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Objectives

After studying the material of this chapter, you should be able to :

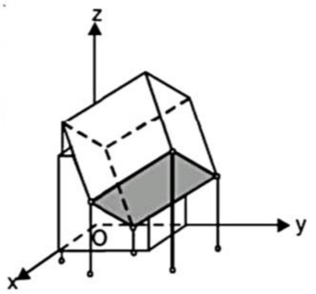
- Understand the direction -ratios and direction cosines of a line.
- Understand to find the equations of lines and planes in space under various conditions.
- Understand to find the angle between two lines, two planes and a line & a plane.
- Understand to find the shortest distance between two skew lines.
- Understand to find the distance of a point from a plane.

3D GEOMETRY

Chapter at Glance

INTRODUCTION

In the previous class, we have already studied plane geometry, known as analytic geometry in 2-dimensions. Now we extend our scope of analytic geometry to 3-dimensions. In the case of plane analytic geometry we confined ourselves to co-ordinate methods. In the same class we had elementary idea regarding 3-dimensional geometry in Cartesian form. Now we shall see that the study of 3-dimensional geometry becomes very simple with the help of vectors. We shall obtain most of the results in vector form by using the techniques of vector algebra. Nevertheless, we shall also translate them in the Cartesian form, which presents a better geometric and analytic picture in many situations.



In this chapter, we will learn following concept :

- Direction cosines and direction ratios of a line joining two points.
- About the equations of lines and planes in space under different conditions
- Angle between two lines, two planes, a line and a plane.
- Shortest distance between two skew lines.
- Distance of a point from a plane.

SUB CHAPTER 11.0 Review

11.1. THREE DIMENSIONAL CARTESIAN FORM

Let us take three axes in such a way that they form a *right–handed* system. This means if a screw, placed at the origin, is turned in the sense from positive *x-axis* to positive *y*-axis, it moves in the direction of positive *z-axis. The three perpendicular co-ordinate axes define the co-ordinate planes.*

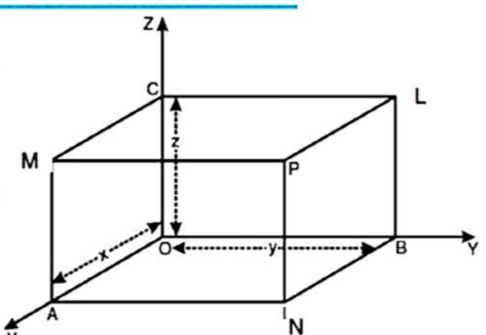


Fig.

The plane, passing through OX and OY, is called **XY-plane** (or XOY plane or Z-plane).

Similarly, YZ- and ZX-planes.

It is obvious that these co-ordinate planes are mutually perpendicular.

Let us associate any point P with the co-ordinates (x, y, z) in the following way :

Thro' P, draw a perpendicular, meeting XY-plane in N. We take PN = z. *From N, draw perpendiculars on the X-axis and Y-axis meeting them in A and B respectively. Take NB = x and NA = y.*

Alternatively, thro' P, draw perps. on the X, Y, Z-axes meeting them in A, B, C respectively. Then OA = x, OB = y and OC = z. *The co-ordinates* x, y, z are taken as +ve or – ve according as the respective points are on the + ve or – ve side of the corresponding co-ordinate axes.

Now we have associated an ordered triplet (x, y, z) with any point P in space in a unique way. Thus there exists one–one correspondence between the points in space and the ordered triplets of real numbers.

11.1.1. DEFINITIONS

With the help of cartesian co–ordinates, we have a better picture with regard to fundamental concepts.

(*a*) Rectangular Axes.

Let X′OX, Y′OY and Z′OZ be three mutually perpendicular straight lines.

(I) The common point O is called the *origin.*

(II) X′OX is called the ***X-axis*** (or axis of X)

(III) Y′OY is called the ***Y-axis*** (or axis of Y)

(IV) Z′OZ is called the ***Z-axis*** (or axis of Z).

These three, taken together, are called *co-ordinate axes* (or simply *axes*).

Note. Since the axes are mutually perpendicular, therefore, they are called **rectangular axes.**

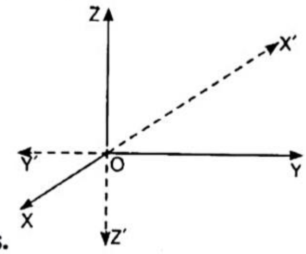


Fig.

KEY POINT

In the whole of this chapter, the treatment is with regard to rectangular axes. Thus by axes, we shall mean rectangular axes.

(*b*) Co-ordinate Planes.

(I) XOY, the plane containing X and Y-axes is called ***XY-plane.***

(II) YOZ, the plane containing Y and Z-axes is called ***YZ-plane.***

(III) ZOX, the plane containing Z and X-axes is called ***ZX-plane.***

These three, taken together, are called **co-ordinate planes.**

(*c*) Convention for Signs.

(I) Distances measured *upwards* of XY-plane are taken as +ve and *downwards* as – ve.

(II) Distances measured *in front* of YZ-plane are taken as +ve and *back* of it as – ve.

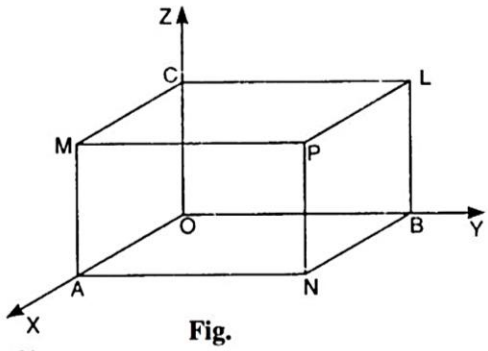


Fig.

(III) Distances measured to the *right* of ZX-plane are taken as +ve and *left* of it as – ve.

The three co-ordinate planes divide the whole space into eight compartments, known as **octants.**

(*d*) Co-ordinates of a point.

Let P be *any* point in space. Thro' P, draw three planes parallel to co–ordinate planes and meeting the axes in A, B and C respectively [(see Fig. of part (*b*)]. If x, y, z be the directed distances OA, OB, OC respectively, then the ordered triplet (x, y, z) are called cartesian rectangular co-ordinates of P and is denoted by P(x, y, z). From the definitions, we observe that : "*Given any point P in space, the ordered triplet (x, y, z) of real numbers is determined uniquely.*"

Conversely, given the ordered triplet (x, y, z) of real numbers, we are to find the point of which these are the co-ordinates. Cut off from O on the co-ordinate axes OX, OY, OZ distances x, y, z respectively and find the respective points A, B, C. Thro' these points, draw planes parallel to YZ, ZX, XY-planes respectively. The point of intersection of these planes is the required point P whose co-ordinates are (x, y, z).

KEY POINT

There is one-one correspondence between the set of points in space and the set of ordered triplets of real numbers.

USEFUL FACTS

(I) The co-ordinates of the origin O are (0, 0, 0).

(II) If the point P (x, y, z) lies in the YZ-plane, then $x = 0$.

Conversely, if $x = 0$, then P lies in the YZ-plane.

Thus the **equation of YZ-plane is x = 0.**

Similarly, the **equation of ZX-plane is y = 0** and the **equation of XY-plane is z = 0.**

(III) Since X-axis is the common line of ZX and XY-planes,

∴ **equations of X-axis are y = 0 and z = 0.**

Similarly, the **equations of Y-axis are z = 0 and x = 0** and the **equations of Z-axis are x = 0 and y = 0.**

11.2. DISTANCE BETWEEN TWO POINTS

The distance between two distinct points whose co–ordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) is :

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$ **(In Cartesian Co–ordinates)**

NOTATIONS

(*i*) | AB | denotes the **distance between A and B**

(*ii*) [AB] denotes the **line segment AB.**

11.3. SECTION FORMULAE

The co–ordinates of the point, which divides the line segment joining two distinct points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m_1 : m_2$ $(m_1 + m_2 \neq 0)$ are $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$.

GUIDE–LINES

Step (i) *Multiply m_1 by x_2 and m_2 by x_1 .*

Step (ii) *Add these two and divide the sum by $m_1 + m_2$.*

This gives *x*, the *x*-co-ordinate. Similarly, *y* and *z*-co-ordinates can be found.

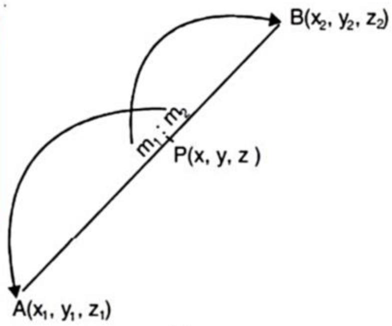


Fig.

Cor. 1. If Q divides the join of A and B in the ratio $m_1 : m_2$ externally, then the co-ordinates of Q are :

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right), \text{ where } m_1 - m_2 \neq 0 \text{ i.e. } m_1 \neq m_2.$$

Cor. 2. Mid-point Formula.

If C (x, y, z) is the mid-point of [AB] with A (x_1, y_1, z_1) and B (x_2, y_2, z_2),

then the co-ordinates of C are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Cor. 3. Putting $\lambda = \frac{m_1}{m_2}, \lambda \neq -1, x = \frac{\lambda x_2 + x_1}{\lambda + 1}, y = \frac{\lambda y_2 + y_1}{\lambda + 1}, z = \frac{\lambda z_2 + z_1}{\lambda + 1}$. These are **parametric representations.**

11.4. CENTROID OF A TRIANGLE

Definition

Centroid of a triangle is the point, which divides all the medians in the ratio 2 : 1.

This is also called as **Centre of gravity** of the triangle.

The co-ordinates of the centroid (centre of gravity) of a triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are :

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

KEY POINT

Medians of a triangle are concurrent.

SUB CHAPTER 11.1

Direction-Cosines

11.5. INTRODUCTION

In this section, we shall deal with direction-ratios and direction-cosines. Also we shall find the angle between the lines whose direction-ratios or direction-cosines are given.

11.6. DIRECTION-COSINES AND DIRECTION-RATIOS OF A LINE

Here we shall study the direction-cosines and direction-ratios of a line.

11.6.1. DIRECTION-COSINES OF A LINE

Let AB be a line in space. Through O, draw a line QP parallel to the line AB. Let the ray OP make angles α, β, γ with the rays OX, OY and OZ respectively.

Then the ray AB also makes same angles with the positive directions of the co-ordinate axes.

The cosines of these angles *i.e.* $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction-cosines of the ray AB.

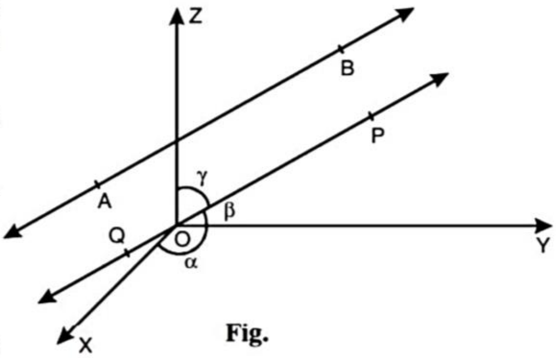


Fig.

Notation. The direction-cosines are usually denoted by $< l, m, n >$.

Thus $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

Observation. Clearly the ray OQ makes angles $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$ with the rays OX, OY and OZ respectively.

Thus the direction-cosines of the ray BA are :

$$< \cos(\pi - \alpha), \cos(\pi - \beta), \cos(\pi - \gamma) >$$

i.e., $< -\cos\alpha, -\cos\beta, -\cos\gamma >$ *i.e.*, $< -l, -m, -n >$.

Cor. The direction-cosines of the axes of co-ordinates. ***(N.C.E.R.T.)***

The x-axis makes angles 0°, 90°, 90° with the co-ordinate axes, its direction-cosines are $< \cos 0°, \cos 90°, \cos 90° >$

i.e., $\mathbf{< 1, 0, 0 >}$.

Similarly, the direction-cosines of y-axis are $\mathbf{< 0, 1, 0 >}$ and the direction-cosines of z-axis are $\mathbf{< 0, 0, 1 >}$.

11.6.2. RELATION BETWEEN DIRECTION-COSINES OF A LINE

Let AB be a line with direction-cosines $< l, m, n >$.

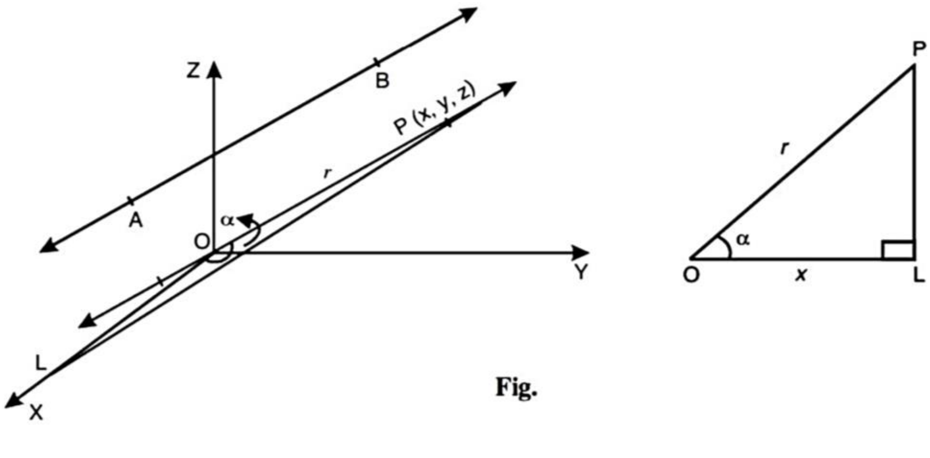


Fig.

Through O, draw a line parallel to AB.

Let P (x, y, z) be *any* point on this line.

From P, draw PL, perpendicular on the x-axis.

If $\quad OP = r$, then $\cos\alpha = \dfrac{OL}{OP} \quad \Rightarrow \quad l = \dfrac{x}{r}$

$$\Rightarrow \quad x = lr.$$

Similarly, $y = mr$ and $z = nr$.

Squaring and adding, $x^2 + y^2 + z^2 = r^2 (l^2 + m^2 + n^2)$

$\Rightarrow \quad r^2 = r^2 (l^2 + m^2 + n^2).$

Hence, $\boxed{\mathbf{l^2 + m^2 + n^2 = 1.}}$

In Words : *Sum of the squares of the direction-cosines of any line is equal to 1.*

Another Form. If α, β, γ be the angles, which OP makes with the axes, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

11.6.3. DIRECTION-RATIOS OF A LINE

Definition

Any three numbers, which are proportional to the direction-cosines of a line, are called the direction-ratios of the line.

If $< a, b, c >$ are direction-ratios of a line, then $< ka, kb, kc >$ $(k \neq 0)$ are also direction-ratios of the line.

Thus there are infinitely many sets of direction-ratios of a line.

11.6.4. CONVERSION OF DIRECTION-RATIOS TO DIRECTION-COSINES

If the direction-ratios of a line are < a, b, c >, to find its direction-cosines.

Let $< l, m, n >$ be the direction-cosines (*d.c.'s*) of the line.

Then, $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c} = k$ (say), where $k \neq 0$

$\Rightarrow \quad l = ak, m = bk, n = ck$ ...(1)

But $\quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad a^2k^2 + b^2k^2 + c^2k^2 = 1 \quad \Rightarrow \quad k = \pm \dfrac{1}{\sqrt{a^2 + b^2 + c^2}}.$

Hence, from (1), the direction-cosines of the line are :

$< \pm \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}, \pm \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}, \pm \dfrac{c}{\sqrt{a^2 + b^2 + c^2}} >$, where signs should be taken all positive or all negative.

Divide each of a, b, c by $\pm\sqrt{a^2 + b^2 + c^2}$.

KEY POINT

$\mathbf{l^2 + m^2 + n^2 = 1}$ **but** $\mathbf{a^2 + b^2 + c^2 \neq 1}$.

11.6.5. DIRECTION-RATIOS OF LINE JOINING TWO POINTS

To find the direction-ratios of the line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be any two points on the line whose direction-cosines are $< l, m, n >$.

Then PQ cos $\alpha = x_2 - x_1$, PQ cos $\beta = y_2 - y_1$, PQ cos $\gamma = z_2 - z_1$, where α, β, γ are the angles, which the line makes with the axes

$$\Rightarrow \quad PQ = \frac{x_2 - x_1}{\cos\alpha}, \; PQ = \frac{y_2 - y_1}{\cos\beta}, \; PQ = \frac{z_2 - z_1}{\cos\gamma}$$

$$\Rightarrow \quad \frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma} \; (= PQ)$$

$$\Rightarrow \quad \frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} \; (= PQ)$$

$$\therefore \quad l = \frac{x_2 - x_1}{PQ} = \frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}. \qquad \left[\because PQ = \sqrt{\Sigma(x_2 - x_1)^2}\right]$$

Similarly, $$m = \frac{y_2 - y_1}{PQ} = \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}} \text{ and } n = \frac{z_2 - z_1}{PQ} = \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}.$$

Hence, the direction-ratios of the line PQ are :

$$< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$$

and the direction-cosines are : $\left\langle \frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}} \right\rangle$.

11.7. ANGLE BETWEEN TWO LINES

(*a*) To determine the angle between two lines L_1 and L_2 whose direction-cosines are $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ respectively.

Let PQ ($\equiv L_1$) and ST ($\equiv L_2$) be two lines whose direction-cosines are given by : $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ respectively.

Let 'θ' be the required angle between these lines.

Through O, draw OA (= r) || to PQ and OB (= r) || to ST.

Thus $\angle AOB = \theta$.

[$\because$ *Angle between two lines = Angle between their parallels*]

$\therefore$ The co-ordinates of A and B are (l_1r, m_1r, n_1r) and (l_2r, m_2r, n_2r) respectively.

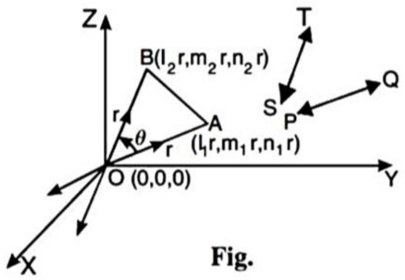


Fig.

$$\therefore \quad |AB| = \sqrt{(l_2r - l_1r)^2 + (m_2r - m_1r)^2 + (n_2r - n_1r)^2}$$

$$= r\sqrt{(l_2 - l_1)^2 + (m_2 - m_1)^2 + (n_2 - n_1)^2}$$

$$= r\sqrt{l_2^2 + l_1^2 - 2l_2l_1 + m_2^2 + m_1^2 - 2m_2m_1 + n_2^2 + n_1^2 - 2n_2n_1}$$

$$= r\sqrt{(l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) - 2(l_1l_2 + m_1m_2 + n_1n_2)}$$

$$= r\sqrt{1 + 1 - 2(l_1l_2 + m_1m_2 + n_1n_2)} = r\sqrt{2 - 2(l_1l_2 + m_1m_2 + n_1n_2)}.$$

Now, by *Cosine-Formula* in ΔOAB, we have :

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2|OA|.|OB|} = \frac{r^2 + r^2 - r^2[2 - 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2r\,.\,r}$$

$$= \frac{1 + 1 - [2 - 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2} = \frac{2(l_1l_2 + m_1m_2 + n_1n_2)}{2}.$$

Hence, $\boldsymbol{\cos\theta = |\,l_1l_2 + m_1m_2 + n_1n_2\,|.}$

i.e. $\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2)$.

Sine Form :

Since $\sin^2\theta = 1 - \cos^2\theta$,

$\therefore$ $\sin^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$ $[\because \cos\theta = |\,l_1l_2 + m_1m_2 + n_1n_2\,|]$

$= \boldsymbol{(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2}$ ***(Note this step)***

$= l_1^2l_2^2 + l_1^2m_2^2 + l_1^2n_2^2 + m_1^2l_2^2 + m_1^2m_2^2 + m_1^2n_2^2 + n_1^2l_2^2 + n_1^2m_2^2 + n_1^2n_2^2$
$\quad - (l_1^2l_2^2 + m_1^2m_2^2 + n_1^2n_2^2 + 2l_1l_2m_1m_2 + 2\,m_1m_2n_1n_2 + 2\,n_1n_2l_1l_2)$

$= (m_1^2n_2^2 + m_2^2n_1^2 - 2\,m_1m_2n_1n_2) + (n_1^2l_2^2 + n_2^2l_1^2 - 2\,n_1n_2l_1l_2)$
$\quad + (l_1^2m_2^2 + l_2^2m_1^2 - 2\,l_1l_2m_1m_2)$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2.$

But $0 \le \theta < \pi$, so that $\sin\theta > 0$.

Hence, $\boldsymbol{\sin\theta = \sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}.}$

Tangent Form :

$$\boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}}.$$

Cor. 1. Condition of Perpendicularity.

The two lines are perpendicular

iff $\theta = \dfrac{\pi}{2}$

iff $\boldsymbol{l_1l_2 + m_1m_2 + n_1n_2 = 0.}$

Cor. 2. Conditions of Parallelism.

The two lines are parallel

iff $\theta = 0°$ iff $\sin\theta = 0$

$\Leftrightarrow$ $\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} = 0$

$\Leftrightarrow$ $(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 0$

$\Leftrightarrow$ $m_1n_2 - m_2n_1 = 0,\ n_1l_2 - n_2l_1 = 0,\ l_1m_2 - l_2m_1 = 0$

$\Leftrightarrow$ $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} \Leftrightarrow \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} = \dfrac{\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{l_2^2 + m_2^2 + n_2^2}}$

$$\Leftrightarrow \quad \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{1}{1} \quad \Leftrightarrow \quad \boxed{l_1 = l_2,\ m_1 = m_2 \text{ and } n_1 = n_2.}$$

(*b*) If $< a_1, b_1, c_1 >$ and $< a_2, b_2, c_2 >$ are direction-ratios of lines L_1 and L_2 respectively, then the angles between them are given by :

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Since the direction-ratios of L_1 and L_2 are $< a_1, b_1, c_1 >$ and $< a_2, b_2, c_2 >$ respectively,

$\therefore$ the direction-cosines of L_1 and L_2 are :

$$< \pm\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \pm\frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \pm\frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} >$$

and

$$< \pm\frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \pm\frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \pm\frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} >.$$

[The signs, to be taken, are all + ve or all – ve]

$\therefore$ The angles between the lines are given by :

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

KEY POINT

If the lines L_1 and L_2 are non-perpendicular, then the acute angle between them is given by :

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and obtuse angle between them is given by :

$$\cos\theta = -\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Sine Form :

Since $\quad \sin^2\theta = 1 - \cos^2\theta,$

$$\therefore \quad \sin^2\theta = 1 - \frac{(a_1a_2 + b_1b_2 + c_1c_2)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}$$

$$= \frac{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}$$

$$= \frac{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}.$$

But $0 \le \theta < \pi$, so that $\sin\theta > 0$.

Hence, $$\sin\theta = \sqrt{\frac{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}{(a_1^2 + b_1^2 + c_1^2)\,(a_2^2 + b_2^2 + c_2^2)}}.$$

Tangent Form :

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\sqrt{\frac{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}{a_1a_2 + b_1b_2 + c_1c_2}}.$$

Cor. 1. Condition of Perpendicularity.

The two lines L_1 and L_2 are perpendicular

iff $\theta = \dfrac{\pi}{2}$ iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Cor. 2. Condition of Parallelism.

The two lines L_1 and L_2 are parallel

iff $\theta = 0°$ iff $\sin\theta = 0$

$$\Leftrightarrow \sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2} = 0$$

$$\Leftrightarrow (a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 = 0$$

$$\Leftrightarrow a_1b_2 - a_2b_1 = 0,\ b_1c_2 - b_2c_1 = 0,\ c_1a_2 - c_2a_1 = 0$$

$$\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

11.8. PROJECTION

(a) Projection of a point on a line.

Definition

The projection of a point P on a line L is defined as P′, the foot of the perpendicular from P on L.

Fig.

(b) Projection of a point on a plane.

Definition

Let π be any plane and P be a given point, not on the plane π, then P′, the foot of perpendicular on the plane, is called orthogonal projection of P on the plane π.

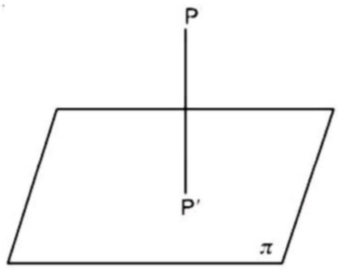


Fig.

(c) Projection of a line segment on a line.

Definition

The projection of the line segment [PQ] on a line L is the segment [P′Q′], where P′, Q′ are the feet of perpendiculars from P, Q respectively on the line L.

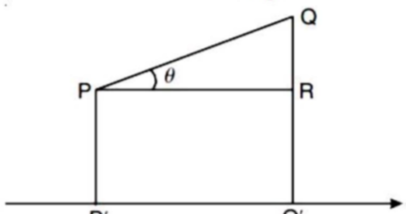


Fig.

If 'θ' is the angle between the line segment [PQ] and the line L, then ∠QPR = θ, where PR || L, meeting QQ′ in R.

∴ P′Q′ = PR = PQ cos θ.

Hence, the projection of line segment [PQ] on the line L is **PQ cos θ**, where 'θ' is the angle between the line segment [PQ] and the line L.

11.9. PROJECTION OF A SEGMENT

To find the projection of a line segment [AB] on line with direction-cosines <l, m, n>, where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the co-ordinates of A and B respectively.

Let [AB] be the given segment. Then [A′B′] is its projection on the line L having direction-cosines < *l, m, n* >.

The direction-cosines of AB are :

$$<\frac{x_2-x_1}{|AB|}, \frac{y_2-y_1}{|AB|}, \frac{z_2-z_1}{|AB|}>.$$

Now | A′ B′ | = | AB | cos θ ...(1),

where 'θ' is the angle between L and AB.

$$\therefore \cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$$

$$= \left|\left(\frac{x_2-x_1}{|AB|}\right)l + \left(\frac{y_2-y_1}{|AB|}\right)m + \left(\frac{z_2-z_1}{|AB|}\right)n\right|$$

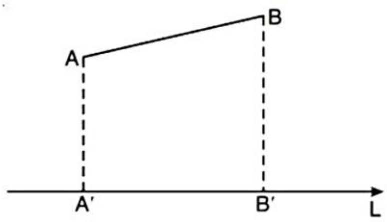


Fig.

$\Rightarrow |AB|\cos\theta = |(x_2-x_1)\,l + (y_2-y_1)\,m + (z_2-z_1)n|$.

Hence, $|\mathbf{A'B'}| = |(\mathbf{x_2 - x_1})\,\mathbf{l} + (\mathbf{y_2 - y_1})\,\mathbf{m} + (\mathbf{z_2 - z_1})\,\mathbf{n}|$. [*Using (1)*]

Frequently Asked Questions

FAQs

Example 1. If a line makes angles of 90°, 60° and 30° with the positive x, y and z-axis respectively, find its direction-cosines. *(N.C.E.R.T.)*

Solution. Direction-cosines are :

$$< \cos 90°, \cos 60°, \cos 30° > \text{ i.e. } <0, \frac{1}{2}, \frac{\sqrt{3}}{2}>.$$

Example 2. Find the acute angle which the line with direction-cosines $<\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, n>$ makes wtith positive direction of *z*–axis. *(C.B.S.E. Sample Paper 2019)*

Solution. Since, $l^2 + m^2 + n^2 = 1$,

$$\therefore \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + n^2 = 1$$

$$\Rightarrow \frac{1}{3} + \frac{1}{6} + n^2 = 1 \quad \Rightarrow \quad \frac{1}{3} + \frac{1}{6} + n^2 = 1$$

$$\Rightarrow n^2 = 1 - \frac{1}{2} \quad \Rightarrow \quad n^2 = \frac{1}{2}$$

$$\Rightarrow n = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \cos\gamma = \frac{1}{\sqrt{2}},$$

where 'γ' is the angle, which the line makes with *z*–axis.

Hence, $\gamma = 45°$ or $\frac{\pi}{4}$.

Example 3. If a line has direction-cosines : $<-\frac{9}{11}, \frac{6}{11}, -\frac{2}{11}>$, then what are its direction-ratios ? *(N.C.E.R.T.)*

Solution. Given : Direction-cosines are :

$$<\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}>.$$

Clearly, < – 9, 6, – 2 > is one set of direction-ratios.

All sets of direction-ratios are given by :

< – 9*k*, 6*k*, – 2*k* >, where $k \neq 0$.

Example 4. Find the direction-cosines of the line joining the points (– 2, 4, – 5) and (1, 2, 3). *(N.C.E.R.T.)*

Solution. We know that the direction-cosines of the line joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are :

$$<\frac{x_2-x_1}{|PQ|}, \frac{y_2-y_1}{|PQ|}, \frac{z_2-z_1}{|PQ|}>,$$

where $|PQ| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$.

Here P is (– 2, 4, – 5) and Q is (1, 2, 3).

$$\therefore |PQ| = \sqrt{(1-(-2))^2 + (2-4)^2 + (3-(-5))^2}$$

$$= \sqrt{9+4+64} = \sqrt{77}.$$

Hence, the direction-cosines of the line are :

$$<\frac{1-(-2)}{\sqrt{77}}, \frac{2-4}{\sqrt{77}}, \frac{3-(-5)}{\sqrt{77}}>$$

$$\text{i.e. } <\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}>.$$

Example 5. If α, β, γ are direction-angles of a line, prove that cos 2α + cos 2β + cos 2γ + 1 = 0. *(N.C.E.R.T.)*

Solution. Since α, β, γ are direction-angles of a line,

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\Rightarrow \quad \frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\gamma}{2} = 1$$

$$\Rightarrow \quad 1 + \cos 2\alpha + 1 + \cos 2\beta + 1 + \cos 2\gamma = 2$$

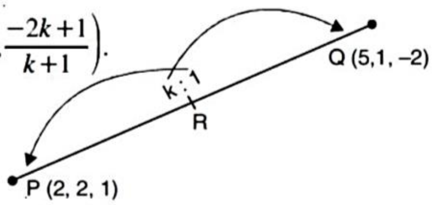
$$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0,$$

which is true.

Example 6. The x-cordinate of a point on the line joining the points P(2, 2, 1) and Q(5, 1, – 2) is 4. Find its z-co-ordinate. *(A.I.C.B.S.E. 2017)*

Solution. Any point R on [PQ] is :

$$\left(\frac{5k+2}{k+1}, \frac{k+2}{k+1}, \frac{-2k+1}{k+1}\right).$$



By the question, $\frac{5k+2}{k+1} = 4$

$\Rightarrow 5k + 2 = 4k + 4 \Rightarrow k = 2.$

Hence, the z-co-ordinate is : $\frac{-2(2)+1}{2+1}$ *i.e.* $\frac{-4+1}{3}$ *i.e.* – 1.

Example 7. Show that the points :

A(1, – 2, – 8); B(5, 0, – 2) and C(11, 3, 7) are collinear, and find the ratio in which B divides AC. *(Jammu B. 2012)*

Solution. (*i*) The direction-ratios of AB are :

< 5 – 1, 0 + 2, – 2 + 8 > *i.e.* < 4, 2, 6 > *i.e.* < 2, 1, 3 >

and the direction-ratios of BC are :

< 11–5, 3 – 0, 7 + 2 > *i.e.* < 6, 3, 9 > *i.e.* < 2, 1, 3 >.

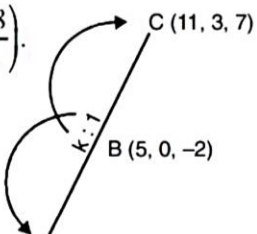
Thus AB, BC are either parallel or coincident lines.

But B is the common point.

Hence, the lines are coincident lines and consequently the three points A, B and C are collinear.

(*ii*) Let B divide [AC] in the ratio k : 1.

$$\therefore \text{B is} \left(\frac{11k+1}{k+1}, \frac{3k-2}{k+1}, \frac{7k-8}{k+1}\right).$$



But B is (5, 0, –2). [*Given*]

$$\therefore \quad \frac{11k+1}{k+1} = 5, \frac{3k-2}{k+1} = 0 \text{ and } \frac{7k-8}{k+1} = -2$$

$\Rightarrow$ $11k + 1 = 5k + 5$, $3k - 2 = 0$ and $7k - 8 = -2k - 2$

$\Rightarrow$ $6k = 4$, $3k = 2$ and $9k = 6$

$\Rightarrow$ $k = \frac{2}{3}$, $k = \frac{2}{3}$ and $k = \frac{2}{3}$.

Hence, the reqd. ratio is $\frac{2}{3}$: 1 *i.e.* 2 : 3.

Example 8. Find the acute angle between the lines whose direction-ratios are :

< 1, 1, 2 > and < – 3, – 4, 1 >.

Solution. If 'θ' be the reqd. angle between the lines, then :

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$= \frac{|(1)(-3)+(1)(-4)+(2)(1)|}{\sqrt{1+1+4}\sqrt{9+16+1}}$$

$$= \frac{|-3-4+2|}{\sqrt{6}\sqrt{26}} = \frac{5}{\sqrt{156}}.$$

Hence, $\theta = \cos^{-1}\left(\frac{5}{\sqrt{156}}\right)$.

Example 9. Find the angle between the lines whose direction-cosines are given by the equations :

3l + m + 5n = 0, 6mn – 2nl + 5lm = 0.

Solution. We have : $3l + m + 5n = 0$ ...(1)

and $6mn - 2nl + 5lm = 0$ ...(2)

From (1), $m = -(3l + 5n)$ ...(3)

Putting in (2), we get :

$$-6(3l + 5n)n - 2nl - 5l(3l + 5n) = 0$$

$$\Rightarrow \quad -18ln - 30n^2 - 2nl - 15l^2 - 25nl = 0$$

$$\Rightarrow \quad -30n^2 - 45nl - 15l^2 = 0$$

$$\Rightarrow \quad 2n^2 + 3nl + l^2 = 0$$

$$\Rightarrow \quad (2n + l)(n + l) = 0.$$

$\therefore$ Either $2n + l = 0$ or $n + l = 0$.

(I) When $2n + l = 0$ *i.e.* $l = -2n$.

From (3), $m = -(-6n + 5n) = n$.

(II) When $n + l = 0$ *i.e.* $l = -n$.

From (3), $m = -(-3n + 5n) = -2n$.

$\therefore$ Direction–ratios of two lines are :

< $-2n, n, n$ > and < $-n, -2n, n$ >

i.e. < – 2, 1, 1 > and < 1, 2, –1 >.

If 'θ' be the angle between the two lines,

$$\text{then } \cos\theta = \frac{|(-2)(1)+(1)(2)+(1)(-1)|}{\sqrt{4+1+1}\sqrt{1+4+1}} = \frac{1}{6}.$$

Hence, $\theta = \cos^{-1}\frac{1}{6}$.

Example 10. Find the length of the projection of the line segment joining the points P (3, – 1, 2) and Q (2, 4, –1) on the line with direction-ratios < – 1, 2, –2 >.

Solution. The direction-ratios of the line are < –1, 2,– 2 >.

Its direction-cosines are :

$$< \frac{-1}{\sqrt{1+4+4}}, \frac{2}{\sqrt{1+4+4}}, \frac{-2}{\sqrt{1+4+4}} >$$

$$\text{i.e. } < \frac{-1}{3}, \frac{2}{3}, \frac{-2}{3} >.$$

$\therefore$ The length of projection of [PQ] on the given line

$= |(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n|$

$$= \left|(2-3)\left(-\frac{1}{3}\right) + (4+1)\left(\frac{2}{3}\right) + (-1-2)\left(\frac{-2}{3}\right)\right|$$

$$= \left|\frac{1}{3} + \frac{10}{3} + \frac{6}{3}\right| = \left|\frac{17}{3}\right| = \frac{17}{3} \text{ units.}$$

Example 11. Find the area of the triangle ABC whose vertices are :
A (1, 2, 4); B (–2, 1, 2) and C (2, 4, –3).

Solution. Area of ΔABC

$$= \frac{1}{2} |AB| |AC| \sin A \quad ...(1)$$

Now $|AB| = \sqrt{(-2-1)^2+(1-2)^2+(2-4)^2}$

$= \sqrt{9+1+4} = \sqrt{14}$

and $|AC| = \sqrt{(2-1)^2+(4-2)^2+(-3-4)^2}$

$= \sqrt{1+4+49} = \sqrt{54}$.

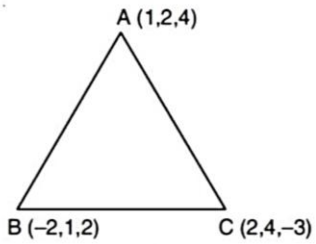


Fig.

Now direction-ratios of AB are :
$< -2-1, 1-2, 2-4 >$ *i.e.* $< 3, 1, 2 >$
and direction-ratios of AC are :
$< 2-1, 4-2, -3-4 >$ *i.e.* $< 1, 2, -7 >$.

$$\therefore \sin A = \frac{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$= \frac{\sqrt{(-7-4)^2+(2+21)^2+(6-1)^2}}{\sqrt{9+1+4}\sqrt{1+4+49}}$$

$$= \frac{\sqrt{121+529+25}}{\sqrt{14}\sqrt{54}} = \frac{\sqrt{675}}{\sqrt{14}\sqrt{54}}.$$

From (1), area of $\Delta ABC = \frac{1}{2} \times \sqrt{14} \times \sqrt{54} \times \frac{\sqrt{675}}{\sqrt{14}\sqrt{54}}$

$$= \frac{15\sqrt{3}}{2} \text{ sq. units.}$$

Definition

(a) (i) Parallelopiped. *It is a figure bounded by three parallel planes. Thus the parallelopiped is as shown in the adjoining figure :*

It has *six faces, viz.* ‖ gms.

OCLB, AMPN, OBNA, CLPM, OAMC, BNPL.

It has *four diagonals viz.* OP, AL, BM, CN.

(ii) Rectangular Parallelopiped.

When the faces are rectangles, then the parallelopiped is a rectangular one.

(b) Cube. *It is a parallelopiped with all its faces as squares.*

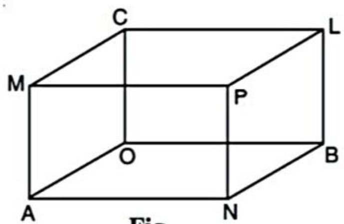


Fig.

Example 12. A line makes angles α, β, γ, δ with the four diagonals of a cube, prove that :

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}.$$

(N.C.E.R.T.)

Solution. Let O be the origin and OA, OB, OC (each = *a*) be the axes.

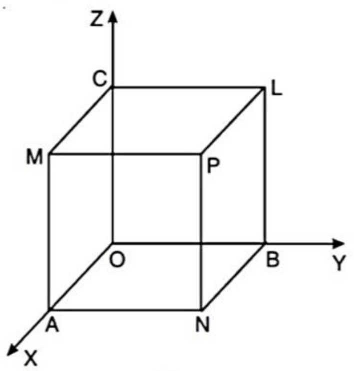


Fig.

Thus the co-ordinates of the points are :
O (0, 0, 0), A (a, 0, 0), B (0, a, 0), C (0, 0, a),
P (a, a, a), L (0, a, a), M (a, 0, a), N (a, a, 0).

Here OP, AL, BM and CN are four diagonals.

Let $< l, m, n >$ be the direction-cosines of the given line.

Now direction-ratios of OP are $< a-0, a-0, a-0 >$
i.e. $< a, a, a >$ *i.e.* $< 1, 1, 1 >$,

direction-ratios of AL are :
$< 0-a, a-0, a-0 >$ *i.e.* $< -a, a, a >$ *i.e.* $< -1, 1, 1 >$,

direction-ratios of BM are :
$< a-0, 0-a, a-0 >$ *i.e.* $< a, -a, a >$ *i.e.* $< 1, -1, 1 >$

and direction-ratios of CN are :
$< a-0, a-0, 0-a >$ *i.e.* $< a, a, -a >$ *i.e.* $< 1, 1, -1 >$.

Thus the direction-cosines of OP are :

$$< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >;$$

the direction-cosines of AL are $< \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$;

the direction-cosines of BM are $< \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$

and the direction-cosines of CN are :

$$< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} >.$$

If the given line makes an angle 'α' with OP, then :

$$\cos\alpha = \left| l\left(\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \right|.$$

$$\therefore \quad \cos\alpha = \frac{|l+m+n|}{\sqrt{3}} \quad ...(1)$$

If the given line makes an angle 'β' with AL, then :

$$\cos\beta = \left| l\left(-\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \right|$$

$$\therefore \quad \cos\beta = \frac{|-l+m+n|}{\sqrt{3}} \quad ...(2)$$

Similarly, $\cos\gamma = \frac{|l-m+n|}{\sqrt{3}}$ ...(3)

and $\cos\delta = \frac{|l+m-n|}{\sqrt{3}}$ ...(4)

Squaring and adding (1), (2), (3) and (4), we get :

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\,[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}\,[4\,(l^2+m^2+n^2)] = \frac{1}{3}\,[4\,(1)].$$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

EXERCISE 11 (a)

Fast Track Answer Type Questions — FTATQ

1. (*a*) Direction-cosines of (*i*) *x*-axis (*ii*) *y*-axis (*iii*) *z*-axis are .... **(Fill in the blank).** *(Kashmir B. 2017, 16)*

(*b*) Find the distance of the point (2, 3, 4) from the *x*-axis. *(C.B.S.E. 2010 C)*

2. (*i*) If a line makes angles 90°, 60° and θ with *x, y* and *z*-axis respectively, where θ is acute, then find 'θ'. *(C.B.S.E. 2015)*

(*ii*) If a line makes angles 90° and 60° respectively, with the positive directions of *x* and *y* axes, find the angle which it makes with the positive direction of *z*-axis. *(C.B.S.E. 2017)*

3. If a line has direction-cosines $< \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} >$, then find the direction-ratios.

4. If a line has direction-ratios < 2, – 1, – 2 >, determine its direction-cosines. *(N.C.E.R.T.; Jharkhand B. 2016; Uttarakhand B. 2013, 15; C.B.S.E. 2012)*

5. (*a*) Find the direction-cosines of a line passing through the points (1, 0, 0) and (0, 1, 1). *(A.I.C.B.S.E. 2011)*

(*b*) Find the direction-cosines of the lines joining the points :
(– 1, – 1, – 1) and (2, 3, 4). *(Rajasthan B. 2012)*

(*c*) Find the direction ratios and direction cosines of the vector joining the points (4, 7, 2) and (5, 11, – 4). *(Meghalaya. B. 2018; Nagaland. B. 2016)*

(*d*) Find the direction cosines of a line segment joining the points A(2, 5, 7) and B(3, 2, 9). *(Nagaland. B. 2018)*

6. Write the direction-cosines of the vector :

(*i*) $-2\hat{i} + \hat{j} - 5\hat{k}$ *(C.B.S.E. 2011)*

(*ii*) $\hat{i} + 2\hat{j} + 3\hat{k}$. *(Assam B. 2015; Karnataka B. 2014)*

7. Find the length of the projection of the line segment joining (3, 4, 5) and (4, 6, 3) on the straight line :

$$\frac{x-4}{2} = \frac{y-5}{3} = \frac{z-6}{6}.$$ *(Tripura B. 2016)*

Very Short Answer Type Questions — VSATQ

8. Show that the following points are collinear :
(1, 2, 7) ; (2, 6, 3) ; (3, 10, –1). *(Jammu B. 2015,13)*

9. Find the acute angle between two lines whose direction-ratios are :
< 2, 3, 6 > and < 1, 2, – 2 >.

10. Find the obtuse angle between two lines whose direction-ratios are :
< 3, – 6, 2 > and <1, – 2, – 2 >.

11. Find the angle between the lines whose direction-ratios are :
$< a, b, c >$ and $< b-c, c-a, a-b >$. *(N.C.E.R.T.)*

Short Answer Type Questions — SATQ

12. Find the direction-cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2). *(N.C.E.R.T.)*

13. Show that the lines with direction-cosines :

$$< \frac{12}{13}, -\frac{3}{13}, -\frac{4}{13} >;\ < \frac{4}{13}, \frac{12}{13}, \frac{3}{13} >;\ < \frac{3}{13}, \frac{-4}{13}, \frac{12}{13} >$$

are mutually perpendicular.

14. Find the angle between the lines whose direction-cosines are given by :

(*i*) $l + m + n = 0,\ l^2 + m^2 - n^2 = 0$

(*ii*) $2l - m + 2n = 0,\ mn + nl + lm = 0.$

15. Find the area of the triangle whose vertices are : A (1, 2, 3) ; B (2, –1, 4) and C (4, 5, –1). *(C.B.S.E. 2017, 13)*

16. Show that the join of the points (4, 7, 8) and (2, 3, 4) is parallel to the join of the points (–1, –2, 1) and (1, 2, 5).

17. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line through the points (3, 5, – 1) and (4, 3, – 1).

18. Determine the value of '*k*' so that the line joining the points A (*k*, 1, – 1), B (2, 0, 2*k*) is perpendicular to the line joining the points C (4, 2*k*, 1) and D (2, 3, 2).

19. Prove that the angle between any two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

Long Answer Type Questions

LATQ

20. Find the projection of the line segment joining the points :

(*i*) (2, –3, 0), (0, 4, 5) on the line with direction-cosines $<\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}>$

(*ii*) (1, 2, 3), (4, 3, 1) on the line with direction-ratios $<3, -6, 2>$.

21. If the edges of a rectangular parallelopiped are *a*, *b* and *c*, show that the angles between the four diagonals are given by :

$$\cos^{-1}\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}.$$

Answers

1. (*a*) (*i*) <1,0,0> (*ii*) <0,1,0> (*iii*) <0,0,1> (*b*) 5.

2. (*i*) $\theta = 30°$ (*ii*) 30°. **3.** $<2k, -k, -2k>$; $k \neq 0$.

4. $<\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}>$.

5. (*a*) $<-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}>$ (*b*) $<\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}>$

(*c*) $<1, 4, -6>$; $<\frac{1}{\sqrt{53}}, \frac{4}{\sqrt{53}}, \frac{-6}{\sqrt{53}}>$

(*d*) $<\frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}>$.

6. (*i*) $<-\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{5}{\sqrt{30}}>$ (*ii*) $<\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}>$.

7. $\frac{4}{7}$. **9.** $\cos^{-1}\frac{4}{21}$.

10. $\cos^{-1}\left(-\frac{11}{21}\right)$. **11.** 90°.

12. $<\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}>$, $<\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}>$, $<\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}>$.

14. (*i*) 60º (*ii*) 90º. **15.** $\sqrt{\frac{137}{2}}$ sq. units.

18. $k = 1$. **20.** (*i*) $\frac{13}{7}$ (*ii*) $\frac{1}{7}$.

Hints to Selected Questions

14. (*i*) $l + m + n = 0 \Rightarrow n = -(l + m)$.

$l^2 + m^2 - n^2 = 0 \Rightarrow l^2 + m^2 - (l + m)^2 = 0 \Rightarrow l = 0, m = 0$.

When $l = 0$, then $m + n = 0 \Rightarrow m = -n$.

When $m = 0$, then $l + n = 0 \Rightarrow l = -n$.

$\therefore$ Direction cosines of two lines are :

$<0, -n, n>$ and $<-n, 0, n>$

i.e. $<0, -1, 1>$ and $<-1, 0, 1>$.

$$\therefore \cos\theta = \frac{|(0)(-1)+(-1)(0)+(1)(1)|}{\sqrt{0+1+1}\sqrt{1+0+1}} = \frac{1}{2} \Rightarrow \theta = 60°.$$

15. As in **Ex. 11.**

19. Refer **Ex. 12.**

If 'θ' be the angle between OP and AL,

then $\cos\theta = \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3}$; etc.

21. As in **Ex. 12.**

SUB CHAPTER 11.2

Straight Line in Space

11.10. INTRODUCTION

We know that a straight line is uniquely determined in space if it :

(*a*) passes through a given point and has a given direction or (*b*) passes through two given points.

We shall determine the vector equation as well as cartesian equation of a straight line under different conditions.

11.11. SYMMETRICAL (CANONICAL) FORM

(*a*) To find the equation of a straight line passing through a fixed point A and parallel to a given vector $\vec{m}$.

Let $\vec{a}$ be the position vector of the fixed point A and $\vec{r}$, the position vector of *any* point P, where O is the origin.

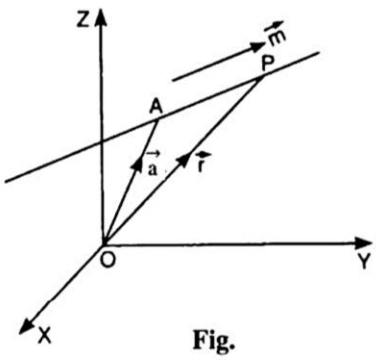


Fig.

Since $\overrightarrow{AP}$ is parallel to $\vec{m}$,

$\therefore\ \overrightarrow{AP} = \lambda\,\vec{m}$ for some scalar λ

$\Rightarrow\quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda\,\vec{m}$

$\Rightarrow\quad \vec{r} - \vec{a} = \lambda\,\vec{m}$

$$\Rightarrow\quad \boxed{\vec{\mathbf{r}} = \vec{\mathbf{a}} + \lambda\,\vec{\mathbf{m}}} \qquad ...(1),$$

which is the reqd. equation.

Cor. *The vector equation of a straight line through the origin and parallel to the vector* $\vec{m}$ *is* $\vec{r} = \lambda\,\vec{m}$.

Cartesian Form.

Let A (x_1, y_1, z_1) be the fixed point and let $< a, b, c >$ be the direction–ratios of the line.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$.

Putting in (1), $(x\hat{i} + y\hat{j} + z\hat{k}) = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$.

Comparing coeffs. of $\hat{i}, \hat{j}, \hat{k}$, $x = x_1 + \lambda a,\ y = y_1 + \lambda b,\ z = z_1 + \lambda c$ (*Parametric Form*)

$$\Rightarrow\quad \frac{\mathbf{x} - \mathbf{x_1}}{\mathbf{a}} = \frac{\mathbf{y} - \mathbf{y_1}}{\mathbf{b}} = \frac{\mathbf{z} - \mathbf{z_1}}{\mathbf{c}}\ (= \lambda) \qquad \textit{(Symmetrical Form)}$$

Cor. *The equations of a straight line whose direction cosines are* $< l, m, n >$ *and passing through the point* (x_1, y_1, z_1) *are* :

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

This result follows the fact that the direction-cosines of a line may also be direction-ratios of the line.

(b) To find the equation of a straight line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let $\vec{a}, \vec{b}$ be the position vectors of the fixed points A and B respectively.

Let $\vec{r}$ be the position vector of *any* point P.

Since A, B, P are collinear,

$\therefore \quad \overrightarrow{AP} = \lambda \overrightarrow{AB}$ for some scalar λ

$\Rightarrow \quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda (\overrightarrow{OB} - \overrightarrow{OA})$

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda (\vec{b} - \vec{a})$

$\Rightarrow \quad \boxed{\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})}$ ...(1),

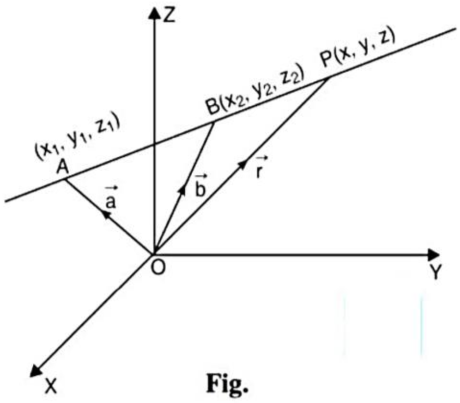


Fig.

which is the reqd. equation.

Cartesian Form :

Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

Putting in (1), $x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda((x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k})$.

Comparing coeffs. of $\hat{i}, \hat{j}, \hat{k}$, $x = x_1 + \lambda(x_2 - x_1), y = y_1 + \lambda(y_2 - y_1), z = z_1 + \lambda(z_2 - z_1)$

$$\Rightarrow \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \ (= \lambda).$$

Cor. COLLINEARITY OF THREE POINTS

Find the equation of the st. line through any two given points. If the remaining third point satisfies the equation, then the three points are collinear.

11.12. ANGLE BETWEEN TWO LINES

(a) Vectorially :

Let $\vec{r} = \vec{a} + \lambda \vec{b}$ ...(1)

and $\vec{r} = \vec{a}' + \mu \vec{b}'$ ...(2)

be two straight lines in space.

Clearly, (1) and (2) are st. lines in the directions of $\vec{b}$ and $\vec{b}'$ respectively.

If 'θ' be the angle between the lines (1) and (2), then 'θ' is the angle between the directions of $\vec{b}$ and $\vec{b}'$.

[∵ *We know that the angle between two st. lines depends on their directions and not on their positions.*]

Now $\vec{b}.\vec{b}' = |\vec{b}||\vec{b}'|\cos\theta$, which gives $\cos\theta = \dfrac{\vec{b}.\vec{b}'}{|\vec{b}||\vec{b}'|}$ ...(A)

(b) Cartesian Form :

Let $\dfrac{x - x_1}{b_1} = \dfrac{y - y_1}{b_2} = \dfrac{z - z_1}{b_3}$ ...(1)

and $\dfrac{x - x_1'}{b_1'} = \dfrac{y - y_1'}{b_2'} = \dfrac{z - z_1'}{b_3'}$ ...(2)

be two straight lines in space.

Then $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{b'} = b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}$, so that :

$$\vec{b}.\vec{b'} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}).(b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}) = b_1b_1' + b_2b_2' + b_3b_3'$$

and $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$ and $|\vec{b'}| = \sqrt{b_1'^2 + b_2'^2 + b_3'^2}$.

The result (A) of part (*a*) becomes : $\cos\theta = \dfrac{b_1b_1' + b_2b_2' + b_3b_3'}{\sqrt{b_1^2 + b_2^2 + b_3^2}\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}$ ...(B)

Cor. 1. If < *l, m, n* > and < *l', m', n'* > be the direction–cosines of two lines, then :

$$l = \frac{b_1}{\sqrt{b_1^2 + b_2^2 + b_3^2}},\quad m = \frac{b_2}{\sqrt{b_1^2 + b_2^2 + b_3^2}},\quad n = \frac{b_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

and $l' = \dfrac{b_1'}{\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}, m' = \dfrac{b_2'}{\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}, n' = \dfrac{b_3'}{\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}$.

The result (B) of part (*b*) becomes $\cos\theta = ll' + mm' + nn'$. ...(C)

Cor. 2. Condition of Perpendicularity (Orthogonality).

When the angle between two lines is $\theta = \dfrac{\pi}{2}$, then from (B), $b_1b_1' + b_2b_2' + b_3b_3' = 0$

and from (C), $ll' + mm' + nn' = 0$. $\left[\because \cos\dfrac{\pi}{2} = 0\right]$

ILLUSTRATIVE EXAMPLES

Example 1. Find the direction-cosines of the line :

$$\frac{x-1}{2} = -y = \frac{z+1}{2}.$$ (*C.B.S.E. Sample Paper 2019*)

Solution. The given line is $\dfrac{x-1}{2} = \dfrac{y}{-1} = \dfrac{z+1}{2}$.

Its direction-ratios are <2, –1, 2>.

Hence, the direction-cosines of the line are :

$$< \frac{2}{\sqrt{4+1+4}}, \frac{-1}{\sqrt{4+1+4}}, \frac{2}{\sqrt{4+1+4}} >$$

i.e., $< \dfrac{2}{\sqrt{9}}, \dfrac{-1}{\sqrt{9}}, \dfrac{2}{\sqrt{9}} >$

i.e., $< \dfrac{2}{3}, -\dfrac{1}{3}, \dfrac{2}{3} >$ or $< -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{-2}{3} >$.

Example 2. If the cartesian equations of a line are :

$$\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4},$$

write the vector equation for the line. (*A.I.C.B.S.E. 2014*)

Solution. The given line is $\dfrac{3-x}{5} = \dfrac{y+4}{7} = \dfrac{2z-6}{4}$

$\Rightarrow \dfrac{x-3}{-5} = \dfrac{y-(-4)}{7} = \dfrac{z-3}{2}$.

Its vector equation is :

$$\vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k}).$$

Example 3. Find the equation of the line, which passes through the point (1, 2, 3) and is parallel to the vector :

$$3\hat{i} + 2\hat{j} - 2\hat{k}.$$ (*Kashmir B. 2017*)

Solution. The equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$

i.e., $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$.

Example 4. Find the vector of the line joining (1, 2, 3) and (–3, 4, 3) and show that it is perpendicular to the *z*–axis. (*C.B.S.E. Sample Paper 2019*)

Solution. The given points are $\vec{a}$ = (1, 2, 3) and $\vec{b}$ = (– 3, 4, 3)

∴ Vector equation of the line joining (1, 2, 3) and (– 3, 4, 3) is :

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda((-3\hat{i} + 4\hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}))$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-4\hat{i} + 2\hat{j}) \quad ...(1)$$

Equation of z–axis is $\vec{r} = \mu \hat{k}$ ...(2)

Since $(-4\hat{i} + 2\hat{j}).\hat{k} = 0$

∴ Hence, (1) is perpendicular to z–axis.

Example 5. Find the vector equation of the line through (4, 3, – 1) and parallel to the line :

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + 4\hat{k}).$$ *(J. & K. B. 2011)*

Solution. The equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$

i.e. $\vec{r} = (4\hat{i} + 3\hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j} + 4\hat{k})$.

Example 6. Find the angle between the following pair of lines :

$$\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}),$$
$$\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \mu(3\hat{i} + \hat{j} - 2\hat{k}).$$

Solution. The given pair of lines is :

$$\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$$

and $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \mu(3\hat{i} + \hat{j} - 2\hat{k})$.

Here $\vec{b} = \hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{b}' = 3\hat{i} + \hat{j} - 2\hat{k}$,

where $|\vec{b}| = \sqrt{1+9+4} = \sqrt{14}$

and $|\vec{b}'| = \sqrt{9+1+4} = \sqrt{14}$

so that $\vec{b}\cdot\vec{b}' = (\hat{i} - 3\hat{j} + 2\hat{k})\cdot(3\hat{i} + \hat{j} - 2\hat{k})$

$= (1)(3) + (-3)(1) + (2)(-2)$

$= 3 - 3 - 4 = -4$.

If 'θ' be the required angle, then :

$$\cos\theta = \frac{\vec{b}\cdot\vec{b}'}{|\vec{b}||\vec{b}'|} = \frac{-4}{\sqrt{14}\sqrt{14}} = \frac{-4}{14} = \frac{-2}{7}.$$

Hence, $\theta = \cos^{-1}\left(\frac{-2}{7}\right) = \pi - \cos^{-1}\left(\frac{2}{7}\right)$.

Example 7. Find the angle between the following pair of lines :

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$$

and check whether the lines are parallel or perpendicular. *(C.B.S.E. 2011)*

Solution. The given lines can be rewritten as :

$$\frac{x-2}{2} = \frac{y-1}{7} = \frac{z+3}{-3} \quad ...(1)$$

and $$\frac{x+2}{-1} = \frac{y-4}{2} = \frac{z-5}{4} \quad ...(2)$$

Here < 2, 7, – 3 > and < – 1, 2, 4 > are direction-ratios of lines (1) and (2) respectively.

$$\therefore \quad \cos\theta = \frac{(2)(-1) + (7)(2) + (-3)(4)}{\sqrt{4+49+9}\sqrt{1+4+16}}$$

$$= \frac{-2+14-12}{\sqrt{62}\sqrt{21}} = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}.$$

Hence, the given lines are perpendicular.

Example 8. Find the point on the line :

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$$

at a distance 5 from the point (1, 3, 3). *(A.I.C.B.S.E. 2010)*

Solution. Any point on the line :

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} \quad ....(1)$$

is P $(3k-2, 2k-1, 2k+3)$ ...(2)

The given point is A (1, 3, 3).

By the question, | AP | = 5

$$\Rightarrow \sqrt{(3k-2-1)^2 + (2k-1-3)^2 + (2k+3-3)^2} = 5$$

$$\Rightarrow (3k-3)^2 + (2k-4)^2 + 4k^2 = 25$$

$$\Rightarrow 9k^2 - 18k + 9 + 4k^2 - 16k + 16 + 4k^2 = 25$$

$$\Rightarrow \quad 17k^2 - 34k = 0 \Rightarrow k = 0, 2.$$

Putting in (2), the reqd. point is :

(– 2, – 1, 3) or (6 – 2, 4 – 1, 4 + 3) *i.e.* (4, 3, 7).

Example 9. Find the equations of the line passing through the point (–1, 3, –2) and perpendicular to the lines :

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}.$$ *(C.B.S.E. 2012)*

Solution. The given lines are $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ ...(1)

and $$\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5} \quad ...(2)$$

Any line through (–1, 3, –2) is $\frac{x+1}{a} = \frac{y-3}{b} = \frac{z+2}{c}$ ...(3)

Since (3) is perpendicular to (1) and (2),

∴ $a(1) + b(2) + c(3) = 0 \Rightarrow a + 2b + 3c = 0$ ...(4)

and $a(-3) + b(2) + c(5) = 0 \Rightarrow 3a - 2b - 5c = 0$ ...(5)

Solving (4) and (5), $\frac{a}{-10+6} = \frac{b}{9+5} = \frac{c}{-2-6}$

$$\Rightarrow \quad \frac{a}{-4} = \frac{b}{14} = \frac{c}{-8} \Rightarrow \frac{a}{2} = \frac{b}{-7} = \frac{c}{4} \quad ...(6)$$

From (3) and (6), $\frac{x+1}{2}=\frac{y-3}{-7}=\frac{z+2}{4}$,

which are the reqd. equations.

Example 10. Find the Vector and Cartesian equations of the line through the point (1, 2, – 4) and perpendicular to the lines:

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

Solution. Let the line be :

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}) \quad ...(1)$$

Since (1) is perpendicular to :

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$,

$\therefore \quad a(3) + b(-16) + c(7) = 0 \quad ...(2)$

and $\quad a(3) + b(8) + c(-5) = 0 \quad ...(3)$

Solving, $\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$

$\Rightarrow \quad \frac{a}{24} = \frac{b}{36} = \frac{c}{72}$

$\Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{6} \quad ...(4)$

From (1) and (4), the vector equation of the line is:

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Cartesian Form:

$$x\hat{i} + y\hat{j} + z\hat{k} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$\Rightarrow x = 1 + 2\lambda, \quad y = 2 + 3\lambda \quad$ and $z = -4 + 6\lambda$

$\Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6} (= \lambda)$,

which is the required Cartesian equation.

Example 11. Show that, if the axes are rectangular, the equations of the line through (x_1, y_1, z_1) at right angles to the lines :

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}, \frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$$

are $\frac{x - x_1}{m_1 n_2 - m_2 n_1} = \frac{y - y_1}{n_1 l_2 - n_2 l_1} = \frac{z - z_1}{l_1 m_2 - l_2 m_1}$.

Solution. The two given lines are :

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} \quad ...(1)$$

and $\quad \frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2} \quad ...(2)$

Any line through (x_1, y_1, z_1) is :

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad ...(3)$$

Since (3) is perp. to (1), $\therefore ll_1 + mm_1 + nn_1 = 0 \quad ...(4)$

Since (3) is perp. to (2), $\therefore ll_2 + mm_2 + nn_2 = 0 \quad ..(5)$

Solving (4) and (5),

$$\frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1} \quad ...(6)$$

From (3) and (6), $\frac{x - x_1}{m_1 n_2 - m_2 n_1} = \frac{y - y_1}{n_1 l_2 - n_2 l_1}$

$$= \frac{z - z_1}{l_1 m_2 - l_2 m_1},$$

which are the reqd. equations.

EXERCISE 11 (b)

Very Short Answer Type Questions

VSATQ

1. Write the vector equation of the line :

$\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}$.

(Tripura B. 2016; C.B.S.E. 2010)

2. Show that the three lines with direction-cosines :

$< \frac{12}{13}, \frac{-3}{13}, \frac{-4}{13} >; < \frac{4}{13}, \frac{12}{13}, \frac{3}{13} >; < \frac{3}{13}, \frac{-4}{13}, \frac{12}{13} >$

are mutually perpendicular. *(N.C.E.R.T.)*

3. Express the following equations of the lines into vector form :

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

and $\quad \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$. *(Kerala B. 2013)*

4. (*a*) Find the cartesian as well as the vector equation of the line passing through :

(*i*) (–2, 4, –5) and parallel to the line :

$\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$ *(C.B.S.E. 2013)*

(*ii*) (0, – 1, 4) and parallel to the straight line :

$\frac{-x-2}{1} = \frac{y+3}{7} = \frac{2z-6}{3}$. *(Bihar B. 2014)*

(*iii*) (–1, 2, 3) and parallel to the line :

$\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-1}{6}$. *(H.B. 2013)*

(*b*) The cartesian equations of a line are :

(*i*) $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$

(N.C.E.R.T.; Jammu B. 2015; A.I.C.B.S.E. 2011)

(ii) $\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$. *(N.C.E.R.T.)*

Find the vector equation of the lines.

(c) Find the vector equation of the line passing through the point A(1, 2, – 1) and parallel to the line :

$5x - 25 = 14 - 7y = 35z$. *(C.B.S.E. 2017)*

5. (a) Find the equation of a line parallel to x–axis and passing through the origin. *(N.C.E.R.T.)*

(b) Find the direction-cosines of a line parallel to the line :

$$\frac{2x-5}{4} = \frac{y+4}{3} = \frac{6-z}{6}.$$

(c) Write the direction-cosines of a line parallel to the line :

$\frac{3-x}{3} = \frac{y+2}{-2} = \frac{z+2}{6}$. *(C.B.S.E. 2009 C)*

6. (a) Find the vector and cartesian equations of the line through the point (5, 2, – 4) and which is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$. *(N.C.E.R.T.)*

(b) Find the equation of a line passing through the point P(2, –1, 3) and perpendicular to the lines :

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$$

and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$.

(C.B.S.E. 2012)

7. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction of $\hat{i} + 2\hat{j} - \hat{k}$.

(N.C.E.R.T.)

8. Find the vector equation for the line through the points :

(– 1, 0, 2) and (3, 4, 6).

(N.C.E.R.T.; Assam B. 2018; Kashmir B. 2016, Jammu B. 2016)

9. Find the vector and cartesian equations of the line that passes through :

(i) the origin and (5, – 2, 3) *(N.C.E.R.T.)*

(ii) the points (1, 2, 3) and (2, – 1, 4). *(J. & K. B. 2011)*

10. (a) Find the equation of a st. line through (– 1, 2, 3) and equally inclined to the axes.

(b) Find the equation of a line parallel to x-axis and passing through the origin.

11. Find the angle between the pairs of lines with direction–ratios :

(i) $< 5, -12, 13 >$; $< -3, 4, 5 >$

(ii) $< a, b, c >$; $< b-c, c-a, a-b >$. *(N.C.E.R.T.)*

12. Find the angle between a line with direction–ratios < 2, 2, 1 > and a line joining (3, 1, 4) to (7, 2, 12).

13. Find the angle between the following pairs of lines :

(i) $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$,

$\vec{r} = 5\hat{j} - 2\hat{k} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

(N.C.E.R.T. ; Karnataka B. 2014; Kerala B. 2014; Kashmir B. 2011)

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$

and $\vec{r} = (2\hat{i} - \hat{j} - 56\hat{k}) + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

(H.P.B. 2016)

(iii) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$

and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$.

(Kerala B. 2018; H.P.B. 2017, 16, 13)

(iv) $\frac{x-4}{3} = \frac{y+1}{4} = \frac{z-6}{5}$ and $\frac{x-5}{1} = \frac{2y+5}{-2} = \frac{z-3}{1}$

(Meghalaya B. 2017)

(v) $\frac{5-x}{3} = \frac{y+3}{-4}$, $z = 7$ and $x = \frac{1-y}{2} = \frac{z-6}{2}$

(H.P.B. 2016; Meghalaya B. 2015)

(vi) $\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$.

(N.C.E.R.T.; H.P.B. 2013 S, 13 ; Kashmir B. 2011)

14. Show that the lines :

(i) $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

(N.C.E.R.T.; H.B. 2017, 15; Kashmir B. 2011)

(ii) $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z-2}{4}$ and $\frac{x+2}{2} = \frac{y-4}{4} = \frac{z+5}{2}$

(Meghalaya B. 2013)

are perpendicular to each other.

15. (i) Find the value of 'p' so that the lines :

$$l_1 : \frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2} \text{ and } l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are at right angles. *(Mizoram B. 2018)*

Also find the equations of the line passing through (3, 2, – 4) and parallel to line l_1.

(N.C.E.R.T.; A.I.C.B.S.E. 2014)

(ii) Find 'k' so that the lines :

$$\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-2}{2k} \text{ and } \frac{x+2}{1} = \frac{4-y}{k} = \frac{z+5}{1}$$

are perpendicular to each other. *(Nagaland B. 2015)*

16. Show that the line through the points :

(a) (1, – 1, 2) , (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6)

(b) (4, 7, 8) , (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1) and (1, 2, 5). *(N.C.E.R.T.)*

Short Answer Type Questions

SATQ

17. The cartesian equations of a line are :

$$3x + 1 = 6y - 2 = 1 - z.$$

Find the fixed point through which it passes, its direction-ratios and also its vector equation.

18. The points A (4, 5, 10), B (2, 3, 4) and C (1, 2, – 1) are three vertices of a parallelogram ABCD. Find the vector and cartesian equations for the sides AB and BC and find the co-ordinates of D. *(C.B.S.E. 2010)*

19. Write the equation of a line, parallel to the line $\frac{x-2}{-3}=\frac{y+3}{2}=\frac{z+3}{6}$ and passing through the point (1, 2, 3).

(A.I.C.B.S.E. 2009 C)

20. Find the equation of the line perpendicular to the lines :

$$\vec{r} = (3\hat{i}+2\hat{j}-4\hat{k})+\lambda(\hat{i}+2\hat{j}-2\hat{k})$$

and $\vec{r} = (5\hat{j}-2\hat{k})+\mu(3\hat{i}+2\hat{j}+6\hat{k})$

and passing through the point (1, 1, 1).

(Kerala B. 2014)

21. (*i*) Find the equations of the straight line passing through the point (2, 3, – 1) and is perpendicular to the lines : $\frac{x-2}{2}=\frac{y+1}{1}=\frac{z-3}{-3}$ and $\frac{x-3}{1}=\frac{y+2}{1}=\frac{z-1}{1}$.

(P.B. 2012, 10 S)

(*ii*) Find the equation of the line which intersects the lines :

$$\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4} \text{ and } \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

perpendicularly and passes through the point (1, 1, 1). *(C.B.S.E. Sample Paper 2018; W. Bengal 2018)*

22. Find the equation in vector and cartesian form of the line passing through the point :

(*i*) (2, – 1, 3) and perpendicular to the lines :

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k}) \text{ and}$$
$$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$$

(A.I.C.B.S.E. 2014; C.B.S.E. 2012)

(*ii*) (2, – 1, 3) and perpendicular to the lines :

$$\vec{r} = (\hat{i}-\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+3\hat{k}) \text{ and}$$
$$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\mu(\hat{i}+\hat{j}+\hat{k}).$$

(P.B. 2011)

23. Prove that the points (1, 2, 3), (4, 0, 4), (– 2, 4, 2) and (7, – 2, 5) are collinear.

24. Show that the following points whose position vectors are given are collinear :

(*i*) $5\hat{i}+5\hat{k}, 2\hat{i}+\hat{j}+3\hat{k}$ and $-4\hat{i}+3\hat{j}-\hat{k}$

(*ii*) $-2\hat{i}+3\hat{j}+5\hat{k}, \hat{i}+2\hat{j}+3\hat{k}$ and $7\hat{i}-\hat{k}$.

(P.B. 2014 S)

25. Find the points on the line through the points A(1, 2, 3) and B (5, 8, 15) at a distance of 14 units from the mid-point of AB. *(Meghalaya B. 2016)*

26. Find the equations of the perpendicular from the point (3, –1, 11) to the line :

$$\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}.$$

27. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points : (3, 5, – 1), (4, 3, – 1). *(N.C.E.R.T.)*

Long Answer Type Questions

LATQ

28. (*i*) Find the vector and cartesian equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines :

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

(N.C.E.R.T.; C.B.S.E. 2017, 12; H.P.B. 2015; Jammu B. 2015, 13, 12; Meghalaya B. 2015; P.B. 2012)

(*ii*) Find the vector and cartesian equations of the line passing through the point (2, 1, 3) and perpendicular to the lines :

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3} \text{ and } \frac{x}{-3}=\frac{y}{2}=\frac{z}{5}.$$

(Type : Assam B. 2017, A.I.C.B.S.E. 2014)

29. (*i*) Find the vector equation of a line passing through a point with position vector $2\hat{i}-\hat{j}+\hat{k}$ and parallel to the line joining the points with position vectors $-\hat{i}+4\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+2\hat{k}$. Also, find the cartesian equivalent of the equation.

(*ii*) Find the vector equation of a line passing through the point with position vector $\hat{i}-2\hat{j}-3\hat{k}$ and parallel to the line joining the points with position vectors $\hat{i}-\hat{j}+4\hat{k}$ and $2\hat{i}+\hat{j}+2\hat{k}$. Also find the cartesian form of the equation.

HOTS

Answers

1. $\vec{r} = (5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}-2\hat{k})$.

3. $\vec{r} = (3\hat{i}+8\hat{j}+3\hat{k})+\lambda(3\hat{i}-\hat{j}+\hat{k})$ and

$\vec{r} = (-3\hat{i}-7\hat{j}+6\hat{k})+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$.

4. (*a*) (*i*) $\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}$;

$\vec{r} = (-2\hat{i} + 4\hat{j} - 5\hat{k}) + \lambda(3\hat{i} - 5\hat{j} + 6\hat{k})$

(ii) $\frac{x}{-1} = \frac{y+1}{7} = \frac{2(z-4)}{3}$;

$\vec{r} = (-\hat{j} + 4\hat{k}) + \lambda\left(-\hat{i} + 7\hat{j} + \frac{3}{2}\hat{k}\right)$

(iii) $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$;

$\vec{r} = (-\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

(b) (i) $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

(ii) $\vec{r} = -3\hat{i} + 5\hat{j} - 6\hat{k} + \lambda(2\hat{i} + 4\hat{j} + 2\hat{k})$

(c) $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k})$.

5. (a) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ (b) $< \frac{2}{7}, \frac{3}{7}, \frac{-6}{7} >$

(c) $< -\frac{3}{7}, \frac{-2}{7}, \frac{6}{7} >$.

6. (a) $\vec{r} = (5+3\lambda)\hat{i} + (2+2\lambda)\hat{j} + (-4-8\lambda)\hat{k}$;

$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$

(b) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$.

7. $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$;

$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$.

8. $\vec{r} = -\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$.

9. (i) $\vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$; $\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$

(ii) $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + \hat{k})$;

$\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z-3}{1}$.

10. (a) $x + 1 = y - 2 = z - 3$ (b) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$.

11. (i) $\cos^{-1}\left(\frac{1}{65}\right)$ (ii) $\frac{\pi}{2}$.

12. $\cos^{-1}\frac{2}{3}$.

13. (i) $\cos^{-1}\left(\frac{19}{21}\right)$ (ii) $\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$

(iii) $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$ (iv) $\cos^{-1}\left(\frac{4}{5\sqrt{6}}\right)$

(v) $\cos^{-1}\left(\frac{5}{9}\right)$ (vi) $\cos^{-1}\left(\frac{8}{5\sqrt{2}}\right)$.

15. (i) $p = \frac{70}{11}$; $\frac{3-x}{3} = \frac{11y-22}{140} = \frac{z+4}{2}$

(ii) $k = 2$.

17. $\left(-\frac{1}{3}, \frac{1}{3}, 1\right)$; $< 2, 1, -6 >$;

$\vec{r} = (-\frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} - 6\hat{k})$.

18. $\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \lambda(\hat{i} + \hat{j} + 3\hat{k})$;

$4 - x = 5 - y = \frac{10-z}{3}$;

$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + (\hat{i} + \hat{j} + 5\hat{k})$;

$2 - x = 3 - y = \frac{4-z}{5}$; (3, 4, 5).

19. $\frac{x-1}{-3} = \frac{y-2}{2} = \frac{z-3}{6}$.

20. $\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \lambda(16\hat{i} - 12\hat{j} - 4\hat{k})$.

21. (i) $\frac{x-2}{4} = \frac{y-3}{-5} = \frac{z+1}{1}$

(ii) $\frac{x-1}{4} = \frac{y-1}{-4} = \frac{z+1}{1}$.

22. (i) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$;

$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-2}$

(ii) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(4\hat{i} - 5\hat{j} + \hat{k})$;

$\frac{x-2}{4} = \frac{y+1}{-5} = \frac{z-3}{1}$.

25. (7, 11, 21) and (–1, –1, –3).

26. $\frac{x-3}{2} = \frac{y+1}{5} = \frac{z-11}{7}$.

28. (i) $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$;

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$

(ii) $\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k})$;

$\frac{x-2}{2} = \frac{y-1}{-7} = \frac{z-3}{4}$.

29. (i) $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$;

$\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$

(ii) $\vec{r} = \hat{i} - 2\hat{j} - 3\hat{k} + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$;

$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z+3}{-2}$.

Hints to Selected Questions

4. (*b*) (*i*) The line passes through (5, – 4, 6) and has direction-cosines < 3, 7, 2 >.

10. (*a*) Equations of the line are $\frac{x+1}{1}=\frac{y-2}{1}=\frac{z-3}{1}$

$\Rightarrow\ x+1=y-2=z-3.$

15. (*i*) Lines are $l_1 : \frac{x-1}{-3}=\frac{y-2}{2p/7}=\frac{z-3}{2}$

and $l_2 : \frac{x-1}{-3p/7}=\frac{y-5}{1}=\frac{z-6}{-5}$.

Lines are perpendicular

if $(-3)\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)(1)+(2)(-5)=0$

if $p=\frac{70}{11}$.

18. Mid–points of [AC] and [BD] are same.

23. Equations of AB are $\frac{x-1}{3}=\frac{y-2}{-2}=\frac{z-3}{1}$.

C lies on it.

29. (*i*) Here $\vec{b}=(\hat{i}+\hat{i})+(2\hat{j}-4\hat{j})+(2\hat{k}-\hat{k})$

$=2\hat{i}-2\hat{j}+\hat{k}$ and $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$.

∴ The equation of the line is :

$\vec{r}=2\hat{i}-\hat{j}+\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$; etc.

11.13. PERPENDICULAR DISTANCE OF A POINT FROM A LINE

To find the perpendicular distance of the point (α , β , γ) from the line :

$$\mathbf{\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}}.$$

The given line L is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(1)

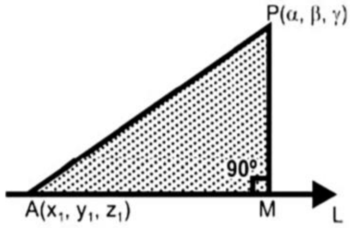


Fig.

Let P (α, β, γ) be the given point.

Let A be the point (x_1, y_1, z_1).

Join AP. Draw PM perp. on the line L.

In ΔAPM, *by Pythagoras' Theorem,* we have :

$PM^2 + AM^2 = AP^2$ *i.e.* $PM^2 = AP^2 - AM^2$ ...(2)

But $AP^2=(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2$

and | AM | = Projection [AP] on the line L = $(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n$,

where < *l, m, n* > are direction-cosines of L

$\Rightarrow AM^2=[(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n]^2$.

Putting in (2), we get : $PM^2=[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2]-[(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n]^2$,

which gives the required perpendicular distance.

GUIDE LINES

Step (i) *Find | AP |, by distance formula, and hence find AP^2.*
Step (ii) *Find | AM |, by projection formula, and hence find AM^2.*
Step (iii) *Find | PM |, from the relation $PM^2 = AP^2 - AM^2$.*

ILLUSTRATIVE EXAMPLES

Example 1. Find the length of the perpendicular from the point (3, 4, 5) on the line $\frac{x-2}{2}=\frac{y-3}{5}=\frac{z-1}{3}$.

Solution. Let P (3, 4, 5) be the given point and L the given line : $\frac{x-2}{2}=\frac{y-3}{5}=\frac{z-1}{3}$.

Let A be a point (2, 3, 1) on L.

The direction-ratios of L are < 2, 5, 3 >.

Its direction-cosines are :

$$<\frac{2}{\sqrt{4+25+9}}, \frac{5}{\sqrt{4+25+9}}, \frac{3}{\sqrt{4+25+9}}>$$

i.e. $<\frac{2}{\sqrt{38}}, \frac{5}{\sqrt{38}}, \frac{3}{\sqrt{38}}>$.

Now $PM^2 = AP^2 - AM^2$ ...(1)

[*Pythagoras' Theorem*]

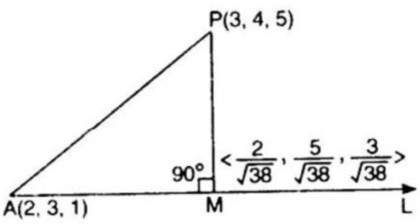


Fig.

But $|AP| = \sqrt{(3-2)^2+(4-3)^2+(5-1)^2}$

[*Distance Formula*]

$$= \sqrt{1+1+16} = \sqrt{18}$$

and | AM | = Projection of [AP] on AL

$$= (3-2)\frac{2}{\sqrt{38}} + (4-3)\frac{5}{\sqrt{38}} + (5-1)\frac{3}{\sqrt{38}}$$

[*Using* $(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$]

$$= \frac{2}{\sqrt{38}} + \frac{5}{\sqrt{38}} + \frac{12}{\sqrt{38}} = \frac{19}{\sqrt{38}}.$$

Putting these values in (1), we get :

$$PM^2 = (\sqrt{18})^2 - \left(\frac{19}{\sqrt{38}}\right)^2 = 18 - \frac{361}{38} = 18 - \frac{19}{2} = \frac{17}{2}.$$

Hence, $|PM| = \sqrt{\frac{17}{2}}$ units,

which is the required perpendicular distance.

Example 2. Find the co-ordinates of the foot of the perpendicular drawn from the point A (1, 8, 4) to the line joining B (0, – 1, 3) and C (2, – 3, – 1). *(A.I.C.B.S.E. 2016)*

Solution.

Any point on BC, which divides [BC] in the ratio *k* : 1, is :

$$\left(\frac{2k}{k+1}, \frac{-3k-1}{k+1}, \frac{-k+3}{k+1}\right) \quad ...(1)$$

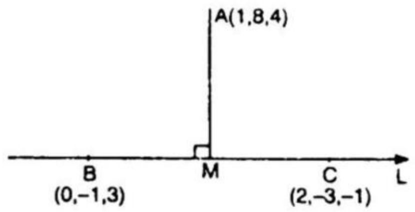


Fig.

This becomes M, the foot of perp. from A on BC

if $AM \perp BC$ ...(2)

But direction-ratios of BC are :

< 2 – 0, – 3 + 1, – 1 – 3 > *i.e.* < 2, – 2, – 4 >

i.e. < 1, – 1, – 2 >

and direction-ratios of AM are :

$$<\frac{2k}{k+1} - 1, \frac{-3k-1}{k+1} - 8, \frac{-k+3}{k+1} - 4>$$

i.e. $< k-1, -11k-9, -5k-1 >$

∴ Due to (2), $(1)(k-1) + (-1)(-11k-9) + (-2)(-5k-1) = 0$

$\Rightarrow k - 1 + 11k + 9 + 10k + 2 = 0$

$\Rightarrow 22k + 10 = 0 \Rightarrow k = -\frac{5}{11}$.

∴ From (1), the co-ordinates of M, the foot of perp. are :

$$\left(\frac{\frac{-10}{11}}{\frac{-5}{11}+1}, \frac{\frac{15}{11}-1}{\frac{-5}{11}+1}, \frac{\frac{5}{11}+3}{\frac{-5}{11}+1}\right)$$

i.e. $\left(\frac{-10}{6}, \frac{4}{6}, \frac{38}{6}\right)$ *i.e.* $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$.

Example 3. Find the vector equation of the line parallel to the line :

$$\frac{x-1}{5}=\frac{3-y}{2}=\frac{z+1}{4}$$

and passing through (3, 0, – 4).

Also, find the distance between these two lines.

Solution. (*i*) The given line is $\frac{x-1}{5}=\frac{y-3}{-2}=\frac{z-(-1)}{4}$ ...(1)

Its direction–ratios are < 5, – 2, 4 >.

The vector equation of the line through (3, 0, – 4) parallel to (1) is :

$$\vec{r}=\left(3\hat{i}-4\hat{k}\right)+\lambda\left(5\hat{i}-2\hat{j}+4\hat{k}\right) \quad ...(2)$$

(*ii*) The cartesian equations of (2) are :

$$\frac{x-3}{5}=\frac{y-0}{-2}=\frac{z+4}{4} \quad ...(3)$$

To find the distance between (1) and (3) :

A (1, 3, – 1) is a point on (1).

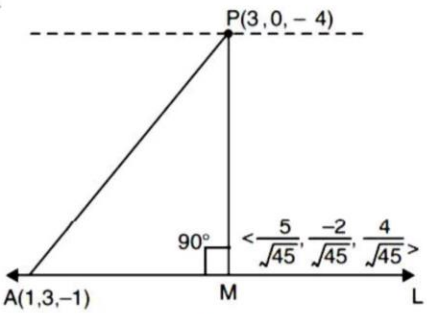


Fig.

The direction-cosines of (1) are :

$$\left\langle\frac{5}{\sqrt{25+4+16}},\frac{-2}{\sqrt{25+4+16}},\frac{4}{\sqrt{25+4+16}}\right\rangle$$

i.e. $\left\langle\frac{5}{\sqrt{45}},\frac{-2}{\sqrt{45}},\frac{4}{\sqrt{45}}\right\rangle$.

Now $PM^2 = AP^2 - AM^2$ ...(4)

[*Pythagoras' Theorem*]

But $|AP| = \sqrt{(3-1)^2+(0-3)^2+(-4+1)^2}$

$= \sqrt{4+9+9}=\sqrt{22}$

and $|AM| = (3-1)\left(\frac{5}{\sqrt{45}}\right)+(0-3)\left(-\frac{2}{\sqrt{45}}\right)+(-4+1)\left(\frac{4}{\sqrt{45}}\right)$

$= \frac{10+6-12}{\sqrt{45}}=\frac{4}{\sqrt{45}}$.

Putting in (4), $PM^2 = 22-\frac{16}{45}=\frac{990-16}{45}=\frac{974}{45}$.

Hence, $|PM| = \sqrt{\frac{974}{45}}$ units.

Example 4. Find the equations of the perpendicular drawn from the point P (2, 4, – 1) to the line :

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}.$$

Also, write down the co-ordinates of the foot of the perpendicular from P to the line.

Solution. The given line AB is $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$ ...(1)

Any point on line (1) is :
$(k-5, 4k-3, -9k+6)$.

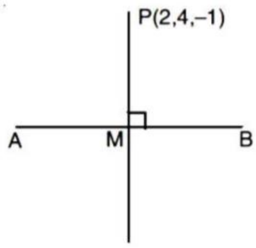


Fig.

For some value of k, this point is M such that PM is perp. to line (1).

Now direction-ratios of the line are < 1, 4, – 9 >

and direction-ratios of PM are :

$< k-5-2, 4k-3-4, -9k+6+1 >$

i.e. $< k-7, 4k-7, -9k+7 >$.

$\therefore$ (1) $(k-7)$ + (4) $(4k-7)$ + (– 9) $(-9k+7)$ = 0

$\Rightarrow k-7+16k-28+81k-63=0$

$\Rightarrow 98k = 98 \Rightarrow k = 1.$

$\therefore$ M is $(1-5, 4-3, -9+6)$ *i.e.* (– 4, 1, – 3).

$\therefore$ The equations of PM are :

$$\frac{x-2}{-4-2}=\frac{y-4}{1-4}=\frac{z+1}{-3+1} \Rightarrow \frac{x-2}{-6}=\frac{y-4}{-3}=\frac{z+1}{-2}$$

i.e. $\frac{x-2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$.

Hence, the equations of PM are :

$$\frac{x-2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$$

and foot of perpendicular is (– 4, 1, –3).

REFLECTION OR IMAGE OF A POINT ON A STRAIGHT LINE

If the perpendicular PM from P on the st. line AB be produced to P' such that PM = MP', then P' is called the image or reflection of P in the given line.

[For fig., see Ex. 5]

Example 5. (a) Find the image of the point (1, 6, 3) in the line :

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}.$$

(*C.B.S.E. 2010 C*)

(b) Also, write the equation of the line joining the given point and its image and find the length of the segment joining the given point and its image. *(C.B.S.E. 2010 C)*

Solution. (*a*) Let P be the given point (1, 6, 3) and M, the foot of perpendicular from P on the given line AB :

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3} \quad (=k \text{ (say)})$$

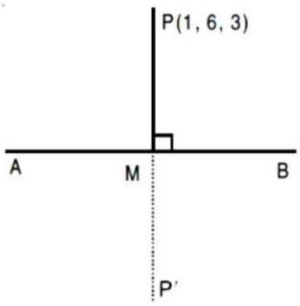


Fig.

Any point on the given line is :

$$(k,\ 1+2k,\ 2+3k).$$

For some value of k, let the point be M.

$\therefore$ Direction-ratios of PM are :

$$< k-1,\ 1+2k-6,\ 2+3k-3 >$$

i.e. $< k-1,\ 2k-5,\ 3k-1 >$.

Since PM $\perp$ AB ,

$\therefore$ (1) $(k-1)$ + (2) $(2k-5)$ + (3) $(3k-1) = 0$

$\Rightarrow\ k-1+4k-10+9k-3=0$

$\Rightarrow\ 14k = 14 \qquad \Rightarrow k = 1.$

$\therefore$ Foot of perpendicular M is (1, 1+2, 2+3) *i.e.* (1,3, 5).

Let P′(α, β, γ) be the image of P in the given line.

Then M is the mid-point of [PP′].

$$\therefore \quad \frac{\alpha+1}{2}=1,\ \frac{\beta+6}{2}=3,\ \frac{\gamma+3}{2}=5$$

$\Rightarrow\ \alpha+1=2,\ \beta+6=6,\ \gamma+3=10$

$\Rightarrow\ \alpha=1,\ \beta=0,\ \gamma=7.$

Hence, the reqd. image is (1, 0, 7).

(*b*) (*i*) The equations of the line PP′ are :

$$\frac{x-1}{1-1}=\frac{y-6}{0-6}=\frac{z-3}{7-3}$$

$$i.e. \quad \frac{x-1}{0}=\frac{y-6}{-6}=\frac{z-3}{4}$$

$$\Rightarrow \quad \frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}.$$

(*ii*) Length of segment [PP′]

$$=\sqrt{(1-1)^2+(0-6)^2+(7-3)^2}$$

$$=\sqrt{0+36+16}=\sqrt{52}=2\sqrt{13} \text{ units.}$$

Another Form

Find the image of the point (1, 6, 3) in the line :

$$\vec{r}=\hat{j}+2\hat{k}+\lambda(\hat{i}+2\hat{j}+3\hat{k}).$$

Solution. The cartesian equations of given vector equation are :

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}.$$

Now it is same as above.

Example 6. Find the co-ordinates of the foot of perpendicular and the length of the perpendicular drawn from the point P (5, 4, 2) to the line :

$$\vec{r}=-\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}-\hat{k}).$$

Also, find the image of P in this line.

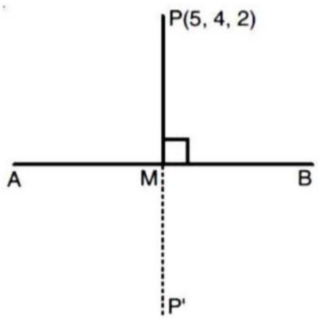
(Mizoram B. 2016; A.I.C.B.S.E. 2012)

Solution. (*i*) The given line is :

$$\vec{r}=-\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}-\hat{k})$$

$$i.e. \quad \frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1} \quad \ldots(1)$$

The given point is P (5, 4, 2).



Let M be the foot of perpendicular from P on the given line AB.

Any point on (1) is $(-1+2k,\ 3+3k,\ 1-k)$.

For some value of k, let the point be M.

$\therefore$ Direction-ratios of PM are :

$< -1+2k-5,\ 3+3k-4,\ 1-k-2>$

i.e. $< 2k-6,\ 3k-1,\ -k-1>$.

Since PM $\perp$ AB,

$\therefore\ 2(2k-6)+3(3k-1)+(-1)(-k-1)=0$

$\Rightarrow\ 4k-12+9k-3+k+1=0$

$\Rightarrow\ 14k-14=0 \Rightarrow k=1.$

$\therefore$ Foot of perpendicular M is $(-1+2,\ 3+3,\ 1-1)$ *i.e.* (1, 6, 0).

(*ii*) Length of Perpendicular

$$=\sqrt{(5-1)^2+(4-6)^2+(2-0)^2}$$

$$=\sqrt{16+4+4}=\sqrt{24}=2\sqrt{6} \text{ units.}$$

(*iii*) Let P′ (α, β, γ) be the image of P in the given line.

Then M is the mid-points of [PP′].

$$\therefore \quad \frac{\alpha+5}{2} = 1, \frac{\beta+4}{2} = 6, \frac{\gamma+2}{2} = 0$$

$$\Rightarrow \quad \alpha + 5 = 2, \beta + 4 = 12, \gamma + 2 = 0$$

$$\Rightarrow \quad \alpha = -3, \beta = 8, \gamma = -2.$$

Hence, the reqd. image is (–3, 8, –2).

EXERCISE 11 (c)

Short Answer Type Questions

SATQ

1. Find the distance of the point (1, 2, 3) from the line joining the points (– 1, 2, 5) and (2, 3, 4).

2. Find the distance of the point (1, 2, 3) from the co-ordinate axes.

3. Find the distance of (– 1, 2, 5) from the line passing through the point (3, 4, 5) and whose direction-ratios are < 2, – 3, 6 >.

4. Find the perpendicular distance of the point (1, 0, 0) from the line :

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}.$$

Also find the co-ordinates of the foot of the perpendicular.

5. (*a*) Find the length of the perpendicular from the point (1, 2, 3) to the line :

$$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}.$$

(*b*) Find the perpendicular distance from the point (1, 2, 3) to the line :

$$\vec{r} = 6\hat{i} + 7\hat{j} + 7\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k}).$$

6. (*a*) Find the foot of the perpendicular from the point (*i*) (2, – 1, 5) on the line :

$$\frac{x-11}{10} = \frac{y+2}{-5} = \frac{z+8}{11}$$ *(C.B.S.E. 2009 C)*

(*ii*) (0, 2, 3) on the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$.

(J. & K.B. 2011; C.B.S.E. 2009)

(*b*) Also find the length of perpendicular in part (*ii*). *(C.B.S.E. 2009)*

7. Find the co-ordinates of the foot of the perpendicular from the point A (1, 0, 3) to the line joining B (4, 7, 1) and C (3, 5, 3).

8. A (1, 0, 4), B (0, – 11, 3), C (2, – 3, 1) are three points and D is the foot of the perpendicular from A on BC. Find the co-ordinates of D.

9. Find the perpendicular distance of an angular point of a cube from a diagonal, which does not pass through that angular point.

Long Answer Type Questions

LATQ

10. Find the image of the point :

(*a*) (*i*) (2, 0, 1) in the line $\frac{x-3}{1} = \frac{y+2}{-2} = \frac{z-3}{5}$ *(P.B. 2014)*

(*ii*) (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. *(H.B. 2018)*

(*b*) Find the image of the point A(–1, 8, 4) in the line joining the points B (0, –1, 3) and C (2, –3, –1). *(A.I.C.B.S.E. 2016)*

11. Let the point P (5, 9, 3) lie on the top of Qutub Minar, Delhi. Find the image of the point on the line :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}.$$

12. Find the foot of the perpendicular from the point (1, 2, 3) to the line joining the points (6, 7, 7) and (9, 9, 5). *(Nagaland B. 2018)*

13. Find the length and the foot of the perpendicular drawn from the point (2, – 1, 5) to the line :

$$\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}.$$ *(H.B. 2018)*

14. Find the equations of the perpendicular drawn from the point (2, 4, – 1) to the line :

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}.$$

15. Find the perpendicular distance of the point (2, 3, 4) from the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.

Also find the co-ordinates of the foot of the perpendicular. *(C.B.S.E. 2009 C)*

16. Find the equations of the perpendicular from the point (3, – 1, 11) to the line :

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}.$$

Also, find the foot of the perpendicular and the length of the perpendicular. *(H.B. 2018)*

17. A line passing through the point A with position vector $\vec{a} = 4\hat{i} + 2\hat{j} + 2\hat{k}$ is parallel to the vector $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Find the length of the perpendicular drawn on this line from a point P with position vector $\vec{r}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$. *(A.I.C.B.S.E. 2015)*

Answers

1. $\sqrt{\frac{24}{11}}$. **2.** $\sqrt{13}, \sqrt{10}, \sqrt{5}$.

3. $\frac{\sqrt{976}}{7}$. **4.** $2\sqrt{6}$, (3, – 4, – 2).

5. (*a*) – (*b*) 7.

6. (*a*) (*i*) $\left(\frac{531}{41}, \frac{-122}{41}, \frac{-240}{41}\right)$

(*ii*) (2, 3, – 1) (*b*) $\sqrt{21}$.

7. $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$. **8.** $\left(\frac{22}{9}, -\frac{1}{9}, \frac{5}{9}\right)$.

9. $a\sqrt{\frac{2}{3}}$, where a is the edge of the cube.

10. (*a*) (*i*) (3, –2, 0) (*ii*) (1, 0, 7) (*b*) (– 3, – 6, 10).

11. (3, 5, 7).

(*b*) (– 3, –6, 10).

12. (3, 5, 9)

13. $\sqrt{14}$ units; (1, 2, 3).

14. $\frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$.

15. $\left(\frac{170}{49}, \frac{78}{49}, \frac{10}{49}\right); \frac{3}{7}\sqrt{101}$.

16. $\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$; (2, 5, 7) ; $\sqrt{53}$.

17. $\frac{1}{7}\sqrt{530}$ units.

11.14. COPLANAR LINES

Let the two given lines be $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} = r_1$ (say) ...(1)

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} = r_2$ (say) ...(2)

When the lines are coplanar i.e. they lie in the same plane, then either they are parallel or they intersect.

If $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$, then the two lines are parallel and consequently they lie in the same plane.

In other words, two parallel lines are always coplanar. In case the two lines are not parallel, they are coplanar if they intersect.

If two non-parallel lines do not intersect, they are not coplanar. Such lines are called **skew lines.**

Definition

Two lines, which are neither parallel nor intersecting, are called skew lines.

11.15. SHORTEST DISTANCE BETWEEN TWO LINES

Definition

(*i*) **Line of Shortest Distance**

If L_1 and L_2 are two skew lines, then there is one and only one line which is perpendicular to both and is known as the line of shortest distance.

(*ii*) **Shortest Distance.**

The shortest distance between two lines L_1 and L_2 is the distance | PQ |, where P, Q are points at which the line of shortest distance meets L_1 and L_2 respectively.

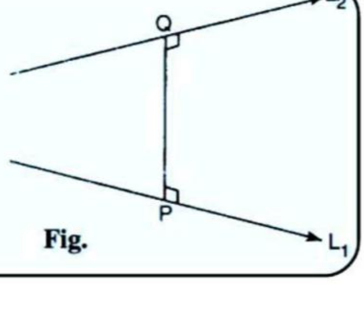


Fig.

KEY POINT

If two lines in space intersect at a point, then the shortest distance between them is zero.

(a) To find the shortest distance between two skew lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$.

Let two skew lines L_1 and L_2 be $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ ...(1)

and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ ...(2)

Take any point S $(\vec{a_1})$ on L_1 and T $(\vec{a_2})$ on L_2.

Let $\overrightarrow{PQ}$ be the shortest distance vector between them.

By *def.*, $\overrightarrow{PQ}$ is perp. to (1) and (2)

$\Rightarrow$ $\overrightarrow{PQ}$ is perp. to both $\vec{b_1}$ and $\vec{b_2}$

$\Rightarrow$ $\overrightarrow{PQ}$ is parallel to $\vec{b_1} \times \vec{b_2}$.

The unit vector $\hat{n}$ along PQ is given by $\hat{n} = \dfrac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|}$.

Let $\overrightarrow{PQ} = d\,\hat{n}$, where d is the magnitude of the shortest distance vector.

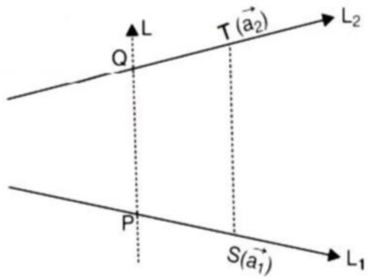


Fig.

Clearly, PQ is projection of $\overrightarrow{ST}$ on $\overrightarrow{PQ}$.

Now if 'θ' be the angle between $\overrightarrow{PQ}$ and $\overrightarrow{ST}$, then PQ = ST cos θ.

But $\cos\theta = \dfrac{\overrightarrow{PQ} \cdot \overrightarrow{ST}}{|\overrightarrow{PQ}||\overrightarrow{ST}|} = \dfrac{d\,\hat{n} \cdot (\vec{a_2} - \vec{a_1})}{d\,|\vec{b_1} \times \vec{b_2}|}$.

Hence, $d = PQ = ST\cos\theta = \dfrac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}$.

Since d is always to be taken as positive, $\therefore$ $\mathbf{d} = \left| \dfrac{(\vec{\mathbf{b_1}} \times \vec{\mathbf{b_2}}) \cdot (\vec{\mathbf{a_2}} - \vec{\mathbf{a_1}})}{|\vec{\mathbf{b_1}} \times \vec{\mathbf{b_2}}|} \right|$.

Cor. If two lines intersect, then $d = 0$

i.e. $(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1}) = 0$.

(b) To find the shortest distance between two parallel lines : $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu \vec{b}$.

Let two parallel lines L_1 and L_2 be : $\vec{r} = \vec{a_1} + \lambda \vec{b}$ ...(1)

and $\vec{r} = \vec{a_2} + \lambda \vec{b}$ ...(2)

These are clearly coplanar.

Clearly, either of L_1 and L_2 is parallel to $\vec{b}$ and they pass through the points S $(\vec{a_1})$ and T $(\vec{a_2})$.

Let $\overrightarrow{PQ}$ be the shortest distance vector between them.

$\therefore \quad d = PQ = ST\cos(90° - \theta) = ST\sin\theta$

$$= (ST)\frac{|\vec{b} \times \overrightarrow{ST}|}{|\vec{b}|(ST)} = \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|}.$$

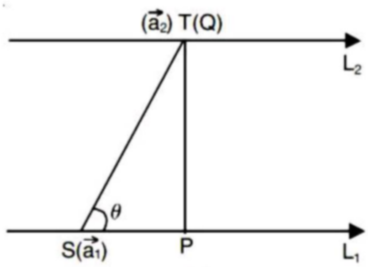


Fig.

Since d is always to be taken as positive, $\therefore$ $\mathbf{d} = \left|\frac{\vec{\mathbf{b}} \times (\vec{\mathbf{a}}_2 - \vec{\mathbf{a}}_1)}{|\vec{\mathbf{b}}|}\right|$.

(c) To find the shortest distance between two straight lines whose equations are :

$$\frac{\mathbf{x - x_1}}{\mathbf{l_1}} = \frac{\mathbf{y - y_1}}{\mathbf{m_1}} = \frac{\mathbf{z - z_1}}{\mathbf{n_1}}$$

and $$\frac{\mathbf{x - x_2}}{\mathbf{l_2}} = \frac{\mathbf{y - y_2}}{\mathbf{m_2}} = \frac{\mathbf{z - z_2}}{\mathbf{n_2}}.$$

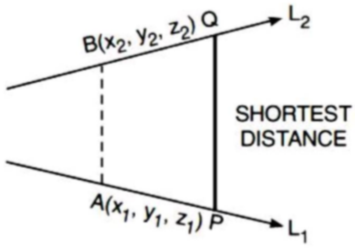


Fig.

Let PQ be the S.D.

Let $< l, m, n >$ be its direction-cosines.

Then $\quad ll_1 + mm_1 + nn_1 = 0$ ...(3)

and $\quad ll_2 + mm_2 + nn_2 = 0$ ...(4)

Solving, $\dfrac{l}{m_1n_2 - m_2n_1} = \dfrac{m}{n_1l_2 - n_2l_1} = \dfrac{n}{l_1m_2 - l_2m_1}$.

$\therefore$ Direction-ratios of PQ are :

$$< m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1,\ l_1m_2 - l_2m_1 > .$$

$\therefore$ Direction-cosines of PQ are :

$$< \frac{m_1n_2 - m_2n_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \frac{n_1l_2 - n_2l_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \frac{l_1m_2 - l_2m_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}} >.$$

$\therefore$ **Length of the S.D.** = | PQ |= Projection of [AB] on PQ

$$= \frac{(x_2 - x_1)(m_1n_2 - m_2n_1) + (y_2 - y_1)(n_1l_2 - n_2l_1) + (z_2 - z_1)(l_1m_2 - l_2m_1)}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}.$$

Cor. If two lines intersect, then :

$(x_2 - x_1)(m_1n_2 - m_2n_1) + (y_2 - y_1)(n_1l_2 - n_2l_1) + (z_2 - z_1)(l_1m_2 - l_2m_1) = 0$

$$\Rightarrow \quad \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

11.16. CO-PLANARITY OF TWO LINES

Consider the two lines :

$$\vec{r} = \vec{a_1} + \lambda\vec{b_1} \quad ...(1)$$

and $$\vec{r} = \vec{a_2} + \mu\vec{b_2} \quad ...(2)$$

(1) passes thro' A having position vector $\vec{a_1}$ and is parallel to $\vec{b_1}$ and

(2) passes thro' B having position vector $\vec{a_2}$ and is parallel to $\vec{b_2}$.

Thus $\overrightarrow{AB}$ = p.v. of B – p.v. of A = $\vec{a_2} - \vec{a_1}$.

The given lines are coplanar if and only if $\overrightarrow{AB}$ is perp. to $\vec{b_1} \times \vec{b_2}$

i.e. $\overrightarrow{AB} \cdot (\vec{b_1} \times \vec{b_2}) = 0$ $\Rightarrow (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0$

i.e. **d, the shortest distance between (1) and (2) = 0.**

CARTESIAN FORM :

Let A and B have co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Let $\vec{b_1}$ and $\vec{b_2}$ have direction-ratios :

$< a_1, b_1, c_1 >$ and $< a_2, b_2, c_2 >$ respectively.

$\therefore$ $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$,

where $\vec{b_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

and $\vec{b_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

The given lines are coplanar if and only if $\overrightarrow{AB} \cdot (\vec{b_1} \times \vec{b_2}) = 0$ [*As above*]

This can be expressed, in cartesian form, as :

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Frequently Asked Questions

FAQs

Example 1. The vector equations of two lines are :

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \text{ and}$$
$$\vec{r} = 2\hat{i} + 4\hat{j} + 5\hat{k} + \mu(3\hat{i} + 4\hat{j} + 5\hat{k}).$$

Find the shortest distance between these lines.

(P.B. 2016)

Solution. Comparing given equations with :

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, we have :

$\vec{b_1} = 2\hat{i} + 3\hat{j} + 4\hat{k}$,

$\vec{b_2} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

and $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$,

$\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$.

Now $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$= \hat{i}(15 - 16) - \hat{j}(10 - 12) + \hat{k}(8 - 9)$

$= -\hat{i} + 2\hat{j} - \hat{k}$.

$\therefore$ $|\vec{b_1} \times \vec{b_2}| = \sqrt{(-1)^2 + (2)^2 + (-1)^2}$

$= \sqrt{1 + 4 + 1} = \sqrt{6}$.

Also, $\vec{a_2} - \vec{a_1} = (2 - 1)\hat{i} + (4 - 2)\hat{j} + (5 - 3)\hat{k}$

$= \hat{i} + 2\hat{j} + 2\hat{k}$.

$\therefore$ *d*, the shortest distance between the given lines is given by :

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \left| \frac{(-\hat{i} + 2\hat{j} - \hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{6}} \right|$$

$$= \left| \frac{(-1)(1) + (2)(2) + (-1)(2)}{\sqrt{6}} \right|$$

$$= \left| \frac{-1 + 4 - 2}{\sqrt{6}} \right| = \left| \frac{1}{\sqrt{6}} \right| = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6} \text{ units.}$$

Example 2. Find the distance between the lines L_1 and L_2 given by :

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ and}$$

$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k}).$$ (N.C.E.R.T.)

Solution. Clearly, L_1 and L_2 are parallel.

Comparing given equations with :

$\vec{r} = \vec{a_1} + \lambda\vec{b}$ and $\vec{r} = \vec{a_2} + \mu\vec{b}$, we have :

$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, so that

$$|\vec{b}| = \sqrt{4+9+36} = \sqrt{49} = 7$$

and $\vec{a_1} = \hat{i} + 2\hat{j} - 4\hat{k}$; $\vec{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k}$.

Now $\vec{a_2} - \vec{a_1} = (3-1)\hat{i} + (3-2)\hat{j} + (-5+4)\hat{k}$

$$= 2\hat{i} + \hat{j} - \hat{k}.$$

$$\therefore \vec{b} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-3-6) - \hat{j}(-2-12) + \hat{k}(2-6)$$

$$= -9\hat{i} + 14\hat{j} - 4\hat{k}.$$

$\therefore$ d, the distance between the given lines is given by :

$$d = \left|\frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|}\right|$$

$$= \left|\frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{7}\right|$$

$$= \frac{1}{7}|-9\hat{i} + 14\hat{j} - 4\hat{k}|$$

$$= \frac{1}{7}\sqrt{81+196+16}$$

$$= \frac{1}{7}\sqrt{293} \text{ units.}$$

Example 3. Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect.

Find also the point of intersection of these lines.

(Mizoram B. 2017; Meghalaya B. 2014; P.B. 2012)

Solution. The given lines are :

$$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad ...(1)$$

and $$L_2 : \frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1} \quad ...(2)$$

Any point on L_1 is $(2\lambda + 1, 3\lambda + 2, 4\lambda + 3)$ ...(3)

Any point on L_2 is $(5\mu + 4, 2\mu + 1, \mu)$ ...(4)

The lines L_1 and L_2 will intersect iff points (3) and (4) coincide

iff $2\lambda + 1 = 5\mu + 4$, $3\lambda + 2 = 2\mu + 1$, $4\lambda + 3 = \mu$

Taking first two, $2\lambda - 5\mu = 3$ ...(5)

Taking middle two, $3\lambda - 2\mu = -1$ ...(6)

Taking last two, $4\lambda - \mu = -3$ ...(7)

Solving (5) and (6), $\lambda = -1$ and $\mu = -1$.

Putting in (7), $4(-1) + 1 = -3$

$\Rightarrow -3 = -3$, which is true.

Hence, the given lines L_1 and L_2 intersect.

Putting $\lambda = -1$ in (3), [or $\mu = -1$ in (4)],

we get the reqd. point of intersection as $(-1, -1, -1)$.

Example 4. Show that the lines :

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and } \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$$

are coplanar. Also, find the equation of the plane.

(N.C.E.R.T.)

Solution. Comparing the given equations with

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2},$$

we get :

$x_1 = -3, y_1 = 1, z_1 = 5$; $x_2 = -1, y_2 = 2, z_2 = 5$

and $a_1 = -3, b_1 = 1, c_1 = 5$; $a_2 = -1, b_2 = 2, c_2 = 5$.

$$\text{Now} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} -1+3 & 2-1 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}$$

$= 2(5-10) - 1.(-15+5) + 0 = -10 + 10 + 0 = 0.$

Hence, the given lines are coplanar.

(*ii*) The equation of the plane containing given lines is :

$$\begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0 \quad \text{[Ref. Art. 11.17 (e)]}$$

$\Rightarrow (x+3)(5-10) - (y-1)(-15+5) + (z-5)(-6+1) = 0$

$\Rightarrow -5x - 15 + 10y - 10 - 5z + 25 = 0$

$\Rightarrow -5x + 10y - 5z = 0 \Rightarrow x - 2y + z = 0.$

Example 5. Find whether the lines :

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j})$$

and $\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{j} - \hat{k})$

intersect or not. If intersecting, find their point of intersection.

Solution. The given lines are :

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j})$$

and $$\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{j} - \hat{k})$$

i.e. $$\vec{r} = (1 + 2\lambda)\hat{i} + (-1 + \lambda)\hat{j} - \hat{k} \quad \text{....(1)}$$

and $$\vec{r} = (2 + \mu)\hat{i} + (-1 + \mu)\hat{j} - \mu\hat{k} \quad \text{...(2)}$$

If the lines (1) and (2) intersect, then for some values of λ and μ, we have :

$$1 + 2\lambda = 2 + \mu \quad \text{...(3)}$$

$$-1 + \lambda = -1 + \mu \quad \text{...(4)}$$

and $$-1 = -\mu \Rightarrow \mu = 1 \quad \text{...(5)}$$

Putting in (4), $-1 + \lambda = -1 + 1 \Rightarrow \lambda = 1.$

Thus $\lambda = 1$ and $\mu = 1$.

These also satisfy (3). [$\because$ *1 + 2 (1) = 2 + 1 i.e. 3 = 3*]

Hence, the lines intersect.

Putting $\lambda = 1$ in (1),

$$\vec{r} = (1 + 2)\hat{i} + (-1 + 1)\hat{j} - \hat{k} = 3\hat{i} - \hat{k}.$$

Putting $\mu = 1$ in (2),

$$\vec{r} = (2 + 1)\hat{i} + (-1 + 1)\hat{j} - \hat{k} = 3\hat{i} - \hat{k}.$$

Hence, the point of intersection is (3, 0, – 1).

Example 6. Find the shortest distance and the vector equation of the line of shortest distance between the lines given by :

$$\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$\vec{r} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}).$$

Solution. The given equations in the cartesian form are :

$$L_1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} (= \lambda) \quad \text{...(1)}$$

and $$L_2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} (= \mu) \quad \text{...(2)}$$

Any point on L_1 is $(3\lambda + 3, -\lambda + 8, \lambda + 3)$.

Any point on L_2 is $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$.

If the line of shortest distance intersects (1) in P and (2) in Q, then the direction-ratios of $\overrightarrow{PQ}$ are :

$< -3\mu - 3 - 3\lambda - 3, 2\mu - 7 + \lambda - 8, 4\mu + 6 - \lambda - 3 >$

i.e. $< 3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3 >$.

Since PQ is perp. to line (1),

$\therefore (3)(3\lambda + 3\mu + 6) + (-1)(-\lambda - 2\mu + 15) + (1)(\lambda - 4\mu - 3) = 0$

$\Rightarrow \quad 11\lambda + 7\mu = 0 \quad \text{...(3)}$

Since PQ is perp. to line (2),

$\therefore (-3)(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

$\Rightarrow \quad -7\lambda - 29\mu = 0 \quad \text{...(4)}$

Solving (3) and (4), $\lambda = 0, \mu = 0$.

$\therefore$ Points P and Q are (3, 8, 3) and (– 3, – 7, 6) respectively.

$\therefore$ S.D. = | PQ |

$$= \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2}$$

$$= \sqrt{36 + 225 + 9} = \sqrt{270} = 3\sqrt{30} \text{ units}$$

and the vector equation of line of S.D. is :

$$\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \mu(-6\hat{i} - 15\hat{j} + 3\hat{k}).$$

[*Using* $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$]

EXERCISE 11 (d)

Short Answer Type Questions

SATQ

Find the shortest distance between the following (1 – 4) lines whose vector equations are :

1. $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

(N.C.E.R.T. ; Kerala B. 2016; Kashmir B. 2016, 12, 11; P.B. 2015, 10; H.P.B. 2013 S, 11; H.B. 2010)

2. (*i*) $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$

(P.B. 2012)

(*ii*) $(\hat{i} - \hat{j} + 2\hat{k}) + \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$

and $(2\hat{i} + 3\hat{j} - \hat{k}) + \mu(3\hat{i} - 2\hat{j} + 2\hat{k})$.

(Assam B. 2016)

(*iii*) $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$

and $\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$.

3. (*i*) $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$

and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$

(N.C.E.R.T.; H.P.B. 2017, 15; P.B. 2013 ; Jammu B. 2012 ; C.B.S.E. (F) 2011)

(*ii*) $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$

and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$

(C.B.S.E. 2018; H.B. 2010)

(*iii*) $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

and $\vec{r} = (3\hat{i} + 3\hat{j} + 5\hat{k}) + \mu(-2\hat{i} + 3\hat{j} + 6\hat{k})$

(H.P.B. 2017; H.B. 2017, 15)

(*iv*) $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$.

(Jharkhand B. 2016)

4. (*i*) $\vec{r} = (\lambda - 1)\hat{i} + (\lambda - 1)\hat{j} - (1 + \lambda)\hat{k}$

and $\vec{r} = (1 - \mu)\hat{i} + (2\mu - 1)\hat{j} + (\mu + 2)\hat{k}$

(*ii*) $\vec{r} = (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (1 + \lambda)\hat{k}$

and $\vec{r} = 2(1 + \mu)\hat{i} - (1 - \mu)\hat{j} + (-1 + 2\mu)\hat{k}$.

5. Consider the equations of the straight lines given by :

$L_1 : \vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$

$L_2 : \vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

If $\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$,

$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$, $\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$, then find :

(*i*) $\vec{a}_2 - \vec{a}_1$ (*ii*) $\vec{b}_2 - \vec{b}_1$

(*iii*) $\vec{b}_1 \times \vec{b}_2$ (*iv*) $\vec{a}_1 \times \vec{a}_2$

(*v*) $(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)$

(*vi*) the shortest distance between L_1 and L_2.

(Jammu B. 2018, 16, 14, 12, 11; Assam B. 2018; Karnataka B. 2017; H.B. 2017; H.P.B. 2016; Bihar 2014)

Find the shortest distance between the following (6 – 7) lines whose vector equations are :

6. (*i*) $\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$

and $\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$

(N.C.E.R.T.; Kashmir B. 2018, 11; Jammu B. 2016, 15W, 14, 13; H.P.B. 2013, 12, 10 S, 10; A.I.C.B.S.E. 2011)

(*ii*) $\vec{r} = (3 - t)\hat{i} + (4 + 2t)\hat{j} + (t - 2)\hat{k}$

and $\vec{r} = (1 + s)\hat{i} + (3s - 7)\hat{j} + (2s - 2)\hat{k}$,

where t and s are scalars.

(H.P.B. Model Paper 2018; H.P.B. 2018, 17, 16, 14, 13; Kerala B. 2018)

7. (*i*) $\vec{r} = (8 + 3\lambda)\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k}$

and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

(*ii*) $\vec{r} = 3\hat{i} - 15\hat{j} + 9\hat{k} + \lambda(2\hat{i} - 7\hat{j} + 5\hat{k})$

and $\vec{r} = (2\mu - 1)\hat{i} + (1 + \mu)\hat{j} + (9 - 3\mu)\hat{k}$.

8. Find the S.D. between the lines :

(*i*) $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+4}{2}$

(H.B. 2016)

(*ii*) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{2}$ and $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-1}{5}$

(Jammu B. 2013)

(*iii*) $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

(H.P.B. 2018, P.B. 2017)

(*iv*) $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

(Kerala B. 2013)

Determine whether or not the following (9 – 11) pairs of lines intersect :

9. $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + \hat{j} - 2\hat{k})$

and $\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} + 2\hat{j} - 2\hat{k})$.

10. $\vec{r} = (2\lambda + 1)\hat{i} - (\lambda + 1)\hat{j} + (\lambda + 1)\hat{k}$.

and $\vec{r} = (3\mu + 2)\hat{i} - (5\mu + 5)\hat{j} + (2\mu - 1)\hat{k}$.

11. $\frac{x-1}{2} = \frac{y+1}{3} = z\,;\ \frac{x+1}{5} = \frac{y-2}{1};\ \frac{z-2}{0}$.

12. Prove that the lines : $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar.

Long Answer Type Questions

13. Find the shortest distance and the equation of the shortest distance between the following two lines :

$$\vec{r} = (-\hat{i} + \hat{j} + 9\hat{k}) + \lambda(2\hat{i} + \hat{j} - 3\hat{k})$$

and $\vec{r} = (3\hat{i} - 15\hat{j} + 9\hat{k}) + \mu(2\hat{i} - 7\hat{j} + 5\hat{k})$.

14. Find the shortest distance and the vector equation of the line of shortest distance between the lines given by :

(*i*) $\vec{r} = (-4\hat{i} + 4\hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} - \hat{k})$

and $\vec{r} = (-3\hat{i} - 8\hat{j} - 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 3\hat{k})$

(*ii*) $\vec{r} = (-\hat{i} + 5\hat{j}) + \lambda(-\hat{i} + \hat{j} + \hat{k})$

and $\vec{r} = (-\hat{i} - 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} + 2\hat{j} + \hat{k})$.

15. Write the vector equations of the following lines and hence determine the distance between them :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6};\ \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}.$$

(C.B.S.E. 2010)

16. Show that the lines : $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$

and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$

intersect each other. Also, find their point of intersection. *(C.B.S.E. 2014; P.B. 2012)*

17. Show that the lines :

(*i*) $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$

and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$ *(A.I.C.B.S.E. 2013)*

(*ii*) $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$

and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ *(C.B.S.E. 2014)*

are intersecting. Hence, find their point of intersection.

LATQ

18. Show that the lines :

(*a*) $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$ and $\frac{x-2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$

(*b*) $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{k})$

and $\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{k} - \hat{k})$

do not intersect. *(Meghalaya B. 2015)*

19. Find the S.D. between the lines :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5},$$

find also its equations. *(P.B. 2014 S)*

20. Show that the following lines are coplanar :

(*i*) $\frac{5-x}{-4} = \frac{y-7}{-4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$

(C.B.S.E. 2014)

(*ii*) $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$.

(Jammu B. 2018; Uttrakhand B. 2013 ; Assam B. 2013)

21. Show that the lines :

$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and

$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar.

(N.C.E.R.T.)

22. Find the equations of the lines joining the following pair of vertices and then find its shortest distance between the lines :

(*i*) (0, 0, 0), (1, 0, 2) (*ii*) (1, 3, 0), (0, 3, 0).

Answers

1. $\frac{10}{\sqrt{59}}$.

2. (i) $3\sqrt{30}$ (ii) $\frac{57}{\sqrt{234}}$ (iii) $\frac{3}{19}\sqrt{19}$.

3. (i) 9 (ii) $\frac{6\sqrt{5}}{5}$ (iii) $\frac{3\sqrt{5}}{5}$ (iv) 0.

4. (i) $\frac{5\sqrt{2}}{2}$ (ii) $\frac{3\sqrt{2}}{2}$.

5. (i) $\hat{i}-3\hat{j}-2\hat{k}$ (ii) $\hat{i}+2\hat{j}+\hat{k}$ (iii) $-3\hat{i}+3\hat{k}$

(iv) $-\hat{i}+3\hat{j}-5\hat{k}$ (v) -9 (vi) $\frac{3}{2}\sqrt{2}$ units.

6. (i) $\frac{8\sqrt{29}}{29}$ (ii) $\sqrt{35}$.

7. (i) 14 (ii) $4\sqrt{3}$.

8. (i) $\frac{1}{3}\sqrt{3}$ (ii) $\frac{\sqrt{2336}}{81}$ (iii) – (iv) $3\sqrt{30}$.

9. No. **10.** No. **11.** No.

13. $4\sqrt{3}$; $\vec{r}=(3\hat{i}+3\hat{j}-3\hat{k})-\mu(4\hat{i}+4\hat{j}+4\hat{k})$.

14. (i) $\sqrt{62}$; $\vec{r}=(-5\hat{i}+3\hat{j}+2\hat{k})+\mu(\hat{i}-2\hat{j}+3\hat{k})$

(ii) $\sqrt{42}$, $\vec{r}=(\hat{i}+3\hat{j}-2\hat{k})+\mu(\hat{i}-4\hat{j}+5\hat{k})$.

15. $\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$

and $\vec{r}=\left(3\hat{i}+3\hat{j}-5\hat{k}\right)+\mu\left(4\hat{i}+6\hat{j}+12\hat{k}\right)$;

$\frac{1}{7}\sqrt{293}$ units.

16. $\left(\frac{1}{2},\frac{-1}{2},\frac{-3}{2}\right)$.

17. (i) (–1, –6, –12) (ii) (4, 0, – 1).

19. $\frac{1}{\sqrt{6}}$; $6x-9=10-3y=6z-25$.

22. (i) $\vec{r}=\lambda(\hat{i}+2\hat{k})$; $\frac{x}{1}=\frac{y}{0}=\frac{z}{2}$

(ii) $\vec{r}=(\hat{i}+3\hat{j})-\mu\hat{i}$; $\frac{x-1}{-1}=\frac{y-3}{0}=\frac{z}{0}$.

S.D. = 3 units.

Hints to Selected Questions

9. Show that S.D. ≠ 0.

16. Show that S.D. = 0.

SUB CHAPTER 11.3 The Plane

INTRODUCTION

A plane is a surface such that if we take any two distinct points on it, then the line joining these points lies wholly on it.

We can specify a particular plane in several ways as :

(i) *One and only one plane can be drawn through three non–collinear points.*

Thus three given non–collinear points specify a particular plane.

(ii) *One and only one plane can be drawn to contain two concurrent lines.*

Thus two given concurrent lines specify a particular plane.

(iii) *One and only one plane can be drawn perpendicular to given direction at a given distance from the origin.*

Thus the normal to the plane and the distance of the plane from the origin specify a particular plane.

(iv) *One and only one plane can be drawn through a given point and perpendicular to a given direction.*

Thus a point on the plane and a normal to the plane specify a particular plane.

Of the above, (iii) and (iv) are most useful.