VECTORS

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VECTORS

IN THIS CHAPTER ....

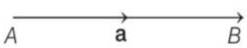
Scalar and Vector Quantities

Those quantities which have only magnitude and which are not related to any fixed direction in space, also do not follow the rules of vector algebra, are called **scalar quantities** or briefly scalars. e.g. Mass, volume, density, work, temperature etc.

Vectors are those which have both magnitude and direction and follow the rules of vector algebra are called **vector quantities**. e.g. Displacement, velocity, acceleration, momentum, weight, force etc.

Representation of a Vector

A vector is generally represented by a direction line segment, say **AB**. A is called the **initial point** and B is called the **terminal point**. The positive number representing the measure of the length of the line segment denoting the vector is called the **modulus** or **magnitude** of the vector and the arrow head indicates its direction.



The modulus of vector **AB** is denoted by $|\mathbf{AB}|$.

Types of Vector

Zero Vector or Null Vectors

A vector of zero magnitude i.e. which has the same initial and terminal points, is called a zero vector. It is denoted by **0**.

Unit Vector

A vector whose magnitude is one unit is called a unit vector. The unit vector in the direction of **a** and is denoted by $\hat{\mathbf{a}}$.

Symbolically, $$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Equal Vector

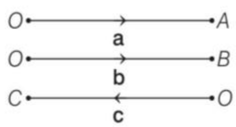
If magnitude and direction of two vectors are same, then those are called equal vectors.

Parallel Vector

Two vectors **a** and **b** are said to be parallel if either both have same line of action or support or there exists a scalar λ such that if $\lambda > 0$, then **a** and **b** have same direction or sense. If $\lambda < 0$, then **a** and **b** have opposite direction or sense.

Like and Unlike Vector

Two parallel vectors are said to be like when they have same sense of direction i.e. angle between them is zero. Otherwise vectors are said to be unlike vectors and angle between them is π.



From figure, vectors **a** and **b** are like vectors and vectors **a** and **c**, **b** and **c** are unlike vectors.

Negative Vector

A vector having the same magnitude as that of a given vector and direction opposite to that given vector is called negative vector.

Coinitial and Coterminal Vectors

The vectors which have the same initial point are coinitial vectors. Similarly, the vectors which have the same terminal point are called coterminal vectors.

Free Vector

The vector whose initial point or tail is not fixed is called free or non-localised vector.

Collinear Vector

Two or more vectors are known as collinear vectors, if they are parallel to a given straight line. The magnitude of collinear vectors can be different.

Coplanar Vector

Vectors are said to be coplanar, if they occur in same or common plane.

Reciprocal of a Vector

A vector having the same direction as that of a given vector **a** but magnitude equal to the reciprocal of the given vector is known as the reciprocal of **a** and is denoted by $\mathbf{a}^{-1}$.

Localized and Free Vectors

A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector.

Force acting on a rigid body is a localized vectors as its effect depends on the line of action of the force.

Position Vector

Let O be a fixed origin, then the position vector of a point P is the vector **OP**, if **a** and **b** are position vectors of two points A and B, then

$$\mathbf{AB} = \mathbf{b} - \mathbf{a}$$
$$= \text{Position vector of } \mathbf{b} - \text{Position vector of } \mathbf{a}$$

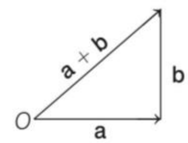
Addition of Vectors

The addition of two vectors **a** and **b** is denoted by $\mathbf{a} + \mathbf{b}$ and it is known as resultant of **a** and **b.**

There are three methods of addition of vectors.

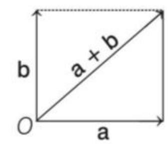
Triangle Law

If two vectors **a** and **b** lie along the two sides of a triangle in consecutive order (as shown in the figure), then third side represents the sum (resultant) $\mathbf{a} + \mathbf{b}$.



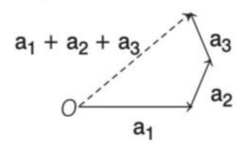
Parallelogram Law

If two vectors lie along two adjacent sides of a parallelogram (as shown in the figure), then diagonal of the parallelogram through the common vertex represents their sum.



Polygon Law

If $(n-1)$ sides of a polygon represents vector $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots$, in consecutive order, then nth side represents their sum (as shown in the figure).



Properties of Vector Addition

(i) Vector addition is commutative,

i.e. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(ii) Vector addition is associative

i.e. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

(iii) $\mathbf{a} + \mathbf{0} = \mathbf{a} = \mathbf{0} + \mathbf{a}$, where **0** is additive identify.

(iv) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0} = (-\mathbf{a}) + \mathbf{a}$ where $(-\mathbf{a})$ is additive inverse.

(v) $|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$, equality holds when **a** and **b** are like vectors.

(vi) | **a** + **b** | ≥ || **a** | − | **b** ||, equality holds when **a** and **b** are unlike vectors.

Note If **a** and **b** are two vectors, then the subtraction of **b** from **a** is defined as the vector sum **a** and − **b**, *i.e.*, reverse the direction of **b** and add it to **a**.

Multiplication of a Vector by a Scalar

If **a** is a vector and λ is a scalar, then λ **a** is a vector parallel to **a** whose modulus is | λ | times that of **a**. This multiplication is called scalar multiplication.

If λ and μ be two scalars, then

(i) $\lambda(\mu\,\mathbf{a}) = \lambda\mu\,\mathbf{a}$

(ii) $(\lambda + \mu)\,\mathbf{a} = \lambda\,\mathbf{a} + \mu\,\mathbf{a}$

(iii) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\,\mathbf{a} + \lambda\,\mathbf{b}$

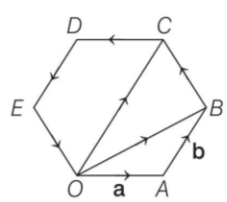
Example 1. *If* **a** *and* **b** *are the vectors determined by two adjacent sides of a regular hexagon, then vector* **EO** *is*

(a) (**a** + **b**) (b) (**a** − **b**)
(c) 2 **a** (d) 2 **b**

Sol. (b) *OABCDE* is a regular hexagon.

Let **OA** = **a** and **AB** = **b**.

Join *OB* and *OC*, we have



$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \mathbf{a} + \mathbf{b}$$

Since, *OC* is parallel to *AB* and double of *AB*.

∴ $\mathbf{OC} = 2\mathbf{AB} = 2\mathbf{b}$

Now, $\mathbf{BC} = \mathbf{OC} - \mathbf{OB}$

$= 2\mathbf{b} - (\mathbf{a} + \mathbf{b}) = \mathbf{b} - \mathbf{a}$

$\mathbf{CD} = -\,\mathbf{OA} = -\,\mathbf{a}$

and $\mathbf{DE} = -\,\mathbf{AB} = -\,\mathbf{b}$

Also, $\mathbf{EO} = -\,\mathbf{BC}$

$= -(\mathbf{b} - \mathbf{a})$

$= \mathbf{a} - \mathbf{b}$

Component of a Vector

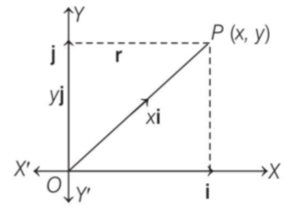
The process of splitting a vector is called resolution of a vector. In simpler language it would mean, determining the effect of a vector in a particular direction. The parts of the vector obtained after splitting the vectors are known as the components of the vector.

Components of a Vector in 2D and 3D

If a point *P* lies in a plane, say *XY*-plane and has coordinates (x, y). Then, $\mathbf{OP} = x\mathbf{i} + y\mathbf{j}$, where **i** and **j** are unit vectors along *OX* and *OY*-axes respectively.

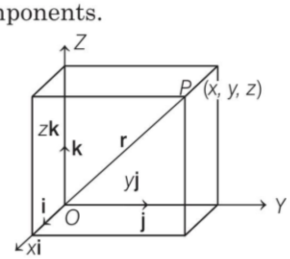
The length of any vector **r** is given by

$$|\mathbf{r}| = |x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$$



Also, if point *P* lies in a space, has coordinates (x, y, z) and **i**, **j** and **k** are unit vectors along *OX*, *OY* and *OZ*-axes respectively. Then, the position vectors of *P* with respect to *O* is given by **OP** (or **r**) $= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

This form of vector **OP** is called component form. Here, x, y and z are called the scalar components and $x\mathbf{i}$, $y\mathbf{j}$ and $z\mathbf{k}$ are called the vector components of **OP** (or **r**) along the respective axes. Sometimes, x, y and z are also termed as rectangular components.



The length of any vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is given by

$$|\mathbf{r}| = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$$

Note

- It two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal, their resolved parts will also equal i.e. $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.
- The resolved parts of a resultant vector of addition of two vectors are equal to the sum of resolved parts of those vectors.

Linear Combinations of Vectors

Let **a**, **b**, **c**, ... , be vectors and x, y, z ... be scalars, then the expression $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots$ is called a linear combination of vectors a, b, c, ...

Linearly Dependent and Independent System of Vectors

(i) The system of vectors **a**, **b**, **c**, ... is said to be linearly dependent, if there exists some scalars $x, y, z, \ldots$ not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots = 0$.

(ii) The system of vectors **a**, **b**, **c**, ... is said to be linearly independent, if

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = 0 \Rightarrow x = y = z = t \ldots = 0$$

Note

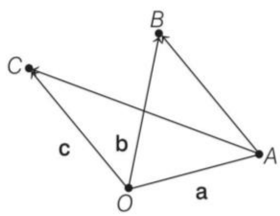
- Two non-zero, non-collinear vectors **a** and **b** are linearly independent.
- Three non-zero, non-coplanar vectors **a**, **b** and **c** are linearly independent.
- More than three vectors are always linearly dependent.

Collinearity of Points or Vectors

Collinearity of Three Points

Three points with position vectors, **a**, **b** and **c** are collinear if and only if there exists three scalars x, y, z not all simultaneously such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$ together with $x + y + z = 0$

Three points representing three position vectors can be represent two vectors in the plane.

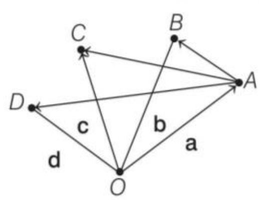


Let A, B and C are three points represented by three position vectors **a, b** and **c** respectively. Now, these three position vectors can represent two vectors **AB** and **AC**.

Now, the vectors **AB** and **AC** are collinear, if there exists a linear relation between the two, such that $\mathbf{AB} = \lambda\ \mathbf{AC}$.

Collinearity of Four Points

Let A, B, C and D be four points represented by four position vectors **a**, **b c** and **d**, respectively.



Now, these four position vectors can represent three vectors **AB, AC** and **AD.**

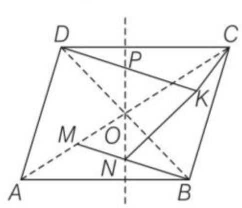
The vectors **AB, AC** and **AD** are collinear, if

$\mathbf{AC} = m\ \mathbf{AB}$ and $\mathbf{AD} = n\ \mathbf{AB}$ and similar other cases for more than four vectors.

Example 2. *Let O be the point of intersection of diagonals of a parallelogram ABCD. The points M, N, K and P are the mid-points of OA, MB, NC and KD respectively, then N, O and P are*

(a) collinear (b) non-collinear
(c) can't say (d) None of these

Sol. (a) Now, $M \equiv \dfrac{\mathbf{a}}{2}, N \equiv \dfrac{\dfrac{\mathbf{a}}{2} + \mathbf{b}}{2} = \dfrac{\mathbf{a} + 2\mathbf{b}}{4}$



$$K \equiv \frac{\dfrac{\mathbf{a} + 2\mathbf{b}}{4} - \mathbf{a}}{2} = \frac{2\mathbf{b} - 3\mathbf{a}}{8}$$

$$P \equiv \frac{-\mathbf{b} + \dfrac{2\mathbf{b} - 3\mathbf{a}}{8}}{2} = \frac{-6\mathbf{b} - 3\mathbf{a}}{16}$$

$$\Rightarrow \quad \mathbf{OP} = -\frac{3}{16}(2\mathbf{b} + \mathbf{a})$$

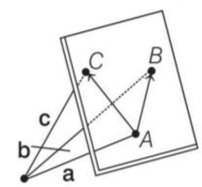
Also, $\mathbf{ON} = \dfrac{1}{4}(\mathbf{a} + 2\mathbf{b}) = -\dfrac{3}{4}(\mathbf{OP})$

Hence, points N, O and P are collinear.

Coplanarity of Points or Vectors

Coplanarity of Three Points

Three points A, B and C represented by position vectors **a**, **b** and **c** respectively represent two vectors **AB** and **AC** and from the figure, two vectors are always coplanar i.e. two vectors always from their own plane.



Coplanarity of Four Points

The necessary and sufficient condition that four points with position vectors **a**, **b**, **c** and **d** should be coplanar is that there exist four scalars x, y, z, t not all zero, such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = 0,\ x + y + z + t = 0$$

The prove that the four points A, B, C and D having position vectors as **a**, **b**, **c** and **d** are coplanar.

***Step* I** Find the vectors **AB**, **AC** and AD having the reference point as A.

***Step* II** Express one of these vectors as the linear combination of the other two

$$\mathbf{AB} = \lambda\ \mathbf{AC} + \mu\ \mathbf{AD}$$

***Step* III** Now, compare the coefficients on LHS and RHS in respective manner and thus find the respective value of λ and μ.

***Step* IV** If real values of the scalars λ and μ exist, then the three vectors representing four points are coplanar otherwise not.

Note Three vectors $a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ are coplanar, if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

Example 3. *The vectors*

$$5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c},\ 7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c} \text{ and } 3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}$$

are (where **a**, **b**, **c** *are three non-coplanar vectors.)*

(a) collinear (b) coplanar
(c) non-coplanar (d) None of these

Sol. (b) Let $\mathbf{A} = 5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c}$, $\mathbf{B} = 7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c}$
and $\mathbf{C} = 3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}$

A, **B** and **C** are coplanar.

$\Rightarrow \quad x\mathbf{A} + y\mathbf{B} + z\mathbf{C} = 0$

must have a real solution for x, y and z other than (0, 0, 0).

Now, $x(5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c}) + y(7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c}) + z(3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}) = \mathbf{0}$

$\Rightarrow \quad (5x + 7y + 3z)\,\mathbf{a} + (6x - 8y + 20z)\mathbf{b} + (7x + 9y + 5z)\mathbf{c} = \mathbf{0}$

$5x + 7y + 3z = 0,\ 6x - 8y + 20z = 0,\ 7x + 9y + 5z = 0$

[as **a**, **b** and **c** are non-coplanar vectors]

$$\text{Now,} \quad D = \begin{vmatrix} 5 & 7 & 3 \\ 6 & -8 & 20 \\ 7 & 9 & 5 \end{vmatrix} = 0$$

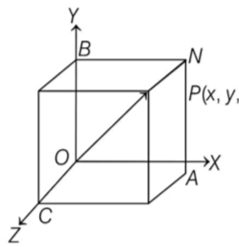
So, the three linear simultaneous equation in x, y and z have a non-trivial solution. Hence, **A**, **B** and **C** are coplanar vectors.

Orthogonal System of Unit Vectors

Let OX, OY and OZ be three mutually perpendicular straight lines. Given any point $P(x, y, z)$ in space, we can construct the rectangular parallelopiped of which OP is a diagonal and $OA = x, OB = y, OC = z$.

Here, A, B and C are $(x, 0, 0), (0, y, 0)$ and $(0, 0, z)$ respectively and L, M and N are $(0, y, z), (x, 0, z)$ and $(x, y, 0)$ respectively.

Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ denote unit vectors along OX, OY and OZ respectively. We have, $\mathbf{r} = \mathbf{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ as $\mathbf{OA} = x\hat{\mathbf{i}}$, $\mathbf{OB} = y\hat{\mathbf{j}}$ and $\mathbf{OC} = z\hat{\mathbf{k}}$.



$$\mathbf{ON} = \mathbf{OA} + \mathbf{AN}$$

$$\mathbf{OP} = \mathbf{ON} + \mathbf{NP}$$

$$\text{So,} \quad \mathbf{OP} = \mathbf{OA} + \mathbf{OB} + \mathbf{OC} \ (\mathbf{NP} = \mathbf{OC}, \mathbf{AN} = \mathbf{OB})$$

$$r = |\mathbf{r}| = |\mathbf{OP}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$$

$$\Rightarrow \quad \mathbf{r} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$$

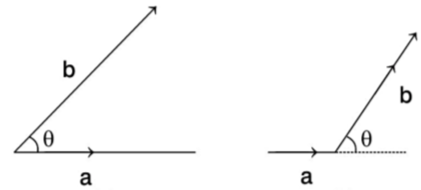
Product of Two Vectors

There are two methods of products of two vectors.

Scalar Product or Dot Product

Let **a** and **b** be two non-zero vectors inclined at an angle θ. Then, the scalar product of **a** and **b** is denoted by $\mathbf{a} \cdot \mathbf{b}$ and defined as

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta,\ 0 \le \theta \le \pi$$



(a) (b)

Since, scalar product of two vectors is a scalar quantity, so it is called scalar product.

Properties of Scalar Product

While solving problems based on properties of scalar product or dot product, always keep in mind the concept of scalar product

(i) The scalar product of two vectors is commutative i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

(ii) If m and n be any two scalars and **a** and **b** be any two vectors, then $(m\,\mathbf{a}) \cdot (n\,\mathbf{b}) = (n\mathbf{a}) \cdot (m\,\mathbf{b})$

(iii) $\mathbf{a} \cdot (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} \pm \mathbf{a} \cdot \mathbf{c}$ (distributive law)

and $(\mathbf{b} \pm \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} \pm \mathbf{c} \cdot \mathbf{a}$

(iv) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

(v) If two vectors **a** and **b** are perpendicular to each other, then $\mathbf{a} \cdot \mathbf{b} = 0$

(a) $\mathbf{a} \cdot \mathbf{b} < 0$, iff **a** and **b** are inclined at an obtuse angle.

(b) $\mathbf{a} \cdot \mathbf{b} > 0$, iff **a** and **b** are inclined at an acute angle.

(vi) If $\mathbf{a} \cdot \mathbf{b} = 0$, then either $\mathbf{a} = 0$, $\mathbf{b} = 0$ or $\mathbf{a} \perp \mathbf{b}$

(vii) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$,

then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

(viii) $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$

(ix) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then

$$\cos\theta = \left[\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right] = \left[\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right]$$

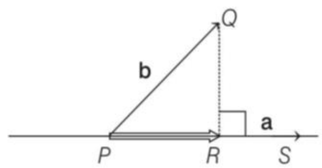
If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then both vectors are perpendicular to each other and if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$,

then both vectors are parallel to each other.

(x) $|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$

(xi) Projection of **b** along $\mathbf{a} = \dfrac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}$

and projection of **a** along $\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$ and perpendicular to $\mathbf{b} = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$



If **b** represents a force, then projection of **b** along **a** represents the effective force in the direction of **a**. Total work done = $\mathbf{F}\cdot\mathbf{d} = F\,d\cos\theta$ where **F** is the force and **d** is the displacement.

(xii) Maximum value of $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$

(xiii) Minimum value of $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$

(xiv) Any vector **a** can be written as,

$$\mathbf{a} = (\mathbf{a}\cdot\hat{\mathbf{i}})\,\hat{\mathbf{i}} + (\mathbf{a}\cdot\hat{\mathbf{j}})\,\hat{\mathbf{j}} + (\mathbf{a}\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}}$$

(xv) If **r** is a vector making angles α, β and γ with OX, OY and OZ respectively, then

$$\cos\alpha = \mathbf{r}\cdot\hat{\mathbf{i}}, \cos\beta = \mathbf{r}\cdot\hat{\mathbf{j}}, = \cos\gamma = \mathbf{r}\cdot\hat{\mathbf{k}}$$

$$\mathbf{r} = |\mathbf{r}|\cos\alpha\hat{\mathbf{i}} + |\mathbf{r}|\cos\beta\,\hat{\mathbf{j}} + |\mathbf{r}|\cos\gamma\,\hat{\mathbf{k}}$$

If **r** a unit vector then,

$$\mathbf{r} = (\cos\alpha)\hat{\mathbf{i}} + (\cos\beta)\,\hat{\mathbf{j}} + (\cos\gamma)\,\hat{\mathbf{k}}$$

(xvi) If **a, b** and **c** are non-coplanar vectors, in space, any vector **r** in space can be written as $\hat{\mathbf{r}} = (\hat{\mathbf{r}}\cdot\hat{\mathbf{a}})\hat{\mathbf{a}} + (\hat{\mathbf{r}}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} + (\hat{\mathbf{r}}\cdot\hat{\mathbf{c}})\hat{\mathbf{c}}$, where $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors along **a, b** and **c**, respectively.

(xvii) If $\hat{\mathbf{r}}$ is a non-zero vector in space and **a**, **b** and **c** are three vectors, $\mathbf{r}\cdot\mathbf{a} = \mathbf{r}\cdot\mathbf{b} = \mathbf{r}\cdot\mathbf{c} = 0$

$\Rightarrow$ **a, b** and **c** are coplanar.

(xviii) If **r** is a non-zero coplanar to two given vectors **a** and **b**, then $\mathbf{r}\cdot\mathbf{a} = \mathbf{r}\cdot\mathbf{b} = 0 \Rightarrow$ **a** and **b** are collinear.

Example 4. *Let* $\mathbf{a} = 2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = 4\hat{\mathbf{i}} + (3-\lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ *and* $\mathbf{c} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + (\lambda_3 - 1)\hat{\mathbf{k}}$ *be the three vectors such that* $\mathbf{b} = 2\mathbf{a}$ *and* **a** *is perpendicular to* **c**. *Then, a possible value of* $(\lambda_1, \lambda_2, \lambda_3)$ *is* ***(JEE Main 2019)***

(a) (1, 3, 1) (b) (1, 5, 1)

(c) $\left(-\frac{1}{2}, 4, 0\right)$ (d) $\left(\frac{1}{2}, 4, -2\right)$

Sol. (c) We have, $\mathbf{a} = 2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$; $\mathbf{b} = 4\hat{\mathbf{i}} + (3-\lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

and $\mathbf{c} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + (\lambda_3 - 1)\hat{\mathbf{k}}$.

Now, $\mathbf{b} = 2\mathbf{a}$

$\Rightarrow \quad 4\hat{\mathbf{i}} + (3-\lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}} = 2(2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

$\Rightarrow \quad 4\hat{\mathbf{i}} + (3-\lambda_2)\hat{\mathbf{j}} + 6\hat{\mathbf{k}} = 4\hat{\mathbf{i}} + 2\lambda_1\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

$\Rightarrow \quad (3 - 2\lambda_1 - \lambda_2)\hat{\mathbf{j}} = 0$

$\Rightarrow \quad 3 - 2\lambda_1 - \lambda_2 = 0$

$\Rightarrow \quad 2\lambda_1 + \lambda_2 = 3$ ...(i)

Also, as **a** is perpendicular to **c**, therefore $\mathbf{a}\,.\,\mathbf{c} = 0$

$\Rightarrow \quad (2\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + (\lambda_3 - 1)\hat{\mathbf{k}}) = 0$

$\Rightarrow \quad 6 + 6\lambda_1 + 3(\lambda_3 - 1) = 0$

$\Rightarrow \quad 6\lambda_1 + 3\lambda_3 + 3 = 0$

$\Rightarrow \quad 2\lambda_1 + \lambda_3 = -1$ ... (ii)

Now, from Eq. (i), $\lambda_2 = 3 - 2\lambda_1$

and from Eq. (ii), $\lambda_3 = -2\lambda_1 - 1$

$\therefore \quad (\lambda_1, \lambda_2, \lambda_3) \equiv (\lambda_1, 3 - 2\lambda_1, -2\lambda_1 - 1)$

If $\lambda_1 = -\frac{1}{2}$, then $\lambda_2 = 4$, and $\lambda_3 = 0$

Thus, a possible value of $(\lambda_1, \lambda_2, \lambda_3) = \left(-\frac{1}{2}, 4, 0\right)$

Example 5. *The projection of the vector* $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ *on the vector* $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ *is*

(a) 0 (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{1}{2}$ (d) None of these

Sol. (a) Use formula for projection of **a** on **b** = $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

Let $\quad \mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}}, \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$

Projection of **a** on **b** is given by,

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{(\hat{\mathbf{i}} - \hat{\mathbf{j}})\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}})}{\sqrt{1^2 + 1^2}} = \frac{1\times 1 + (-1)\times 1}{\sqrt{2}} = 0$$

Hence, the projection of vector **a** on **b** is 0.

Example 6. *If* **a** *is a non-zero vector of magnitude a and* λ *is a non-zero scalar, then* λ **a** *is unit vector if*

(a) $\lambda = 1$ (b) $\lambda = -1$ (c) $a = |\lambda|$ (d) $a = 1/|\lambda|$

Sol. (d) Vector $\lambda\,\mathbf{a}$ is a unit vector, if $|\lambda\,\mathbf{a}| = 1$

Now, $|\lambda\,\mathbf{a}| = 1 \Rightarrow |\lambda||\mathbf{a}| = 1 \Rightarrow |\lambda||\mathbf{a}| = 1 \Rightarrow |\mathbf{a}| = \frac{1}{|\lambda|}$ $\quad [\because \lambda \neq 0]$

$\Rightarrow \quad a = \frac{1}{|\lambda|}$ $\quad$ [given $|\mathbf{a}| = a$]

Example 7. *Let* **a** *and* **b** *be two unit vectors. If the vectors* $\mathbf{c} = a + 2\mathbf{b}$ *and* $\mathbf{d} = 5\mathbf{a} - 4\mathbf{a}$ *are perpendicular to each other, then the angle between* **a** *and* **b** *is*

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

Sol. (c) Given that,

(i) **a** and **b** are unit vectors,
i.e. $|\mathbf{a}| = |\mathbf{b}| = 1$

(ii) $\mathbf{c} = \mathbf{a} + 2\,\mathbf{b}$ and $\mathbf{d} = 5\,\mathbf{a} - 4\,\mathbf{b}$

(iii) **c** and **d** are perpendicular to each other.
i.e. $\mathbf{c}\cdot\mathbf{d} = 0$

To find Angle between **a** and **b**.

Now, $\mathbf{c}\cdot\mathbf{d} = 0 \Rightarrow (\mathbf{a} + 2\,\mathbf{b})\cdot(5\,\mathbf{a} - 4\,\mathbf{b}) = 0$

$\Rightarrow \quad 5\,\mathbf{a}\cdot\mathbf{a} - 4\,\mathbf{a}\cdot\mathbf{b} + 10\,\mathbf{b}\cdot\mathbf{a} - 8\,\mathbf{b}\cdot\mathbf{b} = 0$

$\Rightarrow \quad 6\,\mathbf{a}\cdot\mathbf{b} = 3$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \frac{1}{2}$

So, the angle between **a** and **b** is $\frac{\pi}{3}$.

Example 8. *Two forces* $\mathbf{f}_1 = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\mathbf{f}_2 = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ *acting on a particle at A move it to B. The work done, if the position vector of A and B are* $-2\hat{\mathbf{i}} + 5\hat{\mathbf{k}}$ *and* $3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, *is*

(a) 20 units (b) 7 units
(c) 25 units (d) None of these

Sol. (c) Let **R** be the resultant of two forces $\mathbf{f}_1$ and $\mathbf{f}_2$ and **d** be the displacement.

Then, $\mathbf{R} = \mathbf{f}_1 + \mathbf{f}_2 = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) + (\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$
$= 4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

and $\mathbf{d} = (3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - (-2\hat{\mathbf{i}} + 5\hat{\mathbf{k}}) = 5\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

$\therefore$ The total work done $= \mathbf{R} \cdot \mathbf{d}$
$= (4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \cdot (5\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$
$= 20 - 7 + 12 = 25$ units

Vector Product or Cross Product

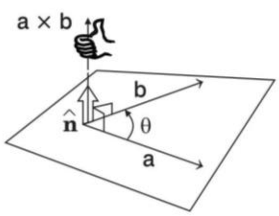
The vector product of two vectors **a** and **b** is a vector and is given by

$$\mathbf{a} \times \mathbf{b} = ab \sin\theta\, \hat{\mathbf{n}}$$

where, θ be the angle between **a** and **b** and $\hat{\mathbf{n}}$ is a perpendicular unit vector to both **a** and **b** such that **a**, **b** and $\hat{\mathbf{n}}$ form a right handed system.

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

Also, $\hat{\mathbf{n}} = \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$, where, $\hat{\mathbf{n}}$ indicates direction of $\mathbf{a} \times \mathbf{b}$.



Properties of Vector Product

(i) Vector product is not commutative,
i.e. $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$
But $\mathbf{b} \times \mathbf{a} = -\,\mathbf{a} \times \mathbf{b}$

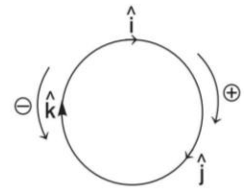
(ii) If m and n be two scalars and **a** and **b** be two vectors, then
$$(m\,\mathbf{a}) \times (n\mathbf{b}) = mn(\mathbf{a} \times \mathbf{b}) = (n\mathbf{a}) \times (m\mathbf{b})$$

(iii) $\mathbf{a} \times (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \times \mathbf{b} \pm \mathbf{a} \times \mathbf{c}$ (distributive law)
and $(\mathbf{b} \pm \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} \pm \mathbf{c} \times \mathbf{a}$

(iv) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then
$$\mathbf{a} \times \mathbf{b} = [(a_2b_3 - a_3b_2)\hat{\mathbf{i}} + (a_3b_1 - a_1b_3)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}]$$
$$\text{or } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

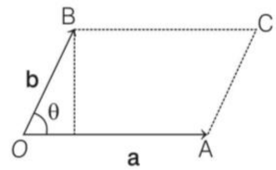
(v) $\mathbf{a} \times \mathbf{b} = \mathbf{0}$
If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or **a** and **b** are two collinear vectors.
or $\mathbf{a} \parallel \mathbf{b}$, where **a** and **b** are non-null vectors.

(vi) $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$



(vii) $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$
and $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\,\hat{\mathbf{k}}, \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\,\hat{\mathbf{i}}, \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$

(viii) Area of parallelogram $OACB$

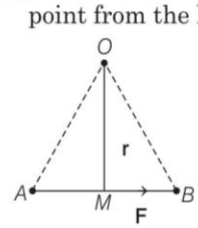


$= OA \times BM$
$= ab \sin\theta$
$= |\mathbf{a} \times \mathbf{b}|$

(ix) (a) Area of $\Delta ABC = \dfrac{1}{2}|\,\mathbf{AB} \times \mathbf{AC}\,|$

(b) If **a**, **b** and **c** are position vectors of A, B and C respectively, then area of
$$\Delta ABC = \frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$$

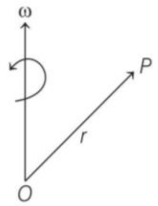
(x) **Moment of a Force about a Point** Let F be the magnitude of force acting at a point A of the rigid body along the line AB and $OM = p$ is the perpendicular distance of fixed point O from AB, then the moment of force **F** about O is
$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$
= Force × Perpendicular distance of the point from the line of action of force



(xi) Area of quadrilateral $OABC$ is $\dfrac{1}{2}|\,\mathbf{OC} \times \mathbf{BA}|$, where **OC** and **BA** are diagonals.

(xii) The unit vector perpendicular to the plane of a and b are $\pm\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$ and **a** vector of magnitude λ perpendicular to the plane of (**a** and **b** or **b** and **a**) is $\dfrac{\lambda\,(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$.

(xiii) **Rotating body** A rigid body is spinning with angular velocity ω about an axis through θ.
The velocity of a point P in the body is $\mathbf{v} = \omega \times \mathbf{OP} = \omega \times \mathbf{r}$



(xiv) **Lagranges Identity** For any two vectors a and b;

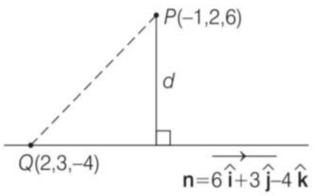
$$(\mathbf{a} \times \mathbf{b})^2 = |\mathbf{a}|^2 \cdot |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix}.$$

Example 9. *The distance of the point having position vector $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ from the straight line passing through the point (2, 3, – 4) and parallel to the vector, $6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ is*

(JEE Main 2019)

(a) $2\sqrt{13}$ (b) $4\sqrt{3}$ (c) 6 (d) 7

Sol. (d) Let point P whose position vector is $(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$ and a straight line passing through $Q(2, 3, -4)$ parallel to the vector $\mathbf{n} = 6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$.



$\because$ Required distance d = Projection of line segment PQ perpendicular to vector $\mathbf{n}$.

$$= \frac{|\mathbf{PQ} \times \mathbf{n}|}{|\mathbf{n}|}$$

Now, $\mathbf{PQ} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 10\hat{\mathbf{k}}$, so

$$\mathbf{PQ} \times \mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & -10 \\ 6 & 3 & -4 \end{vmatrix} = 26\hat{\mathbf{i}} - 48\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

So,
$$d = \frac{\sqrt{(26)^2 + (48)^2 + (3)^2}}{\sqrt{(6)^2 + (3)^2 + (4)^2}}$$
$$= \sqrt{\frac{676 + 2304 + 9}{36 + 9 + 16}} = \sqrt{\frac{2989}{61}}$$
$$= \sqrt{49} = 7 \text{ units}$$

Example 10. *A unit vector perpendiculat to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ is*

(JEE Main 2019)

(a) $\frac{1}{3}\hat{\mathbf{i}} - \frac{1}{3}\hat{\mathbf{j}} - \frac{1}{3}\hat{\mathbf{k}}$ or $-\frac{1}{3}\hat{\mathbf{i}} + \frac{1}{3}\hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}}$

(b) $\frac{4}{3}\hat{\mathbf{i}} - \frac{4}{3}\hat{\mathbf{j}} - \frac{4}{3}\hat{\mathbf{k}}$ or $-\frac{4}{3}\hat{\mathbf{i}} + \frac{4}{3}\hat{\mathbf{j}} + \frac{4}{3}\hat{\mathbf{k}}$

(c) $\frac{2}{3}\hat{\mathbf{i}} - \frac{2}{3}\hat{\mathbf{j}} - \frac{1}{3}\hat{\mathbf{j}}$ or $-\frac{2}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}}$

(d) None of the above

Sol. (c) Given that, $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$\therefore$ $\mathbf{a} + \mathbf{b} = (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$
$= 4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$

and $\mathbf{a} - \mathbf{b} = (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = 2\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$

Now, $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

$$= \hat{\mathbf{i}}(16 - 0) - \hat{\mathbf{j}}(16 - 0) + \hat{\mathbf{k}}(0 - 8)$$
$$= 16\hat{\mathbf{i}} - 16\hat{\mathbf{j}} - 8\hat{\mathbf{k}}$$

$\Rightarrow$ $|(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})| = \sqrt{(16)^2 + (-16)^2 + (-8)^2}$
$= \sqrt{256 + 256 + 64} = \sqrt{576} = 24$

$\therefore$ A unit vector, perpendicular to both $(\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - \mathbf{b})$ is

$$\pm \frac{(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})}{|(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})|} = \pm \frac{16\hat{\mathbf{i}} - 16\hat{\mathbf{j}} - 8\hat{\mathbf{k}}}{24}$$
$$= \pm \frac{8(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 1\hat{\mathbf{k}})}{24}$$
$$= \pm \frac{1}{3}(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

$\therefore$ Required vector is either $\frac{2}{3}\hat{\mathbf{i}} - \frac{2}{3}\hat{\mathbf{j}} - \frac{1}{3}\hat{\mathbf{k}}$

or $-\frac{2}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}}$

Example 11. *Area of a rectangle having vertices $A\left(-\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right), B\left(\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}]\right), C\left(\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right)$ and $D\left(-\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right)$ is*

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4

Sol. (c) $\mathbf{AB}$ = PV of B – PV of $A = \left(\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right) - \left(-\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right)$

$$= [1 - (-1)]\hat{\mathbf{i}} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{\mathbf{j}} + (4 - 4)\hat{\mathbf{k}} = 2\hat{\mathbf{i}}$$

and $\mathbf{AD}$ = Position vector of D – Position vector of A

$$= \left(-\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right) - \left(-\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right)$$
$$= [-1 - (-1)]\hat{\mathbf{i}} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{\mathbf{j}} + (4 - 4)\hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

$$\therefore \mathbf{AB} \times \mathbf{AD} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = -2\hat{\mathbf{k}}$$

Area of rectangle $ABCD = |\mathbf{AB} \times \mathbf{AD}| = \sqrt{(-2)^2} = 2$ sq units

[now, it is known that the area of a parallelogram whose adjacent sides are $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$]

Hence, the area of the rectangle is $|\mathbf{AB}| \times |\mathbf{AD}| = 2$ sq units

Example 12. *Let* $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ *and* $\mathbf{b} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$. *If* $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{r}$, $\mathbf{r} \cdot (\alpha\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 3$ *and* $\mathbf{r} \cdot (2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \alpha\,\hat{\mathbf{k}}) = -1$, $\alpha \in R$, *then the value of* $\alpha + |\mathbf{r}|^2$ *is equal to* ***(JEE Main 2021)***

(a) 9 (b) 15 (c) 13 (d) 11

Sol. (b) We have, $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{r}$

$$\mathbf{r} \times (\mathbf{a} + \mathbf{b}) = 0$$

$$\mathbf{r} = \lambda(\mathbf{a} + \mathbf{b})$$

$$\mathbf{r} = \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}} + 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$$

$$\mathbf{r} = \lambda(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \qquad \text{...(i)}$$

$$\mathbf{r} \cdot (\alpha\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 3 \qquad \text{...(ii)}$$

Puting the value of **r** from Eq. (i) in Eq. (ii), we get

$$\alpha\lambda = 1 \qquad \text{...(iii)}$$

Also, $\mathbf{r} \cdot (2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \alpha\hat{\mathbf{k}}) = -1$ ...(iv)

Put **r** from Eqs. (i) and (iv), we get

$$2\lambda\alpha - \lambda = 1 \qquad \text{...(v)}$$

Solving Eqs. (iii) and (v) we get

$$\alpha = 1,\ \lambda = 1$$

$$\Rightarrow \quad \mathbf{r} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

$$|\mathbf{r}|^2 = 14 \text{ and } \alpha = 1$$

$$\therefore \quad \alpha + |\mathbf{r}|^2 = 1 + 14 = 15$$

Example 13. *The area of the triangle with vectices* $A(1, 1, 2)$, $B(2, 3, 5)$ *and* $C(1, 5, 5)$ *is*

(a) $\sqrt{61}$ sq units (b) $\frac{1}{2}\sqrt{31}$ sq units

(c) $\frac{1}{2}\sqrt{61}$ sq units (d) $2\sqrt{61}$ sq units

Sol. (c) To determine the area of triangle, use formula

$$\Delta = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|.$$

The vertices of ΔABC are given as $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$.

First, we find vectors **AB** and **AC**.

Now, $\mathbf{AB} = \text{PV of } B - \text{PV of } A = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$$= (2-1)\hat{\mathbf{i}} + (3-1)\hat{\mathbf{j}} + (5-2)\hat{\mathbf{k}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

and $\mathbf{AC} = \text{PV of } C - \text{PV of } A = (\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$$= (1-1)\hat{\mathbf{i}} + (5-1)\hat{\mathbf{j}} + (5-2)\hat{\mathbf{k}} = 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

$$\therefore \quad \mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(6-12) - \hat{\mathbf{j}}(3-0) + \hat{\mathbf{k}}(4-0) = -6\hat{\ } - 3\hat{\ } + 4\hat{\ }$$

$$\therefore |\mathbf{AB} \times \mathbf{AC}| = \sqrt{(-6)^2 + (-3)^2 + (4)^2}$$

$$= \sqrt{36 + 9 + 16} = \sqrt{61}$$

Area of $\Delta ABC = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{1}{2} \times \sqrt{61} = \frac{\sqrt{61}}{2}$ sq units

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ sq units.

Example 14. *The moment about the point* $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ *of a force represented by* $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ *acting through the point* $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$, *is*

(a) $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ (b) $3\hat{\mathbf{i}} + \hat{\mathbf{j}}$ (c) $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ (d) $3\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$

Sol. (d) Here, $\mathbf{r} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

$$\Rightarrow \quad \mathbf{r} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}} \text{ and } \mathbf{F} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Then, the required moment is given by

$$\mathbf{r} \times \mathbf{F} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$$

$\therefore$ Moment about given point $= 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$

Product of Three Vectors

There are two methods of products of three vectors.

Scalar Triple Product (Box Product)

The scalar triple product of three vectors **a**, **b** and **c** is defined as $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = |\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin\theta \cos\phi$ where, θ is the angle between **a** and **b** and ϕ is the angle between $\mathbf{a} \times \mathbf{b}$ and **c**. It is also defined as $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$

$$\text{Then, } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\mathbf{i}}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\mathbf{j}}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Scalar Triple Product

(i) The value of scalar triple product does not depend upon the position of dot and cross.
i.e. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(ii) If **a**, **b** and **c** are cyclically permuted. The value of scalar product remains same.
The change of cyclic order of vectors in scalar triple product changes the sign of the scalar but not the magnitude.
i.e. $[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}]$
and $[\mathbf{abc}] = -[\mathbf{bac}] = -[\mathbf{cba}] = -[\mathbf{acb}]$

(iii) The scalar triple product of three vectors is zero, if any two of them are equal.

(iv) The scalar triple product of three vectors is zero, if two of them are parallel or collinear.

(v) If three vectors **a**, **b** and **c** are coplanar, then [**abc**] = 0

(vi) [**abc** + **d**] = [**abc**] + [**abd**]

(vii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by **a**, **b** and **c** i.e. $V = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(viii) If **a, b, c** are non-coplanar, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] > 0$ for right handed system and $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] < 0$ for left handed system.

(ix) $[\hat{\mathbf{i}}\ \hat{\mathbf{j}}\ \hat{\mathbf{k}}] = 1$

(x) $[k\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = k\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, where k is any scalar.

(xi) Volume of tetrahedron with 0 as origin and the position vectors of A, B and C being **a**, **b** and **c** respectively, is given by

$$V = \frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

(xii) Four points with position vectors **a, b, c** and **d** will be coplanar, if

$$[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

(xiii) $[\mathbf{a}\ \mathbf{a}\ \mathbf{b}] = 0$

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$; $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and

$$\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}, \text{ then } [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

In general, if $\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n}$; $\mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n}$ and $\mathbf{c} = c_1\mathbf{l} + c_2\mathbf{m} + c_3\mathbf{n}$, then

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{l}\ \mathbf{m}\ \mathbf{n}]; \text{ where } \mathbf{l}, \mathbf{m} \text{ and } \mathbf{n} \text{ are}$$

non-coplanar vectors.

(xiv) The position vector of the centroid of a tetrahedron if the PV's of its angular vertices are **a**, **b**, **c** and **d** are given by $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$.

Remember that $[\mathbf{a} - \mathbf{b}, \mathbf{b} - \mathbf{c}\ \mathbf{c} - \mathbf{a}] = 0$ and $[\mathbf{a} + \mathbf{b}\ \mathbf{b} + \mathbf{c}\ \mathbf{c} + \mathbf{a}] = 2\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Example 15. *Let x_0 be the point of local maxima of $f(x) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, where and $\mathbf{c} = 7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + x\hat{\mathbf{k}}$. Then the value of $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$ at $x = x_0$ is* **(JEE Main 2020)**

(a) 14 (b) −4 (c) −22 (d) −30

Sol. (c) Given vectors $\mathbf{a} = x\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = -2\hat{\mathbf{i}} + x\hat{\mathbf{j}} - \hat{\mathbf{k}}$

and $\mathbf{c} = 7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + x\hat{\mathbf{k}}$

And, it is given that

$$f(x) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} x & -2 & 3 \\ -2 & x & -1 \\ 7 & -2 & x \end{vmatrix}$$

$$= x(x^2 - 2) + 2(-2x + 7) + 3(4 - 7x)$$

$$= x^3 - 27x + 26$$

It is also given that $f(x)$ has local maxima at $x = x_0$.

So, $f'(x_0) = 0 \Rightarrow 3x_0^2 - 27 = 0 \quad \Rightarrow \quad x_0^2 = 9$

$\Rightarrow \quad x_0 = \pm 3$, but maximum at $x_0 = -3$.

Now, $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$

$$= -2x - 2x - 3 - 14 - 2x - x + 7x + 4 + 3x = 3x - 13$$

$\therefore$ The value of $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$ at $x = x_0 = -3$, is −22

Hence, option (c) is correct.

Vector Triple Product

The vector triple product of three vactors **a**, **b** and **c** is defined as $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. If atleast one **a**, **b** and **c** is a zero vector or **b** and **c** are collinear vectors or **a** is perpendicular to both **b** and only **c**, then $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{0}$. In all other cases $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ will be a non-zero vector in the plane of non-collinear vectors and perpendicular to the vector **a**.

Thus, we can take $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \lambda\mathbf{b} + \mu\mathbf{c}$, for some scalar λ and μ. Since, $\mathbf{a} \perp \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{a}\cdot[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})] = 0$

$$\Rightarrow \qquad \lambda(\mathbf{a}\cdot\mathbf{b}) + \mu(\mathbf{ac}) = 0$$

$\lambda(\mathbf{a}\cdot\mathbf{c})\alpha\,\mu = -(\mathbf{a}\cdot\mathbf{b})$ for same scalar α.

Hence, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, for any vectors **a**, **b** and **c** satisfying the conditions given in the beginning.

In particular, if we take, $\mathbf{a} = \mathbf{b} = \mathbf{i}$, $\mathbf{c} = \mathbf{j}$, then $\alpha = 1$.

Hence, $\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

and $\quad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$

but $\quad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

Example 16. *Let **a**, **b** and **c** be three unit vectors, out of which vectors **b** and **c** are non-parallel. If α and β are the angles which vector **a** makes with vectors **b** and **c** respectively and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}\mathbf{b}$, then $|\alpha - \beta|$ is equal to* **(JEE Main 2019)**

(a) 30° (b) 45° (c) 90° (d) 60°

Sol. (a) Given, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}\mathbf{b}$

$$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \frac{1}{2}\mathbf{b} \qquad [\because \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}]$$

On comparing both sides, we get

$$\mathbf{a}\cdot\mathbf{c} = \frac{1}{2} \qquad \text{...(i)}$$

and $$\mathbf{a}\cdot\mathbf{b} = 0 \qquad \text{...(ii)}$$

$\because$ **a**, **b** and **c** are unit vectors, and angle between **a** and **b** is α and angle between **a** and **c** is β, so

$$|\mathbf{a}||\mathbf{c}|\cos\beta = \frac{1}{2} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \cos\beta = \frac{1}{2} \qquad [\because |\mathbf{a}| = 1 = |\mathbf{c}|]$$

$$\Rightarrow \quad \beta = \frac{\pi}{3} \qquad \left[\because \cos\frac{\pi}{3} = \frac{1}{2}\right] \quad \text{...(iii)}$$

and $$|\mathbf{a}||\mathbf{b}|\cos\alpha = 0 \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad \alpha = \frac{\pi}{2} \qquad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$|\alpha - \beta| = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} = 30^\circ$$

Reciprocal System of Vectors

Let **a**, **b** and **c** be a system of three non-coplanar vectors. Then, the system of vectors **a′**, **b′** and **c′** which satisfies $\mathbf{a}\cdot\mathbf{a}'=\mathbf{b}\cdot\mathbf{b}'=\mathbf{c}\cdot\mathbf{c}'=1$ and $\mathbf{a}\cdot\mathbf{b}'=\mathbf{a}\cdot\mathbf{c}'=\mathbf{b}\cdot\mathbf{a}'=\mathbf{b}\cdot\mathbf{c}'=\mathbf{c}\cdot\mathbf{a}'=\mathbf{c}\cdot\mathbf{b}'=0$ is called the reciprocal system to the vectors **a**, **b**, **c**.

In terms of **a**, **b**, **c** the vectors **a′**, **b′**, **c′** are given by

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},$$

$$\mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},$$

$$\mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

Properties of Reciprocal System of Vectors

(i) Scalar product of any vector of one system with the vector of other system is zero

i.e. $\mathbf{a}\cdot\mathbf{b}'=\mathbf{a}\cdot\mathbf{c}'=\mathbf{b}\cdot\mathbf{c}'=\mathbf{b}\cdot\mathbf{a}'=\mathbf{a}\cdot\mathbf{b}'=\mathbf{a}\cdot\mathbf{c}'=0$

(ii) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']=1$

(iii) $\hat{\mathbf{i}}'=\hat{\mathbf{i}},\ \hat{\mathbf{j}}'=\hat{\mathbf{j}},\ \hat{\mathbf{k}}'=\hat{\mathbf{k}}$

(iv) Let **a′ b′ c′** is a reciprocal system **a**, **b**, **c** and **r** is any vector, then

$$\mathbf{r}=(\mathbf{r}\cdot\mathbf{a}')\,\mathbf{a}+(\mathbf{r}\cdot\mathbf{b}')\mathbf{b}+(\mathbf{r}\cdot\mathbf{c}')\,\mathbf{c}$$

$$\mathbf{r}=(\mathbf{r}\cdot\mathbf{a})\,\mathbf{a}'+(\mathbf{r}\cdot\mathbf{b})\ \mathbf{b}'+(\mathbf{r}\cdot\mathbf{c})\,\mathbf{c}'$$

(v) If the system **a**, **b**, **c** are non-coplanar and so are the reciprocal system **a′**, **b′**, **c′**.

(vi) $(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}'+(\mathbf{b}+\mathbf{c})\cdot\mathbf{b}'+(\mathbf{c}+\mathbf{a})\cdot\mathbf{c}'=3$

Example 17. *If* **a**, **b**, **c** *and* **a′**, **b′**, **c′** *are reciprocal system of vectors, then* $\mathbf{a}'\times\mathbf{b}'+\mathbf{b}'\times\mathbf{c}'+\mathbf{c}'\times\mathbf{a}'$ *is equal to*

(a) $\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ (b) $\dfrac{\mathbf{a}+2\mathbf{b}-\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

(c) $\dfrac{\mathbf{a}-\mathbf{b}+3\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ (d) $\dfrac{2\mathbf{a}-\mathbf{b}-\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

Sol. (a) $\mathbf{a}'\times\mathbf{b}'=\dfrac{(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2}$

$$=\frac{\{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}\}\mathbf{c}-\{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}\}\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2}$$

$$=\frac{[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2}=\frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2}$$

$$=\frac{\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \qquad \text{...(i)}$$

Similarly, $\mathbf{b}'\times\mathbf{c}'=\dfrac{\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ ...(ii)

and $\mathbf{c}'\times\mathbf{a}'=\dfrac{\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$\mathbf{a}'\times\mathbf{b}'+\mathbf{b}'\times\mathbf{c}'+\mathbf{c}'\times\mathbf{a}'=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Application of Vectors in Geometry

(i) 'The points A, B and C are collinear' means

(a) Area of ΔABC is zero

(b) $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are collinear vectors

(c) $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are parallel

(d) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\mathbf{0}$

(e) There exist α,β and γ not all zero such that $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}=\mathbf{0}$ and $\alpha+\beta+\gamma=0$

Otherwise A, B and C are not collinear.

(ii) 'A, B, C and D are coplanar' means.

(a) Volume of tetrahedron $ABCD$ is zero.

(b) $\mathbf{b}-\mathbf{a}$, $\mathbf{c}-\mathbf{a}$ and $\mathbf{d}-\mathbf{a}$ are coplanar.

(c) $[\mathbf{b}-\mathbf{a},\ \mathbf{c}-\mathbf{a},\ \mathbf{d}-\mathbf{a}]=0$

(d) There exist α,β,γ and δ not all zero such that $\alpha\,\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=0$ and $\alpha+\beta+\gamma+\delta=0$

Otherwise A, B, C and D are not coplanar.

(iii) If **a** and **b** are the position vector of A and B and **r** be the position vector of the point P which divides to join of A and B in the ratio $m:n$, then

$$\mathbf{r}=\frac{m\mathbf{b}\pm n\mathbf{a}}{m\pm n}$$

'+' sign takes for internal ratio and

'–' sign takes for external ratio.

(iv) If **a**, **b** and **c** be the PV of three vertices of ΔABC and **r** be the PV of the centroid of ΔABC, then

$$\mathbf{r}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$$

Example 18. *Solution of the vector equation* $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$, $\mathbf{r}\cdot\mathbf{c}=0$ *provided that* **c** *is not perpendicular to* **b**, *is*

(a) $\mathbf{r}=\mathbf{a}-\left(\dfrac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{b}\cdot\mathbf{c}}\right)\mathbf{b}$ (b) $\mathbf{r}=\mathbf{b}-\dfrac{(\mathbf{a}\cdot\mathbf{c})}{(\mathbf{b}\cdot\mathbf{r})}\mathbf{a}$

(c) $\mathbf{r}=\mathbf{b}-\left(\dfrac{\mathbf{b}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{c}}\right)\mathbf{a}$ (d) None of these

Sol. (a) We are given;

$$\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$$

$\Rightarrow$ $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$

Hence, $(\mathbf{r}-\mathbf{a})$ and **b** are parallel

$\Rightarrow$ $\mathbf{r}-\mathbf{a}=t\,\mathbf{b}$ ...(i)

and we know $\mathbf{r}\cdot\mathbf{c}=0$,

$\therefore$ Taking dot product of Eq. (i) by **c** we get

$$\mathbf{r}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}=t(\mathbf{b}\cdot\mathbf{c})$$

$$0-\mathbf{a}\cdot\mathbf{c}=t(\mathbf{b}\cdot\mathbf{c})$$

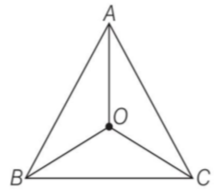
$\Rightarrow$ $t=-\left(\dfrac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{b}\cdot\mathbf{c}}\right)$ ...(ii)

$\therefore$ From Eqs. (i) and (ii) solution of **r** is

$$\mathbf{r}=\mathbf{a}-\left(\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{b}\cdot\mathbf{c}}\right)\mathbf{b}$$

Tetrahedron

A tetrahedron is a three dimensional figure formed by four triangles.



In figure, $OABC \rightarrow$ tetrahedron

$\Delta ABC \rightarrow$ base

$OAB, OBC, OCA \rightarrow$ faces

OA, OB, OC, AB, BC and $CA \rightarrow$ edges

OA, BC, OB, CA, OC and $AB \rightarrow$ pair of opposite edges.

Properties of tetrahedron

(i) A tetrahedron in which all edges are equal is called a regular tetrahedron.

(ii) If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair are also perpendicular to each other.

(iii) The sum of the squares of two opposite edges is the same for each pair of opposite edges.

(iv) Any two opposite edges in a regular tetrahedron are perpendicular.

(v) Volume of a tetrahedron $ABCD$ is

$$\frac{1}{6}\left|[\mathbf{a}-\mathbf{d}, \mathbf{b}-\mathbf{d}, \mathbf{c}-\mathbf{d}]\right|,$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are position vectors.

(vi) volume of a tetrahedren whose three coterminous edges are in the right handed system are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is given by $\frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

(vii) Centroid of tetrahedron is $\frac{[\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}]}{4}$,

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are position vectors.

Example 19. *If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar uni-modular vectors, each inclined with other at an angle 30°, then volume of tetrahedron whose edges are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is* ***(JEE Main 2021)***

(a) $\frac{\sqrt{3\sqrt{3}-5}}{12}$ (b) $\frac{3\sqrt{3}-5}{12}$

(c) $\frac{5\sqrt{2}+3}{12}$ (d) None of these

Sol. (a) Since, the volume of tetrahedron with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Where, $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = 1$

and $\mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = \frac{\sqrt{3}}{2}$ [given]

$$\therefore \quad V = \frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$\Rightarrow \quad V^2 = \frac{1}{36}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \frac{1}{36}\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

$$= \frac{1}{36}\begin{vmatrix} 1 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 1 \end{vmatrix}$$

$$= \frac{1}{36}\left(\frac{3\sqrt{3}}{4} - \frac{5}{4}\right)$$

$$\therefore \quad V = \frac{1}{12}\sqrt{3\sqrt{3}-5}$$

Practice Exercise

ROUND I Topically Divided Problems

Algebra and Modulus of Vectors

1. If vectors $\mathbf{a}_1 = x\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{a}_2 = \hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ are collinear, then a possible unit vector parallel to the vector $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is *(JEE Main 2021)*

(a) $\frac{1}{\sqrt{2}}(-\hat{\mathbf{j}} + \hat{\mathbf{k}})$ (b) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(c) $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$ (d) $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$

2. A unit vector in XY-plane, making an angle of 30° in anti-clockwise direction with the positive direction of X-axis is

(a) $\frac{1}{2}\hat{\mathbf{i}} + \frac{3}{2}\hat{\mathbf{i}}$ (b) $4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

(c) $\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \hat{\mathbf{j}}$ (d) $\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}$

3. If $\mathbf{a} = \mathbf{b} + \mathbf{c}$, then which of the following statements is correct?

(a) $|\mathbf{a}| - |\mathbf{b}| = |\mathbf{c}|$

(b) $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{c}|$

(c) $|\mathbf{a}| = |\mathbf{b}| + |\mathbf{c}|$

(d) None of the above

4. A vector of magnitude 5 units and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, is

(a) $\pm\frac{3}{2}\hat{\mathbf{i}} \pm \frac{\sqrt{10}}{2}\hat{\mathbf{j}}$ (b) $\pm\frac{3\sqrt{10}}{2}\hat{\mathbf{i}} \pm \frac{1}{2}\hat{\mathbf{j}}$

(c) $\pm\frac{3\sqrt{10}}{2}\hat{\mathbf{i}} \pm \frac{\sqrt{10}}{2}\hat{\mathbf{j}}$ (d) None of these

5. If the points $(-1, -1, 2)$, $(2, m, 5)$ and $(3, 11, 6)$ are collinear, then the value of m is

(a) 2 (b) 4 (c) 6 (d) 8

6. A vector $\mathbf{r}$ of magnitude $3\sqrt{2}$ units which makes an angle of $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with Y and Z-axes, respectively is

(a) $\mathbf{r} = \pm 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ (b) $\mathbf{r} = 3\hat{\mathbf{i}} \neq 3\hat{\mathbf{j}}$

(c) $\mathbf{r} = -3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ (d) None of these

7. The two vectors $\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ represents the two sides AB and AC, respectively of a ΔABC. The length of the median through A is

(a) $\frac{\sqrt{34}}{2}$ (b) $\frac{\sqrt{48}}{2}$

(c) $\sqrt{18}$ (d) None of these

8. If $|\mathbf{a}| = 3$ and $-1 \le k \le 2$, then $|k\mathbf{a}|$ lies in the interval

(a) $[0, 6]$ (b) $[-3, 6]$ (c) $[3, 6]$ (d) $[1, 2]$

9. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B, respectively, then the position vector of a point C in BA produced such that $BC = 1.5\,BA$, is

(a) $3\mathbf{a} - \mathbf{b}$ (b) $\mathbf{a} - 3\mathbf{b}$

(c) $0.5(\mathbf{a} - 3\,\mathbf{b})$ (d) $0.5(3\mathbf{a} - \mathbf{b})$

10. The value of λ for which the vectors $3\,\hat{\mathbf{i}} - 6\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $2\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + \lambda\,\hat{\mathbf{k}}$ are parallel is

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{2}{5}$

11. The points $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$, $2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$, $-7\mathbf{b} + 10\,\mathbf{c}$ are

(a) collinear (b) non-collinear

(c) can't say (d) None of these

12. If $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear vectors and $x\mathbf{a} + y\mathbf{b} = 0$

(a) $x = 0$ but y is not necessarily zero

(b) $y = 0$ but x is not necessarily zero

(c) $x = 0$, $y = 0$

(d) None of the above

13. Five points given by A, B, C, D and E are in a plane. Three forces **AC, AD** and **AE** act at A and three forces **CB, DB, EB** act at B. Then, their resultant is

(a) $2\mathbf{AC}$ (b) $3\mathbf{AB}$ (c) $3\mathbf{DB}$ (d) $2\mathbf{BC}$

14. Let $ABCD$ be the parallelogram whose sides AB and AD are represented by the vectors $2\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}} - 5\,\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}$ respectively. Then, if $\mathbf{a}$ is a unit vector parallel to **AC**, then $\mathbf{a}$ equal to

(a) $\frac{1}{3}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (b) $\frac{1}{3}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(c) $\frac{1}{7}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ (d) $\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

15. If D, E and F are respectively the mid-points of AB, AC and BC in ΔABC, then $\mathbf{BE} + \mathbf{AF}$ is equal to

(a) $\mathbf{DC}$ (b) $\frac{1}{2}\mathbf{BF}$ (c) $2\mathbf{BF}$ (d) $\frac{3}{2}\mathbf{BF}$

16. If three points A, B and C have position vectors $(1, x, 3)$, $(3, 4, 7)$ and $(y, -2, -5)$ respectively and if they are collinear, then (x, y) is equal to

(a) $(2, -3)$ (b) $(-2, 3)$
(c) $(2, 3)$ (d) $(-2, -3)$

17. If position vector of a point A is $\mathbf{a} + 2\mathbf{b}$ and any point $P(\mathbf{a})$ divides $\mathbf{AB}$ in the ratio of $2:3$, then position vector of B is

(a) $2\mathbf{a} - \mathbf{b}$ (b) $\mathbf{b} - 2\mathbf{a}$
(c) $\mathbf{a} - 3\mathbf{b}$ (d) $\mathbf{b}$

18. If A, B, C, D and E are five coplanar points, then $\mathbf{DA} + \mathbf{DB} + \mathbf{DC} + \mathbf{AE} + \mathbf{BE} + \mathbf{CE}$ is equal to

(a) $\mathbf{OE}$ (b) $3\mathbf{DE}$
(c) $2\mathbf{DE}$ (d) $4\mathbf{ED}$

19. The vectors $\mathbf{a}(x) = \cos x\hat{\mathbf{i}} + (\sin x)\hat{\mathbf{j}}$ and $\mathbf{b}(x) = x\hat{\mathbf{i}} + \sin x\hat{\mathbf{j}}$ are collinear for

(a) unique value of $x, 0 < x < \frac{\pi}{6}$

(b) unique value of $x, \frac{\pi}{6} < x < \frac{\pi}{3}$

(c) no value of x

(d) infinitely many values of $x, 0 < x < \frac{\pi}{2}$

Scalar or Dot Product of Two Vectors and Its Applications

20. The angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ with magnitudes $\sqrt{3}$ and 2 respectively, having $\mathbf{a} \cdot \mathbf{b} = \sqrt{6}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

21. In a ΔABC, if $|\mathbf{BC}| = 8$, $|\mathbf{CA}| = 7$, $|\mathbf{AB}| = 10$, then the projection of the vector $\mathbf{AB}$ on $\mathbf{AC}$ is equal to *(JEE Main 2021)*

(a) $\frac{25}{4}$ (b) $\frac{85}{14}$ (c) $\frac{127}{20}$ (d) $\frac{115}{16}$

22. If $\mathbf{a} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$ such that $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{c}$, then the value of λ is

(a) 2 (b) 4 (c) 6 (d) 8

23. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then the value of $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$ is

(a) 0 (b) $-\frac{1}{2}$ (c) $-\frac{3}{2}$ (d) 2

24. If the vertices A, B, C of a ΔABC have position vectors $(1, 2, 3)$, $(-1, 0, 0)$, $(0, 1, 2)$ respectively, then $\angle ABC$ ($\angle ABC$ is the angle between the vectors BA and BC), is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$ (d) $\cos^{-1}\left(\frac{1}{3}\right)$

25. Given, two vectors are $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, the unit vector coplanar with the two vectors and perpendicular to first is

(a) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (b) $\frac{1}{\sqrt{5}}(2\hat{\mathbf{i}} + \hat{\mathbf{j}})$

(c) $\pm\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (d) None of these

26. The scalar product of the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ with a unit vector along the sum of vectors $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is equal to one. The value of λ is

(a) 1 (b) 2 (c) 3 (d) 4

27. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ is the angle between them. Then, $\mathbf{a} + \mathbf{b}$ is a unit vector, if

(a) $\theta = \frac{\pi}{4}$ (b) $\theta = \frac{\pi}{3}$ (c) $\theta = \frac{\pi}{2}$ (d) $\theta = \frac{2\pi}{3}$

28. The value of a, for which the points, A, B, C with position vectors $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ respectively are the vertices of a right angled triangle with $C = \frac{\pi}{2}$ are

(a) -2 and -1 (b) -2 and 1
(c) 2 and -1 (d) 2 and 1

29. If $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$, then the value of λ such that $\mathbf{a}$ is perpendicular to $\lambda\mathbf{b} + \mathbf{c}$ is

(a) -1 (b) -2 (c) 1 (d) 2

30. The projection of vector $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ along $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) 2 (d) $\sqrt{6}$

31. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then what is the angle between $\mathbf{a}$ and $\mathbf{b}$ for $\sqrt{3}\,\mathbf{a} - \mathbf{b}$ to be a unit vector ?

(a) 30° (b) 45° (c) 60° (d) 90°

32. If A, B, C and D are the points with position vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $2\hat{\mathbf{i}} - 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, respectively, Then, the projection of $\mathbf{AB}$ along $\mathbf{CD}$ is

(a) $\frac{1}{\sqrt{21}}$ (b) $\sqrt{21}$ (c) $\sqrt{\frac{3}{7}}$ (d) $\frac{2}{\sqrt{21}}$

33. Let **u**, **v** and **w** be vectors such that **u** + **v** + **w** = **0**. If |**u**| = 3, |**v**| = 4 and |**w**| = 5, then **u** · **v** + **v** · **w** + **w** · **u** is equal to

(a) 0 (b) 25 (c) −25 (d) 1

34. If the vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mu\hat{\mathbf{k}}$ are mutually orthogonal, then (λ, μ) is equal to

(a) (−3, 2) (b) (2, −3)
(c) (−2, 3) (d) (3, −2)

35. Let **a** and **b** be two unit vectors such that angle between them is 60°. Then, |**a** − **b**| is equal to

(a) $\sqrt{5}$ (b) $\sqrt{3}$ (c) 0 (d) 1

36. Let *ABCD* be a parallelogram such that **AB** = **q**, **AD** = **p** and ∠*BAD* be an acute angle. If **r** is the vector that coincides with the altitude directed from the vertex *B* to the side *AD*, then **r** is given by

(a) $\mathbf{r} = 3\mathbf{q}\,\frac{3(\mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}\cdot\mathbf{p})}\,\mathbf{p}$ (b) $\mathbf{r} = -\mathbf{q} + \left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$

(c) $\mathbf{r} = \mathbf{q} - \left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$ (d) $\mathbf{r} = -3\mathbf{q}\,\frac{3(\mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}\cdot\mathbf{p})}\,\mathbf{p}$

37. If |**a**| = 3, |**b**| = 4, then a value of λ for which **a** + λ**b** is perpendicular to **a** − λ**b**, is

(a) $\frac{9}{16}$ (b) $\frac{3}{4}$
(c) $\frac{3}{2}$ (d) $\frac{4}{3}$

38. The angle between **a** and **b** is $\frac{5\pi}{6}$ and the projection of **a** in the direction of **b** is $\frac{-6}{\sqrt{3}}$, then |**a**| is equal to

(a) 6 (b) $\frac{\sqrt{3}}{2}$ (c) 12 (d) 4

39. If $\mathbf{a}\cdot\hat{\mathbf{i}} = \mathbf{a}\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}}) = \mathbf{a}\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$, then **a** is equal to

(a) $\hat{\mathbf{i}}$ (b) $\hat{\mathbf{k}}$
(c) $\hat{\mathbf{j}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

40. If the angle between $\hat{\mathbf{i}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + a\hat{\mathbf{k}}$ is $\frac{\pi}{3}$, then the value of a is

(a) 0 or 2 (b) −4 or 0
(c) 0 or −2 (d) 2 or −2

41. Forces acting on a particle have magnitude 5, 3 and 1 unit and act in the direction of the vectors $6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, respectively. They remain constant while the particle is displaced from the points *A*(2, −1, −3) to *B*(5, −1, 1). The work done is

(a) 11 units (b) 33 units
(c) 10 units (d) 30 units

42. If $\sum_{i=1}^{n} \mathbf{a}_i = \mathbf{0}$, where $|\mathbf{a}_i| = 1, \forall\, i$, then the value of $\sum_{1\le i<j\le n} \sum \mathbf{a}_i \cdot \mathbf{a}_j$ is

(a) n^2 (b) $-n^2$ (c) n (d) $-\frac{n}{2}$

43. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ and $\mathbf{c} = 5\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ be three vectors such that the projection vector of **b** on **a** is **a**. If **a** + **b** is perpendicular to **c**, then |**b**| is equal to *(JEE Main 2019)*

(a) 6 (b) 4 (c) $\sqrt{22}$ (d) $\sqrt{32}$

44. Let **a** and **b** be unit vectors inclined at an angle $2\alpha\,(0 \le \alpha \le \pi)$ each other, then |**a** + **b**| < 1, if

(a) $\alpha = \frac{\pi}{2}$ (b) $\alpha < \frac{\pi}{3}$
(c) $\alpha > \frac{2\pi}{3}$ (d) $\frac{\pi}{3} < \alpha < \frac{2\pi}{3}$

45. The length of longer diagonal of the parallelogram constructed on 5**a** + 2**b** and **a** − 3**b**. If it is given that $|\mathbf{a}| = 2\sqrt{2}$, |**b**| = 3 and angle between **a** and **b** is $\frac{\pi}{4}$, is

(a) 15 (b) $\sqrt{113}$ (c) $\sqrt{593}$ (d) $\sqrt{369}$

46. The unit vector perpendicular to $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and coplanar with $\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ is

(a) $\frac{2\hat{\mathbf{i}} - 5\hat{\mathbf{j}}}{\sqrt{2a}}$ (b) $2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ (c) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}}$

47. If **a** and **b** are two unit vectors inclined to *X*-axis at angles 30° and 120°, then |**a** + **b**| is equal to

(a) $\sqrt{\frac{2}{3}}$ (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

48. If **a**, **b**, **c** are the *p*th, *q*th, *r*th terms of an HP and $\mathbf{u} = (q-r)\hat{\mathbf{i}} + (r-p)\hat{\mathbf{j}} + (p-q)\hat{\mathbf{k}}$ and $\mathbf{v} = \frac{\hat{\mathbf{i}}}{a} + \frac{\hat{\mathbf{j}}}{b} + \frac{\hat{\mathbf{k}}}{c}$, then

(a) **u, v** are parallel vectors
(b) **u, v** are orthogonal vectors
(c) **u** · **v** = 1
(d) $\mathbf{u} \times \mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

49. If **a**, **b**, **c** are three non-coplanar vectors and **p**, **q**, **r** are reciprocal vectors, then (l**a** + m**b** + n**c**) · (l**p** + m**q** + n**r**) is equal to

(a) $l + m + n$ (b) $l^3 + m^3 + n^3$
(c) $l^2 + m^2 + n^2$ (d) None of these

50. The sum of two unit vectors is a unit vector. The magnitude of their difference is

(a) 2 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) 1

51. For non-zero vector **a** and **b**, if $|\mathbf{a}+\mathbf{b}|<|\mathbf{a}-\mathbf{b}|$, then **a** and **b** are
(a) collinear
(b) perpendicular to each other
(c) inclined at an acute angle
(d) inclined at an obtuse angle

52. Let there be two points A and B on the curve $y=x^2$ in the plane OXY satisfying $OA\cdot\hat{\mathbf{i}}=1$ and $OB\cdot\hat{\mathbf{i}}=-2$ then the length of the vector $2OA-3OB$ is
(a) $\sqrt{14}$ (b) $2\sqrt{51}$ (c) $3\sqrt{41}$ (d) $2\sqrt{41}$

53. Given that $(\mathbf{x}-\hat{\mathbf{a}})\cdot(\mathbf{x}+\hat{\mathbf{a}})=8$ and $\mathbf{x}\cdot\hat{\mathbf{a}}=2$, then the angle between $(\mathbf{x}-\hat{\mathbf{a}})$ and $(\mathbf{x}+\hat{\mathbf{a}})$ is
(a) $\cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$ (b) $\cos^{-1}\left(\frac{3}{\sqrt{21}}\right)$
(c) $\cos^{-1}\left(\frac{5}{\sqrt{21}}\right)$ (d) $\cos^{-1}\left(\frac{4}{\sqrt{21}}\right)$

Vector Product or Cross Product of Two Vectors and Its Applications

54. Let $\mathbf{a}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ be two vectors. If **c** is a vector such that $\mathbf{b}\times\mathbf{c}=\mathbf{b}\times\mathbf{a}$ and $\mathbf{c}\cdot\mathbf{a}=0$, then $\mathbf{c}\cdot\mathbf{b}$ is equal to *(JEE Main 2020)*
(a) $\frac{1}{2}$ (b) $-\frac{3}{2}$ (c) $-\frac{1}{2}$ (d) -1

55. If **u** and **v** are unit vectors and θ is the acute angle between them, then $2\,\mathbf{u}\times 3\,\mathbf{v}$ is a unit vector for
(a) exactly two values of θ
(b) more than two values of θ
(c) no value of θ
(d) exactly one value of θ

56. Let **a** and **b** be two non-zero vectors perpendicular to each other and $|\mathbf{a}|=|\mathbf{b}|$. If $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|$, then the angle between the vectors $(\mathbf{a}+\mathbf{b}+(\mathbf{a}\times\mathbf{b}))$ and **a** is equal to *(JEE Main 2021)*
(a) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$
(c) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$

57. Let **a**, **b** and **c** be three unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. If $\lambda=\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ and $\mathbf{d}=\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$, then the ordered pair, $(\lambda,\mathbf{d})$ is equal to *(JEE Main 2020)*
(a) $\left(\frac{3}{2},3\mathbf{b}\times\mathbf{c}\right)$ (b) $\left(-\frac{3}{2},3\mathbf{c}\times\mathbf{b}\right)$
(c) $\left(\frac{3}{2},3\mathbf{a}\times\mathbf{c}\right)$ (d) $\left(-\frac{3}{2},3\mathbf{a}\times\mathbf{b}\right)$

58. The two adjacent sides of a parallelogram are $2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}-2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$. Then, the unit vector parallel to its diagonal. Also, its area are
(a) $\frac{3}{7}\hat{\mathbf{i}}-\frac{6}{7}\hat{\mathbf{j}}+\frac{2}{7}\hat{\mathbf{k}}$ and $11\sqrt{5}$ sq units
(b) $\frac{2}{7}\hat{\mathbf{i}}-\frac{6}{7}\hat{\mathbf{j}}+\frac{3}{7}\hat{\mathbf{k}}$ and $11\sqrt{3}$ sq units
(c) $\frac{6}{7}\hat{\mathbf{i}}-\frac{2}{7}\hat{\mathbf{j}}+\frac{3}{7}\hat{\mathbf{k}}$ and $11\sqrt{7}$ sq units
(d) None of the above

59. Let $\mathbf{a}=\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\mathbf{b}=3\hat{\mathbf{i}}-2\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ and $\mathbf{c}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$. Then, a vector **d** which is perpendicular to both **a** and **b** and $\mathbf{c}\cdot\mathbf{d}=15$, is
(a) $\frac{5}{3}(\hat{\mathbf{i}}-32\hat{\mathbf{j}}-14\hat{\mathbf{k}})$ (b) $\frac{1}{3}(\hat{\mathbf{i}}-32\hat{\mathbf{j}}-14\hat{\mathbf{k}})$
(c) $\frac{5}{3}(32\hat{\mathbf{i}}-\hat{\mathbf{j}}-14\hat{\mathbf{k}})$ (d) None of these

60. If $|\mathbf{a}|=8$, $|\mathbf{b}|=3$ and $|\mathbf{a}\times\mathbf{b}|=12$, then value of $\mathbf{a}\cdot\mathbf{b}$ is
(a) $6\sqrt{3}$ (b) $8\sqrt{3}$
(c) $12\sqrt{3}$ (d) None of these

61. Let $\mathbf{a}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+x\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$, for some real x. Then, $|\mathbf{a}\times\mathbf{b}|=r$ is possible if *(JEE Main 2019, 8 April)*
(a) $0<r\le\sqrt{\frac{3}{2}}$ (b) $\sqrt{\frac{3}{2}}<r\le 3\sqrt{\frac{3}{2}}$
(c) $3\sqrt{\frac{3}{2}}<r<5\sqrt{\frac{3}{2}}$ (d) $r\ge 5\sqrt{\frac{3}{2}}$

62. Using vectors, the area of the ΔABC with vertices $A(1,2,3)$, $B(2,-1,4)$ and $C(4,5,-1)$ is
(a) $\sqrt{\frac{137}{2}}$ (b) $\sqrt{137}$
(c) $\frac{1}{2}\sqrt{137}$ (d) $\frac{1}{2}\sqrt{278}$

63. Let $\alpha=3\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\beta=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$. If $\beta=\beta_1-\beta_2$, where β_1 is parallel to α and β_2 is perpendicular to α, then $\beta_1\times\beta_2$ is equal to *(JEE Main 2019)*
(a) $\frac{1}{2}(3\hat{\mathbf{i}}-9\hat{\mathbf{j}}+5\hat{\mathbf{k}})$ (b) $\frac{1}{2}(-3\hat{\mathbf{i}}+9\hat{\mathbf{j}}+5\hat{\mathbf{k}})$
(c) $-3\hat{\mathbf{i}}+9\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ (d) $3\hat{\mathbf{i}}-9\hat{\mathbf{j}}-5\hat{\mathbf{k}}$

64. For any vector **a**, the value of $(\mathbf{a}\times\hat{\mathbf{i}})^2+(\mathbf{a}\times\hat{\mathbf{j}})^2+(\mathbf{a}\times\hat{\mathbf{k}})^2$ is equal to
(a) $\mathbf{a}^2$ (b) $3\mathbf{a}^2$ (c) $4\mathbf{a}^2$ (d) $2\mathbf{a}^2$

65. If $|\mathbf{a}|=10$, $|\mathbf{b}|=2$ and $\mathbf{a}\cdot\mathbf{b}=12$, then value of $|\mathbf{a}\times\mathbf{b}|$ is
(a) 5 (b) 10 (c) 14 (d) 16

66. The number of vectors of unit length perpendicular to the vectors $\mathbf{a}=2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is
(a) one (b) two (c) three (d) infinite

67. The value of $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})\times(\mathbf{a}+\mathbf{b}+\mathbf{c})$ is equal to

(a) 0 (b) $\mathbf{a}$ (c) $\mathbf{b}$ (d) $\mathbf{c}$

68. If the vectors $\mathbf{c}$, $\mathbf{a}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{j}}$ are such that $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{b}$ form a right handed system, then $\mathbf{c}$ is

(a) $z\hat{\mathbf{i}}-x\hat{\mathbf{k}}$ (b) $\mathbf{0}$
(c) $y\hat{\mathbf{j}}$ (d) $-z\hat{\mathbf{i}}+x\hat{\mathbf{k}}$

69. If $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+3\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ and $\mathbf{c}=7\hat{\mathbf{i}}+9\hat{\mathbf{j}}+11\hat{\mathbf{k}}$, then the area of the parallelogram having diagonals $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}+\mathbf{c}$ is

(a) $4\sqrt{6}$ (b) $\frac{1}{2}\sqrt{21}$ (c) $\frac{\sqrt{6}}{2}$ (d) $\sqrt{6}$

70. The moment about the point $M(-2, 4, -6)$ of the force represented in magnitude and position $\mathbf{AB}$, where the points A and B have the coordinates $(1, 2, -3)$ and $(3, -4, 2)$ respectively, is

(a) $8\hat{\mathbf{i}}-9\hat{\mathbf{j}}-14\hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}}-6\hat{\mathbf{j}}+5\hat{\mathbf{k}}$
(c) $-3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ (d) $-5\hat{\mathbf{i}}-8\hat{\mathbf{j}}-8\hat{\mathbf{k}}$

71. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors such that $\mathbf{a}\neq 0$ and $\mathbf{a}\times\mathbf{b}=2\mathbf{a}\times\mathbf{c}$, $|\mathbf{a}|=|\mathbf{c}|=1$, $|\mathbf{b}|=4$ and $|\mathbf{b}\times\mathbf{c}|=\sqrt{15}$. If $\mathbf{b}-2\mathbf{c}=\lambda\mathbf{a}$, then λ is equal to

(a) 1 (b) -4 (c) 3 (d) -2

Scalar Triple Product and Its Application

72. The vectors $\lambda\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$ are coplanar, if

(a) $\lambda=-2$ (b) $\lambda=0$ (c) $\lambda=1$ (d) $\lambda=-1$

73. The value of $[\mathbf{a}-\mathbf{b}\ \ \mathbf{b}-\mathbf{c}\ \ \mathbf{c}-\mathbf{a}]$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

74. For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the value of $[\mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}]$ is equal to

(a) [abc] (b) 2[abc]
(c) 3[abc] (d) None of these

75. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-coplanar vectors and let $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ be vector defined by the relations. $\mathbf{p}=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}$, $\mathbf{q}=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}$ and $\mathbf{r}=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$. Then, the value of the expression $(\mathbf{a}+\mathbf{b})\cdot\mathbf{p}+(\mathbf{b}+\mathbf{c})\cdot\mathbf{q}+(\mathbf{c}+\mathbf{a})\cdot\mathbf{r}$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

76. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\mathbf{a}+2\mathbf{b}+3\mathbf{c}$, $\lambda\mathbf{b}+4\mathbf{c}$ and $(2\lambda-1)\mathbf{c}$ are non-coplanar for

(a) all values of λ
(b) all except one value of λ
(c) all except two values of λ
(d) no value of λ

77. If the volume of parallelopiped formed by the vectors $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$ and $\lambda\hat{\mathbf{i}}+\hat{\mathbf{k}}$ is minimum, then λ is equal to *(JEE Main 2019)*

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $-\sqrt{3}$

78. Let $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\mathbf{c}=x\hat{\mathbf{i}}+(x-2)\hat{\mathbf{j}}-\hat{\mathbf{k}}$. If the vector $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, then x equal to

(a) 0 (b) 1 (c) -4 (d) -2

79. A unit vector coplanar with $\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and perpendicular to $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ is

(a) $\left(\frac{\hat{\mathbf{j}}-\hat{\mathbf{k}}}{\sqrt{2}}\right)$ (b) $\left(\frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}}{\sqrt{3}}\right)$
(c) $\left(\frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}}{\sqrt{6}}\right)$ (d) $\left(\frac{\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}}{\sqrt{6}}\right)$

80. Let $\mathbf{a}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ and $\mathbf{c}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}+(\lambda^2-1)\hat{\mathbf{k}}$ be coplanar vectors. Then, the non-zero vector $\mathbf{a}\times\mathbf{c}$ is *(JEE Main 2019)*

(a) $-10\hat{\mathbf{i}}+5\hat{\mathbf{j}}$ (b) $-10\hat{\mathbf{i}}-5\hat{\mathbf{j}}$
(c) $-14\hat{\mathbf{i}}-5\hat{\mathbf{j}}$ (d) $-14\hat{\mathbf{i}}+5\hat{\mathbf{j}}$

81. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the three vectors mutually perpendicular to each other to form a right handed system and $|\mathbf{a}|=1$, $|\mathbf{b}|=3$ and $|\mathbf{c}|=5$, then $[\mathbf{a}-2\mathbf{b}\ \ \mathbf{b}-3\mathbf{c}\ \ \mathbf{c}-4\mathbf{a}]$ is equal to

(a) 0 (b) -24 (c) 3600 (d) -215

82. If the volume of a parallelopiped, whose coterminus edges are given by the vectors $\mathbf{a}=\hat{\mathbf{i}}+\hat{\mathbf{j}}+n\hat{\mathbf{k}}$, $\mathbf{b}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-n\hat{\mathbf{k}}$ and $\mathbf{c}=\hat{\mathbf{i}}+n\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ $(n\geq 0)$, is 158 cu units, then *(JEE Main 2020)*

(a) $n=9$ (b) $\mathbf{b}\cdot\mathbf{c}=10$
(c) $\mathbf{a}\cdot\mathbf{c}=17$ (d) $n=7$

83. $[\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}]+[\hat{\mathbf{k}}\ \hat{\mathbf{j}}\ \hat{\mathbf{i}}]+[\hat{\mathbf{j}}\ \hat{\mathbf{k}}\ \hat{\mathbf{i}}]$ is equal to

(a) 1 (b) 3 (c) -3 (d) -1

84. If the vectors $p\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+q\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+\hat{\mathbf{j}}+r\hat{\mathbf{k}}$ (where, $p\neq q\neq r\neq 1$) are coplanar, then the value of $pqr-(p+q+r)$ is

(a) -2 (b) 2 (c) 0 (d) -1

85. A tetrahedron has vertical at $O(0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then, the angle between the faces OAB and ABC will be

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{7}{31}\right)$ (c) $30°$ (d) $90°$

86. The sum of the distinct real values of μ, for which the vectors, $\mu\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+\mu\hat{j}+\hat{k}$, $\hat{i}+\hat{j}+\mu\hat{k}$ are coplanar, is *(JEE Main 2019)*

(a) 2 (b) 0 (c) 1 (d) -1

87. If $\mathbf{A}=2\hat{i}+3\hat{j}+4\hat{k}$, $\mathbf{B}=\hat{i}+\hat{j}+5\hat{k}$ and $\mathbf{C}$ from a left handed system, then $\mathbf{C}$ is

(a) $11\hat{i}-6\hat{j}-\hat{k}$ (b) $-11\hat{i}+6\hat{j}+\hat{k}$
(c) $11\hat{i}-6\hat{j}+\hat{k}$ (d) $11\hat{i}+6\hat{j}-\hat{k}$

88. Let the volume of a parallelopiped whose conterminous edges are given by $\mathbf{u}=\hat{i}+\hat{j}+\lambda\hat{k}$, $\mathbf{v}=\hat{i}+\hat{j}+3\hat{k}$ and $\mathbf{w}=2\hat{i}+\hat{j}+\hat{k}$ be 1 cu unit. If θ be the angle between the edges $\mathbf{u}$ and $\mathbf{w}$, then $\cos\theta$ can be *(JEE Main 2020)*

(a) $\frac{5}{3\sqrt{3}}$ (b) $\frac{7}{6\sqrt{3}}$ (c) $\frac{7}{6\sqrt{6}}$ (d) $\frac{5}{7}$

89. Let $\alpha \in R$ and the three vectors

$\mathbf{a}=\alpha\hat{i}+\hat{j}+3\hat{k}$, $\mathbf{b}=2\hat{i}+\hat{j}-\alpha\hat{k}$

and $\hat{\mathbf{c}}=\alpha\hat{i}-2\hat{j}+3\hat{k}$. Then, the set

$S=\{\alpha : \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar$\}$ *(JEE Main 2019)*

(a) is singleton
(b) is empty
(c) contains exactly two positive numbers
(d) contains exactly two numbers only one of which is positive

Vector Triple Product and Its Applications

90. $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]$ is equal to

(a) $(\mathbf{a}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{a})$
(b) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{a})-\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})$
(c) $[\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{a}$
(d) $(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{b}\times\mathbf{a})$

91. If $\mathbf{a}=\hat{i}+\hat{j}+\hat{k}$, $\mathbf{b}=\hat{i}+\hat{j}$, $\mathbf{c}=\hat{i}$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\lambda\mathbf{a}+\mu\mathbf{b}$, then $\lambda+\mu$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

92. If $\mathbf{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\mathbf{b}=\hat{i}\times(\mathbf{a}\times\hat{i})+\hat{j}\times(\mathbf{a}\times\hat{j})+\hat{k}\times(\mathbf{a}\times\hat{k})$, then length of $\mathbf{b}$ is equal to

(a) $\sqrt{12}$ (b) $2\sqrt{12}$
(c) $3\sqrt{14}$ (d) $2\sqrt{14}$

93. Let $\mathbf{a}=\hat{j}-\hat{k}$ and $\mathbf{a}=\hat{i}-\hat{j}-\hat{k}$. Then, the vector $\mathbf{b}$ satisfying $\mathbf{a}\times\mathbf{b}+\mathbf{c}=0$ and $\mathbf{a}\cdot\mathbf{b}=3$, is

(a) $-\hat{i}+\hat{j}-2\hat{k}$ (b) $2\hat{i}-\hat{j}+2\hat{k}$
(c) $\hat{i}-\hat{j}-2\hat{k}$ (d) $\hat{i}+\hat{j}-2\hat{k}$

94. If $\mathbf{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\mathbf{b}=-\hat{i}+2\hat{j}-4\hat{k}$ and $\mathbf{c}=\hat{i}+\hat{j}+\hat{k}$, then $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})$ is

(a) 60 (b) 68 (c) -60 (d) -74

95. If $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-5\mathbf{a}+4\mathbf{b}$ and $\mathbf{a}\cdot\mathbf{b}=3$, then $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is equal to

(a) $5\mathbf{b}-3\mathbf{c}$ (b) $3\mathbf{c}-4\mathbf{b}$
(c) $3\mathbf{b}-5\mathbf{c}$ (d) $4\mathbf{b}-3\mathbf{c}$

96. If $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors such that $\mathbf{a}\cdot\mathbf{b}\neq 0$, $\mathbf{b}\cdot\mathbf{c}\neq 0$, then $\mathbf{a}$ and $\mathbf{c}$ are

(a) inclined at an angle of $\frac{\pi}{6}$ between them
(b) perpendicular
(c) parallel
(d) inclined at an angle of $\frac{\pi}{3}$ between them

97. Let $\mathbf{a}=\hat{i}-\hat{j}$, $\mathbf{b}=\hat{i}+\hat{j}+\hat{k}$ and $\mathbf{c}$ be a vector such that $\mathbf{a}\times\mathbf{c}+\mathbf{b}=\mathbf{0}$ and $\mathbf{a}\cdot\mathbf{c}=4$, then $|\mathbf{c}|^2$ is equal to *(JEE Main 2019)*

(a) 8 (b) $\frac{19}{2}$ (c) 9 (d) $\frac{17}{2}$

ROUND II Mixed Bag

Only One Correct Option

1. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors such that no two of them are collinear and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is the angle between vectors $\mathbf{b}$ and $\mathbf{c}$, then a value of $\sin\theta$ is *(JEE Main 2015)*

(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{-\sqrt{2}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{-2\sqrt{3}}{3}$

2. If V is the volume of the parallelopiped having three coterminus edges, as $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ then the volume of the parallelopiped having three coterminus edges as

$\alpha=(\mathbf{a}\cdot\mathbf{a})\mathbf{a}+(\mathbf{a}\cdot\mathbf{b})\mathbf{b}+(\mathbf{a}\cdot\mathbf{c})\mathbf{c}$
$\beta=(\mathbf{a}\cdot\mathbf{b})\mathbf{a}+(\mathbf{b}\cdot\mathbf{b})\mathbf{b}+(\mathbf{b}\cdot\mathbf{c})\mathbf{c}$
$\gamma=(\mathbf{a}\cdot\mathbf{c})\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})\mathbf{b}+(\mathbf{c}\cdot\mathbf{c})\mathbf{c}$

(a) V^3 (b) $3V$ (c) V^2 (d) $2V$

3. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and r is a real number, then the vectors $\mathbf{a}+2\mathbf{b}+3\mathbf{c}$, $\lambda\mathbf{b}+4\mathbf{c}$ and $(2\lambda-1)\mathbf{c}$ are non-coplanar for

(a) no value of λ
(b) all except one value of λ
(c) all except two values of λ
(d) all values of λ

4. Let **u**, **v** and **w** be three vectors such that $|\mathbf{u}|=1, |\mathbf{v}|=2, |\mathbf{w}|=3$. If the projection of **v** along **u** is equal to that of **w** along **u** , **v** and **w** are perpendicular to each other, then $|\mathbf{u}-\mathbf{v}+\mathbf{w}|$ is equal to

(a) 4 (b) $\sqrt{7}$ (c) $\sqrt{14}$ (d) 2

5. If **a** and **b** are perpendicular vectors, then $\mathbf{a}\times(\mathbf{a}\times(\mathbf{a}\times(\mathbf{a}\times\mathbf{b})))$ is equal to *(JEE Main 2021)*

(a) $\frac{1}{2}|\mathbf{a}|^4\mathbf{b}$ (b) $\mathbf{a}\times\mathbf{b}$

(c) $|\mathbf{a}|^4\mathbf{b}$ (d) $\mathbf{0}$

6. Let **b** and **c** be non-collinear vectors. If **a** is a vector such that $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=4$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(x^2-2x+6)\,\mathbf{b}+\sin y\cdot\mathbf{c}$; then (x, y) lies on the line

(a) $x+y=0$ (b) $x-y=0$ (c) $x=1$ (d) $y=\pi$

7. Let **a**, **b** and **c** be three unit vectors such that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\frac{\sqrt{3}}{2}(\mathbf{b}+\mathbf{c})$. If **b** is not parallel to **c**, then the angle between **a** and **b** is *(JEE Main 2016)*

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

8. Let $|\mathbf{a}|=2\sqrt{2}, |\mathbf{b}|=3$ and the angle between **a** and **b** is $\frac{\pi}{4}$. If a parallelogram is constructed with adjacent sides $2\mathbf{a}-3\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$, then its longer diagonal is of length

(a) 10 (b) 8 (c) $2\sqrt{26}$ (d) 6

9. Let O be the origin. Let $\mathbf{OP}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{OQ}=-\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3x\hat{\mathbf{k}}, x, y\in R, x>0$ be such that $|\mathbf{PQ}|=\sqrt{20}$ and the vector **OP** is perpendicular to **OQ**. If $\mathbf{OR}=3\hat{\mathbf{i}}+z\hat{\mathbf{j}}-7\hat{\mathbf{k}}, z\in R$ is coplanar with **OP** and **OQ**. Then, the value of $x^2+y^2+z^2$ is *(JEE Main 2021)*

(a) 7 (b) 9 (c) 2 (d) 1

10. If the positive numbers a, b and c are the pth, qth and rth terms of GP, then the vectors $\log a\cdot\hat{\mathbf{i}}+\log b\cdot\hat{\mathbf{j}}+\log c\cdot\hat{\mathbf{k}}$ and $(q-r)\hat{\mathbf{i}}+(r-p)\hat{\mathbf{j}}+(p-q)\hat{\mathbf{k}}$ are

(a) equal (b) parallel

(c) perpendicular (d) None of these

11. Let $\mathbf{a}=\hat{\mathbf{i}}-\hat{\mathbf{k}}, \mathbf{b}=x\hat{\mathbf{i}}+\hat{\mathbf{j}}+(1-x)\hat{\mathbf{k}}$ and $\mathbf{c}=y\hat{\mathbf{i}}+x\hat{\mathbf{j}}+(1+x-y)\hat{\mathbf{k}}$. Then, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends on

(a) Neither x nor y (b) Both x and y

(c) Only x (d) Only y

12. If **a** is a unit vector and projection of **x** along **a** is 2 and $\mathbf{a}\times\mathbf{r}+\mathbf{b}=\mathbf{r}$, then **r** is equal to

(a) $\frac{1}{2}[\mathrm{a}-\mathrm{b}+\mathrm{a}\times\mathrm{b}]$ (b) $\frac{1}{2}[2\mathrm{a}+\mathrm{b}+\mathrm{a}\times\mathrm{b}]$

(c) $\mathrm{a}+\mathrm{a}\times\mathrm{b}$ (d) $\mathrm{a}-\mathrm{a}\times\mathrm{b}$

13. Let **a**, **b** and **c** be three unit vectors such that **a** is perpendicular to the plane of **b** and **c**. If the angle between **b** and **c** is $\frac{\pi}{3}$, then $|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}|$ is equal to

(a) 1/3 (b) 1/2 (c) 1 (d) 2

14. If $\mathbf{r}=\alpha\,\mathbf{b}\times\mathbf{c}+\beta\mathbf{c}\times\mathbf{a}+\gamma\mathbf{a}\times\mathbf{b}$ and $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=2$, then $\alpha+\beta+\gamma$ is equal to

(a) $\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})$ (b) $\frac{1}{2}\mathbf{r}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})$

(c) $2\mathbf{r}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})$ (d) 4

15. If $\mathbf{a}+2\mathbf{b}+3\mathbf{c}=0$, then $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=k\mathbf{a}\times\mathbf{b}$, where k is equal to

(a) 0 (b) 1 (c) 2 (d) 3

16. Let $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}, \mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$, $\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$. If $|\mathbf{c}|=1$ and $(a\times b)\times c=0$, then $\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$ is equal to

(a) 0 (b) 1 (c) $|\mathbf{a}|^2|\mathbf{b}|^2$ (d) $|\mathbf{a}\times\mathbf{b}|^2$

17. If **a, b** and **c** are coplanar vectors and λ is a real number, then the vectors $\mathbf{a}+2\mathbf{b}+3\mathbf{c}$, $\lambda\mathbf{b}+\mu\mathbf{c}$ and $(2\lambda-1)\mathbf{c}$ are coplanar for

(a) $r=1$ (b) $\lambda=\frac{1}{2}$

(c) $\lambda=2$ (d) no value of λ

18. A vector **a** has components $3p$ and 1 with respect to a rectangular cartesian system. This system is rotated through a certain angle about the origin in the clockwise sense. If with respect to new system, **a** has components $p+1$ and $\sqrt{10}$, then a value of p is equal to *(JEE Main 2021)*

(a) 1 (b) $-\frac{5}{4}$ (c) $\frac{4}{5}$ (d) -1

19. If $A(-4, 0, 3)$ and $B(14, 2, -5)$, then which one of the following points lie on the bisector of the angle between **OA** and **OB** (O is the origin of reference)?

(a) (2, 2, 4) (b) (2, 12, 6)

(c) (−3, −3, 6) (d) (2, 1, 1)

20. In a four dimensional space where unit vectors along the axes are $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and $\hat{\mathbf{l}}$ and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and $(\lambda-1)(\mathbf{a}_1-\mathbf{a}_2)+\mu(\mathbf{a}_2+\mathbf{a}_3)+\gamma(\mathbf{a}_3+\mathbf{a}_4-2\mathbf{a}_2)+\mathbf{a}_3+\delta\mathbf{a}_4=\mathbf{0}$, then

(a) $\lambda=3$ (b) $\mu=-2/3$ (c) $\gamma=2/3$ (d) $\delta=1/5$

21. Unit vectors **a** and **b** are perpendicular and unit vector **c** is inclined at an angle θ to both **a** and **b**. If $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$, then

(a) $\alpha = \beta$
(b) $\gamma^2 = 1 + 2\alpha^2$
(c) $\gamma^2 = \cos 2\theta$
(d) $\beta^2 = \frac{1 + \cos 2\theta}{3}$

22. **a** and **b** are two given vectors. With these vectors as adjacent sides, a parallelogram is constructed. The vector which is the altitude of the parallelogram and which is perpendicular to **a** is

(a) $\frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{a}|^2}\mathbf{a} + \mathbf{b}$
(b) $\frac{1}{|\mathbf{a}|^2}\{|\mathbf{a}|^2\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}\}$
(c) $\frac{\mathbf{a} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{a}|^3}$
(d) $\frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2}$

23. $a_1, a_2, a_3 \in R - \{0\}$ and $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all $x \in R$, then

(a) vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ are perpendicular to each other
(b) vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ are parallel to each other
(c) if vector $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ is of length $\sqrt{10}$ units, then one of the ordered tripplet $(a_1, a_2, a_3) = (1, -1, -2)$
(d) if $2a_1 + 3a_2 + 6a_3 = 26$, then $|a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}|$ is $5\sqrt{6}$

24. If **a** and **b** are two vectors and angle between them is θ, then

(a) $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = 2|\mathbf{a}|^2|\mathbf{b}|^2$
(b) $|\mathbf{a} \times \mathbf{b}| \neq (\mathbf{a} \cdot \mathbf{b})$, if $\theta = \pi/4$
(c) $\mathbf{a} \times \mathbf{b} = (\mathbf{a} \cdot \mathbf{b})\hat{\mathbf{n}}$, ($\hat{\mathbf{n}}$ is normal unit vector), if $\theta = \pi/6$
(d) None of the above

25. Let **a** and **b** be two non-zero perpendicular vectors. A vector **r** satisfying the equation $\mathbf{r} \times \mathbf{b} = \mathbf{a}$ can be

(a) $\mathbf{b} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$
(b) $2\mathbf{b} - \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$
(c) $|\mathbf{a}|\mathbf{b} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$
(d) $|\mathbf{b}|\mathbf{b} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$

26. If vectors **a** and **b** are non-collinear, then $\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}$ is

(a) a unit vector
(b) in the plane of **a** or **b**
(c) equally inclined to **a** or **b**
(d) perpendicular to $\mathbf{a} \times \mathbf{b}$

Numerical Value Questions

27. If the vectors $\mathbf{p} = (a + 1)\hat{\mathbf{i}} + a\hat{\mathbf{j}} + a\hat{\mathbf{k}}$, $\mathbf{q} = a\hat{\mathbf{i}} + (a + 1)\hat{\mathbf{j}} + a\hat{\mathbf{k}}$ and $\mathbf{r} = a\hat{\mathbf{i}} + a\hat{\mathbf{j}} + (a + 1)\hat{\mathbf{k}}$ $(a \in R)$ are coplanar and $3(\mathbf{p} \cdot \mathbf{q})^2 - \lambda|\mathbf{r} \times \mathbf{q}|^2 = 0$, then the value of λ is .... *(JEE Main 2020)*

28. Let **a**, **b** and **c** be three vectors such that $|\mathbf{a}| = \sqrt{3}$, $|\mathbf{b}| = 5$, $\mathbf{b} \cdot \mathbf{c} = 10$ and the angle between **b** and **c** is $\frac{\pi}{3}$. If **a** is perpendicular to the vector $\mathbf{b} \times \mathbf{c}$, then $|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|$ is equal to .... *(JEE Main 2020)*

29. Let $\alpha = (\lambda - 2)\,\mathbf{a} + \mathbf{b}$ and $\beta = (4\lambda - 2)\,\mathbf{a} + 3\mathbf{b}$ be two given vectors where vectors **a** and **b** are non- collinear. The value of $(-\lambda)$ for which vectors α and β are collinear, is *(JEE Main 2019)*

30. If $[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] = \lambda[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$, then λ is equal to *(JEE Main 2014)*

31. Let the vectors **a**, **b**, **c** be such that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 4$ and $|\mathbf{c}| = 4$. If the projection of **b** on **a** is equal to the projection of **c** on **a** and **b** is perpendicular to **c**, then the value of $|\mathbf{a} + \mathbf{b} - \mathbf{c}|$ is .... *(JEE Main 2020)*

32. If **a** and **b** are unit vectors, then the greatest value of $\sqrt{3}\,|\mathbf{a} + \mathbf{b}| + |\mathbf{a} - \mathbf{b}|$ is .... *(JEE Main 2020)*

33. A particle is acted upon by constant forces $4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ which displace it from a point $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ to the point $5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$. The work done in standard units by the forces is given by ....

34. Let three vectors **a**, **b** and **c** be such that **c** is coplanar with **a** and **b**, $\mathbf{a} \cdot \mathbf{c} = 7$ and **b** is perpendicular to **c**, where $\mathbf{a} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{k}}$, then the value of $2|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2$ is *(JEE Main 2021)*

35. Let **x** be a vector in the plane containing vectors $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$. If the vector **x** is perpendicular to $(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ and its projection on **a** is $\frac{17\sqrt{6}}{2}$, then the value of $|\mathbf{x}|^2$ is equal to *(JEE Main 2021)*

Answers

Round I

1. (c)	2. (d)	3. (d)	4. (c)	5. (d)	6. (a)	7. (a)	8. (a)	9. (d)	10. (a)
11. (a)	12. (c)	13. (b)	14. (d)	15. (a)	16. (a)	17. (c)	18. (b)	19. (b)	20. (a)
21. (b)	22. (d)	23. (c)	24. (c)	25. (a)	26. (a)	27. (d)	28. (d)	29. (b)	30. (a)
31. (a)	32. (b)	33. (c)	34. (a)	35. (d)	36. (b)	37. (b)	38. (d)	39. (a)	40. (b)
41. (b)	42. (d)	43. (a)	44. (d)	45. (c)	46. (c)	47. (b)	48. (b)	49. (c)	50. (b)
51. (d)	52. (d)	53. (d)	54. (c)	55. (d)	56. (b)	57. (d)	58. (a)	59. (c)	60. (c)
61. (d)	62. (a)	63. (b)	64. (d)	65. (d)	66. (b)	67. (a)	68. (c)	69. (a)	70. (a)
71. (b)	72. (a)	73. (a)	74. (b)	75. (d)	76. (c)	77. (b)	78. (d)	79. (a)	80. (a)
81. (d)	82. (b)	83. (d)	84. (a)	85. (a)	86. (d)	87. (b)	88. (b)	89. (d)	90. (d)
91. (a)	92. (d)	93. (a)	94. (d)	95. (d)	96. (c)	97. (b)			

Round II

1. (a)	2. (a)	3. (c)	4. (c)	5. (c)	6. (c)	7. (d)	8. (c)	9. (b)	10. (c)
11. (a)	12. (b)	13. (c)	14. (b)	15. (c)	16. (d)	17. (b)	18. (d)	19. (a)	20. (b)
21. (a)	22. (b)	23. (a)	24. (c)	25. (b)	26. (d)	27. (1)	28. (30)	29. (4)	30. (1)
31. (6)	32. (4)	33. (40)	34. (75)	35. (486)					

Solutions

Round I

1. Given, $\mathbf{a}_1 = x\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{a}_2 = \hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ are collinear

$$\therefore \quad \frac{x}{1} = \frac{-1}{y} = \frac{1}{z} = \lambda \text{ (say)}$$

$$\Rightarrow \quad x = \lambda,\ y = \frac{-1}{\lambda},\ z = \frac{1}{\lambda}$$

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = \lambda\hat{\mathbf{i}} - \frac{1}{\lambda}\hat{\mathbf{j}} + \frac{1}{\lambda}\hat{\mathbf{k}}$$

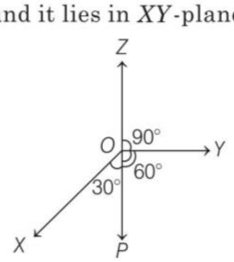
Unit vector parallel to $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is

$$= \pm \frac{\left(\lambda\hat{\mathbf{i}} - \frac{1}{\lambda}\hat{\mathbf{j}} + \frac{1}{\lambda}\hat{\mathbf{k}}\right)}{\sqrt{\lambda^2 + \frac{1}{\lambda^2} + \frac{1}{\lambda^2}}} = \pm \frac{\left(\frac{\lambda^2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\lambda}\right)}{\sqrt{\lambda^2 + \frac{2}{\lambda^2}}}$$

For $\lambda = 1$

It is $\pm \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$

2. Let OP make 30° with X-axis, 60° with Y-axis and 90° with Z-axis and it lies in XY-plane.



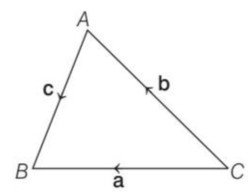
$\therefore$ Direction cosines of **OP** are $\cos 30°$, $\cos 60°$ and $\cos 90°$.

i.e. $\frac{\sqrt{3}}{2}, \frac{1}{2}, 0$. Thus, **OP** is $\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}$,

$$|\mathbf{OP}| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

which is required unit vector in XY-plane.

3. In ΔABC, let $\mathbf{CB} = \mathbf{a}$, $\mathbf{CA} = \mathbf{b}$ and $\mathbf{AB} = \mathbf{c}$



(as shown in the following figure)

Now, by the triangle law of addition, we have $\mathbf{a} = \mathbf{b} + \mathbf{c}$

It is clearly known that $|\mathbf{a}|, |\mathbf{b}|$ and $|\mathbf{c}|$ represent the side of ΔABC.

Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$$\therefore \quad |\mathbf{a}| < |\mathbf{b}| + |\mathbf{c}|$$

Hence, it is not true that $|\mathbf{a}| = |\mathbf{b}| + |\mathbf{c}|$

4. Firstly, determine the resultants of vectors **a** and **b** vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and then find the direction of **c** by finding unit vector in its direction and then multiplying it by 5, we can find the required vector.

Given vectors $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Let **c** be the resultant of **a** and **b**.

$\therefore \mathbf{c} = \mathbf{a} + \mathbf{b} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) + (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) \Rightarrow \mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 0\hat{\mathbf{k}}$

Comparing with $\mathbf{X} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\therefore \quad |\mathbf{c}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 1} = \sqrt{9+1} = \sqrt{10}$

$\therefore$ Unit vector in the direction of $\mathbf{c}$,

$$\hat{\mathbf{c}} = \frac{\mathbf{c}}{|\mathbf{c}|} = \frac{3\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{10}}$$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors **a** and **b** is

$$\pm 5\hat{\mathbf{c}} = \pm 5\frac{1}{\sqrt{10}}(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) = \pm\frac{3\sqrt{10}}{2}\hat{\mathbf{i}} \pm \frac{\sqrt{10}}{2}\hat{\mathbf{j}}$$

5. Let the given points be $A(-1,-1,2), B(2,m,5)$ and $C(3,11,6)$.

Then,

$\mathbf{AB} = (2+1)\hat{\mathbf{i}} + (m+1)\hat{\mathbf{j}} + (5-2)\hat{\mathbf{k}} = 3\hat{\mathbf{i}} + (m+1)\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

and $\mathbf{AC} = (3+1)\hat{\mathbf{i}} + (11+1)\hat{\mathbf{j}} + (6-2)\hat{\mathbf{k}} = 4\hat{\mathbf{i}} + 12\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

Since, A, B and C are collinear, we have $\mathbf{AB} = \lambda\,\mathbf{AC}$,

i.e. $\quad (3\hat{\mathbf{i}} + (m+1)\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = \lambda(4\hat{\mathbf{i}} + 12\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

$\Rightarrow \quad 3 = 4\lambda$ and $m + 1 = 12\lambda$

Therefore, $m = 8$

6. Here, $m = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $n = \cos\frac{\pi}{2} = 0$

Therefore, $\quad l^2 + m^2 + n^2 = 1 \qquad$ (gives)

$$l^2 + \frac{1}{2} + 0 = 1$$

$$\Rightarrow \quad l = \pm\frac{1}{\sqrt{2}}$$

Hence, the required vector $\mathbf{r} = 3\sqrt{2}\,(l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}})$ is given by

$$\mathbf{r} = 3\sqrt{2}\left(\pm\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}} + 0\hat{\mathbf{k}}\right) = \mathbf{r} = \pm 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$$

7. Median **AD** is given by

$$|\mathbf{AD}| = \frac{1}{2}|3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}| = \frac{\sqrt{34}}{2}$$

8. The smallest value of $|k\mathbf{a}|$ will exist at numerically smallest value of k,

i.e. at $k = 0$, which gives $|k\mathbf{a}| = |k|\,|\mathbf{a}| = 0 \times 3 = 0$

The numerically greatest value of k is 2 at which $|k\mathbf{a}| = 6$

9. Given that, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$

Let $\mathbf{c}$ be the position vector of c.

Also given, $\quad \mathbf{BC} = 1.5\,\mathbf{BA}$

$\Rightarrow \quad \mathbf{OC} - \mathbf{OB} = 1.5\,(\mathbf{OA} - \mathbf{OB})$

$\Rightarrow \quad \mathbf{c} - \mathbf{b} = 1.5\,(\mathbf{a} - \mathbf{b})$

$\Rightarrow \quad \mathbf{c} = 1.5\mathbf{a} - 1.5\mathbf{b} + \mathbf{b}$

$\Rightarrow \quad \mathbf{c} = 1.5\mathbf{a} - 0.5\mathbf{b}$

$\Rightarrow \quad \mathbf{c} = 0.5(3\mathbf{a} - \mathbf{b})$

10. Let $\mathbf{a} = 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$

Since, **a** and **b** are parallel, then

$$\mathbf{a} = \mu\mathbf{b}$$

$\Rightarrow \quad (3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}}) = \mu(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}})$

$\Rightarrow \quad 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}} = 2\mu\hat{\mathbf{i}} - 4\mu\hat{\mathbf{j}} + \lambda\mu\hat{\mathbf{k}}$

On comparing, we get

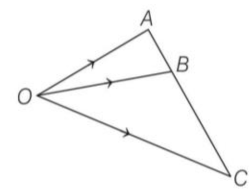
$$2\mu = 3 \Rightarrow \mu = \frac{3}{2}$$

and $\quad \lambda\mu = 1 \Rightarrow \lambda\cdot\frac{3}{2} = 1 \;\Rightarrow\; \lambda = \frac{2}{3}$

11. Let the given points be A, B and C. Let O be the origin.

Then, $\mathbf{OA} = \mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$, $\mathbf{OB} = 2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$

and $\quad \mathbf{OC} = -7\mathbf{b} + 10\mathbf{c}$



Now, $\quad \mathbf{AC} = \mathbf{OC} - \mathbf{OA}$

$= (-7\mathbf{b} + 10\mathbf{c}) - (\mathbf{a} - 2\mathbf{b} + 3\mathbf{c})$

$= -\mathbf{a} - 5\mathbf{b} + 7\mathbf{c}$

$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

$= (2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}) - (\mathbf{a} - 2\mathbf{b} + 3\mathbf{c})$

$= \mathbf{a} + 5\mathbf{b} - 7\mathbf{c}$

$\therefore \quad \mathbf{AB} = -\mathbf{AC} = (-1) \times \mathbf{AC} = \text{scalar} \times \mathbf{AC}$

Hence, the points A, B and C are collinear.

12. If **a** and **b** are two non-zero non-collinear vectors and x, y are two scalars such that $x\mathbf{a} + y\mathbf{b} = 0$, then $x = 0, y = 0$. Because otherwise one will be a scalar multiple of the other and hence collinear, which is a contradiction.

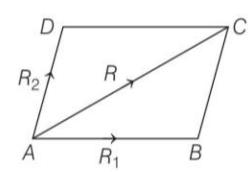
13. All points A, B, C, D, E are in a plane.

$\therefore$ Resultant $= (\mathbf{AC} + \mathbf{AD} + \mathbf{AE}) + (\mathbf{CB} + \mathbf{DB} + \mathbf{EB})$

$= (\mathbf{AC} + \mathbf{CB}) + (\mathbf{AD} + \mathbf{DB}) + (\mathbf{AE} + \mathbf{EB})$

$= \mathbf{AB} + \mathbf{AB} + \mathbf{AB} = 3\mathbf{AB}$

14. $\mathbf{R}_1 = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\mathbf{R}_2 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

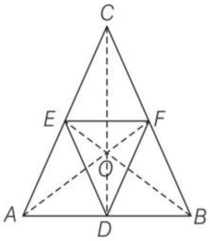


$\therefore \mathbf{R}$ (along $\mathbf{AC}$) $= \mathbf{R}_1 + \mathbf{R}_2 = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$\therefore \hat{\mathbf{a}}$ (unit vector along $\mathbf{AC}$) $= \dfrac{\mathbf{R}}{|\mathbf{R}|} = \dfrac{3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{9+36+4}}$

$= \dfrac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

15. Let O be the origin.



$\therefore \quad \mathbf{BE} + \mathbf{AF} = \mathbf{OE} - \mathbf{OB} + \mathbf{OF} - \mathbf{OA}$

$$= \frac{\mathbf{OA} + \mathbf{OC}}{2} - \mathbf{OB} + \frac{\mathbf{OB} + \mathbf{OC}}{2} - \mathbf{OA}$$

$$= \frac{\mathbf{OC}}{2} + \frac{\mathbf{OC}}{2} + \frac{\mathbf{OA}}{2} - \mathbf{OA} + \frac{\mathbf{OB}}{2} - \mathbf{OB}$$

$$= \mathbf{OC} - \frac{\mathbf{OA} + \mathbf{OB}}{2} = \mathbf{OC} - \mathbf{OD} = \mathbf{DC}$$

16. Given that, $\mathbf{OA} = \hat{\mathbf{i}} + x\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{OB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

and $\quad \mathbf{OC} = y\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

Since, A, B, C are collinear.

Then, $\mathbf{AB} = \lambda\, \mathbf{BC}$

$\Rightarrow \quad 2\hat{\mathbf{i}} + (4 - x)\hat{\mathbf{j}} + 4\hat{\mathbf{k}} = \lambda[(y-3)\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 12\hat{\mathbf{k}}]$

On comparing the coefficient of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ we get

$$2 = (y-3)\lambda \quad \ldots\text{(i)}$$

$$4 - x = -6\lambda \quad \ldots\text{(ii)}$$

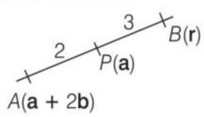
and $\quad 4 = -12\lambda$

$\Rightarrow \quad \lambda = -\frac{1}{3} \quad \ldots\text{(iii)}$

On putting the value of λ is Eqs. (i) and (ii), we get

$$y = -3 \text{ and } x = 2$$

17. Let position vector of B is $\mathbf{r}$.



$\therefore \quad \mathbf{a} = \frac{2\mathbf{r} + 3(\mathbf{a} + 2\mathbf{b})}{2 + 3}$

$\Rightarrow \quad 5\mathbf{a} = 2\mathbf{r} + 3\mathbf{a} + 6\mathbf{b}$

$\Rightarrow \quad 2\mathbf{r} = 2\mathbf{a} - 6\mathbf{b}$

$\therefore \quad \mathbf{r} = \mathbf{a} - 3\mathbf{b}$

18. $\mathbf{DA} + \mathbf{DB} + \mathbf{DC} + \mathbf{AE} + \mathbf{BE} + \mathbf{CE}$

$$= (\mathbf{DA} + \mathbf{AE}) + (\mathbf{DB} + \mathbf{BE}) + (\mathbf{DC} + \mathbf{CE})$$

$$= \mathbf{DE} + \mathbf{DE} + \mathbf{DE} = 3\,\mathbf{DE}$$

19. For collinearity, $\cos x\hat{\mathbf{i}} + \sin x\hat{\mathbf{j}} = \lambda(x\hat{\mathbf{i}} + \sin x\hat{\mathbf{j}})$

$\Rightarrow \quad \cos x = x$

Let $\quad f(x) = \cos x - x$

$\Rightarrow \quad f'(x) = -\sin x - 1 < 0$

$f(x)$ is decreasing function and for $x \geq \frac{\pi}{3}$, $f(x) < 0$ and

for $\frac{\pi}{6} < x < \frac{\pi}{3}$, $f(x) > 0$.

Hence, unique solution exist.

20. To determine the angle between $\mathbf{a}$ and $\mathbf{b}$ we can use the formula $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

It is given that $|\mathbf{a}| = \sqrt{3}$, $|\mathbf{b}| = 2$ and $\mathbf{a}\cdot\mathbf{b} = \sqrt{6}$

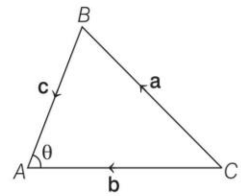
Let θ be the required angle, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\sqrt{6}}{\sqrt{3}\times 2} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\cos\frac{\pi}{4}\right) = \frac{\pi}{4}$

Hence, the angle between the given vectors $\mathbf{a}$ and $\mathbf{b}$ is $\pi/4$.

21.



$$|\mathbf{a}| = 8, |\mathbf{b}| = 7, |\mathbf{c}| = 10$$

$$\cos\theta = \frac{|\mathbf{b}|^2 + |\mathbf{c}|^2 - |\mathbf{a}|^2}{2|\mathbf{b}||\mathbf{c}|} = \frac{17}{28}$$

Projection of $\mathbf{c}$ on $\mathbf{b} = |\mathbf{c}|\cos\theta$

$$= 10 \times \frac{17}{28} = \frac{85}{14}$$

22. The given vectors are $\mathbf{a} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

Now, $\quad (\mathbf{a} + \lambda\mathbf{b}) \perp \mathbf{c} \quad$ (given)

$\Rightarrow \quad (\mathbf{a} + \lambda\mathbf{b})\cdot\mathbf{c} = 0$

($\because$ scalar product of two perpendicular vectors is zero)

$\Rightarrow \quad [(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \lambda(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})]\cdot(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) = 0$

$\Rightarrow \quad [(2-\lambda)\hat{\mathbf{i}} + (2 + 2\lambda)\hat{\mathbf{j}} + (3 + \lambda)\hat{\mathbf{k}}]\cdot(3\hat{\mathbf{i}} + \hat{\mathbf{j}}) = 0$

$\Rightarrow \quad (2-\lambda)3 + (2 + 2\lambda)1 + (3 + \lambda)0 = 0$

$\Rightarrow \quad 6 - 3\lambda + 2 + 2\lambda = 0 \Rightarrow 8 - \lambda = 0 \Rightarrow \lambda = 8$

Hence, the required value of λ is 8.

23. As $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors, so use $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$ and expand $(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c})$ to get the value of $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$.

Given, $\quad |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$ and $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

We have, $\quad (\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c}) = 0$

$\Rightarrow \mathbf{a}\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c}) + \mathbf{b}\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c}) + \mathbf{c}\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c}) = 0$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{c} = 0$

$\Rightarrow \quad |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

($\because \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$)

$\Rightarrow \quad 1 + 1 + 1 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

($\because |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$)

$\Rightarrow \quad 3 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} = -\frac{3}{2}$

24. Here, we have to find $\angle ABC$ i.e. angle between **BA** and **BC**, so first of all, we have to calculate both these vectors after that we can find the angle between them by $\cos\theta = \dfrac{(\mathbf{BA})\cdot(\mathbf{BC})}{|\mathbf{BA}||\mathbf{BC}|}$

We are given the points $A(1,2,3)$, $B(-1,0,0)$ and $C(0,1,2)$. Also, it is given that $\angle ABC$ is the angle between the vectors **BA** and **BC**.

Here, $\mathbf{BA}$ = PV of A – PV of B

$$= (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (-\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}})$$
$$= [\hat{\mathbf{i}} - (-\hat{\mathbf{i}}) + (2\hat{\mathbf{j}} - 0) + (3\hat{\mathbf{k}} - 0)] = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$
$$|\mathbf{BA}| = \sqrt{(2)^2 + (2)^2 + (3)^2} = \sqrt{4+4+9} = \sqrt{17}$$

$\mathbf{BC}$ = PV of C – PV of B

$$= (0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) - (-\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}})$$
$$= [0 - (-\hat{\mathbf{i}}) + (1\hat{\mathbf{j}} - 0) + (2\hat{\mathbf{k}} - 0)]$$
$$= \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$
$$|\mathbf{BC}| = \sqrt{(1)^2 + (1)^2 + (2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

Now, $\mathbf{BA}\cdot\mathbf{BC} = (2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$$= 2\times1 + 2\times1 + 3\times2 = 10$$
$$\cos\theta = \frac{\mathbf{BA}\cdot\mathbf{BC}}{|\mathbf{BA}||\mathbf{BC}|} \Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17}\sqrt{6}}$$
$$\Rightarrow \quad \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

25. Given two vectors lie in xy-plane. So, a vector coplanar with them is

$$\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Since, $\quad \mathbf{a} \perp (\hat{\mathbf{i}} - \hat{\mathbf{j}})$

$\Rightarrow \quad (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\cdot(\hat{\mathbf{i}} - \hat{\mathbf{j}}) = 0$

$\Rightarrow \quad x - y = 0 \Rightarrow x = y$

$\therefore \quad \mathbf{a} = x\hat{\mathbf{i}} + x\hat{\mathbf{j}}$

and $\quad |\mathbf{a}| = \sqrt{x^2 + x^2} = x\sqrt{2}$

$\therefore$ Required unit vector $= \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{x(\hat{\mathbf{i}} + \hat{\mathbf{j}})}{x\sqrt{2}} = \dfrac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

26. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\mathbf{c} = \lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Now, $\mathbf{b} + \mathbf{c} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}} + \lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3$

$$= (2+\lambda)\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$\therefore \quad |\mathbf{b} + \mathbf{c}| = \sqrt{(2+\lambda)^2 + (6)^2 + (-2)^2}$

$$= \sqrt{4 + \lambda^2 + 4\lambda + 36 + 4} = \sqrt{\lambda^2 + 4\lambda + 44}$$

The unit vector along $(\mathbf{b} + \mathbf{c})$,

i.e. $\quad \dfrac{\mathbf{b}+\mathbf{c}}{|\mathbf{b}+\mathbf{c}|} = \dfrac{(2+\lambda)\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{\lambda^2 + 4\lambda + 44}}$

Scalar product of $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ with this unit vector is 1.

$\therefore \quad (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot\dfrac{\mathbf{b}+\mathbf{c}}{|\mathbf{b}+\mathbf{c}|} = 1$

$\Rightarrow \quad (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot\dfrac{(2+\lambda)\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$

$\Rightarrow \quad \dfrac{1(2+\lambda) + 1(6) + 1(-2)}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$

$\Rightarrow \quad \dfrac{(2+\lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$

$\Rightarrow \quad \lambda + 6 = \sqrt{\lambda^2 + 4\lambda + 44}$

$\Rightarrow \quad (\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$

$\Rightarrow \quad \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

$\Rightarrow \quad 8\lambda = 8 \Rightarrow \lambda = 1$

Hence, the value of λ is 1.

27. Let **a** and **b** be two unit vectors and θ be the angle between them.

Then, $\quad |\mathbf{a}| = |\mathbf{b}| = 1$

Now, $(\mathbf{a} + \mathbf{b})$ is a unit vector, if

$|\mathbf{a} + \mathbf{b}| = 1 \Rightarrow (\mathbf{a} + \mathbf{b})^2 = 1$

$\Rightarrow \quad (\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) = 1 \Rightarrow \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} = 1$

$\Rightarrow \quad |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b} = 1 \qquad (\because \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a})$

$\Rightarrow \quad 1^2 + 1^2 + 2\mathbf{a}\cdot\mathbf{b} = 1$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = -\dfrac{1}{2} \Rightarrow |\mathbf{a}||\mathbf{b}|\cos\theta = -\dfrac{1}{2}$

$\Rightarrow \quad 1\times1\times\cos\theta = -\dfrac{1}{2}$

$\Rightarrow \quad \cos\theta = -\dfrac{1}{2} \Rightarrow \theta = \dfrac{2\pi}{3}$

28. Since, position vectors of A, B, C are $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$, respectively.

Now, $\mathbf{AC} = (a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = (a-2)\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$

and $\quad \mathbf{BC} = (a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}) = (a-1)\hat{\mathbf{i}} + 6\hat{\mathbf{k}}$

Since, the ΔABC is right angled at C, then

$$\mathbf{AC}\cdot\mathbf{BC} = 0$$

$\Rightarrow \quad \{(a-2)\hat{\mathbf{i}} - 2\hat{\mathbf{j}}\}\cdot\{(a-1)\hat{\mathbf{i}} + 6\hat{\mathbf{k}}\} = 0$

$\Rightarrow \quad (a-2)(a-1) = 0$

$\therefore \quad a = 1$ and $a = 2$

29. We have, $\lambda\mathbf{b} + \mathbf{c} = \lambda(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) + (\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})$

$$= (\lambda+1)\hat{\mathbf{i}} + (\lambda+3)\hat{\mathbf{j}} - (2\lambda+1)\hat{\mathbf{k}}$$

Since, $\quad \mathbf{a} \perp (\lambda\mathbf{b} + \mathbf{c})$, $\mathbf{a}\cdot(\lambda\mathbf{b} + \mathbf{c}) = 0$

$\Rightarrow \quad (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot[(\lambda+1)\hat{\mathbf{i}} + (\lambda+3)\hat{\mathbf{j}} - (2\lambda+1)\hat{\mathbf{k}}] = 0$

$\Rightarrow \quad 2(\lambda+1) - (\lambda+3) - (2\lambda+1) = 0$

$\therefore \quad \lambda = -2$

30. Projection of a vector **a** and **b** is

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{1+4+4}} = \frac{2}{3}$$

31. We have,

$$(\sqrt{3}\mathbf{a} - \mathbf{b})^2 = 3\mathbf{a}^2 + \mathbf{b}^2 - 2\sqrt{3}\mathbf{a}\cdot\mathbf{b}$$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \dfrac{\sqrt{3}}{2} \Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ$

32. Let $\mathbf{OA} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{OB} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$,

$\mathbf{OC} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{k}}$ and $\mathbf{OD} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Now, $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$

$= \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

$\mathbf{CD} = \mathbf{OD} - \mathbf{OC} = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - 3\hat{\mathbf{k}})$

$= \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

$\therefore$ Projection of **AB** along **CD** is $= \dfrac{\mathbf{AB}\cdot\mathbf{CD}}{|\mathbf{CD}|}$

$$= \frac{(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})}{\sqrt{1+4+16}}$$

$$= \frac{1+4+16}{\sqrt{21}} = \frac{21}{\sqrt{21}} = \sqrt{21}$$

33. $|\mathbf{u}+\mathbf{v}+\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 + 2(\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u})$

$\Rightarrow \quad 0 = 9 + 16 + 25 + 2(\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u})$

$\Rightarrow \quad \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u} = -25$

34. Since, the given vectors are mutually orthogonal, therefore

$\mathbf{a}\cdot\mathbf{b} = 2 - 4 + 2 = 0$

$\mathbf{a}\cdot\mathbf{c} = \lambda - 1 + 2\mu = 0 \quad$ ...(i)

and $\quad \mathbf{b}\cdot\mathbf{c} = 2\lambda + 4 + \mu = 0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$\mu = 2$ and $\lambda = -3$

Hence, $\quad (\lambda, \mu) = (-3, 2)$

35. $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|^2 = 1 + 1 - 2\cos 60^\circ = 2 - 1$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}| = 1$

36. **Given**

(i) A parallelogram *ABCD* such that $\mathbf{AB} = \mathbf{q}$ and $\mathbf{AD} = \mathbf{p}$.

(ii) The altitude from vertex *B* to side *AD* coincides with a vector **r**.

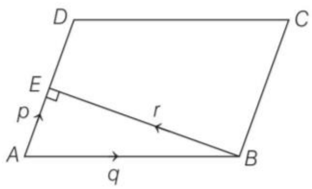
To find The vector **r** in terms of **p** and **q**.

Let *E* be the foot of perpendicular from *B* to side *AD*.

AE = Projection of vector **q** on $\mathbf{p} = \mathbf{q}\cdot\hat{\mathbf{p}} = \dfrac{\mathbf{q}\cdot\mathbf{p}}{|\mathbf{p}|}$

AE = Vector along *AE* of length *AE*

$$= |\mathbf{AE}|\,\hat{\mathbf{AE}} = \left(\frac{\mathbf{q}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\hat{\mathbf{p}} = \frac{(\mathbf{q}\cdot\mathbf{p})\,\mathbf{p}}{|\mathbf{p}|^2}$$



Now, applying triangles law in ΔABE, we get

$\mathbf{AB} + \mathbf{BE} = \mathbf{AE}$

$\Rightarrow \quad \mathbf{q} + \mathbf{r} = \dfrac{(\mathbf{q}\cdot\mathbf{p})\,\mathbf{p}}{|\mathbf{p}|^2}$

$\Rightarrow \quad \mathbf{r} = \dfrac{(\mathbf{q}\cdot\mathbf{p})\,\mathbf{p}}{|\mathbf{p}|^2} - \mathbf{q}$

$\Rightarrow \quad \mathbf{r} = -\mathbf{q} + \left(\dfrac{\mathbf{q}\cdot\mathbf{p}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$

37. Given that, $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{a} - \lambda\mathbf{b}$.

$\therefore \quad (\mathbf{a}+\lambda\mathbf{b})\cdot(\mathbf{a}-\lambda\mathbf{b}) = 0$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b}\lambda + \lambda\mathbf{b}\cdot\mathbf{a} - \lambda^2\mathbf{b}\cdot\mathbf{b} = 0$

$\Rightarrow \quad |\mathbf{a}|^2 - \lambda^2|\mathbf{b}|^2 = 0$

$\Rightarrow \quad \lambda^2 = \dfrac{|\mathbf{a}|^2}{|\mathbf{b}|^2} \Rightarrow \lambda = \dfrac{|\mathbf{a}|}{|\mathbf{b}|} = \dfrac{3}{4}$

38. $\because \quad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\dfrac{5\pi}{6} = -\dfrac{|\mathbf{a}||\mathbf{b}|\sqrt{3}}{2}$

$\therefore \quad -\dfrac{6}{\sqrt{3}} = -\dfrac{|\mathbf{a}||\mathbf{b}|\sqrt{3}}{2|\mathbf{b}|} \quad$ (given condition)

$\Rightarrow \quad |\mathbf{a}| = \dfrac{6\times 2}{3} = 4$

39. Let $\quad \mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\therefore \quad \mathbf{a}\cdot\hat{\mathbf{i}} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{i}} = x$

$\mathbf{a}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}) = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}) = x + y$

and $\mathbf{a}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}) = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}) = x + y + z$

$\because$ Given that, $\mathbf{a}\cdot\hat{\mathbf{i}} = \mathbf{a}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}) = \mathbf{a}\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

$\Rightarrow \quad x = x + y = x + y + z$

Take $\quad x = x + y \Rightarrow y = 0$

and $\quad x + y = x + y + z \Rightarrow z = 0$

$\Rightarrow x$ has any real values.

Now, take $x = 1 \;\therefore\; \mathbf{a} = \hat{\mathbf{i}}$

40. $\cos\dfrac{\pi}{3} = \dfrac{(\hat{\mathbf{i}}+\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+\hat{\mathbf{j}}+a\hat{\mathbf{k}})}{\sqrt{2}\sqrt{1+1+a^2}}$

$\Rightarrow \quad \dfrac{1}{2} = \dfrac{1+a}{\sqrt{2}\sqrt{2+a^2}} \Rightarrow \dfrac{1}{4} = \dfrac{(1+a)^2}{2(2+a^2)}$

$\Rightarrow \quad 2 + a^2 = 2(1 + a^2 + 2a)$

$\Rightarrow \quad a^2 + 4a = 0 \Rightarrow a = 0, -4$

41. $\because \mathbf{F}_1 = \dfrac{5(6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})}{7}$, $\mathbf{F}_2 = \dfrac{3(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{7}$

$\mathbf{F}_3 = \dfrac{1\,(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})}{7}$

and $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$= \dfrac{1}{7}(30\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 15\hat{\mathbf{k}} + 9\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 18\hat{\mathbf{k}} + 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$

$= \dfrac{1}{7}(41\hat{\mathbf{i}} + \hat{\mathbf{j}} + 27\hat{\mathbf{k}})$

and $\quad \mathbf{AB} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}} - 2\hat{\mathbf{i}} + \hat{\mathbf{k}} + 3\hat{\mathbf{k}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$

$\therefore$ Work done $= \dfrac{1}{7}[41\hat{\mathbf{i}} + \hat{\mathbf{j}} + 27\hat{\mathbf{k}}]\cdot[3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}]$

$= \dfrac{1}{7}[123 + 108] = 33$ units

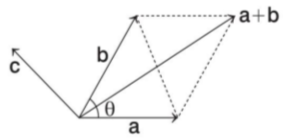
42. $\because \sum_{i=1}^{n} \mathbf{a}_i = \mathbf{0}$

$\therefore \quad \left(\sum_{i=1}^{n} \mathbf{a}_i\right)\left(\sum_{i=1}^{n} \mathbf{a}_j\right) = \sum_{i=1}^{n} |\mathbf{a}_i|^2 + 2\sum\sum_{1 \le i < j \le n} \mathbf{a}_i \cdot \mathbf{a}_j$

$\Rightarrow \quad 0 = n + 2 \sum\sum_{1 \le i < j \le n} \mathbf{a}_i \cdot \mathbf{a}_j$

$\therefore \quad \sum\sum_{1 \le i < j \le n} \mathbf{a}_i \cdot \mathbf{a}_j = -\frac{n}{2}$

43. According to given information, we have the following figure.



Clearly, projection of **b** on **a** $= \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}$

$= \frac{(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}})\,(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}})}{\sqrt{1^2 + 1^2 + (\sqrt{2})^2}}$

$= \frac{b_1 + b_2 + 2}{\sqrt{4}} = \frac{b_1 + b_2 + 2}{2}$

But projection of **b** on **a** $= |\mathbf{a}|$

$\therefore \quad \frac{b_1 + b_2 + 2}{2} = \sqrt{1^2 + 1^2 + (\sqrt{2})^2}$

$\Rightarrow \quad \frac{b_1 + b_2 + 2}{2} = 2 \Rightarrow b_1 + b_2 = 2$ ...(i)

Now, $\mathbf{a} + \mathbf{b} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}}) + (b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}})$

$= (b_1 + 1)\hat{\mathbf{i}} + (b_2 + 1)\hat{\mathbf{j}} + 2\sqrt{2}\hat{\mathbf{k}}$

$\because \ (\mathbf{a} + \mathbf{b}) \perp \mathbf{c}$, therefore $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = 0$

$\Rightarrow \{ (b_1 + 1)\hat{\mathbf{i}} + (b_2 + 1)\hat{\mathbf{j}} + 2\sqrt{2}\hat{\mathbf{k}}\}\,(5\hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}}) = 0$

$\Rightarrow \quad 5(b_1 + 1) + 1(b_2 + 1) + 2\sqrt{2}(\sqrt{2}) = 0$

$\Rightarrow \quad 5b_1 + b_2 = -10$ ...(ii)

From Eqs. (i) and (ii),

$b_1 = -3$ and $b_2 = 5$

$\Rightarrow \quad \mathbf{b} = -3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{b}| = \sqrt{(-3)^2 + (5)^2 + (\sqrt{2})^2} = \sqrt{36} = 6$

44. Given, $|\mathbf{a} + \mathbf{b}| < 1$

$\Rightarrow \quad \sqrt{1 + 1 + 2\cos 2\alpha} < 1$

$\Rightarrow \quad \sqrt{2(1 + \cos 2\alpha)} < 1$

$\Rightarrow \quad \sqrt{4\cos^2\alpha} < 1 \Rightarrow |\cos\alpha| < \frac{1}{2}$

$\Rightarrow \quad \frac{\pi}{3} < \alpha < \frac{2\pi}{3} \quad (\because 0 \le \alpha \le \pi)$

45. Given that, $|\mathbf{a}| = 2\sqrt{2}, |\mathbf{b}| = 3$

The longer vectors is $5\mathbf{a} + 2\mathbf{b} + \mathbf{a} - 3\mathbf{b} = 6\mathbf{a} - \mathbf{b}$

Length of one diagonal $= |6\mathbf{a} - \mathbf{b}|$

$= \sqrt{36\mathbf{a}^2 + \mathbf{b}^2 - 2 \times 6|\mathbf{a}||\mathbf{b}|\cos 45°}$

$= \sqrt{36 \times 8 + 9 - 12 \times 2\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}}}$

$= \sqrt{288 + 9 - 12 \times 6} = \sqrt{225} = 15$

Other diagonal is $4\mathbf{a} + 5\mathbf{b}$.

Its length $= \sqrt{16 \times 8 + 25 \times 9 + 40 \times 6} = \sqrt{593}$

46. Let the unit vector $\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}$ is perpendicular to $\hat{\mathbf{i}} - \hat{\mathbf{j}}$, then we get

$$\frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \cdot (\hat{\mathbf{i}} - \hat{\mathbf{j}})}{\sqrt{2}} = \frac{1 - 1}{\sqrt{2}} = 0$$

$\therefore \ \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}$ is the required unit vector.

47. Clearly, angle between **a** and **b** $= \frac{\pi}{2} \Rightarrow \mathbf{a} \cdot \mathbf{b} = 0$

$\therefore \quad |\mathbf{a} + \mathbf{b}|^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b} = 1 + 1 + 0 = 2$

$\Rightarrow \quad |\mathbf{a} + \mathbf{b}| = \sqrt{2}$

48. $\because$ **a**, **b** and **c** are pth, qth and rth terms of HP, respectively.

$$\frac{1}{a} = A + (p - 1)D, \ \frac{1}{b} = A + (q - 1)D$$

and $\quad \frac{1}{c} = A + (r - 1)D$

$\therefore \quad q - r = \frac{c - b}{bcD}, \ r - p = \frac{a - c}{acD}$

and $\quad q - r = \frac{b - a}{abD}$

$\Rightarrow \quad \frac{(q - r)}{a} + \frac{(r - p)}{b} + \frac{(p - q)}{c} = 0$

$\Rightarrow \quad \mathbf{u} \cdot \mathbf{v} = 0$

49. $\because$ **p**, **q** and **r** are reciprocal vectors of **a**, **b**, **c** respectively.

$\therefore \ \mathbf{p} \cdot \mathbf{a} = 1, \mathbf{p} \cdot \mathbf{b} = 0 = \mathbf{p} \cdot \mathbf{c}$ etc.

$\therefore \ (l\mathbf{a} + m\mathbf{b} + n\mathbf{c}) \cdot (l\mathbf{p} + m\mathbf{q} + n\mathbf{r}) = l^2 + m^2 + n^2$

50. Let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ where **a**, **b** and **c** are all unit vectors

$\therefore \quad \mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{ab}$

$\Rightarrow \quad 2\mathbf{a} \cdot \mathbf{b} = -1$

Again, $|\mathbf{a} - \mathbf{b}|^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b} = 1 + 1 - (-1) = 3$

$\therefore \quad |\mathbf{a} - \mathbf{b}| = \sqrt{3}$

51. Since, $|\mathbf{a} + \mathbf{b}| < |\mathbf{a} - \mathbf{b}|$

$$|\mathbf{a} + \mathbf{b}|^2 < |\mathbf{a} - \mathbf{b}|^2$$

$\Rightarrow \quad \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b} < \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b}$

$\Rightarrow \quad 4\mathbf{a} \cdot \mathbf{b} < 0$

$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} < 0$

$\Rightarrow \cos\theta < 0 \Rightarrow \theta$ is obtuse.

52. Let $\mathbf{OA} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$ and $\mathbf{OB} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$,

since $1 = \mathbf{OA}.\,\hat{\mathbf{i}} = x_1$ and $-2 = \mathbf{OB}.\,\hat{\mathbf{i}} = x_2$

Moreover, $y_1 = x_1^2$ and $y_2 = x_2^2 = 4$, so

$$\mathbf{OA} = \hat{\mathbf{i}} + \hat{\mathbf{j}} \text{ and } \mathbf{OB} = -2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}.$$

Hence, $|2\mathbf{OA} - 3\mathbf{OB}| = |8\hat{\mathbf{i}} - 10\hat{\mathbf{j}}| = \sqrt{164} = 2\sqrt{41}$

53. $(\mathbf{x}-\hat{\mathbf{a}})\cdot(\mathbf{x}+\hat{\mathbf{a}})=8 \Rightarrow \mathbf{x}=3$

To determine $(\mathbf{x}-\hat{\mathbf{a}})$, we have

$(\mathbf{x}-\hat{\mathbf{a}})=9+1-4=6$ so that

$|\mathbf{x}-\hat{\mathbf{a}}|=\sqrt{6}$ and similarly $|\mathbf{x}+\hat{\mathbf{a}}|=\sqrt{14}$

Then, $\quad (\mathbf{x}-\hat{\mathbf{a}})\cdot(\mathbf{x}+\hat{\mathbf{a}})=\sqrt{14}\times\sqrt{6}\cos\theta$

$\Rightarrow \quad 8=\sqrt{14}\times\sqrt{6}\cos\theta$

$\Rightarrow \quad \cos\theta=\dfrac{4}{\sqrt{21}}$

54. Given vectors, $\mathbf{a}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $\mathbf{c}$, such that $\mathbf{b}\times\mathbf{c}=\mathbf{b}\times\mathbf{a}$

$\Rightarrow \quad \mathbf{b}\times(\mathbf{c}-\mathbf{a})=\mathbf{0}$

$\Rightarrow \quad \mathbf{b}\,||\,\mathbf{c}-\mathbf{a}$

$\Rightarrow \quad \mathbf{c}-\mathbf{a}=\lambda\mathbf{b} \Rightarrow \mathbf{c}=\mathbf{a}+\lambda\mathbf{b}$

$\Rightarrow \quad \mathbf{c}=(1+\lambda)\hat{\mathbf{i}}-(2+\lambda)\hat{\mathbf{j}}+(1+\lambda)\hat{\mathbf{k}}$

$\because \quad \mathbf{c}\cdot\mathbf{a}=0$ (given)

$\Rightarrow \quad (1+\lambda)+2(2+\lambda)+(1+\lambda)=0$

$\Rightarrow \quad 4\lambda+6=0$

$\Rightarrow \quad \lambda=-\dfrac{3}{2}$

$\therefore \quad \mathbf{c}=-\dfrac{1}{2}\hat{\mathbf{i}}-\dfrac{1}{2}\hat{\mathbf{j}}-\dfrac{1}{2}\hat{\mathbf{k}}$

So $\quad \mathbf{c}\cdot\mathbf{b}=-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{2}=-\dfrac{1}{2}$

Hence, option (c) is correct.

55. Since, $(2\mathbf{u}\times3\mathbf{v})$ is a unit vector.

$\Rightarrow \quad |2\mathbf{u}\times3\mathbf{v}|=1$

$\Rightarrow \quad 6|\mathbf{u}||\mathbf{v}||\sin\theta|=1$

$\Rightarrow \quad \sin\theta=\dfrac{1}{6} \qquad [\because |\mathbf{u}|=|\mathbf{v}|=1]$

Since, θ is an acute angle, then there is exactly one value of θ for which $(2\mathbf{u}\times3\mathbf{v})$ is a unit vector.

56. We have, $|\mathbf{a}|=|\mathbf{b}|, |\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|$ and $\mathbf{a}\cdot\mathbf{b}=0$

$|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin 90°$

$|\mathbf{a}|=|\mathbf{a}||\mathbf{b}|$

$\Rightarrow \quad |\mathbf{b}|=1=|\mathbf{a}|=|\mathbf{a}\times\mathbf{b}|$

$|\mathbf{a}+\mathbf{b}+(\mathbf{a}\times\mathbf{b})|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{a}\times\mathbf{b}|^2$
$\qquad +2\,(\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})+\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}))$

$\Rightarrow \quad |\mathbf{a}+\mathbf{b}+(\mathbf{a}\times\mathbf{b})|^2=1+1+1+0=3$

Now, angle bwtween $(\mathbf{a}+\mathbf{b}+(\mathbf{a}\times\mathbf{b}))$ and $\mathbf{a}$ is

$$\cos\theta=\frac{(\mathbf{a}+\mathbf{b}+(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}}{|\mathbf{a}+\mathbf{b}+(\mathbf{a}\times\mathbf{b})||\mathbf{a}|}$$

$$\cos\theta=\frac{|\mathbf{a}|^2+\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})}{\sqrt{3}\times1}$$

$$\cos\theta=\frac{1+0+0}{\sqrt{3}}$$

$$\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

57. Given three unit vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that

$$\mathbf{a}+\mathbf{b}+\mathbf{c}=0 \qquad \text{...(i)}$$

If we do dot product in Eq. (i) with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then we get relations

$$\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=0$$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}=-1,\ \mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{c}=-1$

and $\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b}=-1$

$\therefore$ On adding above three obtained relations, we get

$2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})=-3 \qquad (\because \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a})$

$\Rightarrow \lambda=\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=-\dfrac{3}{2}$

If we do cross product in Eq. (i) with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then we get relation

$$0+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$$

$\Rightarrow \quad \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$

and $\quad \mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}=\mathbf{0}$

and $\quad \mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}=0$

$\therefore \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=3(\mathbf{a}\times\mathbf{b})=3(\mathbf{b}\times\mathbf{c})=3(\mathbf{c}\times\mathbf{a})$

$\therefore \quad \mathbf{d}=3\mathbf{a}\times\mathbf{b}=3\mathbf{b}\times\mathbf{c}=3\mathbf{c}\times\mathbf{a}$

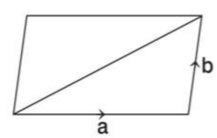
From the given options the ordered pair,

$$(\lambda,\mathbf{d})=\left(-\frac{3}{2}, 3\mathbf{a}\times\mathbf{b}\right)$$

58. Adjacent sides of a parallelogram are given as

$$\mathbf{a}=2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}} \text{ and } \mathbf{b}=\hat{\mathbf{i}}-2\hat{\mathbf{j}}-3\hat{\mathbf{k}}.$$

Then, the diagonal of a parallelogram is given by $\mathbf{v}=\mathbf{a}+\mathbf{b}$.



($\because$ from the figure, it is clear that resultant of adjacent sides of a parallelogram is given by the diagonal)

$\therefore \quad \mathbf{v}=2\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}}+\hat{\mathbf{i}}-2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

$\qquad =(2+1)\hat{\mathbf{i}}+(-4-2)\hat{\mathbf{j}}+(5-3)\hat{\mathbf{k}}=3\hat{\mathbf{i}}-6\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

Comparing with $\mathbf{X}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, we get

$$x=3,\ y=-6,\ z=2$$

$\therefore \quad |\mathbf{v}|=\sqrt{x^2+y^2+z^2}$

$\qquad =\sqrt{(3)^2+(-6)^2+(2)^2}$

$\qquad =\sqrt{9+36+4}=\sqrt{49}=7$

Thus, the unit vector parallel to the diagonal is

$$\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{3\hat{\mathbf{i}}-6\hat{\mathbf{j}}+2\hat{\mathbf{k}}}{7}=\frac{3}{7}\hat{\mathbf{i}}-\frac{6}{7}\hat{\mathbf{j}}+\frac{2}{7}\hat{\mathbf{k}}$$

Also, area of parallelogram $ABCD$, $|\mathbf{a}\times\mathbf{b}|=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & -4 & 5\\ 1 & -2 & -3\end{vmatrix}$

$=|\hat{\mathbf{i}}\,(12+10)-\hat{\mathbf{j}}(-6-5)+\hat{\mathbf{k}}(-4+4)|=|22\hat{\mathbf{i}}+11\hat{\mathbf{j}}+0\hat{\mathbf{k}}|$

$=\sqrt{(22)^2+(11)^2+0^2}=\sqrt{(11)^2(2^2+1)^2}=11\sqrt{5}$ sq units

59. The vector which is perpendicular to both **a** and **b** most be parallel **a** × **b**. Now,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 4 & 2\\ 3 & -2 & 7\end{vmatrix} = \hat{\mathbf{i}}(28+4)-\hat{\mathbf{j}}(7-6)+\hat{\mathbf{k}}(-2-12)$$

$$=32\hat{\mathbf{i}}-\hat{\mathbf{j}}-14\hat{\mathbf{k}}$$

Let $\mathbf{d}=\lambda(\mathbf{a}\times\mathbf{b})=\lambda(32\hat{\mathbf{i}}-\hat{\mathbf{j}}-14\hat{\mathbf{k}})$

Also, $\mathbf{c}\cdot\mathbf{d}=15 \Rightarrow (2\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})\cdot\lambda(32\hat{\mathbf{i}}-\hat{\mathbf{j}}-14\hat{\mathbf{k}})=15$

$\Rightarrow \quad 2\times(32\lambda)+(-1)\times(-\lambda)+4\times(-14\lambda)=15$

$\Rightarrow \quad 64\lambda+\lambda-56\lambda=15$

$\Rightarrow \quad 9\lambda=15$

$\Rightarrow \quad \lambda=\frac{15}{9}=\frac{5}{3}$

∴ Required vector, $\mathbf{d}=\frac{5}{3}(32\hat{\mathbf{i}}-\hat{\mathbf{j}}-14\hat{\mathbf{k}})$.

60. Using the formula $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|\cdot|\mathbf{b}||\sin\theta|$, we get

$\theta=\pm\frac{\pi}{6}$

Therefore, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta=8\times3\times3\times\frac{\sqrt{3}}{2}=12\sqrt{3}$

61. Given vectors are $\mathbf{a}=3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+x\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$

$$\therefore \quad \mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 2 & x\\ 1 & -1 & 1\end{vmatrix}$$

$$=\hat{\mathbf{i}}(2+x)-\hat{\mathbf{j}}(3-x)+\hat{\mathbf{k}}(-3-2)$$

$$=(x+2)\hat{\mathbf{i}}+(x-3)\hat{\mathbf{j}}-5\hat{\mathbf{k}}$$

$$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}|=\sqrt{(x+2)^2+(x-3)^2+25}$$

$$=\sqrt{2x^2-2x+4+9+25}$$

$$=\sqrt{2\left(x^2-x+\frac{1}{4}\right)-\frac{1}{2}+38}$$

$$=\sqrt{2\left(x-\frac{1}{2}\right)^2+\frac{75}{2}}$$

So, $\quad |\mathbf{a}\times\mathbf{b}|\ge\sqrt{\frac{75}{2}} \quad$ [at $x=\frac{1}{2}$, $|\mathbf{a}\times\mathbf{b}|$ is minimum]

$\Rightarrow \quad r\ge5\sqrt{\frac{3}{2}}$

62. Let position vectors of **A**, **B** and **C** are;

$\mathbf{OA}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, $\mathbf{OB}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

and $\quad \mathbf{OC}=4\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\hat{\mathbf{k}}$

Now, $\quad \mathbf{AB}=\mathbf{OB}-\mathbf{OA}=(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$\quad =\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}}$

and $\quad \mathbf{AC}=\mathbf{OC}-\mathbf{OA}=(4\hat{\mathbf{i}}+5\hat{\mathbf{j}}-\hat{\mathbf{k}})-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$\quad =3\hat{\mathbf{i}}+3\hat{\mathbf{j}}-4\hat{\mathbf{k}}$

∴ Area of $\Delta ABC=\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$

$$=\frac{1}{2}|(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})\times(3\hat{\mathbf{i}}+3\hat{\mathbf{j}}-4\hat{\mathbf{k}})|$$

$$=\frac{1}{2}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{matrix}\right\|$$

$$=\frac{1}{2}|\hat{\mathbf{i}}(12-3)-\hat{\mathbf{j}}(-4-3)+\hat{\mathbf{k}}(3+9)|$$

$$=\frac{1}{2}|9\hat{\mathbf{i}}+7\hat{\mathbf{j}}+12\hat{\mathbf{k}}|$$

$$=\frac{1}{2}\sqrt{81+49+144}=\frac{1}{2}\sqrt{274}=\sqrt{\frac{137}{2}}$$

63. Given vectors $\vec{\alpha}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\vec{\beta}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$

and $\vec{\beta}=\vec{\beta}_1-\vec{\beta}_2$ such that $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to α.

So, $\quad \vec{\beta}_1=\lambda\alpha=\lambda(3\hat{\mathbf{i}}+\hat{\mathbf{j}})$

Now, $\quad \vec{\beta}_2=\vec{\beta}_1-\vec{\beta}=\lambda(3\hat{\mathbf{i}}+\hat{\mathbf{j}})-(2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}})$

$\quad =(3\lambda-2)\hat{\mathbf{i}}+(\lambda+1)\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

∵ $\vec{\beta}_2$ is perpendicular to α, so $\vec{\beta}_2\cdot\alpha=0$

[since if non-zero vectors **a** and **b** are perpendicular to each other, then $\mathbf{a}\cdot\mathbf{b}=0$]

$\therefore \quad (3\lambda-2)(3)+(\lambda+1)(1)=0$

$\Rightarrow \quad 9\lambda-6+\lambda+1=0$

$\Rightarrow \quad 10\lambda=5$

$\Rightarrow \quad \lambda=\frac{1}{2}$

So, $\quad \vec{\beta}_1=\frac{3}{2}\hat{\mathbf{i}}+\frac{1}{2}\hat{\mathbf{j}}$

and $\quad \vec{\beta}_2=\left(\frac{3}{2}-2\right)\hat{\mathbf{i}}+\left(\frac{1}{2}+1\right)\hat{\mathbf{j}}-3\hat{\mathbf{k}}=-\frac{1}{2}\hat{\mathbf{i}}+\frac{3}{2}\hat{\mathbf{j}}-3\hat{\mathbf{k}}$

$$\therefore \quad \vec{\beta}_1\times\vec{\beta}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{3}{2} & \frac{1}{2} & 0\\ \frac{-1}{2} & \frac{3}{2} & -3\end{vmatrix}$$

$$=\hat{\mathbf{i}}\left(-\frac{3}{2}-0\right)-\hat{\mathbf{j}}\left(-\frac{9}{2}-0\right)+\hat{\mathbf{k}}\left(\frac{9}{4}+\frac{1}{4}\right)$$

$$=-\frac{3}{2}\hat{\mathbf{i}}+\frac{9}{2}\hat{\mathbf{j}}+\frac{5}{2}\hat{\mathbf{k}}=\frac{1}{2}(-3\hat{\mathbf{i}}+9\hat{\mathbf{j}}+5\hat{\mathbf{k}})$$

64. Let $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$

Now, $(\mathbf{a}\times\hat{\mathbf{i}})=(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})\times\hat{\mathbf{i}}$

$$=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_1 & a_2 & a_3\\ 1 & 0 & 0\end{vmatrix}=-\hat{\mathbf{j}}(-a_3)+\hat{\mathbf{k}}(-a_2)$$

$$=a_3\hat{\mathbf{j}}-a_2\hat{\mathbf{k}}$$

Similarly, $(\mathbf{a}\times\hat{\mathbf{j}}) = (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})\times\hat{\mathbf{j}}$

$= a_1\hat{\mathbf{k}} - a_3\hat{\mathbf{i}}$

and $(\mathbf{a}\times\hat{\mathbf{k}}) = (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})\times\hat{\mathbf{k}}$

$= -a_1\hat{\mathbf{j}} + a_2\hat{\mathbf{i}}$

Now, $(\mathbf{a}\times\hat{\mathbf{i}})^2 = (\mathbf{a}\times\hat{\mathbf{i}})\cdot(\mathbf{a}\times\hat{\mathbf{i}})$

$= (a_3\hat{\mathbf{j}} - a_2\hat{\mathbf{k}})\cdot(a_3\hat{\mathbf{j}} - a_2\hat{\mathbf{k}})$

$= a_3^2 + a_2^2$

$(\mathbf{a}\times\hat{\mathbf{j}})^2 = a_1^2 + a_3^2$

and $(\mathbf{a}\times\hat{\mathbf{k}})^2 = a_1^2 + a_2^2$

$\therefore (\mathbf{a}\times\hat{\mathbf{i}})^2 + (\mathbf{a}\times\hat{\mathbf{j}})^2 + (\mathbf{a}\times\hat{\mathbf{k}})^2$

$= a_3^2 + a_2^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$

$= 2(a_1^2 + a_2^2 + a_3^2)$

$= 2(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})\cdot(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})$

$= 2\mathbf{a}\cdot\mathbf{a} = 2\mathbf{a}^2$

65. Given that, $|\mathbf{a}| = 10, |\mathbf{b}| = 2$ and $\mathbf{a}\cdot\mathbf{b} = 12$

$\because \quad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$\Rightarrow \quad 12 = 10\times 2\cdot\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{3}{5}$

$\because \quad \mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\cdot\hat{\mathbf{n}}$

$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\sin\theta||\hat{\mathbf{n}}| \qquad [\because |\hat{\mathbf{n}}| = 1]$

$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}| = 10\cdot 2\cdot|\sqrt{1-\cos^2\theta}|\cdot 1$

$= 20\cdot|\sqrt{1-9/25}|$

$= 20\times\left|\sqrt{\frac{16}{25}}\right| = 20\times\frac{4}{5}$

$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}| = 16$

66. The number of vectors of unit length perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b} = \pm\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$

$= \pm\frac{(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})\times(\hat{\mathbf{j}} + \hat{\mathbf{k}})}{|(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})\times(\hat{\mathbf{j}} + \hat{\mathbf{k}})|}$

$= \pm\frac{(2\hat{\mathbf{i}} - 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{i}})}{|-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}|}$

$= \pm\frac{(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{1+4+4}}$

$= \pm\frac{(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{9}}$

$= \pm\frac{1}{3}(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$\therefore$ Required number of vectors is 2.

67. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})\times(\mathbf{a}+\mathbf{b}+\mathbf{c})$

$= \mathbf{a}\cdot\{-\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{c}\times\mathbf{b})$

$= \mathbf{a}\cdot(-\mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a}) = 0$

68. Since, the vectors $\mathbf{a} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{j}}$ are such that $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{b}$ form a right handed system.

$\therefore \quad \mathbf{c} = \mathbf{b}\times\mathbf{a} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 1 & 0\\ x & y & z\end{vmatrix} = z\,\hat{\mathbf{i}} - x\,\hat{\mathbf{k}}$

69. Given that, $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

and $\quad \mathbf{c} = 7\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 11\hat{\mathbf{k}}$

Let $\mathbf{A} = \mathbf{a} + \mathbf{b} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + (\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

and $\mathbf{B} = \mathbf{b} + \mathbf{c} = (\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) + (7\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 11\hat{\mathbf{k}})$

$= 8\hat{\mathbf{i}} + 12\hat{\mathbf{j}} + 16\hat{\mathbf{k}}$

If $\mathbf{A}$ and $\mathbf{B}$ are diagonals, then area of parallelogram

$= \frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \frac{1}{2}\left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 4 & 6\\ 8 & 12 & 16\end{matrix}\right\|$

$= \frac{1}{2}|\hat{\mathbf{i}}(64-72) - \hat{\mathbf{j}}(32-48) + \hat{\mathbf{k}}(24-32)|$

$= \frac{1}{2}|-8\hat{\mathbf{i}} + 16\hat{\mathbf{j}} - 8\hat{\mathbf{k}}| = |-4\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 4\hat{\mathbf{k}}|$

$= \sqrt{(-4)^2 + (8)^2 + (-4)^2}$

$= \sqrt{16+64+16} = \sqrt{96} = 4\sqrt{6}$

70. Force $\mathbf{F} = \mathbf{AB} = (3-1)\hat{\mathbf{i}} + (-4-2)\hat{\mathbf{j}} + (2+3)\hat{\mathbf{k}}$

$= 2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

Moment of force $\mathbf{F}$ with respect to $M = \mathbf{MA}\times\mathbf{F}$

$\because \quad \mathbf{MA} = (1+2)\hat{\mathbf{i}} + (2-4)\hat{\mathbf{j}} + (-3+6)\hat{\mathbf{k}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Now, $\mathbf{MA}\times\mathbf{F} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & -2 & 3\\ 2 & -6 & 5\end{vmatrix}$

$= \hat{\mathbf{i}}(-10+18) + \hat{\mathbf{j}}(6-15) + \hat{\mathbf{k}}(-18+4)$

$= 8\hat{\mathbf{i}} - 9\hat{\mathbf{j}} - 14\hat{\mathbf{k}}$

71. Given that, $\quad |\mathbf{a}| = |\mathbf{c}| = 1, |\mathbf{b}| = 4$

Let angle between $\mathbf{b}$ and $\mathbf{c}$ is α, then

$|\mathbf{b}\times\mathbf{c}| = \sqrt{15} \qquad$ (given)

$\Rightarrow \quad |\mathbf{b}||\mathbf{c}|\sin\alpha = \sqrt{15}$

$\Rightarrow \quad \sin\alpha = \frac{\sqrt{15}}{4\times 1} = \frac{\sqrt{15}}{4}$

$\therefore \quad \cos\alpha = \sqrt{1-\sin^2\alpha} = \frac{1}{4}$

We have, $\quad \mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$

On squaring both sides, we get

$(\mathbf{b} - 2\mathbf{c})^2 = \lambda^2(\mathbf{a})^2$

$\Rightarrow \quad \mathbf{b}^2 + 4\mathbf{c}^2 - 4\mathbf{b}\cdot\mathbf{c} = \lambda^2\mathbf{a}^2$

$\Rightarrow \quad 16 + 4 - 4|\mathbf{b}||\mathbf{c}|\cos\alpha = \lambda^2$

$\Rightarrow \quad 16 + 4 - 4\times 4\times 1\times\frac{1}{4} = \lambda^2$

$\Rightarrow \quad \lambda^2 = 16 + 4 - 4 = 16$

$\Rightarrow \quad \lambda = \pm 4$

72. Given that vectors $\lambda\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$ are coplanar, then $\begin{vmatrix}\lambda & 1 & 2\\ 1 & \lambda & -1\\ 2 & -1 & \lambda\end{vmatrix}=0$

$\Rightarrow \quad \lambda(\lambda^2-1)-(\lambda+2)+2(-1-2\lambda)=0$

$\Rightarrow \quad \lambda^3-\lambda-\lambda-2-2-4\lambda=0$

$\Rightarrow \quad \lambda^3-6\lambda-4=0$

$\Rightarrow \quad \lambda^2(\lambda+2)-2\lambda(\lambda+2)-2(\lambda+2)=0$

$\Rightarrow \quad (\lambda+2)(\lambda^2-2\lambda-2)=0$

$\Rightarrow \quad \lambda=-2$

73. $[\mathbf{a}-\mathbf{b}\ \mathbf{b}-\mathbf{c}\ \mathbf{c}-\mathbf{a}]$

$=\{(\mathbf{a}-\mathbf{b})\times(\mathbf{b}-\mathbf{c})\}\cdot(\mathbf{c}-\mathbf{a})$

$=(\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}-\mathbf{a})$

$=(\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}+\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}-\mathbf{a})$

$=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}-(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$

$=[\mathbf{abc}]-[\mathbf{abc}]=0$

74. We have, $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]=\{(\mathbf{a}+\mathbf{b})\times(\mathbf{b}+\mathbf{c})\}\cdot(\mathbf{c}+\mathbf{a})$

$=(\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}+\mathbf{a})$

$=(\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}+\mathbf{a})\quad[\because \mathbf{b}\times\mathbf{b}=0]$

$=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}+(\mathbf{a}\times\mathbf{c})\cdot\mathbf{c}+(\mathbf{b}+\mathbf{c})\cdot\mathbf{c}$
$\quad+(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}+(\mathbf{a}\times\mathbf{c})\cdot\mathbf{a}+(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$

$=[\mathbf{abc}]+0+0+0+0+[\mathbf{bca}]$

$(\because [\mathbf{acc}]=0, [\mathbf{bcc}]=0, [\mathbf{aba})=0, [\mathbf{aca}]=0$

$=[\mathbf{abc}]+[\mathbf{abc}]=2[\mathbf{abc}]$

75. Given, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors and $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ defined by the relations

$\mathbf{p}=\dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}, \mathbf{q}=\dfrac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}$ and $\mathbf{r}=\dfrac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$

$\therefore \quad \mathbf{a}\cdot\mathbf{p}=\dfrac{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}=\dfrac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{abc}]}=1$

and $\quad \mathbf{a}\cdot\mathbf{q}=\mathbf{a}\cdot\dfrac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}=\dfrac{\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{abc}]}=0$

Similarly, $\quad \mathbf{b}\cdot\mathbf{q}=\mathbf{c}\cdot\mathbf{r}=1$

and $\quad \mathbf{a}\cdot\mathbf{r}=\mathbf{b}\cdot\mathbf{p}=\mathbf{c}\cdot\mathbf{q}=\mathbf{c}\cdot\mathbf{p}=\mathbf{b}\cdot\mathbf{r}=0$

$\therefore \quad (\mathbf{a}+\mathbf{b})\cdot\mathbf{p}+(\mathbf{b}+\mathbf{c})\cdot\mathbf{q}+(\mathbf{c}+\mathbf{a})\cdot\mathbf{r}$

$=\mathbf{a}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{q}+\mathbf{c}\cdot\mathbf{q}+\mathbf{c}\cdot\mathbf{r}+\mathbf{a}\cdot\mathbf{r}$

$=1+1+1=3$

76. The three vectors $(\mathbf{a}+2\mathbf{b}+3\mathbf{c}), (\lambda\mathbf{b}+4\mathbf{c})$ and $(2\lambda-1)\mathbf{c}$ are non-coplanar, if $\begin{vmatrix}1 & 2 & 3\\ 0 & \lambda & 4\\ 0 & 0 & 2\lambda-1\end{vmatrix}\neq 0$

$\Rightarrow \quad (2\lambda-1)(\lambda)\neq 0 \Rightarrow \lambda\neq 0, \dfrac{1}{2}$

So, these three vectors are non-coplanar for all except two values of λ.

77. Given vectors are $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{j}}+\lambda\hat{\mathbf{k}}$ and $\lambda\hat{\mathbf{i}}+\hat{\mathbf{k}}$, which forms a parallelopiped.

$\therefore$ Volume of the parallelopiped is

$$V=\left\|\begin{matrix}1 & \lambda & 1\\ 0 & 1 & \lambda\\ \lambda & 0 & 1\end{matrix}\right\|=1+\lambda^3-\lambda$$

$\Rightarrow \quad V=\lambda^3-\lambda+1$

On differentiating w.r.t. λ, we get

$\dfrac{dV}{d\lambda}=3\lambda^2-1$

For maxima or minima, $\dfrac{dV}{d\lambda}=0\Rightarrow\lambda=\pm\dfrac{1}{\sqrt{3}}$

and $\quad \dfrac{d^2V}{d\lambda^2}=6\lambda=\begin{cases}2\sqrt{3}>0, & \text{for } \lambda=\dfrac{1}{\sqrt{3}}\\ 2\sqrt{3}<0, & \text{for } \lambda=-\dfrac{1}{\sqrt{3}}\end{cases}$

$\because \dfrac{d^2V}{d\lambda^2}$ is positive for $\lambda=\dfrac{1}{\sqrt{3}}$, so volume '$V$' is minimum for $\lambda=\dfrac{1}{\sqrt{3}}$

78. Since, given vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

$\therefore \quad \begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ x & x-2 & -1\end{vmatrix}=0$

$\Rightarrow 1\{1-2(x-2)\}-1(-1-2x)+1(x-2+x)=0$

$\Rightarrow \quad 1-2x+4+1+2x+2x-2=0$

$\Rightarrow \quad 2x=-4$

$\Rightarrow \quad x=-2$

79. Let unit vector is $a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$

$\because a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$ is perpendicular to $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$

Then, $\quad a+b+c=0 \quad$ ...(i)

and $a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$, $(\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})$ and $(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})$ are coplanar.

$\therefore \quad \begin{vmatrix}a & b & c\\ 1 & 1 & 2\\ 1 & 2 & 1\end{vmatrix}=0$

$\Rightarrow \quad -3a+b+c=0 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$a=0$ and $c=-b$

$\because a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$ is a unit vector, then

$a^2+b^2+c^2=1$

$\Rightarrow \quad 0+b^2+b^2=1$

$\Rightarrow \quad b=\dfrac{1}{\sqrt{2}}$

$\therefore \quad a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}=\dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}-\dfrac{1}{\sqrt{2}}\hat{\mathbf{k}}$

$=\dfrac{\hat{\mathbf{j}}-\hat{\mathbf{k}}}{\sqrt{2}}$

80. We know that, if **a**, **b**, **c** are coplanar vectors, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

$$\therefore \begin{vmatrix} 1 & 2 & 4 \\ 1 & \lambda & 4 \\ 2 & 4 & \lambda^2 - 1 \end{vmatrix} = 0$$

$$\Rightarrow 1\{\lambda(\lambda^2 - 1) - 16\} - 2((\lambda^2 - 1) - 8) + 4(4 - 2\lambda) = 0$$

$$\Rightarrow \lambda^3 - \lambda - 16 - 2\lambda^2 + 18 + 16 - 8\lambda = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 - 9\lambda + 18 = 0$$

$$\Rightarrow \lambda^2(\lambda - 2) - 9(\lambda - 2) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 - 9) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda + 3)(\lambda - 3) = 0$$

$$\therefore \lambda = 2,\ 3 \text{ or } -3$$

If $\lambda = 2$, then

$$\mathbf{a} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 4 \\ 2 & 4 & 3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(6 - 16) - \hat{\mathbf{j}}(3 - 8) + \hat{\mathbf{k}}(4 - 4) = -10\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$$

If $\lambda = \pm 3$, then $\mathbf{a} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{vmatrix} = 0$

(because last two rows are proportional).

81. Given that, $|\mathbf{a}| = 1, |\mathbf{b}| = 3$ and $|\mathbf{c}| = 5$

$$\therefore [\mathbf{a} - 2\mathbf{b}\ \ \mathbf{b} - 3\mathbf{c}\ \ \mathbf{c} - 4\mathbf{a}]$$

$$= (\mathbf{a} - 2\mathbf{b}) \cdot \{(\mathbf{b} - 3\mathbf{c}) \times (\mathbf{c} - 4\mathbf{a})\}$$

$$= (\mathbf{a} - 2\mathbf{b}) \cdot \{\mathbf{b} \times \mathbf{c} - 4\mathbf{b} \times \mathbf{a} + 12\mathbf{c} \times \mathbf{a}\}$$

$$= (\mathbf{a} - 2\mathbf{b}) \cdot (\mathbf{a} + 4\mathbf{c} + 12\mathbf{b})$$

$$= \mathbf{a} \cdot \mathbf{a} - 24\mathbf{b} \cdot \mathbf{b} = 1 - 24 \times 9$$

$$= 1 - 216 = -215$$

82. As we know, the volume of parallelopiped, where coterminus edges are given by vectors

$$\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + n\hat{\mathbf{k}},\ \mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - n\hat{\mathbf{k}}$$

and $\mathbf{c} = \hat{\mathbf{i}} + n\hat{\mathbf{j}} + 3\hat{\mathbf{k}},\ (n \geq 0)$, is

$$\begin{vmatrix} 1 & 1 & n \\ 2 & 4 & -n \\ 1 & n & 3 \end{vmatrix} = 158 \qquad \text{[given]}$$

$$\Rightarrow 1(12 + n^2) - 1(6 + n) + n(2n - 4) = 158$$

$$\Rightarrow 3n^2 - 5n + 6 = 158$$

$$\Rightarrow 3n^2 - 5n - 152 = 0$$

$$\Rightarrow 3n^2 - 24n + 19n - 152 = 0$$

$$\Rightarrow (3n + 19)(n - 8) = 0$$

$$\Rightarrow n = 8 \text{ as } n \geq 0$$

$$\therefore \mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 8\hat{\mathbf{k}},\ \mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 8\hat{\mathbf{k}}$$

and $\mathbf{c} = \hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$$\therefore \mathbf{a} \cdot \mathbf{c} = 1 + 8 + 24 = 33$$

and $\mathbf{b} \cdot \mathbf{c} = 2 + 32 - 24 = 10$

83. $[\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] + [\hat{\mathbf{k}}\ \hat{\mathbf{j}}\ \hat{\mathbf{i}}] + [\hat{\mathbf{j}}\ \hat{\mathbf{k}}\ \hat{\mathbf{i}}] = [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] + [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] - [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}]$

$$= [\hat{\mathbf{i}}\ \hat{\mathbf{k}}\ \hat{\mathbf{j}}] = \hat{\mathbf{i}} \cdot (\hat{\mathbf{k}} \times \hat{\mathbf{j}})$$

$$= \hat{\mathbf{i}} \cdot (-\hat{\mathbf{i}}) = -1$$

84. Given, $\mathbf{a} = p\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + q\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + r\hat{\mathbf{k}}$ are coplanar and $p \neq q \neq r \neq 1$.

Since, **a**, **b** and **c** are coplanar.

$$\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0 \Rightarrow \begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$$\Rightarrow p(qr - 1) - 1(r - 1) + 1(1 - q) = 0$$

$$\Rightarrow pqr - p - r + 1 + 1 - q = 0$$

$$\therefore pqr - (p + q + r) = -2$$

85. Angle between the faces OAB and ABC is same as angle between normals of the faces OAB and ABC.

Vector along the normals of OAB

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}} = \mathbf{a} \text{ (let)}$$

Vector along normals of ABC

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}} = \mathbf{b} \text{ (let)}$$

$$\therefore \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{5 + 5 + 9}{\sqrt{35}\sqrt{35}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{19}{35}\right)$$

86. Given vectors, $\mu\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \mu\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mu\hat{\mathbf{k}}$ will be coplanar, if

$$\begin{vmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{vmatrix} = 0$$

$$\Rightarrow \mu(\mu^2 - 1) - 1(\mu - 1) + 1(1 - \mu) = 0$$

$$\Rightarrow (\mu - 1)[\mu(\mu + 1) - 1 - 1] = 0$$

$$\Rightarrow (\mu - 1)[\mu^2 + \mu - 2] = 0$$

$$\Rightarrow (\mu - 1)[(\mu + 2)(\mu - 1)] = 0$$

$$\Rightarrow \mu = 1 \text{ or } -2$$

So, sum of the distinct real values of

$$\mu = 1 - 2 = -1.$$

87. Since, **A B C** from a left handed system

$$\therefore [\mathbf{A}, \mathbf{B}, \mathbf{C}] < 0$$

Now, $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 4 \\ 1 & 1 & 5 \end{vmatrix} = 11\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - \hat{\mathbf{k}}$

$$\therefore \mathbf{C} = -11\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + \hat{\mathbf{k}}, \text{ so that}$$

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = -121 - 36 - 1 = -158 < 0$$

88. Since, the volume of parallelopiped whose coterminous edges are $\mathbf{u} = \hat{i} + \hat{j} + \lambda\hat{k}$, $\mathbf{v} = \hat{i} + \hat{j} + 3\hat{k}$ and $\mathbf{w} = 2\hat{i} + \hat{j} + \hat{k}$ is

$$[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{vmatrix} 1 & 1 & \lambda \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 1 \text{ cuc unit} \qquad \text{(given)}$$

$\Rightarrow |1(1-3) - 1(1-6) + \lambda(1-2)| = 1$

$\Rightarrow \quad |-2+5-\lambda| = 1$

$\Rightarrow \quad |\lambda - 3| = 1 \quad \Rightarrow \lambda - 3 = \pm 1$

$\Rightarrow \quad \lambda = 2, 4$

Since, angle between **u** and **w** is θ, so

$$\cos\theta = \frac{|2+1+\lambda|}{\sqrt{1+1+\lambda^2}\sqrt{4+1+1}} = \frac{|\lambda+3|}{\sqrt{\lambda^2+2}\sqrt{6}}$$

for $\lambda = 2, \cos\theta = \dfrac{5}{6}$

for $\lambda = 4, \cos\theta = \dfrac{7}{6\sqrt{3}}$

89. Given three vectors are

$$\mathbf{a} = \alpha\hat{i} + \hat{j} + 3\hat{k}$$

$$\mathbf{b} = 2\hat{i} + \hat{j} - \alpha\hat{k}$$

and $\quad \mathbf{c} = \alpha\hat{i} - 2\hat{j} + 3\hat{k}$

Clearly, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} \alpha & 1 & 3 \\ 2 & 1 & -\alpha \\ \alpha & -2 & 3 \end{vmatrix}$

$= \alpha(3-2\alpha) - 1(6+\alpha^2) + 3(-4-\alpha)$

$= -3\alpha^2 - 18$

$= -3(\alpha^2 + 6)$

$\because$ There is no value of α for which $-3(\alpha^2+6)$ becomes zero, so $= \begin{vmatrix} \alpha & 1 & 3 \\ 2 & 1 & -\alpha \\ \alpha & -2 & 3 \end{vmatrix}$ $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

$\Rightarrow$ vectors **a**, **b** and **c** are not coplanar for any value $\alpha \in R$.

So, the set $S = \{\alpha : \mathbf{a}, \mathbf{b} \text{ and } \mathbf{c} \text{ are coplanar}\}$ is empty set.

90. $\mathbf{a} \times [\mathbf{a} \times (\mathbf{a} \times \mathbf{b})] = \mathbf{a} \times \{(\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}\}$

(expanding by vector triple product)

$= (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\times\mathbf{a}) - (\mathbf{a}\cdot\mathbf{a})(\mathbf{a}\times\mathbf{b})$

$= (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\times\mathbf{a}) \qquad [\because \mathbf{a}\times\mathbf{a} = 0]$

91. $\mathbf{a}\cdot\mathbf{c} = (\hat{i}+\hat{j}+\hat{k})\cdot\hat{i} = 1$ and $\mathbf{b}\cdot\mathbf{c} = (\hat{i}+\hat{j})\cdot\hat{i} = 1$

Now, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a} = \mu\mathbf{b} + \lambda\mathbf{a}$

$\Rightarrow \quad \mu = \mathbf{c}\cdot\mathbf{a}$ and $\lambda = -\mathbf{c}\cdot\mathbf{b}$

$\Rightarrow \quad \mu = 1$ and $\lambda = -1$

$\therefore \quad \mu + \lambda = 1 - 1 = 0$

92. We have, $\quad \mathbf{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\quad \mathbf{b} = \hat{i}\times(\mathbf{a}\times\hat{i}) + \hat{j}\times(\mathbf{a}\times\hat{j}) + \hat{k}\times(\mathbf{a}\times\hat{k})$

$= 3\mathbf{a} - \mathbf{a} = 2\mathbf{a} = 2(\hat{i} + 2\hat{j} + 3\hat{k})$

$\Rightarrow \quad |\mathbf{b}| = \sqrt{4+16+36} = \sqrt{56} = 2\sqrt{14}$

93. We have, $\quad \mathbf{a}\times\mathbf{b} + \mathbf{c} = 0$

$\Rightarrow \quad \mathbf{a}\times(\mathbf{a}\times\mathbf{b}) + \mathbf{a}\times\mathbf{c} = 0$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b})\,\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\,\mathbf{b} + \mathbf{a}\times\mathbf{c} = 0$

$\Rightarrow \quad 3\,\mathbf{a} - 2\,\mathbf{b} + \mathbf{a}\times\mathbf{c} = 0$

$\Rightarrow \quad 2\,\mathbf{b} = 3\,\mathbf{a} + \mathbf{a}\times\mathbf{c}$

$\Rightarrow \quad 2\,\mathbf{b} = 3\hat{j} - 3\hat{k} - 2\hat{i} - \hat{j} - \hat{k}$

$= -2\hat{i} + 2\hat{j} - 4\hat{k}$

$\therefore \quad \mathbf{b} = -\hat{i} + \hat{j} - 2\hat{k}$

94. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = -10\hat{i} + 9\hat{j} + 7\hat{k}$

$\mathbf{a}\times\mathbf{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4\hat{i} - 3\hat{j} - \hat{k}$

$\therefore \quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c}) = -40 - 27 - 7 = -74$

95. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a} = -5\mathbf{a} + 4\mathbf{b}$

$\therefore \quad \mathbf{c}\cdot\mathbf{a} = 4, \mathbf{c}\cdot\mathbf{b} = 5$

$\Rightarrow \quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\Rightarrow \quad 4\mathbf{b} - 3\mathbf{c}$

96. Since, $\quad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

$\therefore \quad (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$

$\Rightarrow \quad (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} = (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$

$\Rightarrow \quad \mathbf{a} = \dfrac{(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{b}\cdot\mathbf{c})}\cdot\mathbf{c}$

Hence, **a** is parallel to **c**.

97. We have, $\quad (\mathbf{a}\times\mathbf{c}) + \mathbf{b} = \mathbf{0}$

$\Rightarrow \quad \mathbf{a}\times(\mathbf{a}\times\mathbf{c}) + \mathbf{a}\times\mathbf{b} = 0$

[taking cross product with **a** both sides]

$\Rightarrow (\mathbf{a}\cdot\mathbf{c})\,\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\,\mathbf{c} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{0}$

$[\because \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}]$

$\Rightarrow \quad 4(\hat{i} - \hat{j}) - 2\mathbf{c} + (-\hat{i} - \hat{j} + 2\hat{k}) = \mathbf{0}$

$[\because \mathbf{a}\cdot\mathbf{a} = (\hat{i}-\hat{j})(\hat{i}-\hat{j}) = 1 + 1 = 2 \text{ and } \mathbf{a}\cdot\mathbf{c} = 4]$

$\Rightarrow \quad 2\mathbf{c} = 4\hat{i} - 4\hat{j} - \hat{i} - \hat{j} + 2\hat{k}$

$\Rightarrow \quad \mathbf{c} = \dfrac{3\hat{i} - 5\hat{j} + 2\hat{k}}{2}$

$\Rightarrow \quad |\mathbf{c}|^2 = \dfrac{9+25+4}{4} = \dfrac{19}{2}$

Round II

1. Given, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$

$\Rightarrow \quad -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$

$\Rightarrow \quad -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$

$$\left[\frac{1}{3}|\mathbf{b}||\mathbf{c}| + (\mathbf{c} \cdot \mathbf{b})\right]\mathbf{a} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

Since, a and b are not collinear.

$$\mathbf{c} \cdot \mathbf{b} + \frac{1}{3}|\mathbf{b}||\mathbf{c}| = 0 \text{ and } \mathbf{c} \cdot \mathbf{a} = 0$$

$\Rightarrow \quad |\mathbf{c}||\mathbf{b}|\cos\theta + \frac{1}{3}|\mathbf{b}||\mathbf{c}| = 0$

$\Rightarrow \quad |\mathbf{b}||\mathbf{c}|\left(\cos\theta + \frac{1}{3}\right) = 0$

$\Rightarrow \quad \cos\theta + \frac{1}{3} = 0 \quad [\because |\mathbf{b}| \neq 0, |\mathbf{c}| \neq 0]$

$\Rightarrow \quad \cos\theta = -\frac{1}{3}$

$\Rightarrow \quad \sin\theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$

2. We have, $|[\mathbf{a} \;\; \mathbf{b} \;\; \mathbf{c}]| = V$

Let V_1 be the volume of the parallelopiped formed by the vectors α, β and γ.

Then, $V_1 = |[\alpha \, \beta \, \gamma]|$

Now, $[\alpha\,\beta\,\gamma] = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix} [\mathbf{a} \; \mathbf{b} \; \mathbf{c}]$

$\Rightarrow \quad [\alpha\,\beta\,\gamma] = [\mathbf{a}\;\mathbf{b}\;\mathbf{c}]^2 [\mathbf{a}\;\mathbf{b}\;\mathbf{c}]$

$\Rightarrow \quad [\alpha\,\beta\,\gamma] = [\mathbf{a}\;\mathbf{b}\;\mathbf{c}]^3$

$\therefore \quad V_1 = |[\alpha\,\beta\,\gamma]| = |[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]^3| = V^3$

3. Let $\alpha = \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \beta = \lambda\mathbf{b} + 4\mathbf{c}$

and $\gamma = (2\lambda - 1)\,\mathbf{c}$

Then, $[\alpha\,\beta\,\gamma] = \begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & (2\lambda - 1) \end{vmatrix} [\mathbf{a}\;\mathbf{b}\;\mathbf{c}]$

$$[\alpha\beta\gamma] = \lambda\,(2\lambda - 1)\,[abc]$$

$\Rightarrow \quad [\alpha\,\beta\,\gamma] = 0,$

If $\quad \lambda = 0, \frac{1}{2} \quad [\because [\mathbf{a}\;\mathbf{b}\;\mathbf{c}] \neq 0]$

Hence, α, β and γ are non-coplanar for all value of λ except two values 0 and $\frac{1}{2}$.

4. We have, projection of $\mathbf{v}$ along $\mathbf{u}$ = projection of $\mathbf{w}$ along $\mathbf{u}$

$\Rightarrow \quad \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{u}|}$

$\Rightarrow \quad \mathbf{v} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u} \quad$ ...(i)

Also, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other

$\therefore \quad \mathbf{v} \cdot \mathbf{w} = 0 \quad$ ...(ii)

Now, $[\mathbf{u} - \mathbf{v} + \mathbf{w}]^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 \;\; |\mathbf{w}|^2$
$- 2(\mathbf{u} \cdot \mathbf{v}) - 2(\mathbf{v} \cdot \mathbf{w}) + 2(\mathbf{u} \cdot \mathbf{w})$

$\Rightarrow \quad [\mathbf{u} - \mathbf{v} + \mathbf{w}]^2 = 1 + 4 + 9$

[from Eqs. (i) and (ii)]

$\Rightarrow \quad |\mathbf{u} - \mathbf{v} + \mathbf{w}| = \sqrt{14}$

5. $\mathbf{a} \times (\mathbf{a} \times ((\mathbf{a} \cdot \mathbf{b})\mathbf{a} - \mathbf{a}^2\mathbf{b}))$

$\mathbf{a} \times (-|\mathbf{a}|^2 (\mathbf{a} \times \mathbf{b})) = -|\mathbf{a}|^2((\mathbf{a} \cdot \mathbf{b})\mathbf{a} - |\mathbf{a}|^2\mathbf{b})$

$= -(\mathbf{a} \cdot \mathbf{b})\mathbf{a}\,|\mathbf{a}|^2 + |\mathbf{a}|^4\,\mathbf{b}$

$= |\mathbf{a}|^4\,\mathbf{b} \quad [\because \mathbf{a} \cdot \mathbf{b} = 0]$

6. By expanding $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, we get

$$\mathbf{a} \cdot \mathbf{c} = x^2 - 2x + 6, \mathbf{a} \cdot \mathbf{b} = -\sin y$$

$\because \quad \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 4$

$\Rightarrow \quad x^2 - 2x + 2 = \sin y$

$\Rightarrow \quad \sin y = x^2 - 2x = 2 + (x-1)^2 + 1 \geq 1$

But $\sin y \leq 1$

$\therefore$ Both sides are equal only for $x = 1$

7. Given, $|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = |\hat{\mathbf{c}}| = 1$

and $\hat{\mathbf{a}} \times (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = \frac{\sqrt{3}}{2}(\hat{\mathbf{b}} + \hat{\mathbf{c}})$

Now, consider $\hat{\mathbf{a}} \times (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = \frac{\sqrt{3}}{2}(\hat{\mathbf{b}} + \hat{\mathbf{c}})$

$\Rightarrow \quad (\hat{\mathbf{a}} \cdot \hat{\mathbf{c}})\hat{\mathbf{b}} - (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})\hat{\mathbf{c}} = \frac{\sqrt{3}}{2}\hat{\mathbf{b}} + \frac{\sqrt{3}}{2}\hat{\mathbf{c}}$

On comparing, we get

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = -\frac{\sqrt{3}}{2} \Rightarrow |\hat{\mathbf{a}}||\hat{\mathbf{b}}|\cos\theta = -\frac{\sqrt{3}}{2}$$

$\Rightarrow \quad \cos\theta = -\frac{\sqrt{3}}{2} \quad [\because |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1]$

$\Rightarrow \quad \cos\theta = \cos\left(\pi - \frac{\pi}{6}\right) \Rightarrow \theta = \frac{5\pi}{6}$

8. Now, $\mathbf{a} \cdot \mathbf{b} = 2\sqrt{2} \cdot 3 \cdot \frac{1}{\sqrt{2}} = 6$

The diagonals are $2\mathbf{a} - 3\mathbf{b} \pm (\mathbf{a} + \mathbf{b})$

$\therefore$ Length of diagonals are

$$|3\mathbf{a} - 2\mathbf{b}|^2 = 9 \cdot 8 + 4 \cdot 9 - 12 \cdot 6 = 36$$

and $\quad |\mathbf{a} - 4\mathbf{b}|^2 = 8 + 16 \cdot 9 - 8 \cdot 6 = 104$

$\therefore$ The length of the longer diagonal is $\sqrt{104}$ i.e. $2\sqrt{26}$.

9. Given $\mathbf{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{OQ} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3x\hat{\mathbf{k}}$

$|\mathbf{PQ}| = \sqrt{20}$ and $\mathbf{OP}$ is perpendicular to $\mathbf{OQ}$

$\mathbf{OR} = 3\hat{\mathbf{i}} + z\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$

Also, $\mathbf{OP}, \mathbf{OQ}, \mathbf{OR}$ are coplanar

$\mathbf{OP} \cdot \mathbf{OQ} = -x + 2y - 3x = 0 \Rightarrow y = 2x$

$|\mathbf{PQ}|^2 = 20 = (x+1)^2 + (y-2)^2 + (1+3x)^2$

$\Rightarrow \quad 20 = (x+1)^2 + (2x-2)^2 + (1+3x)^2$

$\Rightarrow \quad x = 1 \Rightarrow y = 2$

OP, **OQ**, **OR** are coplanar

$$\therefore \quad \begin{vmatrix} x & y & -1 \\ -1 & 2 & 3x \\ 3 & z & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 3 & z & -7 \end{vmatrix} = 0$$

$\Rightarrow \quad (-14-3z) - 2(7-9) - 1(-z-6) = 0 \Rightarrow z = -2$

$\therefore \quad x^2 + y^2 + z^2 = 1 + 4 + 4 = 9$

10. Let first term and common ratio of a GP be α and β. Then, $a = \alpha \cdot \beta^{p-1}, b = \alpha \cdot \beta^{q-1}, c = \alpha \cdot \beta^{r-1}$

$\therefore \quad \log a = \log \alpha + (p-1)\log \beta,$

$\log b = \log \alpha + (q-1)\log \beta$

and $\quad \log c = \log \alpha + (r-1)\log \beta$

The dot product of the given two vectors is

$\Sigma\{\log \alpha + (p-1)\log \beta\}\ (q-r)$

$= (\log \alpha - \log \beta)\ \Sigma(q-r) + \log \beta\ \Sigma p(q-r) = 0$

11. Given, vectors are $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{k}}, \mathbf{b} = x\hat{\mathbf{i}} + \hat{\mathbf{j}} + (1-x)\hat{\mathbf{k}}$

and $\quad \mathbf{c} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + (1+x-y)\hat{\mathbf{k}}$

$$\therefore \quad [\mathbf{a\ b\ c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

Applying $C_3 \to C_3 + C_1$, we get

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} = 1(1+x) - x = 1$$

Thus, $[\mathbf{a\ b\ c}]$ depends upon neither x nor y.

12. Here, $\quad \mathbf{a} \cdot \mathbf{x} = 2$ and $\mathbf{a} \times \mathbf{r} + \mathbf{b} = \mathbf{r} \quad \ldots$ (i)

Dot product of Eq. (i) with **a** to get, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{r} = 2$

Cross product of Eq. (i) with **a** to get,

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{r}) + \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{r} = \mathbf{r} - \mathbf{b}$$

$\Rightarrow \quad 2\mathbf{a} - \mathbf{r} + \mathbf{a} \times \mathbf{b} + \mathbf{r} - \mathbf{b}$

$\therefore \quad \mathbf{r} = \frac{1}{2}[2\mathbf{a} + \mathbf{b} + \mathbf{a} \times \mathbf{b}]$

13. Since, $\quad \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 0, \mathbf{b} \cdot \mathbf{c} = \frac{1}{2}$

$|\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}|^2 = |\mathbf{a} \times (\mathbf{b} - \mathbf{c})|^2$

$= |\mathbf{a}|^2 |\mathbf{b} - \mathbf{c}|^2 - (\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}))^2 = |\mathbf{b} - \mathbf{c}|^2$

$= |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2|\mathbf{b}||\mathbf{c}|\cos\frac{\pi}{3} = 1$

14. We know that,

$$\mathbf{r} = \frac{(\mathbf{r} \cdot \mathbf{a})\ \mathbf{b} \times \mathbf{c} + (\mathbf{r} \cdot \mathbf{b})\ \mathbf{c} \times \mathbf{a} + (\mathbf{r} \cdot \mathbf{c})\ \mathbf{a} \times \mathbf{b}}{[\mathbf{a\ b\ c}]}$$

$\therefore \ \alpha + \beta + \gamma = \frac{\mathbf{r}}{2} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$

15. Taking cross product of the given relation with **a** and **b** in turn, we get

$$2\mathbf{a} \times \mathbf{b} = 3\mathbf{c} \times \mathbf{a},\ \mathbf{a} \times \mathbf{b} = 3\mathbf{b} \times \mathbf{c}$$

$$\therefore \quad \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \left(1 + \frac{1}{3} + \frac{2}{3}\right)(\mathbf{a} \times \mathbf{b})$$

$$= 2\mathbf{a} \times \mathbf{b}$$

16. If $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = 0$, then $\mathbf{c} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$

$\therefore \quad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = |\mathbf{a} \times \mathbf{b}|$

$$\Rightarrow \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = |\mathbf{a} \times \mathbf{b}|^2$$

17. For coplanar vectors $\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & \mu \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} = 0$

$\Rightarrow \quad (2\lambda - 1)\lambda = 0 \Rightarrow \lambda = 0, \frac{1}{2}$

18. $\mathbf{a}_{\text{Old}} = 3p\hat{\mathbf{i}} + \hat{\mathbf{j}}$

$\mathbf{a}_{\text{New}} = (p+1)\hat{\mathbf{i}} + \sqrt{10}\hat{\mathbf{j}}$

$\Rightarrow \quad |\mathbf{a}_{\text{Old}}| = |\mathbf{a}_{\text{New}}|$

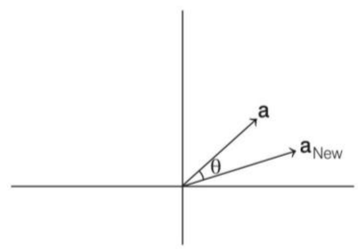
$\Rightarrow \quad 9p^2 + 1 = p^2 + 2p + 1 + 10$

$\Rightarrow \quad 8p^2 - 2p - 10 = 0$

$\Rightarrow \quad 4p^2 - p - 5 = 0$

$\Rightarrow \quad (4p-5)(p+1) = 0$

$\Rightarrow \quad p = \frac{5}{4}, -1$



19. $\mathbf{OA} = -4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}, \mathbf{OB} = 14\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

$\hat{\mathbf{a}} = \frac{-4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{5}; \hat{\mathbf{b}} = \frac{14\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}}{15}$

$\mathbf{r} = \frac{\lambda}{15}[-12\hat{\mathbf{i}} + 9\hat{\mathbf{k}} + 14\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}]$

$\mathbf{r} = \frac{\lambda}{15}[2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}]$

$\mathbf{r} = \frac{2\lambda}{15}[\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}]$

20. $(\lambda - 1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = \mathbf{0}$

i.e. $(\lambda - 1)\mathbf{a}_1 + (1 - \lambda + \mu - 2\gamma)\mathbf{a}_2 + (\mu + \gamma + 1)\mathbf{a}_3 + (\gamma + \delta)\mathbf{a}_4 = \mathbf{0}$

Since, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4$ are linearly independent.

$\lambda - 1 = 0, 1 - \lambda + \mu - 2\gamma = 0, \gamma + y + 1 = 0$ and $\gamma + \delta = 0$

i.e. $\lambda = 1, \mu = 2\gamma, \mu + \gamma + 1 = 0, \gamma + \delta = 0$

i.e. $\lambda = 1, \mu = -\frac{2}{3}, \gamma = -\frac{1}{3}, \delta = \frac{1}{3}$

21. Since, **a**, **b** and **c** are unit vectors inclined at an angle θ.

$|\mathbf{a}| = |\mathbf{b}| = 1$ and $\cos\theta = \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}$

Now, $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})$ ...(i)

$\Rightarrow \mathbf{a}\cdot\mathbf{c} = \alpha(\mathbf{a}\cdot\mathbf{a}) + \beta(\mathbf{ab}) + \gamma\{\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})\}$

$\Rightarrow \cos\theta = \alpha|\mathbf{a}|^2 \quad [\because \mathbf{a}\cdot\mathbf{b} = 0, \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0]$

$\Rightarrow \cos\theta = \alpha$

Similarly, by taking det roduct on both sides of Eq. (i), we get

$\beta = \cos\theta$

$\therefore \alpha = \beta$

Again, $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})$

$\Rightarrow |\mathbf{c}|^2 = |\alpha\,\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a}\times\mathbf{b})|^2$

$= \alpha^2|\mathbf{a}|^2 + \beta^2|\mathbf{b}|^2 + \gamma^2|\mathbf{a}\times\mathbf{b}|^2 + 2\alpha\beta(\mathbf{a}\cdot\mathbf{b})$
$+ 2\alpha\gamma\{\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})\} + 2\beta\gamma[\mathbf{b}\cdot\{\mathbf{a}\times\mathbf{b}\}]$

$\Rightarrow 1 = \alpha^2 + \beta^2 + \gamma^2|\mathbf{a}\times\mathbf{b}|^2$

$\Rightarrow 1 = 2\alpha^2 + \gamma^2\{|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\pi/2\}$

$\Rightarrow 1 = 2\alpha^2 + \gamma^2\{|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\pi/2\}$

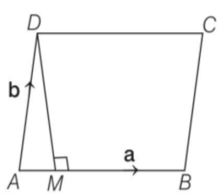
$\Rightarrow 1 = 2\alpha^2 + \gamma^2 \Rightarrow 2\alpha^2 = \frac{1-\gamma^2}{2}$

But $\alpha = \beta = \cos\theta.$

$1 = 2\alpha^2 + \gamma^2 \Rightarrow \gamma^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

$\therefore \beta^2 = \frac{1-\gamma^2}{2} = \frac{1+\cos 2\theta}{2}$

22. We have, AM = Projection of **b** on **a** $= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$



$\therefore \mathbf{AM} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

Now, in ΔADM

$\mathbf{AD} = \mathbf{AM} + \mathbf{MD} \Rightarrow \mathbf{DM} = \mathbf{AM} - \mathbf{AD}$

$\Rightarrow \mathbf{DM} = \frac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}}{|\mathbf{a}|^2} - \mathbf{b}$

Also, $\mathbf{DM} = \frac{1}{|\mathbf{a}|^2}[(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a} - |\mathbf{a}|^2\,\mathbf{b}]$

$\Rightarrow \mathbf{MD} = \frac{1}{|\mathbf{a}|^2}\{|\mathbf{a}|^2\,\mathbf{b} - (\mathbf{ab})\,\mathbf{a}\}$

Now, $\frac{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}|^2} = \frac{1}{|\mathbf{a}|^2}[(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}] = \mathbf{DM}$

23. $a_1 + a_2\cos 2x + a_3\sin^2 x = 0, \forall\, x \in R$

$\Rightarrow (a_1 + a_2) + \sin^2 x(a_3 - 2a_2) = 0\ \forall x \in R$

$\Rightarrow a_1 + a_2 = 0$ and $a_3 - 2a_2 = 0$

$\frac{a_1}{-1} = \frac{a_2}{1} = \frac{a_3}{2} = \lambda\ (\neq 0)$

$\Rightarrow a_1 = -\lambda, a_2 = \lambda, a_3 = 2\lambda$

24. $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$

$\Rightarrow |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$\Rightarrow \sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$ ...(i)

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \Rightarrow \cos\theta = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$ ...(ii)

From Eqs. (i) and (ii),

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow |\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$

If $\theta = \frac{\pi}{4}$, then $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

Therefore, $|\mathbf{a}\times\mathbf{b}| = \frac{|\mathbf{a}||\mathbf{b}|}{\sqrt{2}}$ and $\mathbf{a}\cdot\mathbf{b} = \frac{|\mathbf{a}|\,|\mathbf{b}|}{\sqrt{2}}$

$|\mathbf{a}\times\mathbf{b}| \neq \mathbf{a}\cdot\mathbf{b}$

$\Rightarrow \mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \frac{|\mathbf{a}||\mathbf{b}|}{\sqrt{2}}\,\hat{\mathbf{n}}$

$\Rightarrow \mathbf{a}\times\mathbf{b} = (\mathbf{a}\cdot\mathbf{b})\,\hat{\mathbf{n}}$

25. Since, **a**, **b** and **a** × **b** are non-coplanar,

$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a}\times\mathbf{b})$

$\therefore \mathbf{r}\times\mathbf{b} = \mathbf{a}$

$\Rightarrow x\mathbf{a}\times\mathbf{b} + z\{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\,\mathbf{a}\} = \mathbf{a}$

$\Rightarrow -(1 + z|\mathbf{b}|^2)\,\mathbf{a} + x\mathbf{a}\times\mathbf{b} = 0 \quad$ [since, $\mathbf{a}\cdot\mathbf{b} = 0$]

$\therefore x = 0$ and $z = -\frac{1}{|\mathbf{b}|^2}$

Thus, $\mathbf{r} = y\mathbf{b} - \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{b}|^2}$, where y is the parameter.

26. Obviously, $\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}$ is a vector in the plane of **a** and **b** and hence, perpendicular to **a** × **b**.

It is also equally inclined to **a** and **b** as it is along the angle bisector.

27. The given vectors are

$\mathbf{p} = (a+1)\hat{\mathbf{i}} + a\hat{\mathbf{j}} + a\hat{\mathbf{k}},$

$\mathbf{q} = a\hat{\mathbf{i}} + (a+1)\hat{\mathbf{j}} + a\hat{\mathbf{k}}$

and $\mathbf{r} = a\hat{\mathbf{i}} + a\hat{\mathbf{j}} + (a+1)\hat{\mathbf{k}}$, ($a \in R$) are coplanar,

So, $[\mathbf{p}\ \mathbf{q}\ \mathbf{r}] = 0$

$$\Rightarrow \begin{vmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{vmatrix} = 0$$

$\Rightarrow (a+1)[(a+1)^2 - a^2] - a[a(a+1) - a^2]$
$+ a[a^2 - a(a+1)] = 0$

$\Rightarrow \quad (a+1)[2a+1]-2a[a]=0$

$\Rightarrow \quad 2a^2+3a+1-2a^2=0$

$\Rightarrow \quad a=-\frac{1}{3}$

So, $\mathbf{p}=\frac{2}{3}\hat{\mathbf{i}}-\frac{1}{3}\hat{\mathbf{j}}-\frac{1}{3}\hat{\mathbf{k}}$

$\mathbf{q}=-\frac{1}{3}\hat{\mathbf{i}}+\frac{2}{3}\hat{\mathbf{j}}-\frac{1}{3}\hat{\mathbf{k}}$

and $\mathbf{r}=-\frac{1}{3}\hat{\mathbf{i}}-\frac{1}{3}\hat{\mathbf{j}}+\frac{2}{3}\hat{\mathbf{k}}$

So, $(\mathbf{p}\cdot\mathbf{q})^2=\left(-\frac{2}{9}-\frac{2}{9}+\frac{1}{9}\right)^2=\left(\frac{3}{9}\right)^2=\frac{1}{9}$

and $\mathbf{r}\times\mathbf{q}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{vmatrix}$

$=\hat{\mathbf{i}}\left(\frac{1}{9}-\frac{4}{9}\right)-\hat{\mathbf{j}}\left(\frac{1}{9}+\frac{2}{9}\right)+\hat{\mathbf{k}}\left(-\frac{2}{9}-\frac{1}{9}\right)$

$=-\frac{1}{3}\hat{\mathbf{i}}-\frac{1}{3}\hat{\mathbf{j}}-\frac{1}{3}\hat{\mathbf{k}}$

$\therefore |\mathbf{r}\times\mathbf{q}|^2=\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{3}$

$\because$ It is given that

$3(\mathbf{p}\cdot\mathbf{q})^2-\lambda|\mathbf{r}\times\mathbf{q}|^2=0$

$\Rightarrow \quad 3\left(\frac{1}{9}\right)-\lambda\left(\frac{1}{3}\right)=0 \Rightarrow \lambda=1$

28. There are three vectors given **a**, **b** and **c**, such that $|\mathbf{a}|=\sqrt{3}, |\mathbf{b}|=5$ and $\mathbf{b}\cdot\mathbf{c}=10$

So, $|\mathbf{b}||\mathbf{c}|\cdot\cos\frac{\pi}{3}=10$

[$\because$ it is given angle between **b** and **c** is $\frac{\pi}{3}$]

$\Rightarrow \quad 5|\mathbf{c}|\left(\frac{1}{2}\right)=10 \Rightarrow |\mathbf{c}|=4$

Now, as **a** is perpendicular to the vector $\mathbf{b}\times\mathbf{c}$,

so $|\mathbf{a}\times(\mathbf{b}\times\mathbf{c})|=\left||\mathbf{a}||\mathbf{b}\times\mathbf{c}|\sin\frac{\pi}{2}\right|$

$=|\mathbf{a}||\mathbf{b}\times\mathbf{c}|=|\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\frac{\pi}{3}$

$=(\sqrt{3})(5)(4)\frac{\sqrt{3}}{2}=30$

29. Two vectors **c** and **d** are said to be collinear, if we can write $\mathbf{c}=\lambda\mathbf{b}$ for some non-zero scalar λ.

Let the vectors $\alpha=(\lambda-2)\,\mathbf{a}+\mathbf{b}$

and $\beta=(4\lambda-2)\mathbf{a}+3\mathbf{b}$ are collinear, where **a** and **b** are non-collinear.

$\therefore$ We can write $\alpha=k\beta$, for some $k\in R-\{0\}$

$\Rightarrow \quad (\lambda-2)\mathbf{a}+\mathbf{b}=k[(4\lambda-2)\mathbf{a}+3\mathbf{b}]$

$\Rightarrow \quad [(\lambda-2)-k(4\lambda-2)]\,\mathbf{a}+(1-3k)\mathbf{b}=0$

Now, as **a** and **b** are non-collinear, therefore they are linearly independent and hence

$(\lambda-2)-k(4\lambda-2)=0$ and $1-3k=0$

$\Rightarrow \quad \lambda-2=k(4\lambda-2)$ and $3k=1$

$\Rightarrow \quad \lambda-2=\frac{1}{3}(4\lambda-2) \quad \left[\because 3k=1\Rightarrow k=\frac{1}{3}\right]$

$\Rightarrow \quad 3\lambda-6=4\lambda-2$

$\Rightarrow \quad \lambda=-4$

$\therefore \quad -\lambda=4$

30. Use the formulae

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$,

$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]=[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

and $[\mathbf{a}\ \mathbf{a}\ \mathbf{b}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{b}]=[\mathbf{a}\ \mathbf{c}\ \mathbf{c}]=0$

Further simplify it and get the result.

Now, $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$

$=\mathbf{a}\times\mathbf{b}\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}))$

$=\mathbf{a}\times\mathbf{b}\cdot((\mathbf{k}\times\mathbf{c}\times\mathbf{a}))$ [here, $\mathbf{k}=\mathbf{b}\times\mathbf{c}$]

$=\mathbf{a}\times\mathbf{b}\cdot[(\mathbf{k}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{k}\cdot\mathbf{c})\,\mathbf{a}]$

$=(\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{b}\times\mathbf{c}\cdot\mathbf{c})\,\mathbf{a})$

$=(\mathbf{a}\times\mathbf{b})\cdot([\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{c})-0 \quad [\because [\mathbf{b}\times\mathbf{c}\cdot\mathbf{c}]=0]$

$=\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\,[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$

$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 \quad \{\because [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\}$

Hence, $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]=\lambda\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2=\lambda\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

$\Rightarrow \quad \lambda=1$

31. It is given that projection of **b** on **a** is equal to the projection of **c** on **a**, where $|\mathbf{a}|=2$, $|\mathbf{b}|=4$ and $|\mathbf{c}|=4$,

so $\frac{\mathbf{a}\cdot\mathbf{b}}{2}=\frac{\mathbf{a}\cdot\mathbf{c}}{2}$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$

and **b** is perpendicular to **c**, so $\mathbf{b}\cdot\mathbf{c}=0$

Now, $|\mathbf{a}+\mathbf{b}-\mathbf{c}|^2$

$=|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{c}-2\mathbf{a}\cdot\mathbf{c}$

$=4+16+16=36$

$\therefore |\mathbf{a}+\mathbf{b}-\mathbf{c}|=6$

32. Let angle between unit vectors **a** and **b** is $\theta\in[0,\pi]$.

Then, $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}\cdot\mathbf{b}$

$=1+1+2\cos\theta=2(1+\cos\theta)$

$=4\cos^2\frac{\theta}{2}$

$\Rightarrow \quad |\mathbf{a}+\mathbf{b}|=2\cos\left(\frac{\theta}{2}\right) \quad \left[\because \theta\in[0,\pi]\Rightarrow\cos\left(\frac{\theta}{2}\right)\geq 0\right]$

Similarly, $|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}$

$=1+1-2\cos\theta$

$=2(1-\cos\theta)=4\sin^2\frac{\theta}{2}$

$\Rightarrow \quad |\mathbf{a}-\mathbf{b}|=2\sin\left(\frac{\theta}{2}\right)$

So, $\sqrt{3}|\mathbf{a}+\mathbf{b}|+|\mathbf{a}-\mathbf{b}|=2\left[\sqrt{3}\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right]$

having greatest value $=2\sqrt{3+1}=4$

[$\because$ greatest value of $a\cos\theta+b\sin\theta$ is $\sqrt{a^2+b^2}$]

33. Total force, $\mathbf{F}=(4\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\hat{\mathbf{k}})+(3\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$

$\therefore\quad \mathbf{F}=7\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}}$

The particle is displaced from

$A(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})$ to $B(5\hat{\mathbf{i}}+4\hat{\mathbf{j}}+\hat{\mathbf{k}})$

Now, displacement,

$\mathbf{AB}=(5\hat{\mathbf{i}}+4\hat{\mathbf{j}}+\hat{\mathbf{k}})-(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})=4\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

$\therefore$ Work done $=\mathbf{F}\cdot\mathbf{AB}=(7\hat{\mathbf{i}}+2\hat{\mathbf{j}}-4\hat{\mathbf{k}})\cdot(4\hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}})$

$=28+4+8=40$ units

34. $\mathbf{c}=\lambda(\mathbf{b}\times(\mathbf{a}\times\mathbf{b}))$

$=\lambda((\mathbf{b}\cdot\mathbf{b})\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\mathbf{b})$

$=\lambda(5(-\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+2\hat{\mathbf{i}}+\hat{\mathbf{k}})$

$=\lambda(-3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+6\hat{\mathbf{k}})$

$\mathbf{c}\cdot\mathbf{a}=7\Rightarrow 3\lambda+5\lambda+6\lambda=7$

$\lambda=\frac{1}{2}$

$\therefore\ 2\left|\left(\frac{-3}{2}-1+2\right)\hat{\mathbf{i}}+\left(\frac{5}{2}+1\right)\hat{\mathbf{j}}+(3+1+1)\hat{\mathbf{k}}\right|$

$=2\left(\frac{1}{4}+\frac{49}{4}+25\right)=25+50=75$

35. Let $\mathbf{x}=\lambda\mathbf{a}+\mu\mathbf{b}$ [λ and μ are scalars]

$\mathbf{x}=\hat{\mathbf{i}}(2\lambda+\mu)+\hat{\mathbf{j}}(2\mu-\lambda)+\hat{\mathbf{k}}(\lambda-\mu)$

Since, $\mathbf{x}\cdot(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}-\hat{\mathbf{k}})=0$

$3\lambda+8\mu=0 \quad \text{...(i)}$

Also projection of $\mathbf{x}$ on $\mathbf{a}$ is $\frac{17\sqrt{6}}{2}$

$\frac{\mathbf{x}\cdot\mathbf{a}}{|\mathbf{a}|}=\frac{17\sqrt{6}}{2}$

$6\lambda-\mu=51 \quad \text{...(ii)}$

From Eqs. (i) and (ii), we get

$\lambda=8,\mu=-3$

$\mathbf{x}=13\hat{\mathbf{i}}-14\hat{\mathbf{j}}+11\hat{\mathbf{k}}$

$|\mathbf{x}|^2=486$