 0$, if
(a) $c = 25$ (b) $c = 35$
(c) $c = 65$ (d) c takes any real value

5. $S : x^2 + y^2 + 6x - 14y - 6 = 0$ is a circle and $L : 7x + 3y + 58 = 0$ is a straight line
(a) L is a diameter of S (b) L is a chord of S
(c) L is a tangent to S (d) none of these

6. The angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ is
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

7. A line passes through the point $P(5, 6)$ outside the circle $x^2 + y^2 = 12$ and meets the circle at A and B. The value of $PA \cdot PB$ is equal to
(a) 25 (b) 36
(c) 49 (d) 61

8. The tangent to the circle $x^2 + y^2 = 5$ at $(1, -2)$ touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ at the point
(a) (3, 1) (b) (3, – 1)
(c) (– 3, 1) (d) (1, – 3)

9. Two circles of equal radius of 5 units have their centres at the origin and the point (2, – 3). Equation of the common chord of these circles is
(a) $4x - 6y - 13 = 0$ (b) $2x - 3y + 13 = 0$
(c) $4x - 6y + 12 = 0$ (d) $2x - 3y - 12 = 0$

10. Two circles touch each other externally at the point $(0, k)$ and y-axis is the common tangent to these circles. Centres of these circle lie on the line
(a) $x = k$ (b) $y = k$
(c) $x + y = k$ (d) $x - y = k$

11. A circle has radius 3 units and its centre lies on the line $y = x - 1$. If the circle passes through the point (7, 3), then an equation of the circle is
(a) $x^2 + y^2 - 8x - 6y + 16 = 0$
(b) $x^2 + y^2 + 6x + 8y + 16 = 0$
(c) $x^2 + y^2 - 14x - 12y - 76 = 0$
(d) $x^2 + y^2 + 12x + 14y - 76 = 0$

12. The line $3x - y - 17 = 0$ meets the circle $x^2 + y^2 - 8x + 10y + 5 = 0$ at the point A and B. P is any point on the circle other than A or B, then the triangle APB is
(a) equilateral (b) isosceles
(c) right angled (d) obtuse angled

13. A circle passes through the origin and its centre is on the line $y = x$. If it cuts the circle $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, then the equation of the circle is
(a) $x^2 + y^2 + 2x + 2y = 0$ (b) $x^2 + y^2 - 2x - 2y = 0$
(c) $2x^2 + 2y^2 - x - y = 0$ (d) $2x^2 + 2y^2 + x + y = 0$

14. Equation of the circle on the common chord of the circles $x^2 + y^2 - ax = 0$ and $x^2 + y^2 - ay = 0$ as a diameter is
(a) $2x^2 + 2y^2 - ax - ay = 0$
(b) $x^2 + y^2 - ax - ay = 0$
(c) $2x^2 + 2y^2 + ax + ay = 0$
(d) $x^2 + y^2 - x - y + a = 0$

15. A circle touches the lines $x - y - 1 = 0$ and $x - y + 1 = 0$, the centre of the circle lies on the line
(a) $x + y - 1 = 0$ (b) $x + y + 1 = 0$
(c) $x - y = 0$ (d) none of these

16. The number of common tangents that can be drawn to the circles $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$ is
(a) 0 (b) 1
(c) 2 (d) 3

17. If the circle $(x - 2)^2 + (y - 3)^2 = a^2$ lies entirely with in the circle $x^2 + y^2 = b^2$, then
(a) $a = b$ (b) $a - b > \sqrt{13}$
(c) $b - a > \sqrt{13}$ (d) $b - a = \sqrt{13}$

18. There are four circles each of radius 1 unit touching both the axis. The equation of the smaller circle touching all these circles is
(a) $x^2 + y^2 = 2$ (b) $x^2 + y^2 = (\sqrt{2} - 1)^2$
(c) $x^2 + y^2 = 4$ (d) $x^2 + y^2 = 1$

19. The locus of the point which moves so that the ratio of the lengths of the tangents to the circle $x^2 + y^2 - 4y + 3 = 0$ and $x^2 + y^2 + 6y + 5 = 0$ is
(a) $5x^2 + 5y^2 + 70x - 33 = 0$
(b) $5x^2 + 5y^2 + 60y + 7 = 0$
(c) $5x^2 + 5y^2 + 70y + 33 = 0$
(d) $5x^2 + 5y^2 + 60x + 7 = 0$

20. A circle has two of its diameters along the lines $2x + 3y - 18 = 0$ and $3x - y - 5 = 0$ and touches the line $x + 2y + 4 = 0$. Equation of the circle is
(a) $x^2 + y^2 + 6x - 8y + 16 = 0$
(b) $x^2 + y^2 - 6x + 8y + 16 = 0$
(c) $x^2 + y^2 - 6x - 8y + 9 = 0$
(d) $x^2 + y^2 - 6x - 8y + 16 = 0$

LEVEL 1
Straight Objective Type Questions

21. Equation of the circle with centre $\left(\frac{a}{2},\frac{b}{2}\right)$ and radius $\sqrt{\frac{a^2+b^2}{4}}$ is
(a) $x^2+y^2-ax-by=(a+b)^2$
(b) $x^2+y^2+ax+by=(a-b)^2$
(c) $x^2+y^2-ax-by=\frac{(a^2+b^2)}{4}$
(d) $x^2+y^2-ax-by=0$

22. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if
(a) $a=h=2, b=0$ (b) $b=h=2, a=0$
(c) $a=b=2, h=0$ (d) none of these

23. The radius of the circle passing through the point (6, 2), two of whose diameter, are $x + y = 6$ and $x + 2y = 4$ is
(a) 10 (b) $2\sqrt{5}$
(c) 6 (d) 4

24. An equation of the circle through (1, 1) and the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is
(a) $4x^2+4y^2-30x-10y-32=0$
(b) $4x^2+4y^2+30x-13y-25=0$
(c) $4x^2+4y^2-43x+10y+25=0$
(d) none of these

25. An equation of the normal at the point (2, 3) to the circle $x^2 + y^2 - 2x - 2y - 3 = 0$ is
(a) $2x+y-7=0$ (b) $x+2y-3=0$
(c) $2x-y-1=0$ (d) $x-2y+1=0$

26. The line $x + y \tan\theta = \cos\theta$ touches the circle $x^2 + y^2 = 4$ for
(a) $\theta=\pi/6$ (b) $\theta=\pi/3$
(c) $\theta=\pi/2$ (d) no value of θ

27. If $(x, 3)$ and (3, 5) are the extremities of a diameter of a circle with centre at $(2x, y)$, then the values of x and y are
(a) $x=1, y=4$ (b) $x=4, y=1$
(c) $x=8, y=2$ (d) none of these

28. The circle $(x - r)^2 + (y - r)^2 = r^2$ touches
(a) x-axis at the origin
(b) y-axis at the origin
(c) both the coordinate axes
(d) none of these

29. The number of circles touching the coordinate axes and the line $x + y = 1$ is
(a) exactly one (b) two
(c) three (d) four

30. The centre of circle passing through the points (0 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is the point whose coordinates are
(a) $(1/\sqrt{2}, 2)$ (b) $(1/2, \sqrt{2})$
(c) $(\sqrt{2}, 1/2)$ (d) $(2, 1/\sqrt{2})$

31. Equation of a circle which touches y-axis at (0, 2) and cuts off an intercept of 3 units from the x-axis is $x^2 + y^2 - 2\alpha x - 4y + 4 = 0$ where α^2 =
(a) 5/2 (b) 25/4
(c) 4/25 (d) 29

32. The circles $x^2 + y^2 + 2x - 2y + 1 = 0$ and $x^2 + y^2 - 2x - 2y + 1 = 0$
(a) touch each other externally
(b) touch each other internally
(c) intersect on the y-axis
(d) intersect on x-axis

33. The tangent at any point to the circle $x^2 + y^2 = r^2$ meets the coordinate axes at A and B. If lines drawn parallel to the coordinate axes through A and B intersect at P, the locus of P is
(a) $x^2+y^2=r^{-2}$ (b) $x^{-2}+y^{-2}=r^2$
(c) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{r^2}$ (d) $\frac{1}{x^2}-\frac{1}{y^2}=r^2$

34. Equation of a circle with centre $C(h, k)$ and radius 5 such that $h^2 - 3h + 2 = 0$ and $k^2 + 5k - 6 = 0$ is
(a) $x^2+y^2-2x-4y=0$
(b) $x^2+y^2-2x-2y-6=0$
(c) $x^2+y^2-4x-2y-20=0$
(d) $x^2+y^2-4x-6y-12=0$.

35. If $S \equiv x^2 + y^2 - 2x - 4y - 4 = 0$, $L \equiv 2x + 2y + 15 = 0$ and $P(3, 4)$ represent a circle, a line and a point respectively then
(a) L is a tangent to S at P
(b) L is polar of P with respect to S
(c) L is the chord of contact of P with respect to S
(d) P is inside and L is outside S

36. If PQR is the triangle formed by the common tangents to the circles $x^2 + y^2 + 6x = 0$ and $x^2 + y^2 - 2x = 0$, then the centroid of the triangle is at the point
(a) (1, 0) (b) (0, 0)
(c) (– 1, 0) (d) none of these

37. A rectangle $ABCD$ is inscribed in the circle $x^2 + y^2 + 3x + 12y + 2 = 0$. If the coordinates of A and B are respectively (3, – 2) and (– 2, 0), then the coordinates of the mid-point of CD are
(a) (–3/2, – 6) (b) (– 7/2, 11)
(c) (5/2, – 1) (d) (– 5/2, – 1)

38. A line makes equal intercepts of length a on the coordinate axes. A circle is circumscribed about the triangle which the line makes with the coordinate axes. The sum of the distance of the vertices of the triangle from the tangent to this circle at the origin is
(a) $a/\sqrt{2}$ (b) a
(c) $a\sqrt{2}$ (d) $a + \sqrt{2}$

39. α, β and γ are the parametric angles of three points P, Q and R, respectively, on the circle $x^2 + y^2 = 1$, and A is the point (–1, 0). If the lengths of the chords AP, AQ and AR are in G.P., then cos ($\alpha/2$), cos ($\beta/2$) and cos ($\gamma/2$) are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

40. If P is a point with integral coordinates on the circle $x^2 + y^2 = 9$, Q is a point on the line $7x + y + 3 = 0$, and the line $x - y + 1 = 0$ is the perpendicular bisector of PQ, the coordinates of P are
(a) (3, 0) (b) (0, 3)
(c) (– 72, 21) (d) (– 1, 1)

41. If the circle C_1: $x^2 + y^2 = 16$ intersects another circle C_2 with centre at (a, b) and radius 5 in such a manner that the common chord is of maximum length, then $a^2 + b^2 =$
(a) 9 (b) 12
(c) 15 (d) 25

42. If OA and OB are equal chords of the circle $x^2 + y^2 - 2x + 4y = 0$ perpendicular to each other passing through the origin, then the slopes of OA and OB are the roots of the equation
(a) $2m^2 - 5m - 2 = 0$ (b) $3m^2 - 8m - 3 = 0$
(c) $6m^2 + 5m - 6 = 0$ (d) $m^2 + 1 = 0$

43. Equation of the circle passing through (1, 0) and (0, 1) and having the smallest possible radius is
(a) $x^2 + y^2 - x - y = 0$
(b) $x^2 + y^2 - 2x - 2y + 1 = 0$
(c) $x^2 + y^2 = 1$
(d) none of these

44. From the point A (0, 3) on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that $AM = 2AB$. Equation of the locus of M is
(a) $x^2 + 2x + (y - 3)^2 = 0$
(b) $(x - 3)^2 + 4y + y^2 = 0$
(c) $x^2 + 8x + (y - 3)^2 = 0$
(d) $x^2 + 4x + (y + 3)^2 = 0$

45. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, radius of the circle is
(a) 3/4 (b) 1
(c) 7/8 (d) 11/10

46. If the distances from the origin of the centres of three circles $x^2 + y^2 - 2\lambda_i x = c^2$ (i = 1, 2, 3) are in G.P., then the lengths of the tangents drawn to them from any point on the circle $x^2 + y^2 = c^2$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

47. The distance between the chords of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is
(a) $2\sqrt{g^2 + f^2 - c}$
(b) $(g^2 + f^2 - c)/2\sqrt{g^2 + f^2}$
(c) $(1/2)(g^2 + f^2 + c)$
(d) $g^2 + f^2 - c$

48. A variable circle passes through the fixed point A (p, q) and touches the axis of x, the locus of the other end of the diameter through A is
(a) $(x - p)^2 = 4qy$ (b) $(x - q)^2 = 4py$
(c) $(y - p)^2 = 4qx$ (d) $(y - q)^2 = 4px$

49. The locus of a point, which moves such that the lengths of the tangents from it two concentric circles of radii a and b are inversely proportional to their radii is a circle with radius
(a) $a + b$ (b) $\sqrt{|a^2 - b^2|}$
(c) $\sqrt{a^2 + b^2}$ (d) none of these

50. The circle that can be drawn to touch the coordinate axes and the line $4x + 3y = 12$ cannot lie in
(a) first quadrant (b) second quadrant
(c) third quadrant (d) fourth quadrant

51. Area of an equilateral triangle inscribed in a circle of radius a is
(a) $\sqrt{3}\ a^2/4$ (b) $\pi a^2/3$
(c) $3\sqrt{3}\ a^2/4$ (d) $\sqrt{3}\ \pi a^2/4$

52. A point P moves such that $PA/PB = p$ where A and B are two fixed points with $AB = a$, the locus of P is a circle with radius

(a) $a/(1-p^2)$ (b) $p/(1-a^2)$
(c) $ap/(1-p^2)$ (d) $ap/(1-a^2)$

53. If the lengths of the tangents from two points A and B to a circle are l and l' respectively and points are conjugate with respect to the circle (i.e. each passes through the polar of the other with respect to the circle), then $(AB)^2 =$

(a) $|l^2 - l'^2|$ (b) $(l + l')^2$
(c) $l^2 + l'^2$ (d) $(l - l')^2$

54. If two circles which pass through the points $(a, 0)$ and $(-a, 0)$ touch the line $y = mx + c$ and cut orthogonally then

(a) $c^2 = a^2(1 + m^2)$ (b) $c^2 = a^2(1 + 2m^2)$
(c) $c^2 = a^2(1 + m)^2$ (d) $c^2 = a^2 + m^2$

55. No portion of the circle $x^2 + y^2 - 16x + 18y + 1 = 0$ lies in the

(a) first quadrant (b) second quadrant
(c) third quadrant (d) fourth quadrant

56. The greatest distance of the point $P(10, 7)$ from the points on the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is

(a) 5 (b) $5\sqrt{3}$
(c) 10 (d) 15

57. The length of the longest ray drawn from the point $(4, 3)$ to the circle $x^2 + y^2 + 16x + 18y + 1 = 0$ is equal to

(a) the radius of the circle
(b) the diameter of the circle
(c) circumference of the circle
(d) the distance of the centre of the circle from the origin

58. If $y = \pm a$ is a pair of tangents to the circle $x^2 + y^2 = a^2$ meeting the tangent at any point C on the circle at P and Q, then $CP.CQ =$

(a) 1 (b) a^2
(c) $1/a^2$ (d) none of these

59. The circle $x^2 + y^2 = 9$ is contained in the circle $x^2 + y^2 - 6x - 8y + 25 = c^2$ if

(a) $c = 2$ (b) $c = 3$
(c) $c = 5$ (d) $c = 10$

60. The line $(x - 1)\cos\theta + (y - 1)\sin\theta = 1$, for all values of θ touches the circle

(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 - 2x = 0$
(c) $x^2 + y^2 - 2y = 0$ (d) $x^2 + y^2 - 2x - 2y + 1 = 0$

61. Equation of the circle which cuts each of the circles $x^2 + y^2 + 2gx + c = 0$, $x^2 + y^2 + 2g_1x + c = 0$ and $x^2 + y^2 + 2hx + 2ky + a = 0$ orthogonally is

(a) $h(x^2 + y^2) + (a - c)x - ch = 0$
(b) $k(x^2 + y^2) + (a - c)y - ck = 0$
(c) $x^2 + y^2 + (a - c)x + c = 0$
(d) $x^2 + y^2 + (a - c)y - c = 0$

62. The point at which the circles $x^2 + y^2 + 2x + 6y + 4 = 0$ and $x^2 + y^2 + 6x + 2y + 7 = 0$ subtend equal angles lies on the circle

(a) $x^2 + y^2 + 10x - 2y + 10 = 0$
(b) $x^2 + y^2 - 2x + 10y + 10 = 0$
(c) $x^2 + y^2 + 10x - 2y - 10 = 0$
(d) $x^2 + y^2 - 10x + 10y - 10 = 0$

63. Equation of a circle which passes through the point $(2, 0)$ and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + \alpha)x + 5\alpha^2 y = 1$ as α tends to 1 is

(a) $25(x^2 + y^2) + 20x - 2y - 140 = 0$
(b) $25(x^2 + y^2) - 20x + 2y - 60 = 0$
(c) $9(x^2 + y^2) - 20x + 2y + 4 = 0$
(d) $9(x^2 + y^2) - 2x - 20y + 4 = 0$

64. If the limiting points of the system of circles $x^2 + y^2 + 2gx + \lambda(x^2 + y^2 + 2fy + k) = 0$, where λ is a parameter, subtend a right angle at the origin, then $k/f^2 =$

(a) -1 (b) 1
(c) 2 (d) none of these

65. If the chord of the circle $x^2 + y^2 - 4y = 0$ along the line $x + y = 1$ subtends an angle θ at a point on the circumference of the larger segment then $\cos\theta =$

(a) 1/2 (b) $1/\sqrt{2}$
(c) $\sqrt{3}/2$ (d) $1/2\sqrt{2}$

66. Locus of the centre of the circle touching the line $3x + 4y + 1 = 0$ and having radius equal to 5 units is

(a) a pair of perpendicular lines
(b) a pair of parallel lines
(c) a pair of straight lines
(d) none of these

67. Chord of the circle $x^2 + y^2 = 81$ bisected at the point $(-2, 3)$ meets the diameter $x + 5y = 0$ at a point

(a) on the circle (b) inside the circle
(c) outside the circle (d) none of these

68. An equilateral triangle is inscribed in the circle $x^2 + y^2 = 1$ with one vertex at the point $(1, 0)$. Length of each side of the triangle is

(a) 1 (b) $\sqrt{2}$
(c) $\sqrt{3}/2$ (d) $\sqrt{3}$

69. The curve parametrically described by the equations $x = 2 + 3\cos\theta$, $y = 3 + 3\sin\theta$ represents a circle which touches

(a) x-axis at $(2, 0)$ (b) y-axis at $(0, 3)$
(c) $x + y = 2$ at $(1, 1)$ (d) $x + y = 3$ at $(1, 2)$

70. Consider two circles $x^2 + y^2 = a^2 - \lambda$ and $x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta + \lambda = 0$. Each circle passes through the centre of the other for

(a) all values of λ (b) no value of λ
(c) only one value of λ (d) more than one value of λ

71. Tangents at two points A and B on the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ intersect at the point (1, 8). Equation of the chord AB is

(a) $x + 3y - 15 = 0$ (c) $3x - y + 15 = 0$
(c) $x - 3y + 15 = 0$ (d) $3x + 2y + 11 = 0$

72. Equation of the circle on AB as diameter, where A and B are the points of intersection of the circle $x^2 + y^2 - 8x = 0$ and the curve $4x^2 - 9y^2 = 36$ is

(a) $x^2 + y^2 - 12x + 24 = 0$ (b) $x^2 + y^2 + 12x + 24 = 0$
(c) $x^2 + y^2 + 24x - 12 = 0$ (d) $x^2 + y^2 - 24x - 12 = 0$

73. A polygon of nine sides, each of length 2, is inscribed in a circle with centre at the origin. Equation of the circle is $x^2 + y^2 = r^2$, where $1/r$ is equal to

(a) cos 20° (b) sin 20°
(c) cos 40° (d) sin 40°

Assertion-Reason Type Questions

74. **Statement-1:** The radius of the circle
$2x^2 + 2y^2 - 4x\cos\theta + 4y\sin\theta - 1 - 4\cos\theta - \cos 2\theta = 0$
is $1 + \cos\theta$

Statement-2: The radius of the circle
$(x - \cos\theta)(x - \sin\theta) + (y - \cos\theta)(y - \sin\theta) = 0$ $(\theta \neq \pi/4)$
is $|\cos\theta - \sin\theta|$

75. **Statement-1:** The points of intersection of the two curves whose equations are
S_1: $x^2 + 2y^2 - 6x - 12y + 23 = 0$ and
S_2: $4x^2 + 2y^2 - 20x - 12y + 35 = 0$ lie on a circle with centre at (8/3, 3) and radius equal to $\sqrt{47}/3\sqrt{2}$.

Statement-2: Equation of a curve passing through the intersection of $S_1 = 0$ and $S_2 = 0$ in Statement-1 is $S_1 + \lambda S_2 = 0$ which represents a circle if there exists a real value of λ for which the coefficients of x^2 and y^2 are equal.

76. **Statement-1:** The locus of the point of intersection of the tangents to the circle $x = a\cos\theta$, $y = a\sin\theta$ at points whose parametric angles differ by $\pi/2$ is $x^2 + y^2 = 2a^2$

Statement-2: Tangents at the extremities of a diameter of a circle are parallel.

77. **Statement-1:** The centre of a circle passing through the points (0, 0), (1, 0) and touching the circles $C : x^2 + y^2 = 9$ lies inside the circle C.

Statement-2: If a circle C_1 passes through the centre of the circle C_2 and also touches the circle, the radius of the circle C_2 is twice the radius of the circle C_1.

78. $C_1 : (x - 3)^2 + (y - 4)^2 = a^2$

Statement-1: C_1 touches the axis of x if $a = 4$

Statement-2: C_1 touches the line $y = x$ if $a = 3$.

79. **Statement-1:** The common chord of the circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ is of maximum length if $r^2 = 34$.

Statement-2: The common chord of two circles is of maximum length if it passes through the centre of the circle with smaller radius.

80. **Statement-1:** The equation $x^2 - y^2 - 4x - 4y = 0$ represents a circle with centre (2, 2) passing through the origin.

Statement-2: The equation $x^2 + y^2 + 4x + 6y + 13 = 0$ represents a point.

LEVEL 2

Straight Objective Type Questions

81. The circles $x^2 + y^2 + ax = 0$ and $x^2 + y^2 = c^2$ $(c > 0)$ touch each other if

(a) $a^2 + c^2 = 1$ (b) $a^2 - c^2 = 0$
(c) $a^2 - c^2 = 1$ (d) none of these

82. If $(a\cos\theta_i, a\sin\theta_i)$ $i = 1, 2, 3$ represent the vertices of an equilateral triangle inscribed in a circle, then

(a) $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$
(b) $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 \neq 0$
(c) $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = 0$
(d) $\cot\theta_1 + \cot\theta_2 + \cot\theta_3 = 0$

83. A circle C is drawn on the line joining the centres of the circles C_1: $x^2 + y^2 - 4 = 0$ and C_2: $x^2 + y^2 - 8x + 7 = 0$ as a diameter. The length of the intercept made

on the common chord of the circles C_1 and C_2 by the circle C is

(a) $\sqrt{231}/8$ (b) $\sqrt{231}/4$
(c) $\sqrt{231}/2$ (d) 11/8

84. If the equation $ax^2 + 2(a^2 + ab - 16)xy + by^2 + 2ax + 2by - 4\sqrt{2} = 0$ represents a circle, the radius of the circle is

(a) 2 (b) $2\sqrt{2}$
(c) $\sqrt{2}$ (d) $4\sqrt{2}$

85. If a circle passes through the points of intersection of the coordinate axes with the line $x - \lambda y + 1 = 0$ $(\lambda \neq 0)$ and $x - 2y + 3 = 0$, then the value of λ is a root of the equation

(a) $6\lambda^2 - 7\lambda + 2 = 0$ (b) $7\lambda^2 - 6\lambda - 2 = 0$
(c) $2\lambda^2 - 6\lambda + 1 = 0$ (d) $2\lambda^2 - 7\lambda + 6 = 0$

86. If a square with length of each side equal to a is inscribed in the circle $x^2 + y^2 + 4x + 10y + 21 = 0$, then a is equal to

(a) $\sqrt{2}$ (b) 2
(c) $2\sqrt{2}$ (d) 4

87. The locus of a point which divides the join of A (–1, 1) and a variable point P on the circle $x^2 + y^2 = 4$ in the ratio 3 : 2 is a circle whose centre is at the point

(a) (2/5, 2/5) (b) (–2/5, 2/5)
(c) (2/5, – 2/5) (d) (– 5/14, 5/14)

88. If (α, β) are the roots of the equation $15x^2 - 22x + 8 = 0$ and (α', β') are the roots of the equation $8x^2 - 22x + 15 = 0$, then the equation of the circle on $A(\alpha, \alpha')$ and $B(\beta, \beta')$ as diameter is

(a) $120(x^2 + y^2) - 22(8x + 15y) + 289 = 0$
(b) $120(x^2 + y^2) - 22(15y + 8x) + 289 = 0$
(c) $23(x^2 + y^2) + 22(8x + 15y) - 120 = 0$
(d) none of these

89. If two circles cut a third circle orthogonally, the radical axis of the two circles

(a) touches the third circle
(b) passes through the centre of the third circle
(c) lies out side the third circle
(d) none of these

90. If a circle of constant radius $3k$ passes through the origin and meets the axes at A and B, the locus of the centroid of the triangle OAB is the circle

(a) $x^2 + y^2 + kx + ky = 0$
(b) $x^2 + y^2 = 4k^2$
(c) $x^2 + y^2 = k^2$
(d) $x^2 + y^2 + 2kx - 2ky + k^2 = 0$

91. The locus of a point which moves such that the sum of the squares of its distances from the three vertices of a triangle is a constant, is a circle whose centre is the

(a) incentre of the triangle
(b) circumcentre of the triangle
(c) orthocentre of the triangle
(d) centroid of the triangle

92. The tangents to the circle $x^2 + y^2 = 48$, which are inclined at angle of 60° with the axis of x form a rhombus, the length of whose sides is

(a) $2\sqrt{3}$ (b) 3
(c) $3\sqrt{3}$ (d) 4

93. An equation of the circle described on the chord $x \cos \alpha + y \sin \alpha - p = 0$ of the circle $x^2 + y^2 = a^2$ as diameter is

(a) $x^2 + y^2 + 2px \cos \alpha + 2py \sin \alpha - (2p^2 + a^2) = 0$
(b) $x^2 + y^2 - 2px \cos \alpha - 2py \sin \alpha + 2p^2 - a^2 = 0$
(c) $x^2 + y^2 - 2px \cos \alpha + 2py \sin \alpha - 2p^2 + a^2 = 0$
(d) $x^2 + y^2 + 2px \cos \alpha - 2py \sin \alpha + 2p^2 - a^2 = 0$

94. The locus of a point which moves in a plane so that the sum of the squares of its distances from the line $ax + by + c = 0$ and $bx - ay + d = 0$ is r^2, is a circle of radius

(a) r (b) $r\sqrt{a^2 + b^2}$
(c) $r\sqrt{ab}$ (d) none of these

95. Two circles are such that one is inscribed in and the other is circumscribed about a square $A_1 A_2 A_3 A_4$. If the length of each side of the square is a and P, Q are two points respectively on these circles, then

$$\left|\sum_{i=1}^{4}(PAi)^2 - \sum_{i=1}^{4}(QAi)^2\right| =$$

(a) $a^2/4$ (b) $a^2/2$
(c) a^2 (d) $2a^2$

96. Two tangents T_1 and T_2 are drawn from (– 2, 0) to the circle $C : x^2 + y^2 = 1$. Equation of a circle touching C, having T_1, T_2 as a pair of tangents from (–2, 0) and radius greater than the radius of C is

(a) $x^2 + y^2 - 6x + 5 = 0$
(b) $x^2 + y^2 - 8x + 7 = 0$
(c) $9x^2 + 9y^2 + 24x + 15 = 0$
(d) none of these

97. The locus of the point of intersection of tangents to the circle $x = a \cos \theta$, $y = a \sin \theta$ at points whose parametric angles differ by $\pi/4$ is

(a) $x^2 + y^2 = 2\left(\sqrt{2} - 1\right)^2 a^2$

(b) $x^2 + y^2 = 2(2 - \sqrt{2})a^2$

(c) $x^2 + y^2 = (\sqrt{2}+1)^2 \ a^2$

(d) none of these

98. The locus of the centre of the circle which cuts the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 + y^2 - 4x + 6y + 4 = 0$ orthogonally is

(a) $12x + 8y + 5 = 0$ (b) $8x - 12y + 5 = 0$
(c) $5x - 12y + 8 = 0$ (d) none of these

99. An equation of the tangent at the point (5, 2) of a circle is given by $3x - 2y - 11 = 0$. If the circle passes through the origin, an equation of the circle is

(a) $x^2 + y^2 - 23x - 102y = 0$
(b) $11x^2 + 11y^2 - 102x - 23y = 0$
(c) $11x^4 + 11y^2 - 23x - 102y = 0$
(d) $x^2 + y^2 - 102x - 23y = 0$

100. A circle passes through the origin O and cuts the axes at A $(a, 0)$ and B $(0, b)$. The reflection of origin O in the line AB is the point

(a) $\left(\frac{2ab^2}{a^2+b^2}, \frac{2a^2b}{a^2+b^2}\right)$ (b) $\left(\frac{2a^2b}{a^2+b^2}, \frac{2ab^2}{a^2+b^2}\right)$

(c) $\left(\frac{2ab}{a^2+b^2}, \frac{2ab}{a^2+b^2}\right)$ (d) (a, b)

101. If a circle cuts $x^2 + y^2 = a^2$ orthogonally and passes through the point $\left(\frac{a^2p}{p^2+q^2}, \frac{a^2q}{p^2+q^2}\right)$, then it will also pass through

(a) $(p, 0)$ (b) $(0, q)$
(c) (p, q) (d) (ap, aq)

102. Two circles intersect at the point P (2, 3) and the line joining the other extremity of the two diameter through P makes an angle $\pi/6$ with x-axis, then the equation of the common chord of the two circles is

(a) $x + \sqrt{3}y - (2 + 3\sqrt{3}) = 0$

(b) $x + \sqrt{3}y - (2\sqrt{3} + 2) = 0$

(c) $\sqrt{3}x + y - (2\sqrt{3} + 3) = 0$

(d) $\sqrt{3}x + y - (2 + 3\sqrt{3}) = 0$

103. A system of circles is drawn through two fixed points (–1, 0) and (1, 0); tangents are drawn to these circles parallel to the line $y = x$, the locus of the points of contact is

(a) $x^2 - y^2 - 2xy = 1$
(b) $x^2 + y^2 = 1$
(c) $x^2 + y^2 + 2xy = 0$
(d) $x^2 - y^2 + 2xy + 1 = 0$

104. The radius upto one place of decimal of the smallest circle which touches the line $3x - y = 6$ at (1, –3) and also touches the line $y = x$ is

(a) 1.2 (b) 1.6
(c) 1.8 (d) 2.1

105. Of the two concentric circles the smaller one has the equation $x^4 + y^2 = 4$. If each of the two intercepts on the line $x + y = 2$ made between the two circles is 1, the equation of the larger circle is

(a) $x^2 + y^2 = 5$ (b) $x^2 + y^2 = 5 + 2\sqrt{2}$
(c) $x^2 + y^2 = 7 + 2\sqrt{2}$ (d) $x^2 + y^2 = 11$

106. An equation of the circle situated symmetrically opposite to the circle $x^2 + y^2 - 2x = 0$ with respect to the line $x + y = 2$ is

(a) $x^2 + y^2 + 2y - 4 = 0$
(b) $x^2 + y^2 - 4x - 2y + 4 = 0$
(c) $x^2 + y^2 - x + y - 2 = 0$
(d) $x^2 + y^2 - 3x - y + 2 = 0$

107. If the chord along the line $y - x = 3$ of the circle $x^2 + y^2 = k^2$ subtends an angle of 30º in the major segment of the circle cut off by the chord then k^2 =

(a) 3 (b) 6
(c) 9 (d) 36

108. Three concentric circles of which the biggest is $x^2 + y^2 = 1$ have their radii in A.P. with common difference d (> 0). If the line $y = x + 1$ cuts all the circles in real distinct points, then

(a) $d < \frac{2-\sqrt{2}}{4}$ (b) $d > \frac{2+\sqrt{2}}{4}$

(c) $d > 1 + \frac{1}{\sqrt{2}}$ (d) d is any real number

109. Radius of the circle centred at (3, – 2), of maximum area contained in the circle $x^2 + y^2 - 4x + y = 0$ is

(a) (1/2) $(\sqrt{17} - \sqrt{13})$ (b) $1/\sqrt{2}$

(c) $\sqrt{2}$ (d) none of these

110. Equation of a family of circles, such that each member of the family passes through the origin and makes an intercept on the line $y = 2x$ which is twice the intercept made on the line $x = 2y$ is (λ being a parameter)

(a) $x^2 + y^2 - 2\lambda y = 0$
(b) $x^2 + y^2 - 2\lambda x = 0$
(c) $x^2 + y^2 - 2\lambda x - 2\lambda y = 0$
(d) none of these

Previous Years' AIEEE/JEE Main Questions

1. Two common tangents to the circle $x^2 + y^2 = 2a^2$ and parabola $y^2 = 8ax$ are
   (a) $x = \pm (y + 2a)$ (b) $y = \pm (x + 2a)$
   (c) $x = \pm (y + a)$ (d) $y = \pm (x + a)$ **[2002]**
2. If the chord $y = mx + 1$ of the circle $x^2 + y^2 = 1$ subtends an angle of measure 45° at the major segment of the circle then the value of m is
   (a) 2 (b) $t - 2$
   (c) -1 (d) none of these **[2002]**
3. The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is
   (a) $4 \le x^2 + y^2 \le 64$ (b) $x^2 + y^2 \le 25$
   (c) $x^2 + y^2 \le 25$ (d) $3 \le x^2 + y^2 \le 9$ **[2002]**
4. The centre of the circle passing through (0, 0) and (1, 0) and touching the circle $x^2 + y^2 = 9$ is
   (a) (1/2, 1/2) (b) $(1/2, -\sqrt{2})$
   (c) (3/2, 1/2) (d) (1/2, 3/2) **[2002]**
5. The equation of a circle with origin as centre and passing through equilateral triangle whose medium is of length $3a$ is
   (a) $x^2 + y^2 = 9a^2$ (b) $x^2 + y^2 = 16a^2$
   (c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = a^2$ **[2002]**
6. If two circle $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points then
   (a) $r < 2$ (b) $r = 2$
   (c) $r > 2$ (d) $2 < r < 8$
7. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle having area as 154 sq. units. Then the equation of the circle is
   (a) $x^2 + y^2 + 2x - 2y = 47$
   (b) $x^2 + y^2 - 2x + 2y = 47$
   (c) $x^2 + y^2 - 2x + 2y = 62$
   (d) $x^2 + y^2 + 2x - 2y = 62$ **[2003]**
8. If a circle passes through (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is
   (a) $2ax - 2by + (a^2 + b^2 + 4) = 0$
   (b) $2ax + 2by - (a^2 + b^2 + 4) = 0$
   (c) $2ax + 2by + (a^2 + b^2 + 4) = 0$
   (d) $2ax - 2by - (a^2 + b^2 + 4) = 0$ **[2004]**
9. A variable circle passes through the fixed point A(p, q) and touches x-axis. The locus of the other end of the diameter through A is
   (a) $(y - p)^2 = 4qx$ (b) $(x - q)^2 = 4py$
   (c) $(x - p)^2 = 4qy$ (d) $(y - q)^2 = 4px$ **[2004]**
10. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is
    (a) $x^2 + y^2 + 2x + 2y - 23 = 0$
    (b) $x^2 + y^2 - 2x - 2y - 23 = 0$
    (c) $x^2 + y^2 - 2x + 2y - 23 = 0$
    (d) $x^2 + y^2 + 2x - 2y - 23 = 0$ **[2004]**
11. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as diameter is
    (a) $x^2 + y^2 + x + y = 0$
    (b) $x^2 + y^2 - x + y = 0$
    (c) $x^2 + y^2 - x - y = 0$
    (d) $x^2 + y^2 + x - y = 0$ **[2004]**
12. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q then the line $5x + by - a = 0$ passes through P and Q for
    (a) infinitely many values of a
    (b) exactly two values of a
    (c) exactly one value of a
    (d) no value of a **[2005]**
13. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = p^2$ orthogonally, then the equation of the locus of its centre is
    (a) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - p^2) = 0$
    (b) $2ax + 2by - (a^2 + b^2 + p^2) = 0$
    (c) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - p^2) = 0$
    (d) $2ax + 2by - (a^2 - b^2 + p^2) = 0$ **[2005]**
14. A circle touches the x-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is
    (a) a hyperbola (b) a parabola
    (c) an ellipse (d) a circle **[2005]**
15. Let C be the circle with centre (0, 0) and radius 3 units. The equation of the locus of the mid points of the chords of the circle C that subtends an angle $2\pi/3$ at its centre is
    (a) $x^2 + y^2 = 9/4$ (b) $x^2 + y^2 = 3/2$
    (c) $x^2 + y^2 = 1$ (d) $x^2 + y^2 = 27/4$ **[2006]**
16. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle of area 49π square units, the equation of the circle is

(a) $x^2 + y^2 - 2x + 2y - 47 = 0$
(b) $x^2 + y^2 + 2x - 2y - 47 = 0$
(c) $x^2 + y^2 + 2x - 2y - 62 = 0$
(d) $x^2 + y^2 - 2x + 2y - 62 = 0$ **[2006]**

17. Consider a family of circles with are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) are the coordinates of the centre of the circles, then the set of values of k is given by the interval
(a) $0 < k < 1/2$ (b) $k \geq 1/2$
(c) $-1/2 \leq k \leq 1/2$ (d) $k \leq 1/2$ **[2007]**

18. The point diametrically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is
(a) $(3, 4)$ (b) $(3, -4)$
(c) $(-3, 4)$ (d) $(-3, -4)$ **[2008]**

19. Three distinct points A, B and C are given in the 2-dimensional coordinate plane such that the ratio of the distance of any one of them from the point $(1, 0)$ to the distance from the point $(-1, 0)$ is equal to $1/3$. Then the circumcentre of the triangle ABC is at the point
(a) $(5/2, 0)$ (b) $(5/3, 0)$
(c) $(0, 0)$ (d) $(5/4, 0)$ **[2009]**

20. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and $(1, 1)$ for
(a) all except two values of p.
(b) exactly one value of p
(c) all values of p
(d) all except one value of p **[2009]**

21. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if
(a) $15 < m < 65$
(b) $35 < m < 85$
(c) $-85 < m < -35$
(d) $-35 < m < 15$ **[2010]**

22. The equation of the circle passing through the points $(1, 0)$ and $(0, 1)$ and having the smallest radius is
(a) $x^2 + y^2 - 2x - 2y + 1 = 0$
(b) $x^2 + y^2 - x - y = 0$
(c) $x^2 + y^2 + 2x + 2y - 7 = 0$
(d) $x^2 + y^2 + x + y - 2 = 0$ **[2011]**

23. The two circles $x^2 + y^2 = ax$, $x^2 + y^2 = c^2$ $(c > 0)$ touch each other
(a) $|a| = 2c$ (b) $2|a| = c$
(c) $|a| = c$ (d) $a = 2c$ **[2011]**

24. The length of the diameter of the circle which touches the x-axis at the point $(1, 0)$ and passes through the point $(2, 3)$ is
(a) 6/5 (b) 5/3
(c) 10/3 (d) 3/5 **[2012]**

25. A circle passing through $(1, -2)$ and touching the axis of x at $(3, 0)$ also passes through the point
(a) $(2, -5)$ (b) $(5, -2)$
(c) $(-2, 5)$ (d) $(-5, 2)$ **[2013]**

26. If each of the lines $5x + 8y = 13$ and $4x - y = 3$ contains a diameter of the circle $x^2 + y^2 - 2(a^2 - 7a + 11)x - 2(a^2 - 6a + 6)y + b^3 + 1 = 0$, then
(a) $a = 5$ and $b \notin (-1, 1)$
(b) $a = 1$ and $b \notin (-1, 1)$
(c) $a = 2$ and $b \in (-\infty, 1)$
(d) $a = 5$ and $b \in (-\infty, 1)$ **[2013, online]**

27. If a circle C passing through $(4, 0)$ touches the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ externally at a point $(1, -1)$, then the radius of the circle C is:
(a) 5 (b) $2\sqrt{5}$
(c) 4 (d) $\sqrt{57}$ **[2013, online]**

28. If two vertices of an equilateral triangle are $A(-a, 0)$ and $B(a, 0)$, $a > 0$, and the third vertex C lies above x-axis, then the equation of the circumcircle of ΔABC is:
(a) $3x^2 + 3y^2 - 2\sqrt{3}\ ay = 3a^2$
(b) $3x^2 + 3y^2 - 2ay = 3a^2$
(c) $x^2 + y^2 - 2ay = a^2$
(d) $x^2 + y^2 - \sqrt{3}\ ay = a^2$ **[2013, online]**

29. If the circle $x^2 + y^2 - 6x - 8y + (25 - a^2) = 0$ touches the axis of x, then a equals
(a) 0 (b) ± 4
(c) ± 2 (d) ± 3 **[2013, online]**

30. **Statement-1:** The only circle having radius $\sqrt{10}$ and a diameter along line $2x + y = 5$ is $x^2 + y^2 - 6x + 2y = 0$.
Statement-2: $2x + y = 5$ is a normal to the circle $x^2 + y^2 - 6x + 2y = 0$
(a) Statement-1 is false, Statement 2 is true.
(b) Statement-1 is true, Statement 2 is true, Statement-2 is a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is true, Statement-2 is true. Statement-2. is **not** a correct explanation for Statement-1. **[2013, online]**

31. If a circle of unit radius is divided into two parts by an arc of another circle subtending an angle 60° on the circumference of the first circle, then the radius of the arc is:
(a) $\sqrt{3}$ (b) 1/2
(c) 1 (d) $\sqrt{2}$ **[2013, online]**

32. Let C be the circle with circle with centre at (1, 1) and radius = 1. If T is the circle centre at (0, y), passing through origin and touching the circle C externally, then radius of T is equal to
(a) $\sqrt{3}/\sqrt{2}$ (b) $\sqrt{3}/2$
(c) 1/2 (d) 1/4 **[2014]**

33. If a point (1, 4) lies inside the circle $x^2 + y^2 - 6x - 10y + p = 0$ and the circle does not touch or intersect the coordinate axes, then the set of all possible values of p is the interval:
(a) (0, 25) (b) (25, 39)
(c) (9, 25) (d) (25, 29) **[2014, online]**

34. The set of all real values of λ for which exactly two common tangents can be drawn to the circle $x^2 + y^2 - 4x - 4y + 6 = 0$ and $x^2 + y^2 - 10x - 10y + \lambda = 0$ is the interval
(a) (12, 32) (b) (18,42)
(c) (12,24) (d) (18, 48) **[2014, online]**

35. For the two circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$ there is/are
(a) one pair of common tangents
(b) two pairs of common tangents
(c) three common tangents
(d) no common tangent **[2014, online]**

36. The equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter is
(a) $x^2 + y^2 + 3x + y - 11 = 0$
(b) $x^2 + y^2 + 3x + y + 1 = 0$
(c) $x^2 + y^2 + 3x + y - 2 = 0$
(d) $x^2 + y^2 + 3x + y - 22 = 0$ **[2014, online]**

37. The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ is:
(a) 1 (b) 2
(c) 3 (d) 4

38. If incentre of an equilateral triangle is (1, 1) and the equation of one side is $3x + 4y + 3 = 0$ than the equation of the circumcircle of this triangle is:
(a) $x^2 + y^2 - 2x - 2y - 2 = 0$
(b) $x^2 + y^2 - 2x - 2y - 14 = 0$
(c) $x^2 + y^2 - 2x - 2y + 2 = 0$
(d) $x^2 + y^2 - 2x - 2y - 7 = 0$ **[2015, online]**

39. If a circle passing through the point (– 1, 0) touches y-axis at (0, 2), then length of the chord of the circle along the x-axis is:
(a) $\frac{3}{2}$ (b) $\frac{5}{2}$
(c) 3 (d) 5 **[2015, online]**

40. If $y + 3x = 0$ is the equation of a chord of the circle, $x^2 + y^2 - 30x = 0$, then the equation of the circle with their chord as diameter is:
(a) $x^2 + y^2 + 3x + 9y = 0$ (b) $x^2 + y^2 - 3x + 9y = 0$
(c) $x^2 + y^2 - 3x - 9y = 0$ (d) $x^2 + y^2 + 3x + 9y = 0$
[2015, online]

41. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on
(a) a circle
(b) an ellipse which is not a circle
(c) a hyperbola (d) a parabola **[2016]**

42. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is:
(a) $5\sqrt{2}$ (b) $5\sqrt{3}$
(c) 5 (d) 10 **[2016]**

43. A circle passes through (–2, 4) and touches the y-axis at (0, 2). Which one of the following equations can represent a diameter of this circle?
(a) $2x - 3y + 10 = 0$ (b) $3x + 4y - 3 = 0$
(c) $4x + 5y - 6 = 0$ (d) $5x + 2y + 4 = 0$
[2016, online]

44. Equation of the tangent to the circle, at the point (1, –1), where centre is the point of intersection of the straight lines $x - y = 1$ and $2x + y = 3$ is:
(a) $x + 4y + 3 = 0$ (b) $3x - y - 4 = 0$
(c) $x - 3y - 4 = 0$ (d) $4x + y - 3 = 0$
[2016, online]

Previous Years' B-Architecture Entrance Examination Questions

1. The line $x \sin \alpha - y \cos \alpha = a$ touches the circle $x^2 + y^2 = a^2$, then.
(a) $\alpha \in [0, \pi]$ (b) $\alpha \in [-\pi, \pi]$
(c) α can have any value (d) $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **[2006]**

2. If a circle of area 16π has two of its diameters along the line $2x - 3y + 5 = 0$ and $x + 3y - 11 = 0$, then the equation of the circle is
(a) $x^2 + y^2 - 4x + 6y - 13 = 0$
(b) $x^2 + y^2 - 4x - 6y - 3 = 0$

(c) $x^2 + y^2 - 4x - 6y - 13 = 0$
(d) $x^2 + y^2 - 4x + 6y - 3 = 0$ **[2006]**

3. The circle passing through the distinct points $(1, t)$, $(t, 1)$ and (t, t) for all values if t passes through the point
(a) $(1, -1)$ (b) $(-1, 1)$
(c) $(-1, -1)$ (d) $(1, 1)$ **[2006]**

4. The value of k for which the circle $x^2 + y^2 - 4x + 6y + 3 = 0$ will bisect the circumference of the circle $x^2 + y^2 + 6x - 4y + k = 0$ is
(a) 53 (b) -53
(c) 47 (d) -47 **[2007]**

5. If the point $(2, k)$ lies outside the circles $x^2 + y^2 = 13$ and $x^2 + y^2 + x - 2y - 14 = 0$, then
(a) $K \in]-\infty, -2\ [U]\ 3, \infty[$
(b) $K \in]-3, -2\ [U]\ 3, 4[$
(c) $K \in]-3, -4[$
(d) $K \in]-\infty, -3\ [U]\ 4, \infty[$ **[2007]**

6. The shortest distance between the circles $x^2 + y^2 = 1$ and $(x - 9)^2 + (y - 12)^2 = 4$ is
(a) 9 (b) 11
(c) 12 (d) 14 **[2008]**

7. The midpoint of the chord intercepted by the circle $x^2 + y^2 = 16$ on the line through the point $(1, -2)$ and $(0, -1)$ is
(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, -\frac{3}{2}\right)$
(c) $\left(-\frac{1}{4}, -\frac{1}{4}\right)$ (d) $\left(\frac{3}{4}, \frac{1}{4}\right)$ **[2009]**

8. The circle $x^2 + y^2 - 6x - 10y + p = 0$ does not touch or intersect the axes and the point $(1, 4)$ lies inside the circle for all p in the interval
(a) $(25, 35)$ (b) $(25, 29)$
(c) $(0, 25)$ (d) $(0, 29)$ **[2010]**

9. The equation of a circle of area 22π square units for which each of the two lines $2x + y = 2$ and $x - y = -5$ is diameter, is:
(a) $x^2 + y^2 - 2x + 8y - 5 = 0$
(b) $x^2 + y^2 - 2x - 8y - 5 = 0$
(c) $x^2 + y^2 + 2x - 8y - 5 = 0$
(d) $x^2 + y^2 + 2x + 8y - 5 = 0$ **[2011]**

10. If a chord of a circle $x^2 + y^2 = 4$ with one extremity at $(1, \sqrt{3})$ subtends a right angle at the centre of this circle, then the coordinates of the other extremity of this chord can be:
(a) $(-1\ \sqrt{3})$ (b) $(1, -\sqrt{3})$
(c) $(-\sqrt{3}, -1)$ (d) $(\sqrt{3}, -1)$ **[2012]**

11. Consider $L_1 : 3x + y + \alpha - 2 = 0$ and $L_2 : 3x + y - \alpha + 3 = 0$, where α is a positive real number, and $C : x^2 + y^2 - 2x + 4y - 4 = 0$.

Statement-1: If L_1 is a chord of the circle C, then the line L_2 is not always a diameter of the circle C.

Statement-2: If L_1 is a diameter of the circle C, then the line L_2 is not a chord of the circle C. then
(a) both the statements are true
(b) both the statements are false
(c) Statement-1 is true and statement-2 is false
(d) statement-2 is true and statement-1 is false **[2013]**

12. A circle has two of its diameters along the lines $x + y = 5$ and $x - y = 1$ and has area 9π, then equation of the circle is
(a) $x^2 + y^2 - 6x - 4y - 3 = 0$
(b) $x^2 + y^2 - 6x - 4y - 4 = 0$
(c) $x^2 + y^2 - 6x - 4y + 3 = 0$
(d) $x^2 + y^2 - 6x - 4y + 4 = 0$ **[2014]**

13. The number of interger values of k for which the equation $x^2 + y^2 + (k - 1)x - ky + 5 = 0$ represents a circle whose radius can not excced 3, is:
(a) 10 (b) 11
(c) 4 (d) 5 **[2015]**

14. If the line $ax + by = 2$ $(a \neq 0)$ touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, thus $a + b$ is equal to:
(a) $\frac{-4}{3}$ (b) $\frac{-5}{3}$
(c) $\frac{-1}{3}$ (d) $\frac{1}{4}$ **[2015]**

Answers

Concept-based

1. (c)	**2.** (c)	**3.** (a)	**4.** (b)
5. (b)	**6.** (d)	**7.** (c)	**8.** (b)
9. (a)	**10.** (b)	**11.** (a)	**12.** (c)
13. (b)	**14.** (a)	**15.** (c)	**16.** (d)
17. (c)	**18.** (b)	**19.** (c)	**20.** (d)

Level 1

21. (d)	**22.** (c)	**23.** (b)	**24.** (b)
25. (c)	**26.** (d)	**27.** (a)	**28.** (c)
29. (d)	**30.** (b)	**31.** (b)	**32.** (a)
33. (c)	**34.** (c)	**35.** (d)	**36.** (a)
37. (b)	**38.** (c)	**39.** (b)	**40.** (a)
41. (a)	**42.** (b)	**43.** (a)	**44.** (c)
45. (a)	**46.** (b)	**47.** (b)	**48.** (a)
49. (c)	**50.** (c)	**51.** (c)	**52.** (c)

53. (c)	**54.** (b)	**55.** (b)	**56.** (d)
57. (a)	**58.** (b)	**59.** (d)	**60.** (d)
61. (b)	**62.** (a)	**63.** (a)	**64.** (c)
65. (d)	**66.** (b)	**67.** (b)	**68.** (d)
69. (a)	**70.** (c)	**71.** (c)	**72.** (a)
73. (b)	**74.** (b)	**75.** (a)	**76.** (b)
77. (a)	**78.** (c)	**79.** (a)	**80.** (d)

Level 2

81. (b)	**82.** (a)	**83.** (b)	**84.** (a)
85. (a)	**86.** (d)	**87.** (b)	**88.** (a)
89. (b)	**90.** (b)	**91.** (d)	**92.** (d)
93. (b)	**94.** (a)	**95.** (c)	**96.** (b)
97. (b)	**98.** (b)	**99.** (c)	**100.** (c)
101. (c)	**102.** (c)	**103.** (a)	**104.** (b)
105. (b)	**106.** (b)	**107.** (b)	**108.** (a)
109. (a)	**110.** (a)		

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (c)	**3.** (a)	**4.** (b)
5. (c)	**6.** (d)	**7.** (b)	**8.** (b)
9. (c)	**10.** (c)	**11.** (c)	**12.** (d)
13. (b)	**14.** (b)	**15.** (a)	**16.** (a)
17. (b)	**18.** (d)	**19.** (d)	**20.** (d)
21. (d)	**22.** (b)	**23.** (c)	**24.** (c)
25. (b)	**26.** (d)	**27.** (a)	**28.** (a)
29. (b)	**30.** (a)	**31.** (c)	**32.** (d)
33. (c)	**34.** (b)	**35.** (d)	**36.** (a)
37. (c)	**38.** (b)	**39.** (c)	**40.** (b)
41. (d)	**42.** (b)	**43.** (a)	**44.** (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (b)	**3.** (d)	**4.** (a)
5. (d)	**6.** (c)	**7.** (a)	**8.** (b)
9. (c)	**10.** (d)	**11.** (c)	**12.** (d)
13. (c)	**14.** (c)		

Hints and Solutions

Concept-based

1. Centre of the circle is (1, 3) or (– 1, – 3) and radius is $\sqrt{10}$.

2. Given equation represents a circle if $b = 3$ and the equation is
$x^2 + y^2 + 4x - 6y + 3 = 0$ whose radius $= \sqrt{10}$.

3. Line meets the axes at (4, 0) and (0, 3). Equation of the circle on the line joining this points as a diameter is
$(x - 0)(x - 4) + (y - 0)(y - 3) = 0.$
$\Rightarrow x^2 + y^2 - 4x - 3y = 0.$

4. (1, 2) lies inside the circle
$\Rightarrow 1 + (2)^2 - 7 + 15(2) - c < 0 \Rightarrow c < 28$
(3, 4) lies outside the circle
$\Rightarrow (3)^2 + (4)^2 - 7(3) + 15(4) - c > 0 \Rightarrow c < 64.$

5. Centre of the circle S is (– 3, 7) and its radius is 8
distance of (– 3, 7) from the line is $\left|\frac{-21+21+58}{\sqrt{49+9}}\right|$
$= \sqrt{58} < 8$
So the line L is a chord of S.

6. Let $y = mx$ be a tangent through the origin
$\Rightarrow \frac{m(7)-(-1)}{\sqrt{1+m^2}} = \sqrt{25} \Rightarrow m = \frac{3}{4}$ or $\frac{-4}{3}$
Product of the slopes = – 1 ⇒ The required angle is $\frac{\pi}{2}$.

7. From Geometry $PA \cdot PB = (PC)^2$
$= (5)^2 + (6)^2 - 12 = 49.$

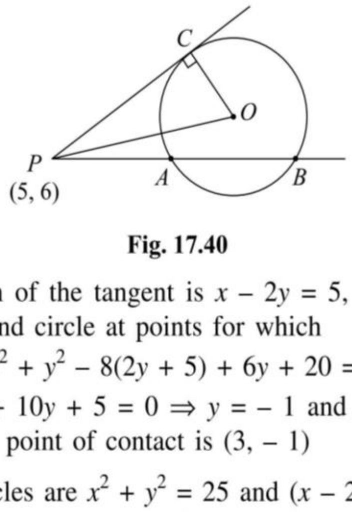

Fig. 17.40

8. Equation of the tangent is $x - 2y = 5$, which meets the second circle at points for which
$(2y + 5)^2 + y^2 - 8(2y + 5) + 6y + 20 = 0$
$\Rightarrow 5y^2 + 10y + 5 = 0 \Rightarrow y = -1$ and $x = 3$ so the required point of contact is (3, – 1)

9. Two circles are $x^2 + y^2 = 25$ and $(x - 2)^2 + (y + 3)^2 = 25$. So the equation of the required chord is
$(x - 2)^2 + (y + 3)^2 - (x^2 + y^2) = 0.$
$\Rightarrow 4x - 6y - 13 = 0$

10. Centres will lie on the line through (0, k) perpendicular to y-axis ($x = 0$) required line $y = k$.

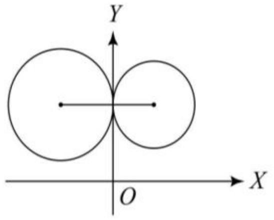

Fig. 17.41

11. Let the centre be $(h, h - 1)$ then the equation of the circle is $(x - h)^2 + (y - h + 1)^2 = 9$. As it passes through (7, 3), $(7 - h)^2 + (3 - h + 1)^2 = 9$.

$\Rightarrow h = 4$ or 7 and the equation of the circle is $x^2 + y^2 - 8x - 6y + 16 = 0$
or $x^2 + y^2 - 14x - 12x + 76 = 0$.

12. The line passes through the centre of the circle, so AB is a diameter of the circle and $\lfloor APB$ being an angle in a semi circle is a right angle.

13. Let the equation of the circle be $x^2 + y^2 + 2gx + 2gy = 0$. As it cuts the given circle orthogonally.
$2g(-2) + 2g(-3) = 10 \Rightarrow g = -1$.

14. Equation of the common chord is $y = x$ and the end points of which are (0, 0) and $(a/2, a/2)$. Equation of the circle on this chord as diameter is

$x(x - a/2) + y(y - a/2) = 0$.

$2x^2 + 2y^2 - ax - ay = 0$.

15. Centre will lie on the line parallel to both the given lines and equidistant from both.

16. Centre of the first circle is (2, 3) and radius 4-centre of the second circle is (– 1, – 1) and radius, distance between the centre = 5 = 4 + 1. So they touch externally and hence 3 common tangents.

17. $r_1 - r_2 > C_1 C_2$ where $r_1 = b$, $r_2 = a$.
$C_1 (0, 0)$, $C_2 (2, 3)$
$\Rightarrow b - a > \sqrt{(0-2)^2 + (0-3)^2} = \sqrt{13}$

18. Equation of the given circle can be $(x \pm 1)^2 + (y \pm 1)^2 = 1$

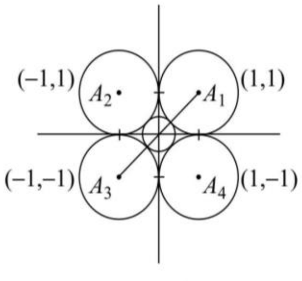

Fig. 17.42

Centre of the required circle is the origin and radius

$= \frac{1}{2}(A_1 A_3 - (1 + 1))$

$= \frac{1}{2}(2\sqrt{2} - 2) = \sqrt{2} - 1$

So its equation is $x^2 + y^2 = (\sqrt{2} - 1)^2$

19. If (h, k) is any point on the locus,

$$\frac{\sqrt{h^2 + k^2 - 4k + 3}}{\sqrt{h^2 + k^2 + 6k + 5}} = \frac{3}{2}$$

$\Rightarrow 5h^2 + 5k^2 + 70k + 33 = 0$

Required locus is $5x^2 + 5y^2 + 70y + 33 = 0$

20. Diameters intersect at the point (3, 4) so the centre of the circle is (3, 4) and the distance of the centre from the line $x + 2y + 4 = 0$ is $\left|\frac{3+2(4)+4}{\sqrt{3^2+4^2}}\right| = 3$.

So the equation of the circle is

$(x - 3)^2 + (y - 4)^2 = 9$ or $x^2 + y^2 - 6x - 8y + 16 = 0$

Level 1

21. $(x - a/2)^2 + (y - b/2)^2 = (a^2 + b^2)/4$
22. Coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$
23. radius = the distance between the given point and the point of intersection of the diameters.
24. $(2x^2 + 2y^2 + 4x - 7y - 25) + \lambda(x^2 + y^2 + 13x - 3y) = 0$ passes through (1, 1)
25. Normal is the line joining the point and the centre of the circle.
26. $\frac{\cos\theta}{\sqrt{1 + \tan^2\theta}} = \pm 2$. (Length of the perpendicular from the centre on the tangent is equal to the radius of the circle). $\Rightarrow \cos\theta = \pm\sqrt{2}$
27. $\frac{x+3}{2} = 2x, \frac{3+5}{2} = y$
28. Distance of the centre from both the axes is equal to the radius.
29. Let the radius of the circle be r (> 0), then the coordinates of the centre are $(\pm r, \pm r)$ as it touches both the axes, since it touches the line $x + y = 1$

$\frac{\pm r \pm r - 1}{\sqrt{2}} = \pm r \Rightarrow (2r \pm 1)^2 = 2r^2$ or $r = 1/\sqrt{2}$.

$\Rightarrow 2r^2 \pm 4r + 1 = 0 \Rightarrow r = \frac{\pm 4 \pm \sqrt{16 - 8}}{4}$, which gives only two positive values of r and hence two circles in Iquadrant. $r = 1/\sqrt{2}$ also gives two circles one in II and the other in IV quadrant.

(Draw the figure and find the answer)

30. As the circle passes through the centre of the given circle, it will touch the given circle internally and so its radius is half the radius of the given circle.
31. $|x_1 - x_2| = 3$ where $x_1 + x_2 = 2\alpha$, $x_1x_2 = 4$.
32. Distance between the centers is equal to the sum of the radii.
33. Equation of a tangent is $x \cos\theta + y \sin\theta = r$, $A\left(\frac{r}{\cos\theta}, 0\right)$ $B\left(0, \frac{r}{\sin\theta}\right)$, $P\left(\frac{r}{\cos\theta}, \frac{r}{\sin\theta}\right) = P(x, y)$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{r^2}$$

34. $h = 1, 2$; $k = 1, -6$; $(h, k) = (1, 1), (1, -6), (2, 1), (2, -6)$ circle is $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - 25 = 0$
35. $P(3, 4)$ does not lie on the line L, $S(3, 4) < 0$ so P is inside the circle length of the perpendicular from the centre $(1, 2)$ on L is greater than the radius of S.
36. $x = 0$ is one of the common tangents and the others two tangents intersect at P on x-axis, the line joining the centres. The triangle being isosceles, the centroid lies on x-axis. Let $y = mx + c$ be a common tangent then $P(-c/m, 0)$

Also $\frac{-3m+c}{\sqrt{1+m^2}} = 3$, $\frac{m+c}{\sqrt{1+m^2}} = 1 \Rightarrow c/m = -3$ and

$P(3, 0)$. Distance of the centroid from the base $x = 0$ is equal to (1/3) the distance of P from the base.
37. C and D are the other ends of the diameters through. A and B respectively.
38. Equation of the circle is $x^2 + y^2 - ax - ay = 0$, the tangent at the origin is $x + y = 0$. Vertices of the triangle are $(a, 0)$, $(0, a)$ and $(0, 0)$.
39. $(AP)^2 = (\cos\alpha + 1)^2 + \sin^2\alpha = 2(1 + \cos\alpha)$ $= 4\cos^2\alpha/2$ etc.
40. The points in (*c*) and (*d*) do not lie on the circle. Verify the result for the points in (*a*) and (*b*).
41. Common chord is of maximum length when it passes through the centre of the circle with smaller radius. $C_2 : x^2 + y^2 - 2ax - 2by + a^2 + b^2 - 25 = 0$. Equation of the common chord is $2ax + 2by - a^2 - b^2 + 9 = 0$ which passes through the origin if $a^2 + b^2 = 9$.
42. OA, OB make equal angles $\pi/4$ with the line joining the origin 0 and the centre $(1, -2)$ of the circle.

$$\frac{m+2}{1-2m} = \pm 1 \Rightarrow 3m^2 - 8m - 3 = 0.$$

43. The radius is smallest when the chord is a diameter.
44. B is the mid-point of AM and lies on the circle.
45. Radius is half the distance between the given parallel tangents.
46. $\lambda_1, \lambda_2, \lambda_3$ are in G.P. Length of the tangents are $l_i = \sqrt{2\lambda_i c \cos\theta}$ $(i = 1, 2, 3)$
47. Chords of contacts from the origin and from (g, f) are respectively $gx + fy + c = 0$ and $2gx + 2fy + g^2 + f^2 + c = 0$
48. Let the other end of the diameter be (h, k), then the centre of the circle is $\left(\frac{p+h}{2}, \frac{k+q}{2}\right)$ and as it touches x-axis, its radius is $\frac{k+q}{2}$

so $\left(p - \frac{p+h}{2}\right)^2 + \left(q - \frac{k+q}{2}\right)^2 = \left(\frac{k+q}{2}\right)^2$.
49. Let the centre be at the origin, then

$$\frac{\sqrt{x^2 + y^2 - a^2}}{\sqrt{x^2 + y^2 - b^2}} = \frac{b}{a}.$$

$\Rightarrow x^2 + y^2 = a^2 + b^2$.
50. No part of the line lies in the third quadrant.
51. Let each side of the triangle be x, each side subtends an angle of 120° at the centre of the circle.

so $\cos 120^\circ = \frac{a^2 + a^2 - x^2}{2a^2}$

$\Rightarrow x^2 = 3a^2$ and area of the triangle is $\frac{\sqrt{3}}{4}x^2$.

52. Let $A(0, 0)$, $B(a, 0)$ and $P(h, k)$, then

$$\frac{h^2 + k^2}{(h-a)^2 + k^2} = p^2$$

Locus is $x^2 + y^2 + \frac{2ap^2}{1-p^2}x - \frac{a^2p^2}{1-p^2} = 0$.
53. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and the circle be $x^2 + y^2 = a^2$ $l_1^2 = x_1^2 + y_1^2 - a^2$, $l_2^2 = x_2^2 + y_2^2 - a^2$, and $x_1x_2 + y_1y_2 = a^2$ as A and B are conjugate w.r.t. the circle.
54. Centres of the circles lie on y-axis as x-axis is the common chord. Let the centres be $(0, f_1)$ and $(0, f_2)$. The circles intersect orthogonally, so $\frac{f_1}{a} \times \frac{f_2}{a} = -1$

$\Rightarrow f_1f_2 = -a^2$.

As they touch $y = mx + c$.

$$\frac{f_1 - c}{\sqrt{1+m^2}} = a^2 + f_1^2$$

$\Rightarrow m^2f_1^2 + 2cf_1 + (1 + m^2)a^2 - c^2 = 0$ which gives two values of f_1 say, f_1 and f_2 such that

$$f_1 f_2 = \frac{(1+m^2)\,a^2 - c^2}{m^2} = -a^2$$

$\Rightarrow \quad c^2 = (2m^2 + 1)a^2.$

55. Centre is (8, – 9), radius is 12, origin lies outside the circle. Centre being in the fourth quadrant no part of the circle lies in second quadrant.

56. Centre is C (2, 1), radius is 5, the greatest distance is $PC + 5$.

57. Longest ray is the tangent from (4, 3) to the circle whose length = 12 = radius of the circle.

58. $y = a$ is a tangent at $A(0, a)$ and $y = -a$ is a tangent at $B(0, -a)$, tangent at C ($a\cos\theta$, $a\sin\theta$) meet the line $y = a$ at P such that $PA = \dfrac{a(1-\sin\theta)}{\cos\theta}$, similarly

$QB = \dfrac{a(1+\sin\theta)}{\cos\theta}$.

$PA = PC$ and $QB = QC \quad \Rightarrow \quad PC.QC = a^2$.

59. Centre of the bigger circle is $A(3, 4)$ and radius is c, centre of the smaller circle is $O(0, 0)$ and radius is 3. $c > OA + 3$ i.e., $c > 8$.

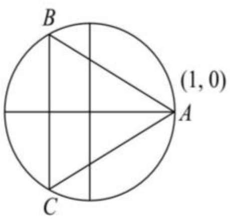

Fig. 17.43

60. $(x-1)\cos\theta + (y-1)\sin\theta = 1$ is a tangent to the circle $(x-1)^2 + (y-1)^2 = 1$.

61. Let the equation of the circle be $x^2 + y^2 + 2Gx + 2Fy + \lambda = 0$. Then $2Gg = \lambda + c = 2Gg_1$

$\Rightarrow \quad G = 0,\ \lambda = -c$ and

$2Gh + 2Fk = \lambda + a \quad \Rightarrow \quad 2F = \dfrac{a-c}{k}$.

62. Let $P(h, k)$ be the point where the common tangents to these circles intersect. Then the circles subtend equal angles at P. Lengths of the tangents from P to two circles are proportional to their radii.

$$\frac{h^2 + k^2 + 6h + 2k + 7}{h^2 + k^2 + 2h + 6k + 4} = \frac{3}{6}$$

$\Rightarrow \quad h^2 + k^2 + 10h - 2k + 10 = 0.$

63. The point of intersection is given by

$(3\alpha^2 - \alpha - 2)x = \alpha^2 - 1 \Rightarrow (3\alpha + 2)x = \alpha + 1\,[\alpha \neq 1]$

$\lim\limits_{\alpha \to 1} \Rightarrow x = \dfrac{2}{5},\ y = \dfrac{1}{25}$.

64. Limiting points are $A\left(-\dfrac{g}{1+\lambda_1}, \dfrac{-\lambda_1 f}{1+\lambda_1}\right)$ and $B\left(-\dfrac{g}{1+\lambda_2}, \dfrac{-\lambda_2 f}{1+\lambda_2}\right)$ where λ_1, λ_2 are the roots of

$\lambda^2(f^2 - k) - k\lambda + g^2 = 0$

OA is perpendicular to OB

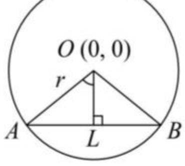

Fig. 17.44

$\Rightarrow \dfrac{\lambda_1 f}{g} \times \dfrac{\lambda_2 f}{g} = -1$

$\Rightarrow \lambda_1 \lambda_2 = -\dfrac{g^2}{f^2}$.

$\Rightarrow \dfrac{g^2}{f^2 - k} = -\dfrac{g^2}{f^2} \Rightarrow \dfrac{k}{f^2} = 2.$

65. Centre is (0, 2), radius is 2, length of the perpendicular from the centre on the line is $1/\sqrt{2}$

so $\cos\theta = \dfrac{1/\sqrt{2}}{2} = \dfrac{1}{2\sqrt{2}}$

66. Required locus is $\dfrac{3x + 4y + 1}{\sqrt{9+16}} = \pm 5$

67. Equation of the chord is

$y - 3 = (2/3)(x + 2) \Rightarrow 2x - 3y = -13$

which intersects the diameter $x + 5y = 0$ at (– 5, 1) whose distance from the centre is $\sqrt{25+1} < 9$, the radius.

68. Equations of AB and AC are

$y = \pm \dfrac{1}{\sqrt{3}}(x - 1)$

which intersect the circle $x^2 + y^2 = 1$ at points for which

$\Rightarrow 3(1 - x^2) = (x - 1)^2$

$\Rightarrow x = 1$ or $x = -1/2$

Taking $x = -1/2$, $B\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$, $C\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$

Length of each side = $\sqrt{3}$.

69. Equation of the circle is

$(x-2)^2 + (y-3)^2 = 3^2$, centre is (2,3) and the radius 3 so it touches x-axis at (2, 0)

70. For $\lambda = 0$, each passes through the centre of the other.

71. AB is the chord of contact of the tangents from (1, 8) to the given circle so its equation is

$x(1) + y(8) - 3(x + 1) - 2(y + 8) - 11 = 0$

$\Rightarrow x - 3y + 15 = 0$

72. Two curves intersect at the points for which

$4x^2 - 9(8x - x^2) = 36$

$\Rightarrow 13x^2 - 72x - 36 = 0$

$\Rightarrow (x - 6)(13x + 6) = 0$

As the circle lies on the positive side of x-axis, $x = 6$

$\Rightarrow y = \pm 2\sqrt{3}$ and the required circle is

$(x - 6)(x - 6) + (y - 2\sqrt{3})(y + 2\sqrt{3}) = 0$

73. Each side subtends an angle $\frac{2\pi}{9}$ at the origin.

$AB = 2 \Rightarrow AL = 1$

$\underline{|AOL} = 20°$

$1 = AL = r \sin 20° \Rightarrow 1/r = \sin 20°$

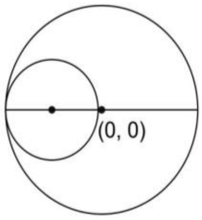

Fig. 17.45

74. Radius in statement-1 is

$$\sqrt{\cos^2\theta + \sin^2\theta + \frac{1 + 4\cos\theta + \cos 2\theta}{2}}$$

$= \sqrt{(1/2)(3 + 4\cos\theta + 2\cos^2\theta - 1)}$

$= \sqrt{(1 + \cos\theta)^2} \Rightarrow$ Statement–1 is true.

In statement-2 the circle is on the line joining $(\cos\theta, \sin\theta)$ and $(\sin\theta, \cos\theta)$ as diameter so the radius is $\frac{1}{2}\sqrt{(\cos\theta - \sin\theta)^2 + (\sin\theta - \cos\theta)^2}$ and the statement is true.

75. Statement-2 is true and using in statement-1, we have $x^2 + 2y^2 - 6x - 12y + 23 + \lambda(4x^2 + 2y^2 - 20x - 12y + 35) = 0$ which represents a circle if $1 + 4\lambda = 2 + 2\lambda$ $\Rightarrow \lambda = 1/2$ and the equation of the circle through the points of intersection of $S_1 = 0$ and $S_2 = 0$ is

$6(x^2 + y^2) - 32x - 36y + 81 = 0$

whose centre is (8/3, 3) and radius is $\sqrt{47}/3\sqrt{2}$. So the statement-1 is also true.

76. $x\cos\theta + y\sin\theta = a$, $x\cos(\pi/2 + \theta) + y\sin(\pi/2 + \theta) = a$

$\Rightarrow (x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 = 2a^2$

$\Rightarrow x^2 + y^2 = 2a^2$

$\Rightarrow$ statement-1 is true, statement-2 is also true but does not lead to statement-1.

77. Statement-2 is true as circle C_1 touches the circle C_2 internally and the diameter of C_1 is equal to the radius of the circle C_2. Using it in statement-1, the centre of the required circle lies inside C.

78. Statement-1 is true as the distance of the centre (3, 4) from x-axis is equal to the radius. Statement-2 is false, because $\frac{4-3}{\sqrt{2}} = a \Rightarrow a = 1/\sqrt{2}$.

79. Statement-2 is True, using it in statement-1, common chord is $10x = r^2 + 16$, which passes through (5, 0) if $r^2 = 34$ (if it passes through (0, 0), $r^2 + 16 = 0$ which is not possible for any real value of r)

80. Statement-1 is not true as the equation represents a pair of straight lines. In statement-2, $(x + 2)^2 + (y + 3)^2 = 0$ which gives a circle of zero radius and hence the point (– 2, – 3). So statement-2 is true.

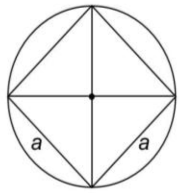

Fig. 17.46

Level 2

81. $x^2 + y^2 + ax = 0$ passes through the centre (0, 0) of the circle $x^2 + y^2 = c^2$. Circles will touch each other if the radius $a/2$ of the first circle is equal to half the radius of the 2nd circle

$\Rightarrow c = \pm a \Rightarrow c^2 - a^2 = 0$

82. Since each side subtends an angle of 120° at the centre (0, 0) of the circle.

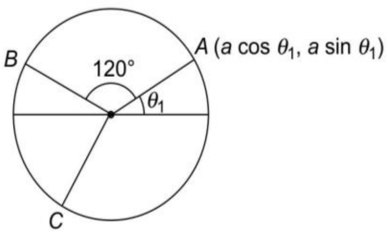

Fig. 17.47

$\theta_2 = 120° + \theta_1$, $\theta_3 = 240° + \theta_1$

So $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$

$= \cos\theta_1 + \cos(120° + \theta_1) + \cos(240° + \theta_1)$

$= \cos\theta_1 + 2\cos(180° + \theta_1)\cos(-120°)$

$= \cos\theta_1 + 2 \times \cos\theta_1 (-1/2) = 0$

Similarly $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0$

83. Equation of the circle $C: x(x - 4) + y^2 = 0$

Equation of the common chord of C_1 and C_2 is $8x - 11 = 0$ which meets the circle C at points for which $x = 11/8$ and $y^2 = \frac{4\times 11}{8} - \left(\frac{11}{8}\right)^2 = \frac{231}{64}$

Hence the required intercept

$= 2\times\sqrt{\frac{231}{64}} = \frac{\sqrt{231}}{4}$

84. We have $a = b$ and $a^2 + ab - 16 = 0$

$\Rightarrow a = b = \pm 2\sqrt{2}$

Equation of the circle is $x^2 + y^2 + 2x + 2y \pm 2 = 0$

Radius of the circle is $\sqrt{(-1)^2 + (-1)^2 \pm 2} = 0$ or 2.

85. Circle passing through the points (–3, 0), (–1, 0) and (0, 3/2) is $x^2 + y^2 + 4x - (7/2)y + 3 = 0$

which passes through the point $(0, 1/\lambda)$ if

$\frac{1}{\lambda^2} - \frac{7}{2\lambda} + 3 = 0 \Rightarrow 6\lambda^2 - 7\lambda + 2 = 0$

86. $2a^2 = (2r)^2$

where $r^2 = (-2)^2 + (-5)^2 - 21$

$\Rightarrow a^2 = 2 \times 8 = 16$

$\Rightarrow a = 4$

87. $A\ (-1, 1)$, $P(2\cos\theta, 2\sin\theta)$

coordinates (x, y) of R which divides the join of A and P in the ratio 3:2 are

$x = \frac{6\cos\theta - 2}{5}$ $y = \frac{6\sin\theta + 2}{5}$

Locus of R is $(5x + 2)^2 + (5y - 2)^2 = 36$

which is a circle with centre (–2/5, 2/5)

88. Equation of the circle is

$(x - \alpha)(x - \beta) + (y - \alpha')(y - \beta') = 0$

$\Rightarrow x^2 + y^2 - (\alpha + \beta)x - (\alpha' + \beta')y + \alpha\beta + \alpha'\beta' = 0$

$\Rightarrow x^2 + y^2 - -\frac{22}{15}x - \frac{22}{8}y + \frac{8}{15} + \frac{15}{8} = 0$

$\Rightarrow 120(x^2 + y^2) - 22(8x + 15y) + 289 = 0$

89. Let $C_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$C_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

$C : x^2 + y^2 + 2gx + 2fy + c = 0$

If C_1 and C_2 cut C orthogonally.

then $2gg_1 + 2ff_1 = c + c_1$

$2gg_2 + 2ff_2 = c + c_2$

$\Rightarrow 2(g_1 - g_2)g + 2(f_1 - f_2)f = c_1 - c_2 \cdots I$

Radical axis of C_1 and C_2 is

$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

which passes through $(-g, -f)$, the centre of C (using I)

90. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy = 0$ s.t $\sqrt{g^2 + f^2} = 3k$

centroid of the triangle OAB is

$(x, y) = \left(\frac{-2g}{3}, \frac{-2f}{3}\right)$

$g^2 + f^2 = 9k^2 \Rightarrow 9(x^2 + y^2) = 4 \times 9k^2$
$\Rightarrow x^2 + y^2 = 4k^2$

91. Let the vertices of the triangle be $A_i\ (x_i, y_i)$, $i = 1, 2, 3$. If (x, y) is any point on the locus then $\sum_{i=1}^{3}\left[(x - x_i)^2 + (y - y_i)^2\right] = c^2$, c being constant.

$$\Rightarrow 3(x^2 + y^2) - 2x\sum_{i=1}^{3} x_i - 2y\sum_{i=1}^{3} y_i + \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} y_i^2 - c^2 = 0$$

Which represents a circle with centre at

$\left(\frac{\sum_{i=1}^{3} x_i}{3}, \frac{\sum_{i=1}^{3} y_i}{3}\right)$, the centroid of the triangle $A_1 A_2 A_3$

92. Equation of a tangent is

$y = \pm\sqrt{3}x \pm 4\sqrt{3}\sqrt{1 + 3}$

which meets the axis of x at $x = \pm 8$

Length of the tangent from (± 8, 0) of the circle $x^2 + y^2 = 48$ is $\sqrt{(\pm 8)^2 - 48} = \sqrt{16} = 4$

which is the required length of the side of the rhombus.

93. Equation of any circle on this chord is

$x^2 + y^2 - a^2 + \lambda(x\cos\alpha + y\sin\alpha - p) = 0$

Chord is a diameter of the circle if the centre $(-\lambda\cos\alpha/2, -\lambda\sin\alpha/2)$ lies on the chord.

$\Rightarrow \lambda = -2p$

94. $(ax + by + c)^2 + (bx - ay + d)^2 = r^2(a^2 + b^2)$

$\Rightarrow (a^2 + b^2)(x^2 + y^2) + 2(ac + bd)x$
$+ 2(bc - ad)y + c^2 + d^2 = r^2(a^2 + b^2)$

which is a circle with radius r

95. Vertices of the square be $(\pm a/2, \pm a/2)$
Equation of the inscribed circle is

$x^2 + y^2 = a^2/4$ (1)

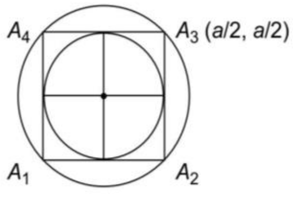

Fig. 17.48

Equation of the circumscribed circle is

$x^2 + y^2 = a^2/2$ (2)

If $P(x, y)$ is a point on (1), then

$\sum_{i=1}^{4}(PA_i)^2 = [x \pm (a/2)]^2 + [y \pm (a/2)]^2$

$= 4(x^2 + y^2) + 4\left(\frac{a^2 + a^2}{4}\right) = a^2 + 2a^2 = 3a^2$

Similarly $\sum_{i=1}^{4}(QA_i)^2 = 2a^2 + 2a^2 = 4a^2$

96. Equation of a tangent from (–2,0) to the circle $x^2 + y^2 = 1$ is $\sqrt{3}y = x + 2$

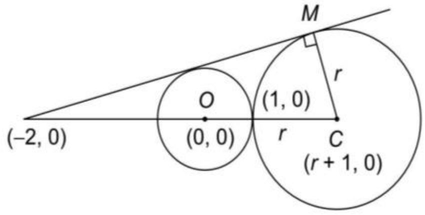

Fig. 17.49

Let the centre of the required circle be $C\ (r + 1, 0)$ and radius r.

Then $CM = \frac{r+1+2-0}{\sqrt{1+3}} = r$

$\Rightarrow r = 3$ and the equation of the circle is $(x - 4)^2 + y^2 = (3)^2 \Rightarrow x^2 + y^2 - 8x + 7 = 0$

97. Let the equations of the two tangents be

$x \cos\theta + y \sin\theta = a$ and
$x \cos(\pi/4 + \theta) + y \sin(\pi/4 + \theta) = a$

$\Rightarrow x\,[\cos\theta - (1/\sqrt{2})(\cos\theta - \sin\theta)]$

$+ y\,[\sin\theta - (1/\sqrt{2})(\cos\theta + \sin\theta)] = 0$

$\Rightarrow x[(\sqrt{2} - 1)\cos\theta + \sin\theta] - [\cos\theta -$
$(\sqrt{2} - 1)\sin\theta]\, y = 0$

$\Rightarrow \frac{x}{\cos\theta - (\sqrt{2}-1)\sin\theta} = \frac{y}{(\sqrt{2}-1)\cos\theta + \sin\theta}$

$= k = \frac{x^2 + y^2}{(\sqrt{2}-1)^2 + 1}$

then from $x \cos\theta + y \sin\theta = a$, we get

$\Rightarrow k \cos\theta\,[\cos\theta - (\sqrt{2} - 1)\sin\theta]$

$+ k \sin\theta\,[(\sqrt{2} - 1)\cos\theta + \sin\theta] = a$

$\Rightarrow k = a$

$\Rightarrow x^2 + y^2 = [(\sqrt{2} - 1)^2 + 1]\, a^2 = 2(2 - \sqrt{2})a^2$

98. Let the equation of the circle cutting the given circle orthogonally be

$x^2 + y^2 + 2gx + 2fy + c = 0$

then $2g(2) + 2f(-3) = c + 9$

and $2g(-2) + 2f(3) = 4 + c$

$\Rightarrow 8g - 12f = 5$

So locus of the centre $(-g, -f)$ is $8x - 12y + 5 = 0$

99. Let the equation of the circle be

$x^2 + y^2 - 2gx - 2fy = 0$

then $(g - 5)^2 + (f - 2)^2 = g^2 + f^2$

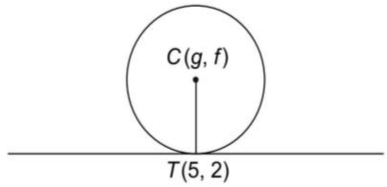

Fig. 17.50

$\Rightarrow 10g + 4f - 29 = 0$ (1)

$C\ (g, f)$ lies on the line through $T(5, 2)$ perpendicular to the

tangent $3x - 2y - 11 = 0$

so $2g + 3f - 16 = 0$ (2)

Solving (1) and (2) we get $g = \frac{23}{22}, f = \frac{51}{11}$

100. Equation of AB is $x/a + y/b = 1$ or $bx + ay = ab$

If $P(h, k)$ is the required point of reflection

then OP is perpendicular to AB and AB is equidistant from O and P.

So $\frac{k}{h} = \frac{a}{b}$ and $\frac{-ab}{\sqrt{a^2+b^2}} = \pm\frac{hb + ak - ab}{\sqrt{a^2+b^2}}$

Taking the –ve sign, we get $hb + ak = 2ab$

Solving, $h = \frac{2ab^2}{a^2+b^2}$, $k = \frac{2a^2b}{a^2+b^2}$

Note + ve sign gives $(h, k) = (0, 0)$

101. Let the equation of the circle be
$x^2 + y^2 + 2gx + 2fy + c = 0$

As it cuts the circle $x^2 + y^2 = a^2$ orthogonally

$c = a^2$ and since it passes through the given point

$\left(\frac{a^2 p}{p^2+q^2}\right)^2 + \left(\frac{a^2 q}{p^2+q^2}\right)^2 + \frac{2ga^2p}{p^2+q^2} + \frac{2fa^2q}{p^2+q^2} + a^2$
$= 0$

$\Rightarrow a^2 + 2gp + 2fq + p^2 + q^2 = 0$

$\Rightarrow p^2 + q^2 + 2\,pg + 2qf + a^2 = 0$

Showing that the circle passes through (p, q).

102. Let the other extremities of the diameter through $P\ (2, 3)$ be $A\ (a_1, b_1)$, $B\ (a_2, b_2)$ then equations of the two circles are

$(x - a_1)\ (x - 2) + (y - b_1)\ (y - 3) = 0$

and $(x - a_2)\ (x - 2) + (y - b_2)\ (y - 3) = 0$

So equation of the common chord is

$(x - 2)\ (a_2 - a_1) + (y - 3)\ (b_2 - b_1) = 0$

Since AB makes an angle $\pi/6$ with x–axis

$$\frac{b_2 - b_1}{a_2 - a_1} = \frac{1}{\sqrt{3}}$$

103. Equation of the circle passing through $(-1, 0)$ and $(1, 0)$ is $x^2 + y^2 + 2fy - 1 = 0$ (f being a parameter)

Let (h, k) be the point of contact of the tangent parallel to $y = x$

Equation of the tangent at (h, k) is

$hx + (k + f)\ y + fk - 1 = 0$

so that $\frac{-h}{k+f} = 1 \Rightarrow f = -\ (h + k)$

since (h, k) lies on the circle

$h^2 + k^2 + 2fk - 1 = 0$

$\Rightarrow h^2 + k^2 - 2k\ (h + k) - 1 = 0$

$\Rightarrow h^2 - k^2 - 2hk - 1 = 0$

Locus of (h, k) is $x^2 - y^2 - 2xy = 1$

104. Line $y = x$ meets the

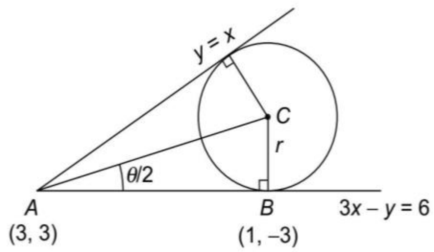

Fig. 17.51

line $3x - y = 6$ at $A(3, 3)$, B be $(1, -3)$, C the centre of the circle, and $\angle CAB = \theta/2$

Then the radius $r = A\ B \tan \theta/2 = \sqrt{40}\ \tan (\theta/2)$

Also $\tan \theta = \frac{3-1}{1+3} = \frac{1}{2} \Rightarrow \tan (\theta/2) = -\ 2 + \sqrt{5}$

$\Rightarrow r = \sqrt{40}\,(-2 + \sqrt{5}) = 1.6.$

105. Let the equation of the larger circle be $x^2 + y^2 = a^2$ and the line $x + y = 2$ meet the smaller circle at A and B, the larger circle at C and D.

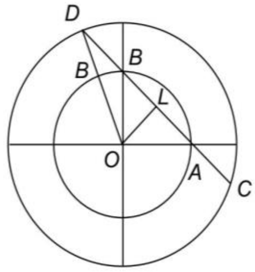

Fig. 17.52

$\Rightarrow OA = OB = 2$ and $OC = OD = a$

Let OL be perpendicular to AB

then $OL = \frac{2}{\sqrt{1+1}} = \sqrt{2}$, $LD = LB + BD$

$= (1/2)\sqrt{4+4} + 1$

$\Rightarrow a^2 = (OD)^2 = (OL)^2 + (LD)^2 = 2 + \left(\sqrt{2}+1\right)^2$

$= 5 + 2\sqrt{2}$.

106. Centre of the required circle is the point of reflection of the centre $(1, 0)$ in the line $x + y = 2$ and its radius is equal to the radius 1 of the given circle. So its equation is $(x - 2)^2 + (y - 1)^2 = 1$

107. If the chord AB makes an angle of 30° in the major segment, it makes an angle of 60° at the centre so that the length of the chord is equal to the radius of the circle. Let OL be perpendicular to AB.

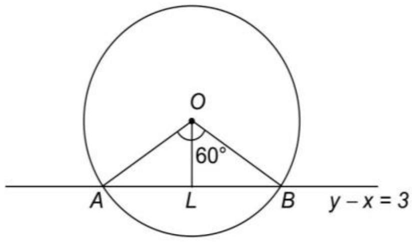

Fig. 17.53

then $(OL)^2 = \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$

$(AL)^2 = (OA)^2 - (OL)^2 = k^2 - (9/2)$

$k^2 = (AB)^2 = 4\ (AL)^2 = 4(k^2 - (9/2))$

$\Rightarrow k^2 = 6$

108. Let the radii of the three circles be

$1,\ 1 - d,\ 1 - 2d$

Line $y = x + 1$ will cut all the three circles in real distinct points if the distance of the centre $(0, 0)$

from the line is less than the radius of the circle with smallest radius.

$$\frac{1}{\sqrt{2}} < 1 - 2d \Rightarrow d < \frac{2-\sqrt{2}}{4}$$

109. Circle with centre at (3, – 2) must touch the given circle internally. If r is the required radius then.

Distance between the centre = difference of the radii

$$\sqrt{(3-2)^2 + (-2+(1/2))^2} = \sqrt{2^2 + (1/2))^2} - r$$

$$\Rightarrow r = (1/2)(\sqrt{17} - \sqrt{13})$$

110. Let the equation of the required circles be

$x^2 + y^2 + 2gx + 2fy = 0$

Intercept on the line $y = 2x$ is $\frac{2g+4f}{5}$

Intercept on the line $x = 2y$ is $\frac{4g+2f}{5}$

$$\frac{2g+4f}{5} = 2 \times \frac{4g+2f}{5} \Rightarrow g = 0$$

and the required equation is

$x^2 + y^2 + 2fy = 0$ or $x^2 + y^2 - 2\lambda y = 0$

where λ is a parameter.

Previous Years' AIEEE/JEE Main Questions

1. Equation of any tangent to the parabola $y^2 = 8ax$ is

$$y = mx + \frac{2a}{m}$$

Since it touches the circle $x^2 + y^2 = 2a^2$

$$\left|\frac{\frac{2a}{m}}{\sqrt{1+m^2}}\right| = \sqrt{2}a$$

$\Rightarrow (m^2 + 2)(m^2 - 1) = 0 \quad \Rightarrow \quad m = \pm 1$

and the required equations of the tangents are

$$y = \pm(x + 2a)$$

2. Chord will subtenes a right angle at the centre (0, 0) of the circle.

Equation of the pair of lines through (0, 0) and the points of intersection of the circle and the chord is

$x^2 + y^2 = (y - mx)^2$

$\Rightarrow \quad (1 - m)^2x^2 + 2mxy = 0$

These are at rightangles if $1 - m^2 = 0$
$\Rightarrow \quad m = \pm 1$

3. Required points will lie on or outside the circle $x^2 + y^2 = 4$ and on or inside the circle $x^2 + y^2 = 64$ and hence $4 \le x^2 + y^2 \le 64$

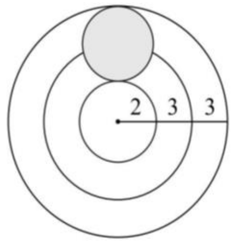

Fig. 17.54

4. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2f = 0$$

Since it passes through (1, 0), $g = -\frac{1}{2}$.

As the circle passes through the centre of (0, 0) of the circle $x^2 + y^2 = 9$, it will touch the circle internally. So the radius of the required circle is 3/2.

$$\Rightarrow \quad g^2 + f^2 = \frac{9}{4} \quad \Rightarrow \quad f^2 = 2 \quad \Rightarrow \quad f = \pm\sqrt{2}$$

and the centre of the required circle in

$$(-g, -f) = \left(\frac{1}{2}, \pm\sqrt{2}\right)$$

5. Radius of the circle = $\frac{2}{3} \times$ length of the median. So the required equation of the circle is

$$x^2 + y^2 = \left(\frac{2}{3} \times 3a\right)^2 \quad \Rightarrow \quad x^2 + y^2 = 4a^2$$

6. Distance between the centre of the given circles

$= \sqrt{(1-4)^2 + (3+1)^2} = 5$

Radius of the given circles are r and $\sqrt{16+1-8} = 3$ circle will intersect in two distinct points if $r - 3 < 5 < r + 3$ or $2 < r < 8$.

7. $\pi r^2 = 154 \quad \Rightarrow \quad r = 7$

Point of intersection of the diameters is (1, –1) which is the centre of the circle and hence its equation is

$(x - 1)^2 + (y + 1)^2 = 49$

or $\quad x^2 + y^2 - 2x + 2y = 47$

8. Let (h, k) be centre of the required circle. As this circle passes through (a, b) its equation is

$(x - h)^2 + (y - k)^2 = (a - h)^2 + (b - k)^2$

or $x^2 + y^2 - 2hx - 2ky - (a^2 + b^2 - 2ha - 2kb) = 0$

This circle will cut

$x^2 + y^2 - 4 = 0$

orthogonally if

$(-2h)(0) + (-2k)(0) = -4 - (a^2 + b^2 - 2ha - 2kb)$

$\Rightarrow \; 2ha + 2kb - (a^2 + b^2 + 4) = 0$

Thus, required is $2ax + 2by - (a^2 + b^2 + 4) = 0$

9. Let the other end of the diameter be $B(h, k)$. Its equation is

$(x - p)(x - h) + (y - q)(y - k) = 0$

As this circle touches the x-axis, i.e. $y = 0$, the equation

$(x - p)(x - h) + kq = 0$ or $x^2 - (p + h)x + ph + kq = 0$ must have equal roots, i.e.

$(p + h)^2 - 4(ph + kq) = 0$

$\Rightarrow \; (p - h)^2 = 4kq$

Thus required locus is $(x - p)^2 = 4qy$

10. As centre lies on both $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$, centre of the circle must be $(1, -1)$. If r is the radius of the circle, we must have

$2\pi r = 10\pi \Rightarrow r = 5$

$\therefore$ Required equation is

$(x - 1)^2 + (y + 1)^2 = 25$

or $\; x^2 + y^2 - 2x + 2y - 23 = 0$

11. Equation of any circle through A and B is

$$x^2 + y^2 - 2x + \lambda(x - y) = 0$$

Its centre is $\left(1 - \dfrac{\lambda}{1}, \dfrac{\lambda}{2}\right)$.

AB will be a diameter if centre lies on AB, that is, if

$$1 - \frac{\lambda}{2} = \frac{\lambda}{2} \text{ or } \lambda = 1.$$

Thus, equation of desired circle is

$$x^2 + y^2 - 2x + x - y = 0$$

or $$x^2 + y^2 - x - y = 0$$

12. P and Q lie on the common chord of the two given circles. An equation of the common chord is $S_1 - S_2 = 0$

or $(x^2 + y^2 + 2ax + cy + a) - (x^2 + y^2 - 3ax + dy - 1) = 0$

or $5ax + (c - d)y + a + 1 = 0$ (1)

Since $5x + by - a = 0$ passes through P and Q. This line must be identical to (1). Therefore

$$\frac{5a}{5} = \frac{c - d}{b} = \frac{a + 1}{-a}$$

$\Rightarrow \; -a^2 = a + 1 \;$ or $\; a^2 + a + 1 = 0$

There is no real value of a satisfying this equation.

13. Let (h, k) be centre of circle passing through (a, b). Then its equation is

$(x - h)^2 + (y - k)^2 = (a - h)^2 + (b - k)^2$

or $x^2 + y^2 - 2hx - 2ky - (a^2 + b^2 - 2ah - 2bk) = 0$

This will meet the circle $x^2 + y^2 - p^2 = 0$ orthogonally if

$(0)(-2h) + (0)(-2k) = -p^2 - (a^2 + b^2 - 2ah - 2bk)$

$\Rightarrow 2ah + 2bk - (a^2 + b^2 + p^2) = 0$

This, required locus is

$2ax + 2by - (a^2 + b^2 + p^2) = 0$

14. Let centre of circle be (α, β). Radius of circle $= \beta$. As the circle touches the circle with centre at $(0, 3)$ and radius 2, distance between centres = sum of the radii

$\Rightarrow \; \sqrt{(\alpha - 0)^2 + (\beta - 3)^2} = \beta + 2$

$\Rightarrow \alpha^2 + \beta^2 - 6\beta + 9 = \beta^2 + 4\beta + 4$

$\Rightarrow \alpha^2 = 10\beta - 5$

$\therefore$ locus of centre $x^2 = 10y - 5$, which is a parabola.

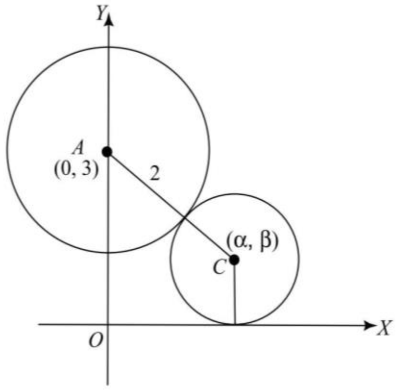

Fig. 17.55

15. Let $M(p, q)$ be the mid-point of a chord AB of the circle subtending an angle of $2\pi/3$ at the centre.

As ΔOAB is an isosceles triangle $OM \perp AB$.

$AM^2 = OA^2 - OM^2$

$= 9 - (p^2 + q^2)$

$\Rightarrow \; AM = \sqrt{9 - (p^2 + q^2)}$

$\Rightarrow \; AB = 2(AM) = 2\sqrt{9 - (p^2 + q^2)}$

By the law of cosines

$$\cos\frac{2\pi}{3}=\frac{OA^2+OB^2-AB^2}{2(OA)(OB)}$$

$\Rightarrow\ -\frac{1}{2}=\frac{9+9-4(9-p^2-q^2)}{2(3)(3)}$

$\Rightarrow\ -9=-18+4\,(p^2+q^2)$

$\Rightarrow\ p^2+q^2=9/4$

Thus, required locus is

$x^2+y^2=9/4$

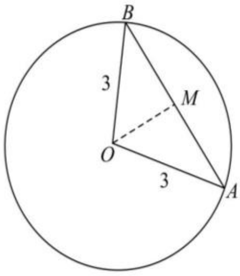

Fig. 17.56

16. Centre is the point of intersection of the diameters

$3x-4y-7=0$

and $2x-3y-5=0$

that is, the centre is (1, – 1). Also, radius of circle is 7.

$\therefore$ Equation of circle is

$(x-1)^2+(y+1)^2=7^2$

or $x^2+y^2-2x+2y-47=0$

17. Radius of circle = $|k|$

Equation of circle

$(x-h)^2+(y-k)^2=|k|^2$

As it passes through (– 1, 1), we get

$(-1-h)^2+(1-k)^2=k^2$

$\Rightarrow\ h^2+2h+2(1-k)=0$

As h is real,

$\Rightarrow\ 4-8\,(1-k)\geq 0$

$\Rightarrow\ k>\frac{1}{2}$

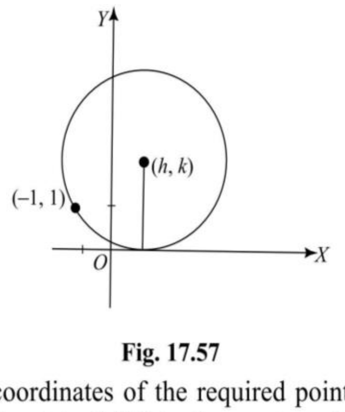

Fig. 17.57

18. Let the coordinates of the required point Q be (α,β) then mid-point of PQ is the centre of the circle so

$\frac{\alpha+1}{2}=-1,\ \frac{\beta}{2}=-2$

$\Rightarrow\ \alpha=-3,\ \beta=-4$

So the required point is (– 3, – 4)

19. Let S be (1, 0) and R be (– 1, 0)

Let $P(x, y)$ be a point such that

$$\frac{PS}{PR}=\frac{1}{3} \qquad (1)$$

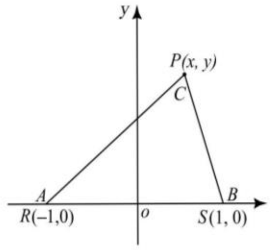

Fig. 17.58

Note that A, B, C lie on the locus of P. Equation (1) gives

$9\,PS^2=PR^2$

$\Rightarrow 9[(x-1)^2+y^2]=(x+1)^2+y^2$

$\Rightarrow 8x^2+8y^2-20x+8=0$

$$\Rightarrow x^2+y^2-\frac{5}{2}x+1=0 \qquad (2)$$

Centre of this circle is (5/4, 0)

As A, B, C lie on (2), circumcentre of ΔABC is (5/4, 0)

20. Equation of any circle through P and Q is

$S_1+\lambda\,(S_2-S_1)=0$

$\Rightarrow x^2+y^2+3x+7y+2p-5$

$+\lambda\,(-x-5y-p^2-2p+5)=0$

It will pass through (1, 1) is $(1+1+3+7+2p-5)$ $+\,l(-1-5-p^2-2p+5)=0$

$\Rightarrow (7+2p=)-\lambda(p+1)^2=0$

$\Rightarrow\ \lambda=\frac{7+2p}{(p+1)^2}$

Thus, there exist a circle through P, Q and (1, 1) for all values of p except – 1.

21. The line $3x-4y=m$ will intersect the circle in two distinct points if length of perpendicular from the centre (2, 4) to the line < radius

$\Rightarrow\ \frac{|(3)(2)-(4)(4)-m|}{\sqrt{9+16}}<\sqrt{4+16+5}$

$\Rightarrow\ |m+10|<25\Rightarrow -25<m\ +10<25$

$\Rightarrow 35<m<15$

22. Let equation of circle be

$x^2+y^2+2gx+2fy+c=0$

As it passes through (1, 0) and (0, 1), we get

$1 + 2g + c = 0, \quad 1 + 2f + c = 0$

If r is radius of the circle, then

$r^2 = g^2 + f^2 - c$

$= \frac{1}{4}(1+c)^2 + \frac{1}{4}(1+c)^2 - c$

$= \frac{1}{2}(1+c^2)$

Note that r^2 will be least if $c = 0$.

$\therefore \quad g = f = -1/2, c = 0$

Thus, equation of required circle is $x^2 + y^2 - x - y = 0$

23. $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ can touch each other only internally. See figures. This is possible if $a = c$ or $c = -a$, that is, if $c = |a|$.

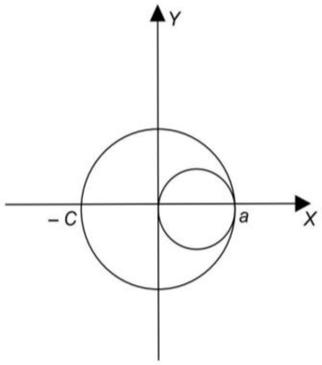

Fig. 17.59

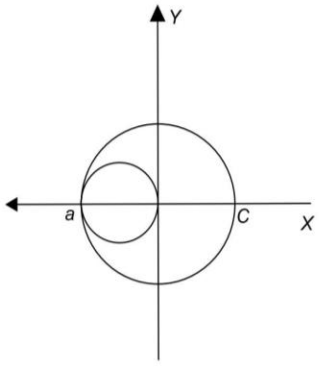

Fig. 17.60

24. As the circle touches the x-axis at (1, 0) its centre is $(1, k)$ where $|k|$ is radius of the circle. Equation of the circle is

$$(x-1)^2 + (y-k)^2 = k^2$$

As it passe through (2, 3), we get $1^2 + (3-k)^2 = k^2$ $\Rightarrow 10 - 6k = 0$

$\Rightarrow 2k = 10/3$

Thus, diameter of the circle is 10/3.

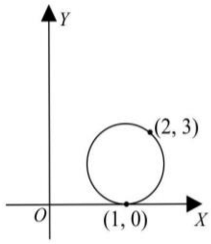

Fig. 17.61

25. Centre of circle lies on the line $x = 3$. Let centre be $(3, k)$, then radius of circle is $|k|$. We have

$(3-1)^2 + (k+2)^2 = |k|^2$

$\Rightarrow k^2 + 4k + 8 = k^2 \Rightarrow k = -2.$

This circle passes through (5, –2) as $(5-3)^2 + (-2 + 2)^2 = 4 = (\text{radius})^2$.

Alternatively, note that (5, – 2) is the other end point of the diameter with one end point at (1, – 2).

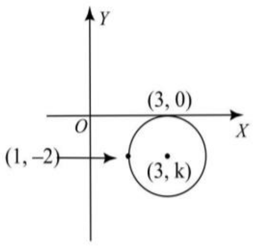

Fig. 17.62

26. Lines intersect at (1, 1)

$\Rightarrow \quad a^2 - 7a + 11 = 1$ and $a^2 - 6a + 6 = 1$

$\Rightarrow \quad a = 5$

So the equation of the circle is

$(x-1)^2 + (y-1)^2 = 1 - b^3.$

Radius of the circle is $1 - b^3 > 0 \Rightarrow \quad b^3 < 1$

$\Rightarrow \quad b \in (-\infty, 1)$

so $\quad a = 5, b \in (-\infty, 1)$

27. Let the centre of the circle c be (h, k) then $(h-1)^2 + (k+1)^2 = (h-4)^2 + k^2$

$\Rightarrow \quad 3h + k = 7 \qquad (1)$

Equation of the tangent at (1, – 1) to the given circle is

$x - y + 2(x+1) - 3(y-1) - 12 = 0$

$\Rightarrow \quad 3x - 4y - 7 = 0$

(h, k) lies on the line $\perp$ to this tangent.

$\Rightarrow \quad 4h + 3k - 1 = 0 \qquad (2)$

Solving (1) and (2) we get $h = 4$, $k = -5$ so that the radius of the circle is 5.

28. Length of each side is $2a$.

Length of the median

$= \sqrt{(4a)^2 - a^2} = \sqrt{3}a$

centre of the circumcircle is

$\left(0, \frac{1}{3}(\sqrt{3}a)\right) = \left(0, \frac{a}{\sqrt{3}}\right)$

Hence the equation of the circumcirele is

$x^2 + \left(y - \frac{a}{\sqrt{3}}\right)^2 = a^2 + \left(\frac{a}{\sqrt{3}}\right)^2$

$\Rightarrow 3x^2 + 3y^2 - 2\sqrt{3}ay = 3a^2$

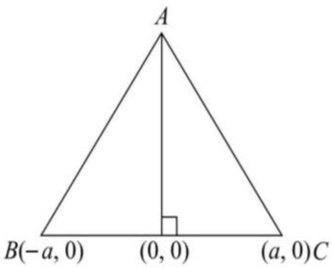

Fig. 17.63

29. y-coordinate of the centre of the circle is equal to its radius

$\Rightarrow \quad 4 = (3)^2 + (4)^2 - (25 - a^2)$

$\Rightarrow \quad a^2 = 16 \quad \Rightarrow \quad a = \pm 4$

30. Statement-2 is true, as every diameter of the circle is a normal to the circle and $2x + y = 5$ is a diameter. Statement-1 is false as any point on the diameter can be taken as the centre of the circle.

31. Let the equation of the circle be $x^2 + y^2 = 1$ and the common chord QR of the two circles subtend an angle of 60º at $P(-1, 0)$

Equation of PQ is

$y = \frac{1}{\sqrt{3}}(x+1)$

which meets the circle $x^2 + y^2 = 1$ at $Q\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Line through Q perpendicular to PQ is

$y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x - \frac{1}{2}\right)$

which meets x-axis at (1, 0), the centre of the arc.

Hence the radius of the arc $= \sqrt{\left(1-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

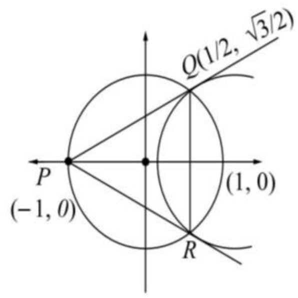

Fig. 17.64

32. $A(0, y)$ and $B(1, 1)$ be the given centres of the circles

AB = sum of the radii

$= y + 1$

Also $\quad AB = \sqrt{(y-1)^2 + 1}$

$\Rightarrow \quad (y + 1)^2 = (y - 1)^2 + 1 \quad \Rightarrow \quad y = \frac{1}{4}$

So the required radius of T is $\frac{1}{4}$

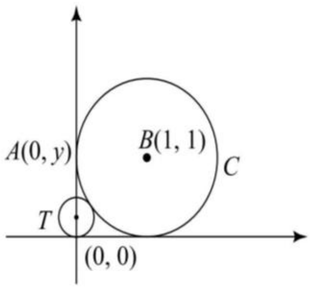

Fig. 17.65

33. The given circle is

$(x - 3)^2 + (y - 5)^2 = 34 - p \qquad (1)$

Its centre is (3, 5) and radius is $\sqrt{34 - p}$ As (1) does not touch or intersect the axes, $\sqrt{34-p} <$ min {3, 5}

$\Rightarrow 25 < p$.

Also, (1, 4) lies inside (1),

$(1 - 3)^2 + (4 - 5)^2 < 34 - p$

$\Rightarrow p < 29$

$\therefore p \in (25, 29)$

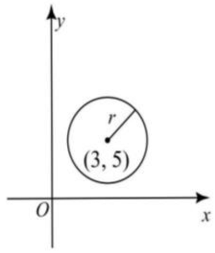

Fig. 17.66

34. There will be exactly two tangents if the two circles intersect, that is, if

$$|r_1 - r_2| < C_1 C_2 < r_1 + r_2 \quad (1)$$

where $r_1 = \sqrt{2}$

$r_2 = \sqrt{50-\lambda}$

$C_1 = (2, 2)$

$C_2 = (5, 5)$

Thus, (1) gives

$$\left|\sqrt{50-\lambda} - \sqrt{2}\right| < \sqrt{9+9} < \sqrt{50-\lambda} + \sqrt{2}$$

$\Rightarrow \quad -3\sqrt{2} < \sqrt{50-\lambda} - \sqrt{2} < 3\sqrt{2} < \sqrt{50-\lambda} + \sqrt{2}$

$\Rightarrow \quad 0 \le \sqrt{50-\lambda} < 4\sqrt{2}, \sqrt{50-\lambda} > 2\sqrt{2}$

$\Rightarrow \quad 50 - \lambda < 32, \; 50 - \lambda > 8$

$\Rightarrow \quad 18 < \lambda < 42 \Rightarrow \quad \lambda \in (18, 42)$

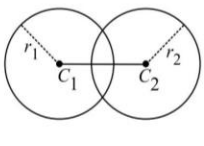

Fig. 17.67

35. We can write $x^2 + y^2 - 2y = 0$ as $x^2 + (y-1)^2 = 1$

This circle lies inside the $x^2 + y^2 = 4$

$\therefore$ There is no common tangent to the two circles.

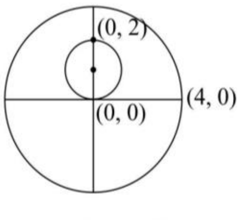

Fig. 17.68

36. An equation of circle passing through intersection of

$$x^2 + y^2 = 16 \quad (1)$$

and $\quad 3x + y + 5 = 0 \quad (2)$

is $\quad x^2 + y^2 - 16 + \lambda(3x + y + 5) = 0$

Its centre is $\left(-\frac{3}{2}\lambda, -\frac{1}{2}\lambda\right)$

Line (2) will be a diameter if centre lies on L, that is, if

$$3\left(-\frac{3}{2}\lambda\right) + \left(-\frac{1}{2}\lambda\right) + 5 = 0 \Rightarrow \quad \lambda = 1$$

$\therefore \quad$ An equation of required circle is

$$x^2 + y^2 + 3x + y - 11 = 0$$

37. Centres and radii of two circles are:

$$C_1(2, 3);\; r_1 = 5$$

and $C_2(-3, -9);\; r_2 = 8$

As $C_1C_2 = 13 = r_1 + r_2$

Two circles touch externally, therefore they have three tangents in common.

38. In an equilateral triangle

$$\frac{r}{R} = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$\Rightarrow \quad R = 2r$

Here $r = \dfrac{|3+4+3|}{\sqrt{9+16}} = 2$

Thus, $R = 4$

Also, in an equilateral triangle incentre and circumcentre coincide.

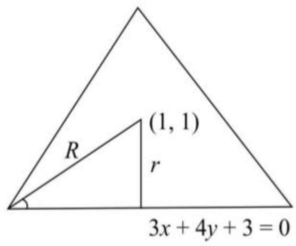

Fig. 17.69

Thus, equation of circumcircle is

$$(x-1)^2 + (y-1)^2 = 4^2$$

or $\qquad x^2 + y^2 - 2x - 2y - 14 = 0.$

39. Centre of the circle lies on the line $y = 2$. Let centre of circle be $(a, 2)$, then its radius is $|a|$.

Thus, $(a+1)^2 + 2^2 = |a|^2$

$\Rightarrow a = -5/2$

An equation of circle

$$(x + 5/2)^2 + (y-2)^2 = 25/4$$

For point of intersection with the x-axis, put $y = 0$, so that

$$(x + \frac{5}{2})^2 + 4 = \frac{25}{4} \Rightarrow \; x = -\frac{5}{2} \pm \frac{3}{2} = -4, -1$$

Thus, length of chord of the along the x-axis is $-1 - (-4) = 3$.

40. An equation of circle through intersection of $x^2 + y^2 - 30x = 0$ and $3x + y = 0$ is

$$x^2 + y^2 - 30 + \lambda(3x + y) = 0 \quad (1)$$

where λ is some real number. Centre of (1) is $\left(15 - \frac{3}{2}\lambda, -\frac{1}{2}\lambda\right)$.

The line $3x + y = 0$ will be a diameter of (1) if

$$3\left(15 - \frac{3}{2}\lambda\right) + \left(-\frac{1}{2}\lambda\right) = 0$$

$\Rightarrow 45 - 5\lambda = 0$ or $\lambda = 9$.

Thus, (1) can be written as

$x^2 + y^2 - 3x + 9y = 0$

41. Centre and radius of circle

$x^2 + y^2 - 8x - 8y - 4 = 0$ (1)

is (4, 4) and 6.

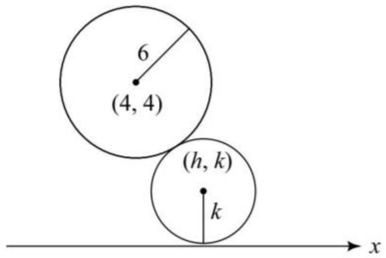

Fig. 17.70

Let (h, k) be centre of the circle touching (1) and the x-axis. Then its radius is k. We have

$$\sqrt{(h-4)^2 + (k-4)^2} = k + 6$$

$(h - 4)^2 = (k + 6)^2 - (k - 4)^2 = 10(2k + 2)$

Thus, equation of required locus is

$(x - 4)^2 = 20(y + 1)$ which is a parabola.

42. An equation of circle with centre at $(-3, 2)$ and radius r is

$(x + 3)^2 + (y - 2)^2 = r^2$

or $x^2 + y^2 + 6x - 4y + 13 - r^2 = 0$ (1)

Common chord of (1) and

$x^2 + y^2 - 4x + 6y - 12 = 0$ (2)

is $10x - 10y + 25 - r^2 = 0$

It will be diameter of (2) if it passes through centre of (2), that is, if it passes through $(2, -3)$. Therefore $10(2) - 10(-3) + 25 - r^2$

$$\Rightarrow r^2 = 75 \quad \Rightarrow r = 5\sqrt{3}$$

43. Note that centre of circle lies on $y = 2$. Let centre of circle be $(h, 2)$, then radius of circle is $|h|$.

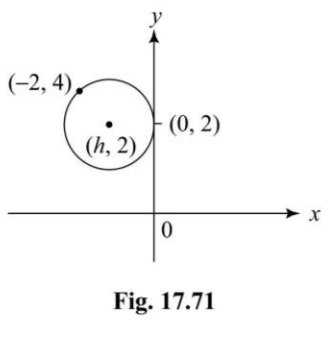

Fig. 17.71

$\therefore$ Equation of circle is $(x - h)^2 + (y - 2)^2 = |h|^2$

As it passes through $(-2, 4)$

$(-2 - h)^2 + (4 - 2)^2 = h^2$

or $(2 + h)^2 + 2^2 = h^2$

$\Rightarrow 4 + 4h + h^2 + 4 = h^2$

$\Rightarrow h = -2$

Thus centre of circle is $(-2, 2)$

A diameter of circle is $2x - 3y + 10 = 0$

44. Centre of circle is (4/3, 1/3)

Thus equation of circle is

$$\left(x - \frac{4}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 = \left(1 - \frac{4}{3}\right)^2 + \left(-1 - \frac{1}{3}\right)^2$$

$$\text{or } x^2 + y^2 - \frac{8}{3}x - \frac{2}{3}y = 0$$

An equation of tangent at $(1, -1)$ is

$$x(1) + (-1)y - \frac{4}{3}(x+1) - \frac{1}{3}(y-1) = 0$$

or $3x - 3y - 4x - 4 - y + 1 = 0$

or $-x - 4y - 3 = 0$ or $x + 4y + 3 = 0$

Previous Years' B-Architecture Entrance Examination Questions

1. Length of the perpendicular from the centre (0, 0) of the circle to the line is equal to a, the radius of the circle

$$\Rightarrow \quad \frac{|a|}{\sqrt{\sin^2\alpha + \cos^2\alpha}} = |a| \quad \text{which is true values of}$$

α.

2. $\pi r^2 = 16\pi \quad \Rightarrow \quad r = 4$

centre of the circle is (2, 3), the point of intersection of the diameters. Hence the required equation is.:

$(x + 2)^2 + (y - 3)^2 = 16$

$\Rightarrow x^2 + y^2 - 4x - 6y - 3 = 0$

3. Equation of the circle passing through $(1, t)$, $(t, 1)$ and (t, t) is $x^2 + y^2 - (t + 1)(x + y) + 2t = 0$

which passes through (1, 1) for all values of t.

4. The common chord of the two circles is a diameter of the circle $x^2 + y^2 + 6x - 4y + x = 0$

equation of the common chord is

$10x - 10y + (k - 3) = 0$ which passes through (3, 2) if

$$10(-3) - 10(2) + (k - 3) = 0 \quad \Rightarrow k = 53$$

5. $2^2 + k^2 > 13 \quad \Rightarrow \quad k^2 > 9$ (1)

$\Rightarrow \quad \left(2+\frac{1}{2}\right)^2 + (k-1)^2 > \left(\frac{1}{2}\right)^2 + (1)^2 + 14 = \frac{61}{4}$

$\Rightarrow \quad (k-1)^2 > 9$ (2)

From (1) and (2) we get

$k \in \,]-\infty, -3[\, \cup \,] \, 4, \infty[$

6. Required shortest distance between the circles is AB

$= C_1C_2 - (C_1A) - (BC_2)$

$= \sqrt{9^2 + 12^2} - 1 - 2$

$= 15 - 1 - 2 = 12.$

Note C_1 (0, 0), $C_2(a, 12)$ are the 1 and 2 are the radii of the two circles respectively.

7. Equation of the line through (1, –2) and (0, –1) is $x + y + 1 = 0$ which meets the circle $x^2 + y^2 = 16$ at points for which $x^2 + (x+1)^2 = 16$

$\Rightarrow \quad \frac{x_1 + x_2}{2} = -\frac{1}{2}$, so x-coordinate of the mid-point of the chord is $-\frac{1}{2}$. Substituting in $x + y + 1 = 0$, y-coordinates of the mid-point is also $-\frac{1}{2}$ and hence the required point is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

8. radius of the circle < min (3, 5)

$\Rightarrow \quad \sqrt{9 + 25 - p} < 3 \quad \Rightarrow 25 < p$

distance of (1, 4) from the centre < radius

$\Rightarrow \quad (3-1)^2 + (4-5)^2 < 34 - p$

$\Rightarrow \quad p < 29.$

Hence $p \in (25, 29)$

9. $\pi r^2 = 22\pi \quad \Rightarrow r = \sqrt{22}$

centre of the circle is (– 1, 4), the point of intersection of the given lines.

Hence the equation of the circle is

$(x+1)^2 + (y-4)^2 = 22$

$\Rightarrow \quad x^2 + y^2 + 2x - 8y - 5 = 0$

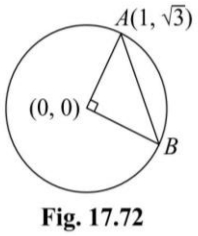

Fig. 17.72

10. Other extremity lies on the line

$y = \frac{1}{\sqrt{3}} x$

$\Rightarrow \quad x = -\sqrt{3} y$

Required point is $(\sqrt{3}, -1)$

11. Centre of the circle C is (1, – 2) and radius = 3

L_1, is a chord of C

if $\frac{3 - 2 - \alpha + 3}{\sqrt{1+9}} < 3 \quad \Rightarrow \quad (\alpha - 1)^2 < 90$

$L2$, is a diameter of c

if $3 - 2 - \alpha + 3 = 0 \quad \Rightarrow \quad \alpha = 4$

Showing that statement-1 is true

Next, L_1 is a diameter of C

if $3 - 2 + \alpha - 2 = 0 \quad \Rightarrow \quad \alpha = 1$

L_2, is a chord of C

if $\frac{3 - 2 - \alpha + 3}{\sqrt{1+9}} < 3$ which is true for $\alpha = 1$

showing that statement-2 is false.

12. $\pi r^2 = 9 \quad \Rightarrow \quad r = 3$

Centre of the circle is (3, 2)

Equation of the circle is $(x-3)^2 + (y-2)^2 = 9$

$\Rightarrow \quad x^2 + y^2 - 6x - 4y + 4 = 0$

13. $r \le 3 \quad \Rightarrow \quad r^2 \le 9$

$\Rightarrow \quad 0 \le \frac{1}{4}(k-1)^2 + \frac{1}{4}k^2 - 5 \le 9$

$\Rightarrow \quad 0 \le k^2 - 2k + 1 + k^2 - 20 \le 36$

$\Rightarrow \quad 19 \le 2k^2 - 2k \le 55$

$\Rightarrow \quad 39 \le (2k-1)^2 \le 111$

As k is an integer,

$2k - 1 = \pm 7, \pm 9$

$\Rightarrow k$ can take four integral values.

14. As $ax + by = 2$ passes through centre (0, 2) of the second circle,

$a(0) + 2b = 2 \quad \Rightarrow \quad b = 1.$

Also, as $ax + by - 2 = 0$ touches

$x^2 + y^2 - 2x - 3 = 0$

$\frac{|a + b(0) - 2|}{\sqrt{a^2 + b^2}} = 2 \quad \Rightarrow \quad |a - 2| = 2\sqrt{a^2 + 1}$

$\Rightarrow a^2 - 4a + 4 = 4a^2 + 4$

$\Rightarrow 3a^2 + 4a = 0 \quad \Rightarrow a = -4/3$ as $a \ne 0.$

Now, $a + b = -\frac{4}{3} + 1 = -\frac{1}{3}$