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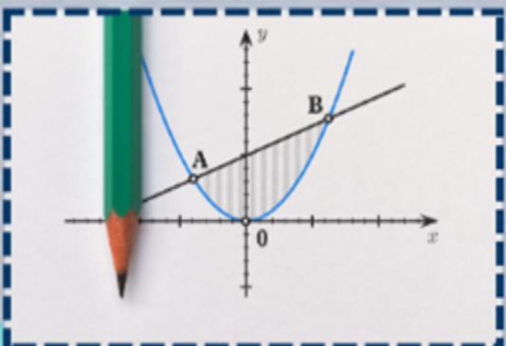
CONIC SECTIONS
PARABOLA



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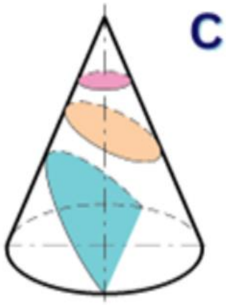

CONIC SECTION

PARABOLA

CONIC SECTION

A *conic* (or a *conic section*) is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line. The fixed point is called the *focus* and the fixed line is called the *directrix* of the conic. The constant ratio is known as the *eccentricity* of the conic and is denoted by the letter e. The conic is an *ellipse*, a *parabola* or a *hyperbola*, according as $e <= \text{or} > 1$.

Geometrically, these curves can be obtained as sections of a right circular cone by a plane in various positions and that is why they are called conic sections. A conic (or a conic section) is given by a second degree equation. Thus a conic means a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola. A circle is a limiting case of an ellipse.

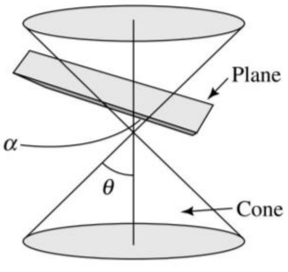

Fig. 18.1

Circle, Ellipse, Parabola and Hyperbola

When the plane cuts the cone with semivertical angle θ at an angle α with the vertical axis of the cone (other than at the vertex) we have the following:

1. If $\alpha = 90°$, the section is *a circle*
2. If $\theta < \alpha < 90°$, the section is *an ellipse*
3. If $\alpha = \theta$, the section is *a parabola*

(In each of the above situations, the plane cuts entirely across one side of the cone above or below the vertex.)

4. If $0 \le \alpha < \theta$, the plane cuts through both the sides of the cone and the curve of intersection is *a hyperbola*.

Equation of a Conic Section

Let the focus of the conic be (α, β) and the directrix be the line $ax + by + c = 0$, e being the eccentricity, then equation of the conic is the locus of the point $P(h, k)$, such that

$$\sqrt{(h-\alpha)^2 + (k-\beta)^2} = e\left|\frac{ah + bk + c}{\sqrt{a^2 + b^2}}\right|$$

$$(x - \alpha)^2 + (y - \beta)^2 = \frac{e^2(ax + by + c)^2}{a^2 + b^2}$$

PARABOLA

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is equal to its distance from a fixed line (called the directrix).

Standard Forms of the Equation of a Parabola.

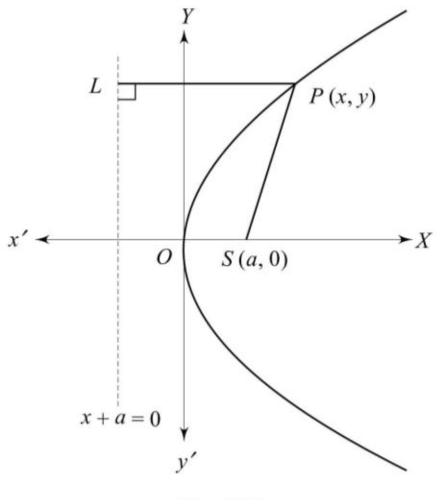

Fig. 18.2

Let the focus S be $(a, 0)$ and directrix be the line $x = -a$, with $a > 0$. If $P(x, y)$ is any point on the locus, then

$$(x - a)^2 + y^2 = (x + a)^2 \Rightarrow y^2 = (x + a)^2 - (x - a)^2 = 4ax$$

$y^2 = 4ax$ is a *standard form* of the equation of a parabola.

Four standard forms of a parabola are

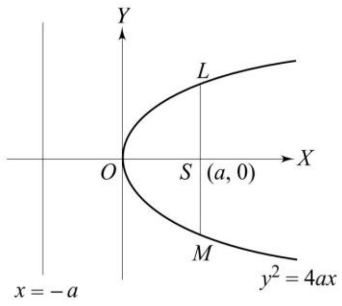

Fig. 18.3 (a)

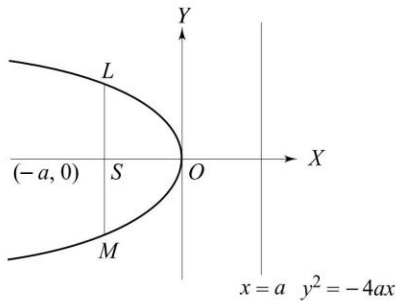

Fig. 18.3 (b)

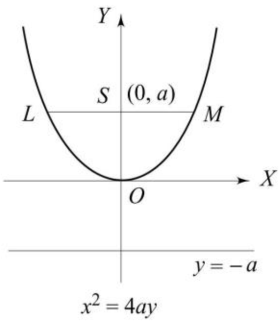

Fig. 18.3 (c)

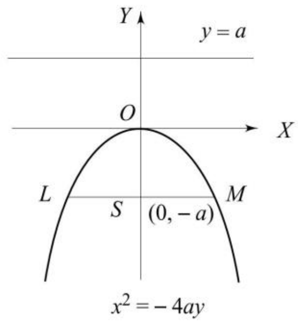

Fig. 18.3 (d)

Illustration 1

Find the equation of the parabola whose focus is $(0, -4)$ and directrix is $y = 4$.

◎ **Solution:** Let $P(x, y)$ be any point on the parabola. $S(0, -4)$ be the focus, then by definition of the parabola.

$$\sqrt{(x-0)^2+(y+4)^2} = \sqrt{(y-4)^2}$$

$$\Rightarrow \quad x^2 = (y-4)^2 - (y+4)^2$$

$$\Rightarrow \quad x^2 = -16y$$

which is the required equation of the parabola.

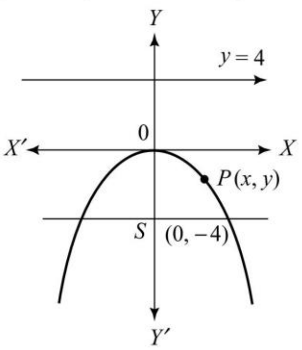

Fig. 18.4

The following terms are used in context of the parabola $y^2 = 4ax$:

1. The point $O(0, 0)$ is the *vertex* of the parabola, and the tangent to the parabola at the vertex is $x = 0$.
2. The line joining the vertex O and the focus $S(a, 0)$ is the *axis* of *the parabola* and its equation is therefore $y = 0$.
3. Any chord of the parabola perpendicular to its axis is called a *double ordinate*.
4. Any chord of the parabola passing through its focus is called a *focal chord*.
5. The focal chord of the parabola perpendicular to its axis is called *its latus rectum*; the length of this latus rectum is therefore $4a$.

Illustration 2

Consider the parabola $y^2 = 12x$. The equation can be written as $y^2 = 4ax$ where $a = 3$. Then

(a) Coordinates of the focus are $(a, 0) = (3, 0)$:

(b) The coordinates of vertex are $(0, 0)$

(c) Equation of the directrix is $x = -a$ i.e. $x = -3$

(d) Equation of the tangent at the vertex is $x = 0$.

(e) Equation of the latus rectum is $x = a$ i.e. $x = 3$

(f) Length of the latus rectum is $4a = 12$.

(g) Let $P(x, 2\sqrt{3x})$ and $P'(x, -2\sqrt{3x})$, $x > 0$ be two points, then for all values of x, PP' is a double ordinate of the parabola.

6. The points on a parabola, the normals at which are concurrent, are called *co-normal points* of the parabola. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

conormal points of the parabola $y^2 = 4ax$, then $y_1 + y_2 + y_3 = 0$.

7. A line which bisects a system of parallel chords of a parabola is called a *diameter* of the parabola.

The following are some *standard results for the parabola* $y^2 = 4ax$:

1. The *parametric equations* of the parabola or the coordinates of any point on it are $x = at^2$, $y = 2at$.
2. The *tangent* to the parabola at (x', y') is $yy' = 2a(x + x')$ and that at $(at^2, 2at)$ is $ty = x + at^2$.
3. The condition that the line $y = mx + c$ is a tangent to the parabola is $c = a/m$ and the equation of any tangent to it (not parallel to the y-axis) is therefore $y = mx + (a/m)$.

Illustration 3

Find the equation of the tangent with slopes 5 to the parabola $y^2 = 20x$, also find the coordinates of the point of contact of the tangent to the parabola.

◎ **Solution:** Equation of the parabola can be written as $y^2 = 4ax$, $a = 5$.
Equation of the tangent with slope 5 is

$$y = 5x + \frac{5}{5} \quad \Rightarrow y = 5x + 1.$$

Parametric coordinates of any point on the parabola are $(5t^2, 10t)$ and the equation of the tangent at this point is

$$ty = x + 5t^2$$

Comparing with $y = 5x + 1$, we get $t = \frac{1}{5}$.

So the coordinates of the point of contact are

$$\left(5\left(\frac{1}{5}\right)^2, 10\left(\frac{1}{5}\right)\right) = \left(\frac{1}{5}, 2\right)$$

4. The *chord of contact* (defined as in circles) of (x', y') w.r.t. the parabola is $yy' = 2a(x + x')$.
5. The *polar* (defined as in circle) of (x', y') w.r.t. the parabola is $yy' = 2a(x + x')$.
6. The *chord with mid-Point* (x', y') of the parabola is $T = S'$, where $T = yy' - 2a(x + x')$ and $S' = y'^2 - 4ax'$.
7. The equation of the *pair of tangents* from (x', y') to the parabola is $T^2 = SS'$. Where $S = y^2 - 4ax$.
8. The *normal* at $(at^2, 2at)$ to the parabola is $y = -tx + 2at + at^3$. If m is the slope of this normal, then its equation is $y = mx - 2am - am^3$, which is the normal to the parabola at $(am^2, -2am)$.

Illustration 4

A normal of slope 4 at a point P on the parabola $y^2 = 28x$, meets the axis of the parabola at Q. Find the length PQ.

◎ **Solution:** $y^2 = 28x = 4ax$, $a = 7$ also $m = 4$, equation of the normal at $P(am^2 - 2am)$ is $y = mx - 2am - am^3$
So coordinates of P are $(112, -56)$ and the equation of the normal is

$$y = 4x - 56 - 448 \quad \text{or} \quad y = 4x - 504.$$

which meets the axis $y = 0$ at Q (126, 0)

and $\quad (PQ)^2 = (126 - 112)^2 + (0 + 56)^2 = 14^2 \times 17$

$\Rightarrow \quad PQ = 14\sqrt{17}$.

9. A *diameter* of the parabola is the locus of the middle points of a system of parallel chords of the parabola and the equation of a diameter is $y = 2a/m$ where m is the slope of the parallel chords which are bisected by it.
10. The equation of a chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.
11. If the chord joining the points having parameters t_1 and t_2 passes through the focus, then $t_1 t_2 = -1$.
12. If the coordinates of one end of a focal chord are $(at^2, 2at)$, then the coordinates of the other end are $(a/t^2, -2a/t)$.

Illustration 5

PP' is a focal chord of the parabola $y^2 = 8x$. If the coordinates of P are (18, 12), find the coordinates of P'.

◎ **Solution:** $y^2 = 8x = 4ax$, $a = 2$.
So, coordinates of the focus are $S(2, 0)$.
Equation of the line joining P and S is $3x - 4y - 6 = 0$ which meets the parabola at points for which $3y^2 - 32y - 48 = 0$
$\Rightarrow \quad y = 12, -4/3$

$y = 12$, represents P. So the coordinates of P' are $\left(\frac{2}{9}, -\frac{4}{3}\right)$

Alternately Coordinates of P are $(at^2, 2at) = (18, 12)$ where $a = 2$, $t = 3$, using the above result. Coordinates of P' are $\left(\frac{a}{t^2}, \frac{-2a}{t}\right) = \left(\frac{2}{9}, -\frac{4}{3}\right)$

13. For the end of the latus rectum, the values of the parameter t are ± 1.
14. The tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2; 2at_2)$ intersect at $(at_1 t_2, a(t_1 + t_2))$.
15. The tangents at the extremities of any focal chord intersect at right angles on the directrix.

Illustration 6

Show that the tangents at the extremities of the focal chord PP' in the Illustration 5 are at right angles and intersect on the directrix of the parabola.

◎ **Solution:** Equations of the tangents at P and P' are

$$12y = 4(x + 18) \text{ and } y\left(-\frac{4}{3}\right) = 4\left(x + \frac{2}{9}\right)$$

$\Rightarrow x - 3y + 18 = 0$ and $3x + y + 2/3 = 0$

which are perpendiculars and intersect at $\left(-2, \frac{16}{3}\right)$ which lies on the directrix $x = -2$ of the parabola.

16. The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.

17. The area of the triangle formed by any three points on the parabola is twice the area of the triangle formed by the tangents at these points.

18. The circle described on any focal chord of a parabola as diameter touches the directrix.

19. If the normal at the point $(at_1^2, 2at_1)$ meets the parabola again at $(at_2^2, 2at_2)$ then $t_2 = -t_1 - 2/t_1$.

20. Three normals can be drawn from a point (x_1, y_1) to the parabola. The points where these normals meet the parabola are called *feet of the normals* or *conormal points*. The sum of the slopes of these normals is zero and the sum of the ordinates of the feet of the normals is also zero.

21. If the normals at 't_1' and 't_2' meet on the parabola then $t_1 t_2 = 2$.

22. The centroid of the triangle formed by the conormal points on the parabola lies on the axis of the parabola.

23. The pole of a focal chord of the parabola lies on its directrix.

24. A diameter of the parabola is parallel to its axis and the tangent at the point where it meets the parabola is parallel to the system of chords bisected by the diameter.

25. The semi-latus rectum of the parabola is the harmonic mean between the segments of any focal chord of the parabola.

26. If the tangent and normal at any point P on the parabola meet the axis of the parabola in T and G respectively, then

(i) $ST = SG = SP$, S being the focus

(ii) $\angle PSK = \pi/2$, where K is the point where the tangent at P meets the directrix.

(iii) The tangent at P is equally inclined to the axis of the parabola and the focal distance of P.

Illustration 7

Find the locus of the intersection of the perpendicular tangents to the parabola $x^2 = 4y$.

Let $x = my + c$ be a tangent to the parabola. Then the roots of the equation $(my + c)^2 = 4y$ are equal, as the line meets the parabola at a single point.

$\Rightarrow (2mc - 4)^2 = 4m^2c^2 \Rightarrow c = \frac{1}{m}$.

So equation of any tangent to the parabola is $x = my + \frac{1}{m}$

and equation of the perpendicular tangent is

$$x = -\frac{1}{m}y - m$$

Solving for the point of intersection we get

$$y + 1 = 0$$

So the perpendicular tangents to the parabola intersect on the line $y + 1 = 0$ which is the directrix of the parabola.

Illustration 8

The normal at a point P (36, 36) on the parabola $y^2 = 36x$ meets the parabola again at a point Q. Find the coordinates of Q.

◎ **Solution:** Parabola is $y^2 = 4ax$, $a = 9$

and $P(36, 36) = (at^2, 2at)$, $t = 2$

Equation of the normal at $(at^2, 2at)$ is

$$y = -tx + 2at + at^3$$

So equation of the normal at P is

$$y = -2x + 36 + 72$$

or $\quad y = -2x + 108$

Which meets the parabola at points for which

$(-2x + 108)^2 = 36x$

$\Rightarrow \quad x^2 - 117x + (54)^2 = 0$

$\Rightarrow \quad x = 36, 81.$

$x = 36$, gives the point P, so other end Q of the normal at P is $(81, -54)$

Note: y-coordinate of Q cannot be +ve

Coordinates of Q are $(at'^2, 2at')$

Where $t' = -3 = -t - \frac{2}{t}$.

(using result 19)

Illustration 9

If the normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ intersect on the parabola, then find the value of $t_1 t_2$.

◎ **Solution:** Equation of normal at $(at^2, 2at)$ to the parabola is $y = -tx + 2at + at^3$.

If it passes through (h, k) then

$$k = -th + 2at + at^3$$

$$\Rightarrow \quad at^3 + (2a - h)t - k = 0 \qquad (1)$$

Which gives three values of t showing that the normal at three points with $t = t_1, t_2, t_3$ pass through (h, k).

So if the normals at P and Q intersect on the parabola, then $h = at_3^2$, $k = 2at_3$.

which proves result 21.

From (7) we have $t_1 t_2 t_3 = \frac{k}{a} = \frac{2at_3}{a}$

$$\Rightarrow \quad t_1 t_2 = 2.$$

Illustration 10

If the tangent and normal at any point P on the parabola meet the axis of the parabola in T and G respectively, then show that the focus S of the parabola is the centre of the circle passing through T, P and G.

◎ **Solution:** Equations of the tangent and normal at $P(at^2, 2at)$ are respectively

$$ty = x + at^2$$
$$y = -tx + 2at + at^3$$

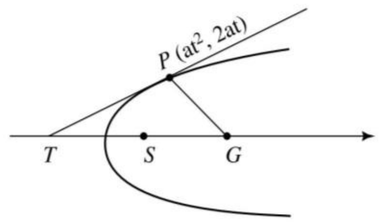

Fig. 18.5

So the coordinates of T are $(-at^2, 0)$ and of G are $(2a + at^2, 0)$

Focus S is $(a, 0)$

$SG = ST = SP = a(1 + t^2)$

Which shows that S is the centre of the circle through T, P and G.

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: The point of intersection of the normals to the parabola $y^2 = 4x$ at the ends of its latus rectum is

(a) (0, 2) (b) (3, 0)
(c) (0, 3) (d) (2, 0)

Ans. (b)

◎ **Solution:** Any point on the parabola is $(at^2, 2at)$ where $a = 1$. For the latus rectum $x = a \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$. So ends of latus rectum are $(1, \pm 2)$.

Equations of the normals at these points is given by $y = -tx + 2at + at^3$

$$\Rightarrow \quad x + y - 3 = 0 \text{ and } x - y - 3 = 0$$

which intersect at (3, 0).

Example 2: Equation of the line joining the foci of the parabolas $y^2 = 4x$ and $x^2 = -4y$ is

(a) $x + y - 1 = 0$ (b) $x - y - 1 = 0$
(c) $x - y + 1 = 0$ (d) $x + y + 1 = 0$

Ans. (b)

◎ **Solution:** Foci of the parabolas are (1, 0) and (0, – 1) and the equation of the line joining them is

$$y = x - 1 \quad \text{or} \quad x - y - 1 = 0$$

Example 3: The tangent at a point P on the parabola $y^2 = 8x$ meets the directrix of the parabola at Q such that distance of Q from the axis of the parabola is 3. Then the coordinates of P cannot be

(a) (2, 4) (b) (8, 8)
(c) (1/2, 2) (d) (8 – 8)

Ans. (a)

◎ **Solution:** Equation of the tangent at $P(2t^2, 4t)$ to the parabola is $ty = x + 2t^2$ which meets the directrix $x = -2$ of the parabola at Q for which $y = \frac{2t^2 - 2}{t}$. So that $\frac{2t^2 - 2}{t} = \pm 3$.

$$\Rightarrow \quad 2t^2 - 3t - 2 = 0 \quad \text{or} \quad 2t^2 + 3t - 2 = 0$$

$$\Rightarrow \quad (2t + 1)(t - 2) = 0 \quad \text{or} \quad (2t - 1)(t + 2) = 0$$

$$\Rightarrow \quad t = \pm 2, \pm \frac{1}{2}$$

So, the coordinates of P are $(8, \pm 8)$ or $\left(\frac{1}{2}, \pm 2\right)$.

Example 4: $y^2 = 16x$ is a parabola and $x^2 + y^2 = 16$ is a circle. Then

(a) circle passes through the vertex of the parabola
(b) circle touches the parabola at the vertex
(c) circle passes through the focus of the parabola
(d) circle lies inside the parabola.

Ans. (c)

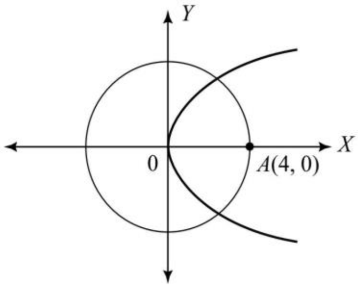

Fig. 18.6

◎ **Solution:** Centre of the circle is the origin, the vertex of the parabola and the focus of the parabola is A (4, 0). The circle passes through A.

◉ **Example 5:** The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$

(c) $\frac{1}{8}$ (d) $\frac{2}{3}$

Ans. (a)

◎ **Solution:** Equation of the tangent at $(t^2, 2t)$ to the parabola $y^2 = 4x$ is $ty = x + t^2$ (1)

and the tangent at $(-16t', -8t'^2)$ to the parabola $x^2 = -32y$ is

$t'x = y - 8t'^2$ (2)

Since (1) and (2) represent the same line, comparing we get

$$\frac{1}{-t'} = \frac{-t}{1} = \frac{t^2}{-8t'^2} \Rightarrow t = 2$$

Hence the required slope of the tangent is $\frac{1}{t} = \frac{1}{2}$.

Alternate Solution

$y = mx + \frac{1}{m}$ is a tangent to the parabola $y^2 = 4x$. If it touches the parabola, $x^2 = -32y$, then the roots of the equation $x^2 + 32\left(mx + \frac{1}{m}\right) = 0$ are equal.

$$\Rightarrow \quad (32m)^2 = 4 \times 32 \times \frac{1}{m} \Rightarrow m = \frac{1}{2}$$

◉ **Example 6:** Let L_1 be the length of the common chord of the curves $x^2 + y^2 = 9$ and $y^2 = 8x$, and L_2 be the length of the latus rectum of $y^2 = 8x$, then

(a) $L_1 > L_2$ (b) $L_1 = L_2$

(c) $L_1 < L_2$ (d) $\frac{L_1}{L_2} = \sqrt{2}$

Ans. (c)

◎ **Solution:** For the points of intersection of the curves $x^2 + y^2 = 9$ and $y^2 = 8x$, we have

$x^2 + 8x - 9 = 0 \quad \Rightarrow \quad x = -9$ or $x = 1$

Since $x > 0$ for $y^2 = 8x$, the points of intersection are $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$ so that the length of the common chord L_1 is $4\sqrt{2}$.

L_2 = length of the latus rectum of $y^2 = 8x$ is 8. So $L_1 < L_2$

◉ **Example 7:** If m is the slope of a common tangent of the parabola $y^2 = 16x$ and the circle $x^2 + y^2 = 8$, then m^2 is equal to

(a) 1 (b) 2

(c) 4 (d) 8

Ans. (a)

◎ **Solution:** Equation of a tangent to the parabola $y^2 = 16x$ is $y = mx + \frac{4}{m}$.

If this touches the circle of $x^2 + y^2 = 8$, then the distance of the centre (0, 0) of the circle from this tangent is equal to the radius $2\sqrt{2}$ of the circle.

$$\Rightarrow \quad \frac{\frac{4}{m}}{\sqrt{1+m^2}} = 2\sqrt{2}$$

$$\Rightarrow \quad 2 = m^2 + m^4$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0 \Rightarrow m^2 = 1.$$

◉ **Example 8:** Equation of a normal to the parabola $y^2 = 32x$ passing through its focus is

(a) $x = 0$ (b) $y = 0$

(c) $x + y - 8 = 0$ (d) $x - y - 8 = 0$

Ans. (b)

◎ **Solution:** Equation of a normal to the parabola is

$y = mx - 2am - am^3$, $4a = 32 \Rightarrow a = 8$

which passes through the focus (8, 0)

if $0 = 8m - 16m - 8m^3 = 0$

$\Rightarrow m^3 + m = 0 \Rightarrow m(m^2 + 1) = 0 \Rightarrow m = 0$

and the required equation of the normal is $y = 0$.

◉ **Example 9:** P_1: $y^2 = 49x$ and P_2: $x^2 = 4ay$ are two parabolas. Equation of a tangent to the parabola P_1 at a point where it intersects the parabola P_2 is:

(a) $2x - y - 4a = 0$ (b) $y = 0$

(c) $x - 2y + 4a = 0$ (d) $x - y = 0$

Ans. (c)

◎ **Solution:** Two parabolas intersect at the points (0, 0) and $(4a, 4a)$.

Equation of the tangent to P_1 at (0, 0) is $x = 0$ and at $(4a, 4a)$ is

$$y(4a) = 2a(x + 4a)$$

$$\Rightarrow \quad x - 2y + 4a = 0$$

◉ **Example 10:** A point R divides the line segment joining the points $P(1, 3)$ and $Q(1, 1)$ in the ratio 1: λ. If R is an interior point of the parabola $y^2 = 4x$, then λ can take a value in the interval.

(a) $-1 < \lambda < \frac{3}{5}$ (b) $-\frac{3}{5} < \lambda < 5$

(c) $-2 < \lambda < 2$ (d) $-\frac{3}{5} < \lambda < 1$

Ans. (d)

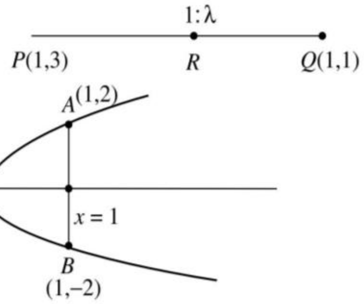

Fig. 18.7

◎ **Solution:** Coordinates of R are

$$= \left(\frac{\lambda+1}{\lambda+1}, \frac{3\lambda+1}{\lambda+1}\right)$$

$$= \left(1, \frac{3\lambda+1}{\lambda+1}\right)$$

⇒ For all values of λ, x-coordinate of R is 1 and y-coordinate of R varies between –2 and 2.

$$\Rightarrow \quad -2 < \frac{3\lambda+1}{\lambda+1} < 2.$$

$$\Rightarrow \quad -\frac{3}{5} < \lambda < 1$$

◉ **Example 11:** A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is

(a) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (b) $(2, -4)$

(c) $(2, 4)$ (d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

Ans. (d)

◎ **Solution:** $y^2 = 18x \Rightarrow 2y\frac{dy}{dx} = 18$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{9}{y}$$

We are given $\frac{dy}{dx} = 2$

so $\frac{9}{y} = 2 \Rightarrow y = \frac{9}{2}$.

and $x = \frac{1}{18}y^2 \Rightarrow x = \frac{9}{8}$

and the required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$

◉ **Example 12:** The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again at the point $(bt_2^2, 2bt_2)$, then

(a) $t_2 = -t_1 + \frac{2}{t_1}$ (b) $t_2 = t_1 - \frac{2}{t_1}$

(c) $t_2 = t_1 + \frac{2}{t_1}$ (d) $t_2 = -t_1 - \frac{2}{t_1}$

Ans. (d)

◎ **Solution:** Equation of the parabola is $y^2 = 4bx$ and the equation of the tangent at $(bt_1^2, 2bt_1)$ is

$$y = -t_1x + 2bt_1 + bt_1^3$$

which passes through $(bt_2^2, 2bt_2)$

$$\Rightarrow \quad 2bt_2 = -t_1(bt_2^2) + 2bt_1 + bt_1^3$$

$$\Rightarrow \quad 2(t_2 - t_1) = -t_1(t_2^2 - t_1^2)$$

$$\Rightarrow \quad t_2 = -t_1 - \frac{2}{t_1}.$$

◉ **Example 13:** If P be the point (1, 0) and Q, a point on the locus $y^2 = 8x$. The locus of the mid point of PQ is:

(a) $x^2 + 4y + 2 = 0$ (b) $x^2 - 4y + 2 = 0$

(c) $y^2 - 4x + 2 = 0$ (d) $y^2 + 4x + 2 = 0$

Ans. (c)

◎ **Solution:** Let the coordinates of Q be $(2t^2, 4t)$

coordinates of the mid point of PQ are $\left(\frac{2t^2+1}{2}, 2t\right)$

Let $h = \frac{2t^2+1}{2}$, $k = 2t$

Eliminating t, we get $4h = k^2 + 2$

So the required locus is $y^2 - 4x + 2 = 0$

◉ **Example 14:** A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then the vertex of the parabola is at

(a) (2, 0) (b) (0, 2)

(c) (1, 0) (d) (0, 1)

Ans. (c)

◎ **Solution:** Vertex is equidistant from the focus and the directrix and hence its coordinates are (1, 0).

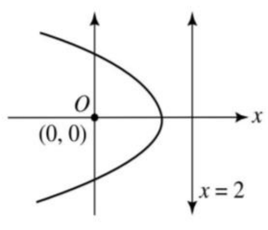

Fig. 18.8

Example 15: A chord is drawn through the focus of the parabola $y^2 = 6x$ such than its distance from the vertex of this parabola is $\frac{\sqrt{5}}{2}$, then its slope can be

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{3}}$

Ans. (a)

Solution: Focus of the parabola is $\left(\frac{3}{2}, 0\right)$

Let the equation of the chord be

$$y = m\left(x - \frac{3}{2}\right)$$

Its distance from the vertex (0, 0) is $\left|\frac{-\frac{3m}{2}}{\sqrt{1+m^2}}\right| = \frac{\sqrt{5}}{2}$

$\Rightarrow \quad 9m^2 = 5(1 + m^2)$

$\Rightarrow \quad m^2 = \frac{5}{4} \Rightarrow m = \frac{\sqrt{5}}{2}$

LEVEL 1

Straight Objective Type Questions

Example 16: The focus of the parabola $4y^2 + 12x - 20y + 67 = 0$ is

(a) (– 7/2, 5/2) (b) (– 3/4, 5/2)

(c) (– 17/4, 5/2) (d) (5/2, – 3/4)

Ans. (c)

Solution: The given equation of the parabola can be written as

$y^2 - 5y = -3x - 67/4 \quad \Rightarrow \quad (y - 5/2)^2 = -3\,(x + 7/2)$

$\Rightarrow \quad Y^2 = 4aX$ where $Y = y - 5/2$, $X = x + 7/2$ and $a = -3/4$

The focus of $Y^2 = 4aX$ is $(X, Y) = (a, 0) = (-3/4, 0)$

$\Rightarrow \quad x + 7/2 = -3/4,\ y - 5/2 = 0 \quad \Rightarrow \quad x = -17/4,\ y = 5/2$

Therefore, required focus is (– 17/4, 5/2)

Example 17: A line bisecting the ordinate PN of a point $P\,(at^2, 2at)$, $t > 0$, on the parabola $y^2 = 4ax$ is drawn parallel to the axis to meet the curve at Q. If NQ meets the tangent at the vertex at the point T, then the coordinates of T are.

(a) $(0, 4at/3)$ (b) $(0, 2at)$

(c) $(at^2/4, at)$ (d) $(0, at)$

Ans. (a)

Solution: Equation of the line parallel to the axis and bisecting the ordinate PN of the point $P\,(at^2, 2at)$ is $y = at$ which meets the parabola $y^2 = 4ax$ at the point $Q\left(\frac{at^2}{4}, at\right)$.

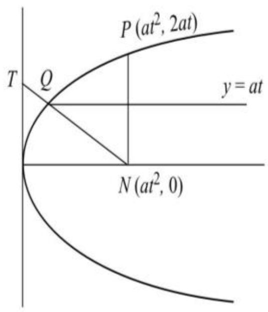

Fig. 18.9

Coordinates of N are $(at^2, 0)$.

Equation of NQ is

$$y = \frac{0 - at}{at^2 - at^2/4}\left(x - at^2\right)$$

which meets the tangent at the vertex, $x = 0$, at the point $y = 4at/3$.

Example 18: If P, Q, R are three points on a parabola $y^2 = 4ax$ whose ordinates are in geometrical progression, then the tangents at P and R meet on

(a) the line through Q parallel to x-axis

(b) the line through Q parallel to y-axis

(c) the line joining Q to the vertex

(d) the line joining Q to the focus.

Ans. (b)

Solution: Let the coordinates of P, Q, R be $(at_i^2, 2at_i)$ $i = 1, 2, 3$ having ordinates in G.P. So that t_1, t_2, t_3 are also in G.P. i.e. $t_1t_3 = t_2^2$. Equations of the tangents at P and R are

$t_1\,y = x + at_1^2$ and $t_3\,y = x + at_3^2$, which intersect

at the point $\frac{x + at_1^2}{t_1} = \frac{x + at_3^2}{t_3} \quad \Rightarrow \quad x = at_1t_3 = at_2^2$

which is a line through Q parallel to y-axis.

Example 19: The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

(a) $x = -a$ (b) $x = -a/2$

(c) $x = 0$ (d) $x = a/2$

Ans. (c)

Solution: The focus of the parabola $y^2 = 4ax$ is $S\,(a, 0)$, let $P\,(at^2, 2at)$ be any point on the parabola then coordinates of the mid-point of SP are given by

$$x = \frac{a(t^2+1)}{2}, \quad y = \frac{2at+0}{2}$$

Eliminating 't' we get the locus of the mid-point as $\quad y^2 = 2ax - a^2$ or $y^2 = 2a(x - a/2) \quad (1)$

which is a parabola of the form $Y^2 = 4AX \quad (2)$

Where $Y = y$, $X = x - a/2$ and $A = a/2$

Equation of the directrix of (2) is $X = -A$

So equation the directrix of (1) is $x - a/2 = -a/2$

$\Rightarrow \quad x = 0$

Example 20: Equation of a common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$ (b) $y = 2x + 1$
(c) $2y = x + 8$ (d) $y = x + 2$

Ans. (d)

Solution: Equation of a tangent at $(at^2, 2at)$ to $y^2 = 8x$ is $ty = x + at^2$ where $4a = 8$ *i.e.* $a = 2$

$\Rightarrow \quad ty = x + 2t^2$ which intersects the curve $xy = -1$ at the points given by $\frac{x(x+2t^2)}{t} = -1$ clearly $t \neq 0$

or $x^2 + 2t^2x + t = 0$ and will be a tangent to the curve if the roots of this quadratic equation are equal, for which $4t^4 - 4t = 0 \Rightarrow t = 0$ or $t = 1$ and an equation of a common tangent is $y = x + 2$.

Example 21: The tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x + b)$ at Q and R, the coordinates of the mid-point of QR are

(a) $(x_1 - a, y_1 + b)$ (b) (x_1, y_1)
(c) $(x_1 + b, y_1 + a)$ (d) $(x_1 - b, y_1 - b)$

Ans. (b)

Solution: Equation of the tangent at $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

or $\quad 2ax - y_1y + 2ax_1 = 0 \quad$ (i)

If $M(h, k)$ is the mid-point of QR, then equation of QR a chord of the parabola $y^2 = 4a(x + b)$ in terms of its mid-point is $ky - 2a(x + h) - 4ab = k^2 - 4a(h + b)$

(using $T = S'$) $\quad$ or $\quad 2ax - ky + k^2 - 2ah = 0 \quad$ (ii)

Since (i) and (ii) represent the same line, we have

$$\frac{2a}{2a} = \frac{y_1}{k} = \frac{2ax_1}{k^2 - 2ah}$$

$\Rightarrow \quad k = y_1$ and $k^2 - 2ah = 2ax_1$

$\Rightarrow \quad y_1^2 - 2ah = 2ax_1 \quad \Rightarrow \quad 4ax_1 - 2ax_1 = 2ah$

(as $P(x_1, y_1)$ lies on the parabola $y^2 = 4ax$)

$\Rightarrow \quad h = x_1$ so that $h = x_1$, $k = y_1$ and the mid point of QR is (x_1, y_1)

Example 22: AB is a chord of the parabola $y^2 = 4ax$ with the end A at the vertex of the given parabola. BC is drawn perpendiculars to AB meeting the axis of the parabola at C. The projection of BC on this axis is

(a) a (b) $2a$
(c) $4a$ (d) $8a$

Ans. (c)

Solution: Draw BD perpendicular to the axis of the parabola. Let the coordinates of B be (x, y) then slope of AB is given by $\tan\theta = y/x$

Projection of BC on the axis of the parabola is

$DC = BD \tan\theta$

$$= y(y/x) = y^2/x = 4ax/x = 4a$$

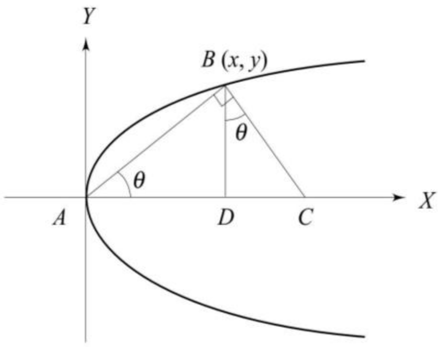

Fig. 18.10

Example 23 Equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is

(a) $\sqrt{3}\ y = 3x + 1$ (b) $\sqrt{3}\ y = -(x + 3)$
(c) $\sqrt{3}\ y = x + 3$ (d) $\sqrt{3}\ y = -(3x + 1)$

Ans. (c)

Solution: Equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + 1/m$.

It will touch the circle $(x - 3)^2 + y^2 = 9$ whose centre is $(3, 0)$ and radius is 3 if

$$\left|\frac{0 + m(3) + (1/m)}{\sqrt{1+m^2}}\right| = 3$$

or if $\quad (3m + 1/m)^2 = 9(1 + m^2)$

or if $\quad 9m^2 + 6 + 1/m^2 = 9 + 9m^2$

or if $\quad m^2 = 1/3$ i.e. $m = \pm 1/\sqrt{3}$

As the tangent is above the x-axis, we take $m = 1/\sqrt{3}$ and thus the required equation is $\sqrt{3}\ y = x + 3$.

Example 24: The point of intersection of the tangents to the parabola $y^2 = 4x$ at the points where the circle $(x - 3)^2 + y^2 = 9$ meets the parabola, other than the origin, is

(a) $(-2, 0)$ (b) $(1, 0)$
(c) $(0, 0)$ (d) $(-1, -1)$

Ans. (a)

◎ **Solution:** The circle meets the parabola at points given by $(x-3)^2 + 4x = 9$

$\Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, x = 2$. But $x = 0$ gives the origin so we take $x = 2$ and $y = \pm 2\sqrt{2}$. Equation of the tangents to the parabola at $(2, 2\sqrt{2})$ and $(2, -2\sqrt{2})$ are respectively.

$$y\,(2\sqrt{2}) = 2(x+2) \text{ and } y(-2\sqrt{2}) = 2(x+2)$$

Solving these we get $y = 0$ and $x = -2$.

◉ **Example 25:** Equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is

(a) $x = -1$ (b) $x = 1$
(c) $x = -3/2$ (d) $x = 3/2$

Ans. (d)

◎ **Solution:** Given equation can be written as

$$(y+2)^2 = -4x + 2 = -4(x - 1/2)$$

which is of the form $Y^2 = 4aX$

where $Y = y + 2, X = x - 1/2, a = -1$

The directrix of the parabola $Y^2 = 4aX$ is $X = -a$

$\Rightarrow \quad x - 1/2 = -(-1) \quad \Rightarrow \quad x = 3/2$

is the equation of the directrix of the given parabola

◉ **Example 26:** If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then it touches the parabola

(a) $y^2 = -36x$ (b) $y^2 = -12x$
(c) $y^2 = -9x$ (d) none of these

Ans. (a)

◎ **Solution:** Since $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$, taking $a = 3$ and $m = -1$ we get

$$y = -x - 2\,(3)(-1) - 3\,(-1)^3$$

$\Rightarrow \quad x + y = 9$ is a normal to the parabola $y^2 = 12x$.

Suppose it touches the parabola $y^2 = 4ax$.

Equation of a tangent to the parabola $y^2 = 4ax$ is

$$y = mx + a/m$$

If it represents the line $x + y = 9$, then

$$m = -1 \text{ and } a/m = 9 \quad \Rightarrow \quad a = -9$$

So an equation of the required parabola is $y^2 = 4(-9)x$ or $y^2 = -36x$

◉ **Example 27:** If the normal chord at a point 't' on the parabola $y^2 = 4ax$ subtends a right angle at the vertex, then a value of t is

(a) 4 (b) $\sqrt{3}$
(c) $\sqrt{2}$ (d) 1

Ans. (c)

◎ **Solution:** Equation of the normal at 't' to the parabola $y^2 = 4ax$ is

$$y = -tx + 2at + at^3 \qquad \text{(i)}$$

The joint equation of the lines joining the vertex (origin) to the points of intersection of the parabola and the line (i) is

$$y^2 = 4ax\left[\frac{y+tx}{2at+at^3}\right]$$

$\Rightarrow \quad (2t + t^3)y^2 = 4x\,(y + tx)$

$\Rightarrow \quad 4t\,x^2 - (2t + t^3)\,y^2 + 4xy = 0$

Since these lines are at right angles co-efficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \quad 4t - 2t - t^3 = 0 \quad \Rightarrow \quad t^2 = 2$

For $t = 0$, the normal line is $y = 0$, i.e. axis of the parabola which passes through the vertex (0, 0).

◉ **Example 28:** The slopes of the normals to the parabola $y^2 = 4ax$ intersecting at a point on the axis of the parabola at a distance $4a$ from its vertex are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Ans. (a)

◎ **Solution:** Equation of any normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3.$$

If it passes through $(4a, 0)$, the point on the axis $y = 0$, at a distance $4a$ from the vertex (0,0) then $m = 0, \pm\sqrt{2}$

Therefore the slopes of the required normals are $-\sqrt{2}$, 0, $\sqrt{2}$; which are in A.P.

◉ **Example 29:** If the focus of a parabola divides a focal chord of the parabola in segments of length 3 and 2, the length of the latus rectum of the parabola is

(a) 3/2 (b) 6/5
(c) 12/5 (d) 24/5

Ans. (d)

◎ **Solution:** Let $y^2 = 4ax$ be the equation of the parabola, then the focus is $S\,(a, 0)$. Let $P\,(at^2{}_1, 2at_1)$ and $Q\,(at^2{}_2, 2at_2)$ be vertices of a focal chord of the parabola, then $t_1\,t_2 = -1$.
Let $SP = 3$, $SQ = 2$

$$SP = \sqrt{a^2(1-t_1^2)^1 + 4a^2t_1^2} = a(1 + t_1^2) = 3 \qquad \text{(i)}$$

and $$SQ = a\left(1 + \frac{1}{t_1^2}\right) = 2 \qquad \text{(ii)}$$

From (*i*) and (ii) we get $t_1^2 = 3/2$ and $a = 6/5$

Hence the length of the latus rectum = 24/5.

◉ **Example 30:** P is a point on the parabola whose ordinate equals its abscissa. A normal is drawn to the parabola at P to meet it again at Q. If S is the focus of the parabola then the product of the slopes of SP and SQ is

(a) -1 (b) 1/2
(c) 1 (d) 2

Ans. (a)

◎ **Solution:** Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$, then $at^2 = 2at \quad \Rightarrow \quad t = 2$ and thus the coordinates of P are $(4a, 4a)$.

Equation of the normal at P is $y = -tx + 2at + at^3$

$\Rightarrow \quad y = -2x + 4a + 8a \Rightarrow 2x + y = 12a \quad$ (i)

which meets the parabola $y^2 = 4ax$ at points given by

$$y^2 = 2a(12a - y) \Rightarrow y^2 + 2ay - 24a^2 = 0$$

$\Rightarrow \quad y = 4a$ or $y = -6a$

$y = 4a$ corresponds to the point P

and $y = -6a \Rightarrow x = 9a$ from (i)

So that the coordinates of Q are $(9a, -6a)$. Since the coordinate of the focus S are $(a, 0)$, slope $SP = 4/3$ and slope of $SQ = -6/8$. Product of the slopes $= -1$.

Example 31: Equation of the directrix of the parabola whose focus is (0, 0) and the tangent at the vertex is $x - y + 1 = 0$ is

(a) $x - y = 0$ (b) $x - y - 1 = 0$
(c) $x - y + 2 = 0$ (d) $x + y - 1 = 0$

Ans. (c)

Solution: Since the directrix is parallel to the tangent at the vertex, let the equation of the directrix be.

$$x - y + \lambda = 0$$

But the distance between the focus and directrix is twice the distance between the focus and the tangent at the vertex.

Therefore $\quad \dfrac{0+0+\lambda}{\sqrt{1+1}} = 2\times\dfrac{0-0+1}{\sqrt{1+1}}$

$\because$ focus lies on the same side of the directrix as the tangent at the vertex of the parabola.

$\Rightarrow \quad \lambda = 2$, and the required equation is $x - y + 2 = 0$

Example 32: The common tangents to the circle $x^2 + y^2 = a^2/2$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola

(a) $x^2 = 4ay$ (b) $x^2 = -4ay$
(c) $y^2 = -4ax$ (d) $y^2 = 4a(x + a)$

Ans. (c)

Solution: Equation of a tangent to the parabola $y^2 = 4ax$ is $y = mx + a/m$.

If it touches the circle $x^2 + y^2 = a^2/2$

$$\frac{a}{m} = \left(\frac{a}{\sqrt{2}}\right)\sqrt{1+m^2} \Rightarrow 2 = m^2(1+m^2)$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0 \Rightarrow (m^2 - 1)(m^2 + 2) = 0$

$\Rightarrow \quad m^2 = 1 \Rightarrow m = \pm 1$

Hence the common tangents are $y = x + a$ and $y = -x - a$ which intersect at the point $(-a, 0)$ which is the focus of the parabola $y^2 = -4ax$.

Example 33: The locus of the vertices of the family of parabolas

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a \text{ is}$$

(a) $xy = 64/105$ (b) $xy = 105/64$
(c) $xy = 3/4$ (d) $xy = 35/16$

Ans. (b)

Solution: Equation of the parabola can be written as

$$\frac{y}{a} = \left(\frac{ax}{\sqrt{3}} + \frac{\sqrt{3}}{4}\right)^2 - \frac{3}{16} - 2$$

or $\quad \left(x + \dfrac{3}{4a}\right)^2 = \dfrac{a^2}{3a}\left(y + \dfrac{35}{16}a\right)$

vertex is $x = -3/4a$, $y = -35a/16$

Locus of the vertex is $xy = 105/64$.

Example 34: An equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is

(a) $(-1, 1)$ (b) $(0, 2)$
(c) $(2, 4)$ (d) $(-2, 0)$

Ans. (d)

Solution: $y = mx + a/m$ is a tangent for all values of m where $a = 2$.

For $m = 1$, $y = x + 2$ so the equation of the tangent perpendicular to the given tangent is $y = -x - 2$. The required point is the point of intersection of these tangents i.e., $(-2, 0)$. Alternately perpendicular tangents to a parabola intersect on the directrix. $x + 2 = 0$

Example 35: If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is

(a) $x = -1$ (b) $2x - 1 = 0$
(c) $x = 1$ (d) $2x + 1 = 0$

Ans. (a)

Solution: Equation of a tangent to the parabola is $y = mx + 1/m$. If it passes through $P(h, k)$, then $k = mh + 1/m$ $\Rightarrow m^2h - mk + 1 = 0$ which gives the slopes of two tangents passing through P. As these are at right angles, product of the slopes $= -1 \Rightarrow 1/h = -1$

Locus of P is $x = -1$ which is the directrix of the parabola.

Example 36: The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is

(a) $\dfrac{3\sqrt{2}}{5}$ (b) $\dfrac{\sqrt{3}}{4}$

(c) $\dfrac{3\sqrt{2}}{8}$ (d) $\dfrac{2\sqrt{3}}{8}$

Ans. (c)

Solution: Any point on the curve is (t^2, t) whose distance from the given line is

$$\left|\frac{t^2 - t + 1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}\left|\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}\right| \geq \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$$

Example 37: If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabola $y^2 = 4ax$ and $x^2 = 4ay$, then

(a) $d^2 + (2b - 3c)^2 = 0$ (b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b + 3c)^2 = 0$ (d) $d^2 + (3b - 2c)^2 = 0$

Ans. (c)

Solution: Given parabolas intersect at (0, 0) and (4*a*, 4*a*). If the given line passes through these points, then

$$d = 0 \text{ and } 2b(4a) + 3c(4a) = 0$$

$\Rightarrow$ $2b + 3c = 0$

Hence $d^2 + (2b + 3c)^2 = 0$.

Example 38: The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the parabola $y^2 = 4ax$ is

(a) $xy = 4a$ (b) $xy = a$
(c) $xy = a^2$ (d) $ax^3 + 2ax + y = 0$

Ans. (b)

Solution: Equation of any tangent with slope *m* to the parabola is $y = mx + \frac{a}{m}$

Comparing with $y = \alpha x + \beta$, we get

$$m = \alpha, \frac{a}{\alpha} = \beta \Rightarrow \alpha\beta = a.$$

and the required locus is $xy = a$.

Example 39: The point of intersection of the normals to the parabola $y^2 = 4x$ at the ends of its latus rectum is

(a) (0, 2) (b) (3, 0)
(c) (0, 3) (d) (2, 0)

Ans. (b)

Solution: Equation of the latus rectum is $x = 1$ and the coordinates of the ends of the latus rectum are (1, 2) and (1, –2). Equations of the normals at these points is $y = -tx + 2at + at^3$, where $t = \pm 1$, $a = 1 \Rightarrow y + x = 3$ and $y - x = -3$ which intersect at (3, 0).

Example 40: Two tangents are drawn from a point (–2, –1) to the curve $y^2 = 4x$. If α is the angle between them, then $|\tan\alpha|$ is equal to:

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) 3.

Ans. (d)

Solution: Equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$.

If it passes through (–2, –1), then

$$-1 = -2m + \frac{1}{m}$$

$$\Rightarrow 2m^2 - m - 1 = 0$$

$$\Rightarrow (2m + 1)(m - 1) = 0 \Rightarrow m = 1, \frac{-1}{2}$$

So $|\tan\alpha| = \left|\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}}\right| = 3$

Example 41: Let *O* be the vertex and *Q* be any point on the parabola $x^2 = 8y$. If the point *P* divides the line segment *OQ* internally in the ratio 1:3, then the locus of *P* is:

(a) $x^2 = y$ (b) $y^2 = x$
(c) $y^2 = 2x$ (d) $x^2 = 2y$

Ans. (d)

Solution: Let the coordinate of *Q* be (x', y') and that of *P* be (*h*, *k*), then

$$h = \frac{1}{4}x', \ k = \frac{1}{4}y'$$

As (x', y') lies on the parabola $x^2 = 8y$

we get $(4h)^2 = 8(4k)$

$\Rightarrow$ $h^2 = 2k$

Locus of *P*(*h*, *k*) is $x^2 = 2y$

Example 42: Let $y^2 = 16x$ be a given parabola and *L* be an extremity of its latus rectum in the first quadrant. If a chord is drawn through *L* with slope-1, then the length of this chord is:

(a) 32 (b) $16\sqrt{2}$
(c) $16\sqrt{3}$ (d) $32\sqrt{2}$

Ans. (d).

Solution: Equation of the latus rectum is $x = 4$ and the coordinates of *L* are (4, 8). Equation of the chord through *L* with slope-1 is $y - 8 = -(x - 4)$

$\Rightarrow x + y = 12$

Solving the equation of the chord and the parabola we get $y^2 = 16(12 - y) \Rightarrow y = 8$ and –24.

So the coordinates of *M* the other end of the chord through *L* is (36, –24).

and $LM = \sqrt{(36-4)^2 + (-24-8)^2} = 32\sqrt{2}$

Example 43: The locus of the mid-points of the chords the parabola $x^2 = 4py$ having slope *m* is a:

(a) line parallel to *x*-axis at a distance $|2pm|$ from it.
(b) line parallel to *y*-axis at a distance $|2pm|$ from it.
(c) line parallel to $y = mx$, $m \neq 0$ at a distance $|2pm|$ from it.

(d) circle with centre at the origin and radius $|2pm|$.

Ans. (b)

◎ **Solution:** Let (h, k) be the mid point of a chord then its equation is

$hx - 2p(y - k) = h^2 - 4pk$ (using $T = S$)

slope of this chord is $\frac{h}{2p} = m \Rightarrow h = 2pm$

Locus of (h, k) is $x = 2pm$

which is line parallel to y-axis at a distance $|2pm|$ from it.

Example 44 Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0,0) to (x, y) in the ratio 2:3. Then locus of P is

(a) $x^2 = y$ (b) $5y^2 = 2x$

(c) $5y^2 = 8x$ (d) $5x^2 = 2y$

Ans. (c)

◎ **Solution:** Let P be (h, k)

then $h = \frac{2x}{5}$, $k = \frac{2y}{5}$

As (x, y) lies on the parabola $y^2 = 4x$.

$$\left(\frac{5k}{2}\right)^2 = 4\left(\frac{5h}{2}\right)$$

$\Rightarrow$ $5k^2 = 8h$

$\Rightarrow$ Locus of P is $5y^2 = 8x$.

Example 45 Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is **not** given by

(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$

(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$

Ans. (c)

◎ **Solution:** The equation of the normal is $y = mx - 2m - m^3$.

As it passes through (9, 6)

$$6 = 9m - 2m - m^3 \Rightarrow m^3 - 7m + 6 = 0$$

$\Rightarrow$ $m = 1, 2, -3$.

Showing that L is not given by (c).

Assertion-Reason Type Questions

Example 46 Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$.

Statement-1: C_1 and C_2 touch each other exactly at two points.

Statement-2: Equation of the tangent at (1, 2) to C_1 and C_2 both is $x - y + 1 = 0$ and at (1,– 2) is $x + y + 1 = 0$.

Ans. (*a*)

◎ **Solution:** Solving for the points of intersection we have

$$x^2 + 4x - 6x + 1 = 0$$

$\Rightarrow$ $(x - 1)^2 = 0$

$\Rightarrow$ $x = 1 \Rightarrow y = \pm 2$

Thus the two curves meet at (1, 2) and (1, –2)

Tangent at (1, 2) to $y^2 = 4x$ is

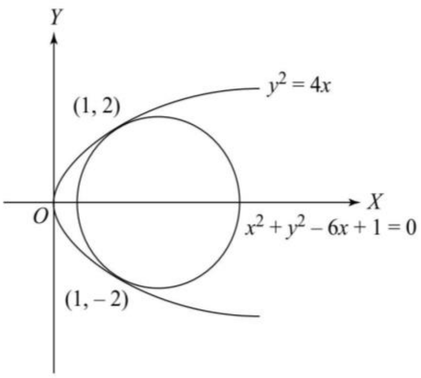

Fig. 18.11

$$y(2) = 2(x + 1) \Rightarrow x - y + 1 = 0$$

Tangent at (1, 2) to the circle C_2 is

$$x + 2y - 3(x + 1) + 1 = 0$$

or $x - y + 1 = 0$ same as the tangent to the curve C_1. Similarly the tangent at the point (1, – 2) to the two curves is $x + y + 1 = 0 \Rightarrow$ statement-2 is true and hence statement-1 is also true.

Example 47: Statement-1: The normal at a point P on the parabola $y^2 = 4x$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M is another parabola whose vertex is at the focus of the given parabola.

Statement-2: The tangent at a point P on the parabola $y^2 = 4x$ meets the x-axis at Q. If R is the mid point of PQ, then the locus of R is the tangent at the vertex of the parabola.

Ans. (*b*)

◎ **Solution:** Normal at the point $P(m^2, -2m)$ on the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$ which meet the x-axis at $Q(2 + m^2, 0)$.

Coordinates of M are $(1 + m^2, -m) = (x, y)$

Locus of M is $y^2 = x - 1$ which is a parabola with vertex (1, 0), the focus of the parabola $y^2 = 4x$. Thus statement-1 is true.

Next, tangent at a point $P(t^2, 2t)$ on the parabola $y^2 = 4x$ is $ty = x + t^2$ which meets the x-axis at $Q(-t^2, 0)$ coordinates of R are $(0, t) = (x, y)$.

Locus of R is $x = 0$, the tangent at the vertex of the parabola $y^2 = 4x$. So statement-2 is also true but does not lead to statement-1.

Example 48: Statement-1: The curve $y = -\frac{x^2}{2} + x + 1$ is symmetrical with respect to the line $x = 1$.
Statement-2: A parabola is symmetric about its axis.
Ans. (a)

Solution: Statement-2 is true, Equation in statement-1 is $(x - 1)^2 = -2(y - 3/2)$ which is a parabola with axis $x - 1 = 0$, using statement-2, statement-1 is also true.

Example 49: Statement-1: A parabola has the origin as its focus and the line $y = 2$ as the directrix, then the vertex of the parabola is at the point (0, 1).
Statement-2: Vertex of a parabola is equidistance from the focus and the directrix and lies on the line through the focus perpendicular to the directrix.
Ans. (a)

Solution: Statement-2 is true and using it in statement-1, the vertex is on the line $x = 0$ at a distance 1 from the focus (0, 0), so the vertex is at the point (0, 1) and the statement-1 is also true.

Example 50: Statement 1: The point $\left(\frac{1}{4}, \frac{1}{2}\right)$ on the parabola $y^2 = x$ is closest to the line $y = x + 1$.
Statement-2: The tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$ to the parabola $y^2 = x$ is parallel to the line $y = x + 1$.
Ans. (b)

Solution: Tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$ to the parabola $y^2 = x$ is $y\left(\frac{1}{2}\right) = \frac{1}{2}\left(x + \frac{1}{4}\right)$ or $y = x + \frac{1}{4}$ which is parallel to y.=.x.+.1.

So statement-2 is true. For Statement-1, any point on the parabola $y^2 = x$ is $\left(\frac{1}{4}t^2, \frac{1}{2}t\right)$ whose distance from the line $y = x + 1$ is $\left|\frac{\frac{1}{2}t - \frac{1}{4}t^2 - 1}{\sqrt{2}}\right| = \frac{(t-1)^2 + 3}{4\sqrt{2}}$

which is minimum when $t = 1$, so the coordinates on the point on the parabola closest to $y = x + 1$ is $\left(\frac{1}{4}, \frac{1}{2}\right)$ and thus statement-1 is also true but does not follow from statement-1.

Example 51: Statement-1 A circle drawn on a focal radii of the parabola $y^2 = 4ax$ as a diameter touches the tangent at the vertex of the parabola.
Statement-2 The portion of the tangent to the parabola $y^2 = 4ax$ intercepted between the point of contact, other than the vertex, and the directrix subtends a right angle at the focus of the parabola.
Ans. (b)

Solution: For statement-2, let the coordinates of a point P on the parabola be $(at^2, 2at)$.
Equation of the tangent at P to the parabola is $ty = x + at^2$

which meets the directrix $x + a = 0$ at $Q\left(-a, a\left(t - \frac{1}{t}\right)\right)$.

Coordinates of the focus S are $(a, 0)$.

Slope of $PS = \frac{2at}{at^2 - a} = m_1$

Slope of $QS = \frac{a\left(t - \frac{1}{t}\right)}{-a - a} = \frac{t^2 - 1}{-2t} = m_2$

$m_1 m_2 = -1 \Rightarrow PQ$ subtends a right angle at the focus and thus statement-2 is true.

Next, for statement-1, equation of the circle on PS as a diameter is

$(x - a)(x - at^2) + (y - 0)(y - 2at) = 0$

which meets the tangent at the vertex $x = 0$ at points for which

$a^2t^2 + y(y - 2at) = 0$

$\Rightarrow (y - at)^2 = 0$

Showing that the circle touches the tangent at the vertex and hence statement-1 is also true but does not follow from statement-2.

Example 52: Statement-1 If $x + 2y = 5$ and $2x - y = 0$ are two tangent to the parabola whose axis is $x + y = 3$, then equation of the directrix of the parabola is $x - y + 1 = 0$.
Statement-2 Perpendicular tangents to a parabola intersect at its directrix.
Ans. (a)

Solution: For statement-2, let the equation of the parabola be $y^2 = 4ax$ and equations of two perpendicular tangents are

$$y = mx + \frac{a}{m} \text{ and } y = -\frac{1}{m}x + \frac{a}{-\frac{1}{m}}$$

Point of intersection of these tangents is given by

$mx + \frac{a}{m} = -\frac{1}{m} - am.$

$\Rightarrow x + a = 0$

which is the directrix of the parabola.

Showing that statement-2 is true.

For statement-1, $x + 2y = 5$ and $2x - y = 0$ are two perpendicular tangents which intersect at the point (1, 2).

So using statement-2, (1, 2) lies on the directrix, also directrix is perpendicular to the axis $x + y = 3$ and hence its equation is $y - 2 = (x - 1)$

$\Rightarrow x - y + 1 = 0$

Showing that the statement-1 is also true.

Example 53: Given: A circle, $2x^2 + 2y^2 = 5$ and a parabola $y^2 = 4\sqrt{5}x$

Statement-1 An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement-2 If the line $y = mx + \frac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.

Ans. (a)

Solution: An equation of the tangent to $y^2 = 4\sqrt{5}x$ is

$y = mx + \frac{\sqrt{5}}{m}$ or $m^2x - my + \sqrt{5} = 0$

which will touch the circle $2x^2 + 2y^2 = 5$

$$\left|\frac{\sqrt{5}}{\sqrt{m^4 + m^2}}\right| = \frac{\sqrt{5}}{\sqrt{2}}$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\Rightarrow \quad m = \pm 1$

$m = \pm 1$ satisfies $m^4 - 3m^2 + 2 = 0$

So statement-2 is true and hence statement-1 is also true.

Example 54: Statement-1: The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of that point.

Statement-2: The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1.

Ans. (b)

Solution: $y^2 = 4ax \Rightarrow 2y\frac{dy}{dx} = 4a$.

$\Rightarrow \frac{dy}{dx} - \frac{2a}{y} = 0$ which is a differential equation of order 1 and degree 1 shows that the statement-2 is true.

For statement-1, slope of the tangent at any point is given by $m = \frac{dy}{dx} = \frac{2a}{y}$.

Showing that the slope m is inversely proportional to the ordinate y of the point and thus statement-1 is also true but does not follow from statement-2.

Example 55: Statement-1 Point of intersection of the tangents drawn to the parabola $x^2 = 4y$ at (4, 4) and (–4, 4) lies on the y-axis.

Statement-2 Tangents drawn at the extremities of the latus rectum of the parabola $x^2 = 4ay$ intersect on the axis of the parabola.

Ans. (b)

Solution: Coordinates of the extremities of the latus rectum of the parabola $x^2 = 4ay$ are $(\pm 2a, a)$ and the equation of the tangents at these extremities are

$x(2a) = 2a(y + a)$ and $x(-2a) = 2a(y + a)$

which intersect on $x = 0$, i.e. y-axis the axis of the parabola. Hence statement-2 is true.

In statement-1, equations of the tangents are $x(4) = 2(y + 4)$ and $x(-4) = 2(y + 4)$ which again intersect on $x = 0$ i.e. y-axis.

So statement-1 is also true but does not follow statement-2 as $(\pm 4, 4)$ are not the extremities of the latus rectum of the parabola $x^2 = 4y$.

LEVEL 2

Straight Objective Type Questions

Example 56: If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, $k \neq 0$ and the parabola intersects the circle $x^2 + y^2 = 4$ in two real distinct points, then the value of k is

(a) – 4 (b) – 8

(c) 4 (d) none of these

Ans. (b)

Solution: The equation of the parabola can be written as

$$y^2 = k(x - 8/k) \text{ which is of the form } Y^2 = 4AX$$

where $Y = y$, $X = x - 8/k$ and $A = k/4$

Equation of the directrix is $X = -A \Rightarrow x - 8/k = -k/4$

which represents the given line $x - 1 = 0$ if $\frac{8}{k} - \frac{k}{4} = 1$

$\Rightarrow k^2 + 4k - 32 = 0 \Rightarrow k = -8$ or 4

For $k = 4$, the parabola is $y^2 = 4(x - 2)$ whose vertex is (2, 0) and touches the circle $x^2 + y^2 = 4$ at the vertex. Therefore $k \neq 4$.

For $k = -8$, the parabola is $y^2 = -8(x + 1)$ which intersects the circle $x^2 + y^2 = 4$ at two real distinct points as the vertex $(-1, 0)$ of the parabola lies inside the circle.

Example 57: M is the foot of the perpendicular from a point P on the parabola $y^2 = 8(x - 3)$ to its directrix and S is the focus of the parabola, if SPM is an equilateral triangle, the length of each side of the triangle is

(a) 2 (b) 3

(c) 4 (d) 8

Ans. (d)

Vertex is (3, 0) focus is (3 + 2, 0) directrix is $x = 3 - 2$.

Solution: Since SPM is an equilateral triangle $\angle SMP = 60°$.

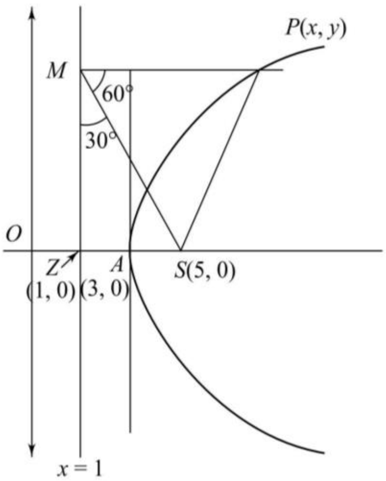

Fig. 18.12

$\Rightarrow \quad \angle SMZ = 30°$

From triangle SZM

$$\frac{SZ}{SM} = \sin 30° \quad \Rightarrow \quad SM = 2SZ$$

$$= 2 \times 4 = 8$$

Hence length of each side of the triangle is 8.

Example 58: PQ is a double ordinate of a parabola $y^2 = 4ax$. The locus of its points of trisection is another parabola length of whose latus rectum is k times the length of the latus rectum of the given parabola, the value of k is

(a) 1/9 (b) 1/3

(c) 2/3 (d) none of these

Ans. (a)

Solution: Let PQ be a double ordinate of the parabola $y^2 = 4ax$ meeting the axis at L, R and S be the points of trisection of PQ. So that $PR = RS = SQ$. (Fig. 18.13)

If the coordinates of R are (h, k), the coordinates of S are $(h, -k)$ the coordinates of P are $(h, 3k)$

Since P lies on the parabola $y^2 = 4ax$

$$(3k)^2 = 4ah \quad \Rightarrow \quad 9k^2 = 4ah$$

The locus of $R(h, k)$ is therefore

$$9y^2 = 4ax \text{ or } y^2 = (4/9)\, ax$$

which is a parabola whose length of the latus rectum = $(4a/9)$

= (1/9) (length of latus rectum of the given parabola)

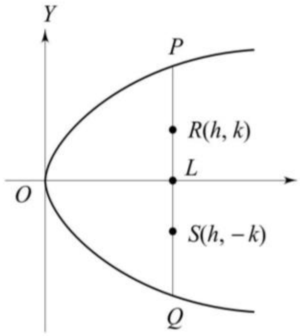

Fig. 18.13

Example 59: An equilateral triangle is inscribed in the parabola $y^2 = 4x$ one of whose vertex is at the vertex of the parabola, the length of each side of the triangle in units is

(a) $\sqrt{3}/2$ (b) $4\sqrt{3}/2$

(c) $8\sqrt{3}/2$ (d) $8\sqrt{3}$

Ans. (d)

Solution: Let A be the vertex and ABC be the equilateral triangle inscribed in the parabola $y^2 = 4x$, AM be perpendicular on BC.

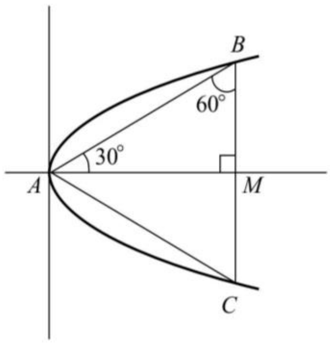

Fig. 18.14

Then if $\quad AB = l,\ AM = l \cos 30°$

and $\quad BM = l \sin 30°$.

Thus the coordinates of B are $(l\sqrt{3}/2,\ l/2)$

Since B lies on the parabola $y^2 = 4x$

$\Rightarrow \quad (l/2)^2 = 4l\sqrt{3}/2$

$\Rightarrow \quad l = 8\sqrt{3}$

Example 60: If PQ is a focal chord of the parabola $y^2 = 4ax$ with focus at S, then $\dfrac{2\,SP.SQ}{SP+SQ}$ =

(a) a (b) $2a$
(c) $4a$ (d) a^2

Ans. (b)

Solution: Let the coordinates of P be $(at_1^2, 2at_1)$ and of Q be $(at_2^2, 2at_2)$. Since PQ is a focal chord, $t_1 t_2 = -1$

Focus is $S(a, 0) \Rightarrow SP = \sqrt{a^2(1-t_1^2)^2 + 4a^2t_1^2} = a(1 + t_1^2)$

and $SQ = a(1 + 1/t_1^2) = \dfrac{a(1+t_1^2)}{t_1^2}$

So that $\dfrac{2\,SP.SQ}{SP+SQ} = \dfrac{2a^2(1+t_1^2)^2}{t_1^2\, a\left[(1+t_1^2)+\left(1+\dfrac{1}{t_1^2}\right)\right]} = 2a$

Example 61: If the tangents at the extremities of a chord PQ of a parabola intersect at T, then the distances of the focus of the parabola from the points P, T, Q are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Ans. (b)

Solution: Let the equation of the parabola be $y^2 = 4ax$ and $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ be the extremities of the chord PQ. The coordinates of T, the point of intersection of the tangents at P and Q are

$$(at_1 t_2, a(t_1 + t_2))$$

Now $SP = a(1 + t_1^2)$

$SQ = a(1 + t_2^2)$

and $ST^2 = (at_1 t_2 - a)^2 + [a(t_1 + t_2) - 0]^2$

$= a^2(t_1^2 + t_2^2 + t_1^2 t_2^2 + 1)$

$= a^2(1 + t_1^2)(1 + t_2^2) = SP.SQ$

So that SP, ST, SQ are in G.P.

Example 62: If the normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum meet the parabola again at the points P and Q then the equation of PQ is

(a) $x = 2a$ (b) $x = 3a$
(c) $x = 6a$ (d) $x = 9a$

Ans. (d)

Solution: The ends of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$. Equation of the normal to the parabola $y^2 = 4ax$ at $(am^2, -2am)$ is $y = mx - 2am - am^3$. Taking $m = -1$ and 1 respectively we get the equations of the normals at L and L' as

$$y = -x + 2a + a \text{ and } y = x - 2a - a$$

or $x + y - 3a = 0$ and $x - y - 3a = 0$

Now $x + y - 3a = 0$ meets the parabola $y^2 = 4ax$ at points given by

$$(3a - x)^2 = 4ax \quad \Rightarrow \quad x^2 - 10ax + 9a^2 = 0$$

$\Rightarrow$ $x = a$ of $x = 9a$. But $x = a$ correspondence to the point L.

Therefore $x = 9a$ and $y = 3a - x = -6a$

and the coordinates of P are $(9a, -6a)$

Similarly the coordinates of Q are $(9a, 6a)$

Hence the equation of PQ is $x = 9a$

Example 63: The orthocentre of the triangle formed by three distinct tangents to the parabola $y^2 = 4ax$ lies on:

(a) $x + a = 0$ (b) $x - a = 0$
(c) $x = 0$ (d) $y = 0$

Ans. (a)

Solution: Equations of the tangents at the points $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$ and $C(at_3^2, 2at_3)$ are

$$t_1 y = x + at_1^2$$
$$t_2 y = x + at_2^2$$
$$t_3 y = x + at_3^2$$

These tangents meet at

$$P(a\,t_2t_3, a(t_3 + t_2))$$
$$Q(a\,t_3t_1, a(t_3 + t_1))$$
$$R(a\,t_1t_2, a(t_1 + t_2))$$

Equation of the altitude through P is $y - a(t_2 + t_3) = -t_1(x - at_2t_3)$

$\Rightarrow y - a(t_1 + t_2 + t_3 + t_1t_2t_3) = -t_1(x + a)$

which passes through the point $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$

From symmetry the other two altitudes also pass through this point and hence the orthocentre of the triangle lies on $x = -a$ or $x + a = 0$.

Example 64: Through the vertex of the parabola $y^2 = 4ax$, chords OA and OB are drawn at right angles to each other. For all positions of the point A, the chord AB meets the axis of the parabola at a fixed point. Coordinates of the fixed point are:

(a) $(a, 0)$ (b) $(-a, 0)$
(c) $(4a, 0)$ (d) $(-4a, 0)$

Ans. (c)

Solution: Let the coordinates of A be $(at_1^2, 2at_1)$ and of B be $(at_2^2, 2at_2)$.

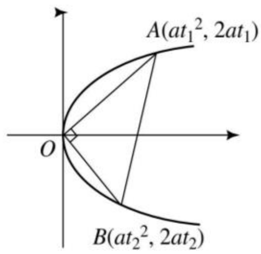

Fig. 18.15

OA is perpendicular to OB

$\Rightarrow \left(\frac{2}{t_1}\right)\left(\frac{2}{t_2}\right) = -1 \Rightarrow t_1t_2 = -4.$

Equation of AB is

$$\frac{x - at_1^2}{at_2^2 - at_1^2} = \frac{y - 2at_1}{2at_2 - 2at_1}$$

$\Rightarrow 2(x - at_1^2) = (t_1 + t_2)(y - 2at_1)$

$\Rightarrow 2(x + a\,t_1t_2) - y(t_1 + t_2) = 0$

$\Rightarrow 2(x - 4a) - y(t_1 + t_2) = 0$

which meets the axis $y = 0$ at the point $(4a, 0)$ for all values of t_1.

Example 65: A is a point on the parabola $y^2 = 4ax$.

The normal at A meets the parabola again at B. If AB subtends a right angle at the vertex of the parabola, then the slope of AB is

(a) ± 1 (b) $\pm\sqrt{2}$

(c) $\pm\frac{1}{\sqrt{2}}$ (d) $\pm 2\sqrt{2}$.

Ans. (b)

Solution: Equation of the normal at $A(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$y = -tx + 2at + at^3$$

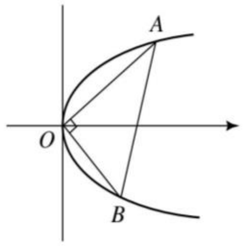

Fig. 18.16

Equation of the pair of lines OA and OB is

$$y^2 = (4ax)\frac{y + tx}{2at + at^3}$$

$\Rightarrow \quad (t^3 + 2t)y^2 = 4tx^2 + 4xy$

$\Rightarrow \quad 4tx^2 + 4xy - (t^3 + 2t)y^2 = 0$

These lines are perpendicular.

$\Rightarrow \quad 4t - t^3 - 2t = 0$

$\Rightarrow \quad t^3 - 2t = 0 \Rightarrow t = 0$ or $t = \pm\sqrt{2}$

EXERCISE

Concept-based Straight Objective Type Questions

1. A chord is drawn through the focus of the parabola $y^2 = 6x$ such that its distance from the vertex of this parabola is $\frac{\sqrt{5}}{2}$, then its slope can be:

   (a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$

   (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{3}}$

2. An equation of the parabola whose focus is $(-3, 0)$ and the directrix is $x + 5 = 0$ is:

   (a) $y^2 = 4(x + 5)$ (b) $y^2 = 4(x + 4)$
   (c) $y^2 = 4(x + 3)$ (d) $y^2 = 4(x - 3)$

3. The centre C of a variable circle passing through a fixed point $(a, 0)$, $a > 0$, touches the line $y = x$. Locus of C is a parabola whose directrix is

   (a) $x + y = 0$ (b) $x + y = a$
   (c) $x - y = 0$ (d) $x - y = a$

4. The parabolas $y^2 = 4x$ and $x^2 = 32y$ intersect at a point P other than the origin. If the angle of intersection is θ, then $\tan\theta$ is equal to

   (a) $\frac{5}{3}$ (b) $\frac{4}{5}$

   (c) $\frac{3}{5}$ (d) 1

5. Length of the common chord of the parabola $y^2 = 8x$ and the circle $x^2 + y^2 - 2x - 4y = 0$ is:

   (a) $\frac{1}{2}\sqrt{5}$ (b) $\sqrt{5}$

   (c) $2\sqrt{5}$ (d) $3\sqrt{5}$

6. If P is the point $(1, 0)$ and Q lies on the parabola $y^2 = 36x$, then the locus of the mid point of PQ is:

   (a) $y^2 = 9(2x - 1)$ (b) $y^2 = 9(x + 2)$
   (c) $y^2 = 2(x - 9)$ (d) $x^2 = 9(y - 2)$

7. Let Q be the foot of the perpendicular from the origin O to the tangent at a point $P(\alpha, \beta)$ on the

parabola $y^2 = 4ax$, and S be the focus of the parabola, then $(OQ)^2$ (SP) is equal to

(a) α (b) $a\alpha^2$
(c) β (d) $a\beta^2$

8. An equation of the latus rectum of the parabola $x^2 + 4x + 2y = 0$ is

(a) $y = \frac{-3}{2}$ (b) $y = \frac{2}{3}$
(c) $y = \frac{3}{2}$ (d) $y = \frac{-2}{3}$

9. If a tangent to the parabola $y^2 = 4x$ makes an angle $\pi/4$ with the positive direction of the axis of x, then the coordinates of the point of contact with the parabola are:

(a) (1, 2) (b) (1, –2)
(c) (4, 4) (d) (4, –4)

10. Distance of a point P on the parabola $y^2 = 48x$ from the focus is l and its distance from the tangent at the vertex is d, then $l - d$ is equal to

(a) 4 (b) 8
(c) 12 (d) 16.

11. Locus of the mid points of the chords of the parabola $y^2 = 8x$ which touch the circle

$x^2 + y^2 = 4$ is

(a) $(y^2 - 4x)^2 = 16(x^2 + 4)$
(b) $(y^2 - 4x^2) = 4(x^2 + a^2)$
(c) $(y^2 - 4x)^2 = 4(16 + y^2)$
(d) $(y^2 - 4x)^2 = 16(4 + y^2)$

12. If θ is the angle between the tangents to the parabola $y^2 = 12x$ passing through the point (–1, 2) then $|\tan\theta|$ is equal to

(a) 2 (b) 3
(c) $\frac{1}{2}$ (d) $\frac{2}{3}$

13. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then equation of L can not be

(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$
(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$

14. Normal at a point P on the parabola $y^2 = 4ax$ meets the axis at Q such that the distance of Q from the focus of the parabola is $10a$. The coordinates of P are:

(a) $(6a, 9a)$ (b) $(6a, -9a)$
(c) $(9a, 6a)$ (d) $(3a, 6a)$

15. $P_1 : y^2 = x$, $P_2 : y^2 = -x$, $P_3 : x^2 = y$, $P_4 : x^2 = -y$ are four parabola. Points of intersection of the parabola P_3 and P_4 with P_1 and P_2 (other than the origin) enclose a square of area (in sq. units).

(a) 2 (b) 4
(c) 8 (d) 16.

LEVEL 1

Straight Objective Type Questions

16. The directrix of the parabola $y^2 + 4x + 3 = 0$ is
(a) $x - 3/4 = 0$ (b) $x + 1/4 = 0$
(c) $x - 1/4 = 0$ (d) $x - 4/3 = 0$

17. The line $x + y = 6$ is a normal to the parabola. $y^2 = 8x$ at the point
(a) (18, – 12) (b) (4, 2)
(c) (2, 4) (d) (3, 3)

18. The length of the chord of the parabola $y^2 = 4ax$ whose equation is

$y - x\sqrt{2} + 4a\sqrt{2} = 0$ is

(a) $2\sqrt{11}\,a$ (b) $4\sqrt{2}\,a$
(c) $8\sqrt{2}\,a$ (d) $6\sqrt{3}\,a$

19. If perpendiculars are drawn on any tangent to a parabola $y^2 = 4ax$ from the points $(a \pm k, 0)$ on the axis. The difference of the squares of their lengths is
(a) 4 (b) $4a$
(c) $4k$ (d) $4ak$

20. If the normal drawn from the point on the axis of the parabola $y^2 = 8ax$ whose distance from the focus is $8a$, and which is not parallel to either axes, makes an angle θ with the axis of x, then θ is equal to
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) none of these

21. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then the locus of the centroid of the triangle PAB as P moves on the circle is

(a) a parabola (b) a circle
(c) an ellipse (d) a pair of straight lines

22. For the parabola $y^2 + 8x - 12y + 20 = 0$, which of the following in not correct
(a) vertex (2, 6)
(b) focus (0, 6)
(c) length of the latus rectum = 4
(d) axis is $y = 6$

23. Tangents at the extremities of a focal chord of a parabola intersect
(a) on the axis of the parabola
(b) on the tangent at the vertex
(c) at the point of intersection of the directrix and the line parallel to the axis of the parabola through the mid-point of the chord.
(d) none of these

24. The coordinates of the end point of the latus rectum of the parabola $(y - 1)^2 = 2(x + 2)$, which does not lie on the line $2x + y + 3 = 0$ are
(a) (– 2, 1) (b) (– 3/2, 1)
(c) (– 3/2, 2) (d) (– 3/2, 0)

25. The point of contact of the tangent to the parabola $y^2 = 9x$ which passes through the point (4, 10) and makes an angle θ with the axis of the parabola such that $\tan\theta > 2$ is
(a) (4/9, 2) (b) (36, 18)
(c) (4, 6) (d) (1/4, 3/2)

26. Equation of the normal at a point on the parabola $y^2 = 36x$, whose ordinate is three times its abcissac is
(a) $2x + 3y + 44 = 0$ (b) $2x - 3y + 44 = 0$
(c) $2x + 3y - 44 = 0$ (d) $2x - 3y = 0$

27. Which of the following parametric equation does not represent a parabola
(a) $x = t^2 + 2t + 1, y = 2t + 2$
(b) $x = a(t^2 - 2t + 1), y = 2at - 2a$
(c) $x = 3\sin^2 t, y = 6\sin t$
(d) $x = a\sin t, y = 2a\cos t$

28. If the line $kx + y = 4$ touches the parabola $y = x - x^2$, then the point of contact is
(a) (– 2, 2) (b) (2, – 2)
(c) (– 2, 6) (d) (2, – 6)

29. If the normal at P to the parabola $y^2 = 4ax$ meets the parabola again at Q such that PQ subtends a right angle at the vertex of the parabola, then the coordinates of P are
(a) $(2a, 2\sqrt{2}a)$ (b) $(-2a, 2\sqrt{2}a)$
(c) $(2\sqrt{2}a, 2a)$ (d) $(2\sqrt{2}a, -2a)$

30. An equation of a common tangent to the parabola $y^2 = 4x$ and $x^2 = 4y$ is
(a) $x - y + 1 = 0$ (b) $x + y - 1 = 0$
(c) $x + y + 1 = 0$ (d) $y = 0$

31. $y = -2x + 12a$ is a normal to the parabola $y^2 = 4ax$ at the point whose distance from the directrix of the parabola is
(a) $4a$ (b) $5a$
(c) $4\sqrt{2}a$ (d) $8a$

32. PN is an ordinate of the parabola $y^2 = 9x$. A straight line is drawn through the mid-point M of PN parallel to the axis of the parabola meeting the parabola at Q. NQ meets the tangent at the vertex A, at a point T, then AT/NP =
(a) 3/2 (b) 4/3
(c) 2/3 (d) 3/4

33. If the area of the triangle inscribed in the parabola $y^2 = 4ax$ with one vertex at the vertex of the parabola and other two vertices at the extremities of a focal chord is $5a^2/2$, then the length of the focal chord is
(a) $3a$ (b) $5a$
(c) $25a/4$ (d) none of these

34. Length of the tangent drawn from an end of the latus rectum of the parabola $y^2 = 4ax$ to the circle of radius a touching externally the parabola at the vertex is
(a) $\sqrt{3}a$ (b) $2a$
(c) $\sqrt{7}a$ (d) $3a$

35. If the tangents at the extremities of a focal chord of the parabola $x^2 = 4ay$ meet the tangent at the vertex at points whose abcissac are x_1 and x_2 then $x_1 x_2$ =
(a) a^2 (b) $a^2 - 1$
(c) $a^2 + 1$ (d) $-a^2$

36. Equation of the tangent at a point P on the parabola $y^2 = 4ax$, the normal at which is at a distance $a\sqrt{5}/4$ from the focus of the parabola is
(a) $4x - 2y + a = 0$ (b) $4x - 8y + 9a = 0$
(c) $2x - y - 12a = 0$ (d) $2x + y - 12a = 0$

37. An isosceles triangle is inscribed in the parabola $y^2 = 4ax$ with its base as the line joining the vertex of the parabola and positive end of the latus rectum of the parabola. If $(at^2, 2at)$ is the vertex of the triangle then
(a) $2t^2 - 8t + 5 = 0$ (b) $2t^2 + 8t - 5 = 0$
(c) $2t^2 + 8t + 5 = 0$ (d) $2t^2 - 8t - 5 = 0$

38. Equation of a family of circles passing through the extremities of the latus rectum of the parabola $y^2 = 4ax$, g being a parametric, is

(a) $x^2 + y^2 + 2g(y - 2a) - 5a^2 = 0$

(b) $x^2 + y^2 + 2g(x + a) - 5a^2 = 0$

(c) $x^2 + y^2 + 2g(x - a) - 5a^2 = 0$

(d) $x^2 + y^2 + 2g(y + 2a) - 5a^2 = 0$

39. A triangle ABC is inscribed in the parabola $y^2 = 4x$ such that A lies at the vertex of the parabola and BC is a focal chord of the parabola with one extremity at (9, 6), the centroid of the triangle ABC lies at

(a) (41/27, 8/9) (b) (82/27, 16/9)
(c) (87/27, 20/9) (d) (80/27, 20/9)

40. P is a point on the parabola $y^2 = 4ax$ whose ordinate is equal to its abscissa and PQ is a focal chord, R and S are the feet of the perpendiculars from P and Q respectively on the tangent at the vertex, T is the foot of the perpendicular from Q to PR, area of the triangle PTQ is

(a) $75\ a^2/4$ (b) $85\ a^2/2$
(c) $75\ a^2/8$ (d) $45\ a^2/2$

41. P is a point on the locus of the mid-points of the chords of the parabola $y^2 = 4ax$ passing through the vertex of the parabola; S is the focus of the parabola and the line joining S and P is produced to meet the parabola at Q. If the ordinate of P is equal to its abscissa, coordinates of Q are

(a) $(4a, 4a)$ (b) $(3a, 3a)$
(c) $(2a, 2a)$ (d) none of these

42. The length of the perpendiculars from the focus and the extremities of a focal chord of a parabola on the tangent at the vertex form

(a) an A.P. (b) a G.P.
(c) an H.P. (d) none of these.

43. Locus of the point of intersection of the normals to the parabola $y^2 = 16x$ which are at right angles is

(a) $y^2 = 4(x - 4)$ (b) $y^2 = 4(x - 8)$
(c) $y^2 = -4(x - 12)$ (d) $y^2 = -8(x - 8)$

44. Equation of a common tangent to the parabola $y = x^2$ and $y = -(x - 2)^2$ is

(a) $y = 4(x - 1)$ (b) $y = 4(x + 1)$
(c) $y = 4(x - 1)$ (d) $y = -4(x + 1)$

45. Let (x_i, y_i), $i = 1, 2, 3, 4$, be the points of intersection of the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, $c \neq 0$, then which of the following is true

(a) $y_1 - y_2 + y_3 - y_4 = 0$

(b) $x_1 + x_2 + x_3 + x_4 = 0$

(c) $y_1y_2y_3y_4 = 16a^2$

(d) $x_1x_2x_3x_4 = c^2$

Assertion-Reason Type Questions

46. **Statement-1:** The tangent to the parabola $y^2 = 4x$ at any point P and perpendicular on it from the focus S meet on the directrix of the parabola.

Statement-2: Tangents and normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square whose area is $8a^2$ sq. units.

47. **Statement-1:** If the vertex of a parabola lies at the point $(a, 0)$ and the directrix is y-axis then the focus of the parabola is at the point $(2a, 0)$.

Statement-2: Length of the common chord of the parabola $y^2 = 12x$ and the circle $x^2 + y^2 = 9$ is equal to the length of the latus rectum of the parabola.

48. **Statement-1:** The tangents at the extremities of a focal chord of a parabola intersect on its directrix.

Statement-2: The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.

49. Let $P : y^2 = 2x$ be a parabola.

Statement-1 If three distinct normals of P pass through $(\alpha, 0)$, then $\alpha > 1$.

Statement-2 If $\alpha \leq 1$, exactly one normal of P passes through $(\alpha, 0)$.

50. **Statement-1** No normal of the parabola $y^2 = 4ax$ with integral slope, can touch the circle $x^2 + y^2 = a^2$.

Statement-2 The equation $x^6 + 4x^4 + 3x^2 - 1 = 0$ has no integral roots.

51. **Statement-1** If the focus of the parabola is (1, –1) and its directrix is $3x + 4y - 9 = 0$, then the length of its latus rectum is 4.

Statement-2 Length of the latus rectum of a parabola is twice the length of the perpendicular from the focus on the directrix.

52. **Statement-1** Number of common tangents to the parabola $y^2 = 2x$ and the circle $x^2 + y^2 - 4x + 3 = 0$ is 2.

Statement-2 The circle $x^2 + y^2 - 4x + 3 = 0$ does not intersect the parabola $y^2 = 2x$.

53. **Statement-1** PQ is a focal chord of the parabola $y^2 = 8x$. If the coordinates are (32, 16) then the coordinates of Q are $\left(\frac{1}{8}, -2\right)$.

Statement-2 If the coordinates of one end of a focal chord of a parabola are $(at^2, 2at)$ then the coordinates of the other end are $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

54. **Statement-1** If the tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ intersect on the axis of the parabola, then $t_1 + t_2 = 0$.

Statement-2 The tangents at the extremities of any focal chord of a parabola intersect on the axis of the parabola and include an angle of $\pi/4$.

55. **Statement-1** Length of a focal chord of the parabola $y^2 = 16x$ can not be a 9 units.

Statement-2 $\left|t + \frac{1}{t}\right| \geq 2$ for all $t \neq 0$.

LEVEL 2

Straight Objective Type Questions

56. P is a point on the axis of the parabola $y^2 = 4ax$; Q and R are the extremities of its latus rectum, A is its vertex. If PQR is an equilateral triangle lying within the parabola and $\angle AQP = \theta$, then $\cos\theta =$

(a) $\frac{2-\sqrt{3}}{2\sqrt{5}}$ (b) $\frac{9}{8\sqrt{5}}$

(c) $\frac{\sqrt{5}-2}{2\sqrt{3}}$ (d) none of these

57. The points of intersection of the circle $x^2 + y^2 = a^2$ with the parabolas $y^2 = 4ax$ and $y^2 = -4ax$ form a rectangle whose area is

(a) $8(\sqrt{5} - 2)a^2$ (b) $8(\sqrt{5} - 2)^{3/2}a^2$

(c) $8(\sqrt{5} + 2)^{3/2}a^2$ (d) none of these

58. The circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$ intersect at P and Q. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. The ratio of the areas of the triangles PQS and PQR is

(a) $1 : \sqrt{2}$ (b) 1 : 2

(c) 1 : 4 (d) 1 : 8

59. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ in sq. units is

(a) 3 (b) 6

(c) 9 (d) 20

60. Length of the chord of contact drawn from the point (–3, 2) to the parabola $y^2 = 4x$ is

(a) 4 (b) $2\sqrt{2}$

(c) $8\sqrt{2}$ (d) $4\sqrt{2}$

61. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose vertex is the point:

(a) $\left(\frac{a}{3}, 0\right)$ (b) $\left(\frac{2a}{3}, 0\right)$

(c) $(a, 0)$ (d) $\left(\frac{4a}{3}, 0\right)$

62. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following are the coordinates of P?

(a) $(4, 2\sqrt{2})$ (b) $(9, 3\sqrt{2})$

(c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (d) $(2, \sqrt{2})$

63. The tangents to the parabola $y^2 = 4ax$ at the points A, B, C, taken in pair intersect at the points P, Q, R. The ratio of the areas of the ΔABC and ΔPQR is

(a) 3 : 2 (b) 2 : 1

(c) 3 : 1 (d) 2 : 3

64. $y = (x - 11)\cos\theta - \cos(3\theta)$ is a normal to the parabola $y^2 = 16x$ for

(a) $\theta = \pi/3$ (b) $\pi/6$

(c) $\theta = \frac{2\pi}{3}$ (d) all values of θ.

65. If P_1Q_1 and P_2Q_2 are two focal chords of the parabola $y^2 = 4ax$, then the lines P_1P_2 and Q_1Q_2 intersect on:

(a) the axis of the parabola.
(b) the directrix of the parabola
(c) the tangent at the vertex of the parabola
(d) latus rectum of the parabola.

Previous Years' AIEEE/JEE Main Questions

1. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$ then

(a) $t_2 = -t_1 + 2/t_1$ (b) $t_2 = t_1 - 2/t_1$
(c) $t_2 = t_1 + 2/t_1$ (d) $t_2 = -t_1 - 2/t_1$ **[2003]**

2. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is

(a) (–9/8, 9/2) (b) (2, – 4)
(c) (2, 4) (d) (9/8, 9/2) **[2004]**

3. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabola $y^2 = 4ax$ and $x^2 = 4ay$, then

(a) $d^2 + (2b - 3c)^2 = 0$
(b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b + 3c)^2 = 0$
(d) $d^2 + (3b - 2c)^2 = 0$ **[2004]**

4. If P be the point (1, 0) and Q a point on the locus $y^2 = 8x$. The locus of the mid point of PQ is

(a) $x^2 + 4y + 2 = 0$ (b) $x^2 - 4y + 2 = 0$
(c) $y^2 - 4x + 2 = 0$ (d) $y^2 + 4x + 2 = 0$ **[2005]**

5. The locus of the vertices of the family of parabola $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is

(a) $xy = 64/105$ (b) $xy = 105/64$
(c) $xy = 3/4$ (d) $xy = 35/16$ **[2006]**

6. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is

(a) (– 1, 1) (b) (0, 2)
(c) (2, 4) (d) (– 2, 0) **[2007]**

7. A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then the vertex of the parabola is at

(a) (2, 0) (b) (0, 2)
(c) (1, 0) (d) (0, 1) **[2008]**

8. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is

(a) $3\sqrt{2}/5$ (b) $\sqrt{3}/4$
(c) $3\sqrt{2}/8$ (d) $2\sqrt{3}/8$ **[2009], [2011]**

9. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is

(a) $x = -1$ (b) $2x - 1 = 0$
(c) $x = 1$ (d) $2x + 1 = 0$ **[2010]**

10. **Statement-1:** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}\,x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement-2: If the line $y = mx + 4\sqrt{3}/m, (m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}\,x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$. **[2012]**

11. Given: A circle, $2x^2 + 2y^2 = 5$ and a parabola $y^2 = 4\sqrt{5}x$.

Statement-1: An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement-2: If the line $y = mx + \sqrt{5}/m\ (m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$ **[2013]**

12. **Statement-1:** The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of the point P.

Statement-2: The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1. **[2013, online]**

13. **Statement-1:** The line $x - 2y = 2$ meets the parabola $y^2 + 2x = 0$ only at the point (–2, –2).

Statement-2: The line $y = mx - 1/2m$ ($m \neq 0$) is a tangent to the parabola, $y^2 = -2x$ at the point $\left(-\frac{1}{2m^2}, -\frac{1}{m}\right)$ **[2013, online]**

14. The point of intersection of the normals to the parabola $y^2 = 4x$ at the ends of its latus rectum is:
(a) (0, 2) (b) (3, 0)
(c) (0, 3) (d) (2, 0) **[2013, online]**

15. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is:
(a) $\frac{1}{2}$ (b) $\frac{3}{2}$
(c) $\frac{1}{8}$ (d) $\frac{2}{3}$ **[2014]**

16. Let L_1 be the length of the common chord of the curves $x^2 + y^2 = 9$ and $y^2 = 8x$, and L_2 be the length of the latus rectum of $y^2 = 8x$, then
(a) $L_1 > L_2$ (b) $L_1 = L_2$
(c) $L_1 < L_2$ (d) $\frac{L_1}{L_2} = \sqrt{2}$
[2014, online]

17. Two tangents are drawn from a point (–2, –1) to the curve $y^2 = 4x$. If α is the angle between them, then $|\tan \alpha|$ is equal to:
(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) 3 **[2014, online]**

18. A chord is drawn through the focus of the parabola $y^2 = 6x$ such that its distance from the vertex of this parabola is $\frac{\sqrt{5}}{2}$, then its slope can be:
(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{3}}$ **[2014, online]**

19. Let O be the vertex and Q be any point on the parabola $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is:
(a) $x^2 = y$ (b) $y^2 = x$
(d) $y^2 = 2x$ (d) $x^2 = 2y$ **[2015]**

20. Let PQ be a double ordinate of the parabola $y^2 = -4x$, where P lies in the second quadrant. If R divides PQ in the ratio 2 : 1, then the locus of R is:
(a) $9y^2 = 4x$ (b) $9y^2 = -4x$
(d) $3y^2 = 2x$ (d) $3y^2 = -2x$
[2015, online]

21. If the tangent to the conic $y - 6 = x^2$ at (2, 10) touches the circle $x^2 + y^2 + 8x - 2y = k$ (for same fixed k) at a point (α, β); then (α, β) is:
(a) $\left(-\frac{6}{17}, \frac{10}{17}\right)$ (b) $\left(-\frac{8}{17}, \frac{2}{17}\right)$
(d) $\left(-\frac{4}{17}, \frac{1}{17}\right)$ (d) $\left(-\frac{7}{17}, \frac{6}{17}\right)$
[2015, online]

22. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre as P is
(a) $x^2 + y^2 - 4x + 8y + 12 = 0$
(b) $x^2 + y^2 - x + 4y - 12 = 0$
(d) $x^2 + y^2 - x/4 + 2y - 24 = 0$
(d) $x^2 + y^2 - 4x + 9y + 18 = 0$ **[2016]**

23. The minimum distance of a point on the curve $y = x^2 - 4$ from the origin is:
(a) $\frac{\sqrt{15}}{2}$ (b) $\sqrt{\frac{19}{2}}$
(d) $\sqrt{\frac{15}{2}}$ (d) $\sqrt{\frac{19}{2}}$ **[2016, online]**

24. P and Q two distinct points on the parabola, $y^2 = 4x$, with parameters t and t_1 respectively. If the normal at P passes through Q, then the minimum value of t_1^2 is:
(a) 8 (b) 4
(d) 6 (d) 2 **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. An equilateral triangle is inscribed in the parabola $y^2 = 8x$ with one of its vertices at the vertex of the parabola. Then the length of its side is
(a) $8\sqrt{3}$ (b) $16\sqrt{3}$
(c) 16 (d) 8 **[2006]**

2. **Statement-1:** The point $\left(\frac{1}{4}, \frac{1}{2}\right)$ on the parabola $y^2 = x$ is closest to the line $y = x + 1$.

Statement-2: The tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$ to the parabola $y^2 = x$ is parallel to the line $y = x + 1$

(a) Statement-1 is true, statement-2 is true, statement-2 is **not** a correct explanation for statement-1.
(b) Statement-1 is true, statement-2 is false.
(c) Statement-1 is false, statement-2 is true.
(d) Statement-1 is true, statement-2 is true, statement-2 is a correct explanation for statement-1. **[2009]**

3. The acute angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is:

(a) $\pi/6$ (b) $\pi/2$
(c) $\pi/3$ (d) $\pi/4$ **[2011]**

4. **Statement-1:** Point of intersection of the tangents drawn to the parabola $x^2 = 4y$ at (4, 4) and (–4, 4) lies on the y-axis.

Statement-2: Tangents drawn at the extremities of the latus rectum of the parabola $x^2 = 4y$ intersect on the axis of the parabola.

(a) Statement-1 is true, statement-2 is true, statement-2 is a correct explanation for statement-1.
(b) Statement-1 is true, statement-2 is true, statement-2 is not a correct explanation for statement-1.
(c) Statement-1 is true, statement-2 is false.
(d) Statement-1 is false, statement-2 is true. **[2012]**

5. Let $y^2 = 16x$ be a given parabola and L be an extremity of its latus rectum in the first quadrant. If a chord is drawn through L with slope –1, then the length of this chord is

(a) 32 (b) $16\sqrt{2}$
(c) $16\sqrt{3}$ (d) $32\sqrt{2}$ **[2013]**

6. The locus of the mid points of the chords of the parabola $x^2 = 4py$ having slope m is a:

(a) line parallel to x-axis at a distance $|2pm|$ from it.
(b) line parallel to y-axis at a distance $|2pm|$ from it.
(c) line parallel to $y = mx$, $m \neq 0$ at a distance $|2pm|$ from it.
(d) circle with centre at the origin and radius $|2pm|$. **[2014]**

7. Let PQ be a focal chord of the parabola $y^2 = 4x$. If the centre of a circle having PQ as its diameter lies on the line $\sqrt{5}\ y + 4 = 0$, then, the length of the chord PQ is:

(a) $\frac{26}{5}$ (b) $\frac{36\sqrt{5}}{5}$
(c) $\frac{26\sqrt{5}}{5}$ (d) $\frac{36}{5}$ **[2016]**

Answers

Concept-based

1. (a)	**2.** (b)	**3.** (c)	**4.** (c)
5. (c)	**6.** (a)	**7.** (b)	**8.** (c)
9. (a)	**10.** (c)	**11.** (c)	**12.** (a)
13. (c)	**14.** (c)	**15.** (b)	

Level 1

16. (c)	**17.** (c)	**18.** (d)	**19.** (d)
20. (c)	**21.** (b)	**22.** (c)	**23.** (c)
24. (c)	**25.** (a)	**26.** (c)	**27.** (d)
28. (b)	**29.** (a)	**30.** (c)	**31.** (b)
32. (c)	**33.** (c)	**34.** (c)	**35.** (d)
36. (a)	**37.** (b)	**38.** (c)	**39.** (b)
40. (c)	**41.** (d)	**42.** (b)	**43.** (c)
44. (a)	**45.** (d)	**46.** (d)	**47.** (c)
48. (a)	**49.** (b)	**50.** (a)	**51.** (a)
52. (d)	**53.** (a)	**54.** (c)	**55.** (a)

Level 2

56. (a)	**57.** (b)	**58.** (c)	**59.** (d)
60. (c)	**61.** (b)	**62.** (a)	**63.** (b)
64. (d)	**65.** (b)		

Previous Years' AIEEE/JEE Main Questions

1. (d)	**2.** (d)	**3.** (c)	**4.** (c)
5. (b)	**6.** (d)	**7.** (c)	**8.** (c)
9. (a)	**10.** (a)	**11.** (a)	**12.** (b)
13. (a)	**14.** (b)	**15.** (a)	**16.** (c)
17. (d)	**18.** (a)	**19.** (d)	**20.** (b)
21. (b)	**22.** (a)	**23.** (c)	**24.** (d)

Previous Years' B-Architecture Entrance Examination Questions

1. (b)	**2.** (a)	**3.** (b)	**4.** (b)
5. (d)	**6.** (b)	**7.** (d)	

Hints and Solutions

Concept-based

1. Let the equation of the chord be.

$$y - 0 = m\left(x - \frac{3}{2}\right)$$

Its distance from the vertex (0, 0) is $\left|\frac{m\left(-\frac{3}{2}\right)}{\sqrt{1+m^2}}\right| = \frac{\sqrt{5}}{2} \Rightarrow m = \pm\frac{\sqrt{5}}{2}$.

2. We have $\sqrt{(x+3)^2 + y^2} = |x + 5|$

$\Rightarrow y^2 = (x + 5)^2 - (x + 3)^2 = 4(x + 4)$

3. C is equidistant from the point $(a, 0)$ and the line $y = x$, so locus of C is a parabola with directrix $y = x$ i.e. $x - y = 0$.

4. Coordinates of P are (16, 8)

$m_1 = \frac{dy}{dx}$ for $y^2 = 4x$ at P

$\Rightarrow m_1 = \left.\frac{2}{y}\right|_{(16,8)} = \frac{1}{4}$

$m_2 = \frac{dy}{dx}$ for $x^2 = 32y$ at P

$m_2 = \left.\frac{x}{16}\right|_{16,8} = 1$

$$\tan\theta = \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}} = \frac{3}{5}.$$

5. Any point on the parabola is $(2t^2, 4t)$ which lies on the circle if $t^4 + 3t^2 - 4t = 0 \Rightarrow t = 0, 1$

$\Rightarrow$ end points of the common chord are (0, 0) and (2, 4) and the required length is $2\sqrt{5}$.

6. Let the coordinates of Q be $(9t^2, 18t)$ coordinates of R the mid point of PQ are $\left(\frac{9t^2+1}{2}, 9t\right) = (x, y)$.

Locus of R is

$$2x = 9\left(\frac{y}{9}\right)^2 + 1$$

$\Rightarrow y^2 = 9(2x - 1)$

7. Let $\alpha = at^2$, $\beta = 2at$.

Equation of the tangent at $P(\alpha, \beta)$ is $ty = x + at^2$

$\Rightarrow OQ = \frac{at^2}{\sqrt{1+t^2}}$,

$SP = a(t^2 + 1)$

So $(OQ)^2\,(SP) = \frac{a^2t^4}{1+t^2} \times a(1 + t^2) = a^3t^4 = a\alpha^2$

8. We can write the equation of the parabola as $(x + 2)^2 = -2(y - 2)$

Shifting the origin to (–2, 2), the equation is $X^2 = -2Y$ where $X = x + 2$, $Y = y - 2$

Equation of the latus rectum is $Y = -\left(\frac{2}{4}\right) = \frac{-1}{2}$

$\Rightarrow y = \frac{3}{2}$

9. Slope of the tangent to the parabola at $(t^2, 2t)$ is $\frac{1}{t} = \tan\frac{\pi}{4} = 1 \Rightarrow t = 1$ and the required point of contact is (1, 2).

10. Any point on the parabola is $P(12t^2, 24t)$ whose distance from the focus (12, 0) is $12(t^2 + 1) = l$ and from y-axis, the tangent at the vertex is $12t^2 = d$.

So $l - d = 12$.

11. Let (h, k) be the mid-point of the chord of the parabola $y^2 = 8x$, then its equation is

$ky - 4(x + h) = k^2 - 8h$

$\Rightarrow 4x - ky + k^2 - 4h = 0$

which touches the circle $x^2 + y^2 = 4$

$\Rightarrow \frac{k^2 - 4h}{\sqrt{16+k^2}} = 2$

$\Rightarrow (k^2 - 4h)^2 = 4(k^2 + 16)$

Locus of (h, k) is $(y^2 - 4x)^2 = 4(16 + y^2)$

12. Equation of any tangent is $y = mx + \frac{3}{m}$ which passes through (–1, 2) if $2 = -m + \frac{3}{m}$

$\Rightarrow m^2 + 2m - 3 = 0$

$\Rightarrow m = 1, -3$

So $|\tan\theta| = \left|\dfrac{\frac{1+3}{1-3}}{}\right|$ = 2.

13. Equation of a normal to the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$ which passes through (9, 6) if $m^3 - 7m + 6 = 0$

$\Rightarrow m = 1, 2, -3.$

So equation of L can not be $y + x - 15 = 0$

14. Equation of the normal at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is $y = -tx + 2at + at^3$ which meets the axis $y = 0$ at $x = a(2 + t^2)$. So coordinates of Q are $(a(2 + t^2), 0)$. Focus is $S(a, 0)$.

$QS = 2a + at^2 - a = a(1+ t^2) = 10a \Rightarrow t^2 = 9$

So the required coordinates are $(9a, \pm 6a)$

15. Points of intersection are $A(1, 1)$, $B(-1, 1)$, $C(-1, -1)$ and $D(1, -1)$

Area of the square $ABCD = 2^2 = 4$

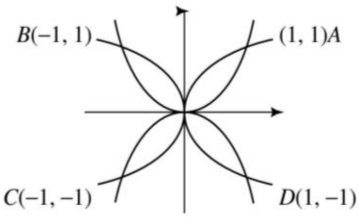

Fig. 18.17

Level 1

16. $y^2 = -4(x + 3/4)$, the directrix is $x + 3/4 - 1 = 0$

17. $y = -x + 6$ is a normal at the point $(am^2, -2am)$ where $a = 2$, $m = -1$ i.e., (2, 4)

18. Any point on the parabola is $(at^2, 2at)$;

$2at - at^2\sqrt{2} + 4a\sqrt{2} = 0$

$\Rightarrow \sqrt{2}\,t^2 - 2t - 4\sqrt{2} = 0$

$\Rightarrow t_1 + t_2 = \sqrt{2}$, $t_1 t_2 = -4$.

$\Rightarrow t_1 - t_2 = \sqrt{18}$

Length of the chord is

$$\sqrt{\left(at_1^2 - at_2^2\right)^2 + \left(2at_1 - 2at_2\right)^2} = a\sqrt{18}\,\sqrt{2+4} = 6\sqrt{3}a$$

19. Any tangent is $y = mx + a/m$. Required difference is

$$\left[\frac{m(a+k)+a/m}{\sqrt{1+m^2}}\right]^2 - \left[\frac{m(a-k)+a/m}{\sqrt{1+m^2}}\right]^2$$

$$= \frac{1}{1+m^2} \times 4(ma + a/m)mk = 4ak.$$

20. Focus is $(2a, 0)$, so point is $P(2a + 8a, 0)$. Equation of any normal is $y = mx - 2(2a)m - (2a)m^3$. If it passes through $P(10a, 0)$, then

$6\,am - 2\,am^3 = 0 \Rightarrow m = 0, \pm\sqrt{3}$.

So the required angle is $\theta = \tan^{-1}\sqrt{3} = \pi/3$.

21. Let (h, k) be the centroid, then

$$h = \frac{r + r\cos\theta}{3},\ k = \frac{r + r\sin\theta}{3}.$$

$\Rightarrow (3h - r)^2 + (3k - r)^2 = r^2.$
$\Rightarrow 9(h^2 + k^2) - 6r(h + k) + r^2 = 0$
Locus of (h, k) is a circle.

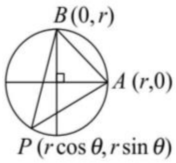

Fig. 18.18

22. $(y - 6)^2 = -8(x - 2)$ or $Y^2 = 4AX$ where $Y = y - 6$, $X = x - 2$, $A = -2$, vertex is (2, 6), focus is $(X = A, Y = 0)$ i.e., (0, 6) axis is $Y = 0$ i.e. $y = 6$ and length of the latus rectum = $|4A|$ = 8.

23. Extremities of a focal chord are $(at^2, 2at)$, $(a/t^2, -2a/t)$ Tangents are $ty = x + at^2$, and $-ty = xt^2 + a$.

They intersect at $\left(-a, \dfrac{a\left(t^2-1\right)}{t}\right)$ or $\left(-a, a\left(t - \dfrac{1}{t}\right)\right)$

which is the point of intersection of the directrix $x = -a$ and the line $y = a\left(t - \dfrac{1}{t}\right)$ which is the line through the mid-point of the chord parallel to the axis.

24. $Y^2 = 4AX$ where $Y = y - 1$, $X = x + 2$, $A = 1/2$.
End of the L.R. is $(X = A, Y = \pm 2A) = (-3/2, 2)$ or $(-3/2, 0)$ but $(-3/2, 0)$ lies on the line.

25. Equation of the tangent at any point $(9t^2/4, 9t/2)$ on the parabola is $ty = x + 9t^2/4$. Since it passes through (4, 10), $10t = 4 + 9t^2/4$ or $9t^2 - 40t + 16 = 0$
$\Rightarrow t = 4/9, 4$. $\tan\theta = 1/t > 2 \Rightarrow t < 1/2$. So $t = 4/9$ and the required point is (4/9, 2).

26. Equation of the normal at $(9m^2, -18m)$ on the parabola is
$y = mx - 2 \times 9m - 9m^3$ where $-18m = 3 \times 9m^2$.
i.e., $m = 0$ or $m = -2/3$. For $m = -2/3$, the equation of the normal is $2x + 3y - 44 = 0$.

27. $x = aT^2$, $y = 2aT$ represents a parabola.
in(a) $a = 1$, $T = t + 1$, in (b) $a = a$, $T = (t - 1)$
in(c) $a = 3$, $T = \sin t$. But in (d) if $2aT = 2a\cos t$
$\Rightarrow T = \cos t$ which does not satisfy $x = aT^2$.

28. $4 - kx = x - x^2$ or $x^2 - (k + 1)x + 4 = 0$ has coincident roots if $(k + 1)^2 - 16 = 0 \Rightarrow k = 3$ or -5. So $x = \pm 2$ and the points of contact are $(2, -2)$ or $(-2, -6)$.

29. Let the coordinates of P be $(am^2, -2am)$ and Q be $(am'^2, -2am')$ then $\frac{-2am}{am^2} \times \frac{-2am'}{am'^2} = -1$

$\Rightarrow mm' = -4$

Normal at P, $y = mx - 2am - am^3$. Since it passes through Q

$-2a\left(-\frac{4}{m}\right) = am\left(\frac{16}{m^2}\right) - 2am - am^3 = 0$

$\Rightarrow m^4 + 2m^2 - 8 = 0 \Rightarrow m = \pm\sqrt{2}$ and the coordinates of P are $(2a, \pm 2\sqrt{2}\ a)$

30. Equation of a tangent to $y^2 = 4x$ is $y = mx + 1/m$. If it touches the parabola $x^2 = 4y$ then the roots of the equation $x^2 = 4(mx + 1/m)$ are coincident. So $(4m)^2 + \frac{16}{m} = 0 \Rightarrow m = -1$ and the equation of the tangent is $x + y + 1 = 0$.

31. $y = -2x + 12a$ is a normal at the point $(a(-2)^2, -2a(-2))$ i.e., $(4a, 4a)$ whose distance from $x = -a$ is $5a$.

32. Let P be $(9t^2/4, 9t/2)$, then $NP = 9t/2$. $MN = 9t/4$, equation of MQ is $y = 9t/4$ which meets the parabola $y^2 = 9x$ at the point $Q(9t^2/16, 9t/4)$, N is $(9t^2/4, 0)$ equation of NQ is

$y = -\frac{9 \times 16}{4 \times 27t}\left(x - \frac{9}{4}t^2\right)$ which meets the tangent $x = 0$ (at the vertex A) at the point for which $y = 3t$ so that $AT = 3t$ and $\frac{AT}{NP} = \frac{3t \times 2}{9t} = \frac{2}{3}$.

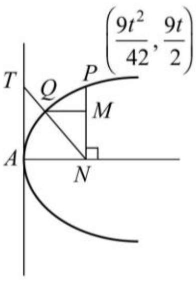

Fig. 18.19

33. Let the vertices be $O(0, 0)$, $A(at^2, 2at)$, $B\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$ then

$$\frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ at^2 & 2at & 1 \\ \frac{a}{t^2} & -\frac{2a}{t} & 1 \end{vmatrix}\right| = \frac{5a^2}{2} \Rightarrow 2t^2 - 5t + 2 = 0$$

$\Rightarrow t = 2$ or $1/2$ so the vertices of a focal chord are $(4a, 4a)$ and $(a/4, -a)$ (Taking $t = 2$) and length of this focal chord is $25\ a/4$.

34. Equation of the circle is $(x + a)^2 + y^2 = a^2$ and an end of the Latus rectum of the parabola is $(a, 2a)$. Required length is $\sqrt{(a+a)^2 + (2a)^2 - a^2} = a\sqrt{7}$.

35. One extremity of the focal chord be $(2at, at^2)$. Equation of the tangent is $tx = y + at^2$ which meets the tangent at the vertex, $y = 0$ at $x = at$ so $x_1 = at$ and $x_2 = a\left(-\frac{1}{t}\right)$ thus $x_1x_2 = -a^2$.

36. Equation of the normal with slope m is $y = mx - 2am - am^3$ whose distance from the focus $(a, 0)$ is $\frac{am - 2am - am^3}{\sqrt{1+m^2}} = \frac{\sqrt{5}}{4}a$

$\Rightarrow 16m^2(1 + m^2) = 5$

$\Rightarrow m^2 = 1/4 \Rightarrow m = \pm 1/2$

Slope of the tangent is ± 2 and the equation of the tangent is

$y = \pm 2x \pm a/2 \Rightarrow 4x \pm 2y + a = 0$.

37. $(at^2)^2 + (2at)^2 = (at^2 - a)^2 + (2at - 2a)^2$, as vertex of the parabola is $(0, 0)$ and positive end of the latus-rectum is $(a, 2a)$.

38. Taking $x = a$ in (c) we get $y = \pm 2a$ for all values of g.

39. If one extremity of the focal chord is $(9, 6)$, the other is $(1/9, -2/3)$ so the centroid of the triangle is $\left(\frac{0+9+1/9}{3}, \frac{0+6-2/3}{3}\right) = \left(\frac{82}{27}, \frac{16}{9}\right)$

40. $P(4a, 4a)$, $Q(a/4, -a)$, $R(0, 4a)$, $S(0, -a)$, $T(a/4, 4a)$. So area of the ΔPTQ is

$\frac{1}{2} \times PT \times QT = \frac{1}{2}\left(4a - \frac{a}{4}\right) \times 5a = \frac{75}{8}a^2$

41. $O(0, 0)$ is the vertex and $R(at^2, 2at)$ is any point on the parabola. $(at^2/2, 2at/2)$ is the mid-point of the chord OR. Whose locus is $y^2 = 2ax$, coordinates of P on this locus are $(2a, 2a)$, focus is $S(a, 0)$. Equation of SP is $y = 2(x - a)$ which meets $y^2 = 4ax$ at the point Q for which $4(x - a)^2 = 4ax$ $\Rightarrow x^2 - 3ax + a^2 = 0$ which is not satisfied by the points in (a), (b) or (c).

42. $y^2 = 4ax$ be the parabola, $P(at^2, 2at)$, $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$ and the focus is $S(a, 0)$. The tangent at the vertex is $x = 0$. Lengths of the perpendicular from P,

Q and S on this tangent are respectively $l_1 = at^2$, $l_2 = a/t^2$, $l_3 = a$ so that $l_1 l_2 = a^2 = l_3^2 \Rightarrow l_1, l_2, l_3$ form a G.P.

43. Let two perpendicular normals to the parabola $y^2 = 16x$ be $y = mx - 8m - 4m^3$ and $y = -\frac{1}{m}x + \frac{8}{m} + \frac{4}{m^3}$.

Solving we get $x = 4\left(m - \frac{1}{m}\right)^2 + 12$

$y = -4\left(m - \frac{1}{m}\right)$

Eliminating m we get the required locus as $y^2 = 4(x - 12)$.

44. Coordinates of any point on $y = x^2$ is (t, t^2)

$\left.\frac{dy}{dx}\right|_{(t,t^2)} = 2t$

Equation of the tangent to the parabola is $y - t^2 = 2t(x - t)$

$\Rightarrow y = 2tx - t^2$

which will touch the parabola $y = -(x-2)^2$ if the roots of the equation $-(x-2)^2 = 2tx - t^2$ are equal.

$\Rightarrow (t-2)^2 - (4 - t^2) = 0 \Rightarrow t = 0$ or 2

For $t = 2$, the required tangent is $y = 4(x - 1)$

45. For the points of intersections of the given parabola and circle, we have $\frac{y^4}{16a^2} + y^2 + 2g \times \frac{y^2}{4a} + 2fy + c = 0$

$\Rightarrow y^4 + 4a(4a + 2g)y^2 + 32a^2fy + 16a^2c = 0$

$\Rightarrow y_1 + y_2 + y_3 + y_4 = 0,\ y_1y_2y_3y_4 = 16a^2c$

Also $x_1x_2x_3x_4 = \frac{(y_1y_2y_3y_4)^2}{(4a)^4} = c^2$

$x_1 + x_2 + x_3 + x_4 = \frac{1}{4a}(y_1^2 + y_2^2 + y_3^2 + y_4^2) \neq 0.$

46. **Statement-1** is false. Equation of any tangent to the parabola is $y = mx + \frac{1}{m}$ and equation of the perpendicular from the focus $S(1, 0)$ on it is

$y = -\frac{1}{m}x + \frac{1}{m}$ and these intersect at $x = 0$, directrix is $x = -1$.

Statement-2 is true. Tangents and normals at $(a, \pm 2a)$ are respectively $x \pm y + a = 0$ and $x \pm y - 3a = 0$ which enclose a square, length of a side $= 2\sqrt{2}a$.

47. In statement-1, focus is on the x-axis at a distance a from the vertex so statement-1 is true.

Statement-2 is false as the length of the latus-rectum of the parabola is 12 which is greater than the diameter of the circle and the common chord is of length less than the diameter.

48. **Statement-2** is true, equations of the perpendicular tangents to the parabola $y^2 = 4ax$ are $y = mx + a/m$, $y = m'x + a/m'$ where $mm' = -1$ they intersect at the point for which

$$(m - m')x = \frac{a}{m'} - \frac{a}{m} = \frac{a(m - m')}{mm'} \Rightarrow x = -a$$

which is the directrix of the parabola.

Since the extremities of a focal chord are $(at^2, 2at)$ and $(at'^2, 2at')$ where $tt' = -1$ and the slopes of the tangents at those points are $1/t$ and $1/t'$ whose product is -1, the tangents are perpendicular and hence by statement-2 they intersect on the directrix.

49. An equation of a normal to the parabola $y^2 = 2x$ is $y = mx - m - \frac{1}{2}m^3$.

It will pass through $(\alpha, 0)$ if $0 = m\alpha - m - \frac{1}{2}m^3$

$\Rightarrow m[m^2 - 2(\alpha - 1)] = 0$

If this equation has three distinct roots then $\alpha > 1$ and thus the statement-1 is true.

Also if $\alpha \leq 1$, we have only one value of m and thus exactly one normal of P passing through $(\alpha, 0)$ and statement-2 is also true but does not justify statement-1.

50. Equation of a normal to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ which touches the circle $x^2 + y^2 = a^2$ if $(-2am - am^3)^2 = a^2(1 + m^2)$

$\Rightarrow m^6 + 4m^4 + 3m^2 - 1 = 0$

Which does not hold for any integral value of m so statement-2 is true and justify that statement-1 is also true.

51. Statement-2 is true, see theory using it in statemenet-1, length of the latus rectum

$= 2\left|\frac{3(1) + (4)(-1) - 9}{\sqrt{9 + 16}}\right| = 4$ and thus statement-1 is also true.

52. From the given equations we have $x^2 + 2x - 4x + 3 = 0 \Rightarrow (x-1)^2 + 2 = 0$ which has no real solution and thus the statement-2 is true.

For statement-1, equation of a tangent to the parabola $y^2 = 2x$ is $y = mx + \frac{1}{2m}$ which will touch the circle if $\left|\frac{2m - 0 + \frac{1}{2m}}{\sqrt{1+m^2}}\right| = \sqrt{2^2 - 3}$

$$\Rightarrow \left(2m + \frac{1}{2m}\right)^2 = 1 + m^2$$

$$\Rightarrow 3m^2 + \frac{1}{4m^2} + 1 = 0$$

which gives no real value of m.

So there is no common tangent to the given curves and the statement-1 is false.

53. Statement-2 is true, see theory.

In statement-1, for the parabola $y^2 = 8x$ $a = 2$,

So $(at^2, 2at) = (32, 16) \Rightarrow t = 4$ and using statement-2, the coordinates of Q are $\left(\frac{a}{t^2}, \frac{-2a}{t}\right) = \left(\frac{2}{16}, \frac{-2\times 2}{2}\right) = \left(\frac{1}{8}, -2\right)$

So statement-2 is also true.

54. Statement-2 is false as these tangents intersect on the directrix and are at right angles.

In statement-1, the tangents intersect at the point $(at_1t_2, a(t_1 + t_2))$ which lies on the axis i.e. $y = 0 \Rightarrow t_1 + t_2 = 0$ and hence statement-1 is true.

55. If $t > 0$, $\left|t + \frac{1}{t}\right| = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 + 2 \geq 2$

If $t < 0$, $\left|t + \frac{1}{t}\right| = \left|\left(-t - \frac{1}{t}\right)\right| \geq 2$

So statement-2 is true.

Length of a focal chord of the parabola $y^2 = 16x$ is $a\left(t + \frac{1}{t}\right)^2$ where $a = 4$

So if $4\left(t + \frac{1}{t}\right)^2 = 9$

$$\Rightarrow \left|t + \frac{1}{t}\right| = \frac{3}{2} < 2$$

Which is not possible by statement-2 and thus statement-1 is also true.

Level 2

56. $PQ = QR = 4a$

$$AQ = \sqrt{a^2 + 4a^2} = \sqrt{5}\,a$$

$$AP = AM + PM$$

$$= a + \frac{\sqrt{3}}{2}(4a) = (1 + 2\sqrt{3})a$$

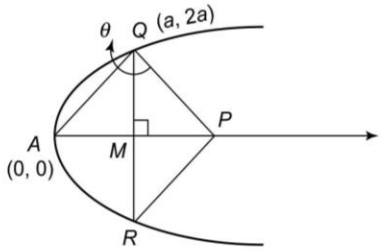

Fig. 18.20

$$\cos\theta = \frac{(AQ)^2 + (PQ)^2 - (AP)^2}{2(AQ)(PQ)}$$

$$= \frac{5 + 16 - \left(1 + 12 + 4\sqrt{3}\right)}{8\sqrt{5}} = \frac{2 - \sqrt{3}}{2\sqrt{5}}$$

57. Points of intersection of $x^2 + y^2 = a^2$ and $y^2 = 4ax$ are $\left(\left(\sqrt{5} - 2\right)a, \pm 2\left(\sqrt{5} - 2\right)^{1/2} a\right)$ and the points of intersection of $y^2 = -4ax$ and $x^2 + y^2 = a^2$ are $\left(\left(2 - \sqrt{5}\right)a, \pm 2\left(\sqrt{5} - 2\right)^{1/2} a\right)$

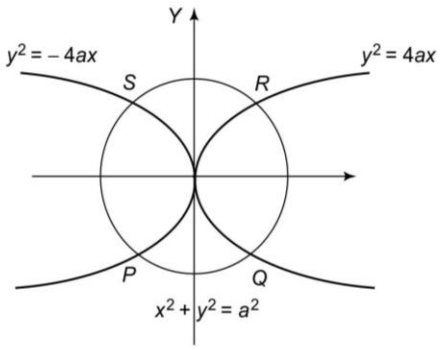

Fig. 18.21

$$\text{Required area} = 2 \times \left(\sqrt{5} - 2\right) \times 2 \times 2\left(\sqrt{5} - 2\right)^{1/2} a^2$$

$$= 8\left(\sqrt{5} - 2\right)^{3/2} a^2$$

58. Coordinates of P are $\left(1, 2\sqrt{2}\right)$ and of Q are $\left(1, -2\sqrt{2}\right)$. Tangent at $P\left(1, 2\sqrt{2}\right)$ to the circle $x^2 + y^2 = 9$ is $x + 2\sqrt{2}y = 9$ which meets x-axis at $R(9, 0)$.

Tangent at P $\left(1, 2\sqrt{2}\right)$ to the parabola $y^2 = 8x$ is $y\left(2\sqrt{2}\right) = 4(x + 1)$ which meets x-axis at $S(-1, 0)$

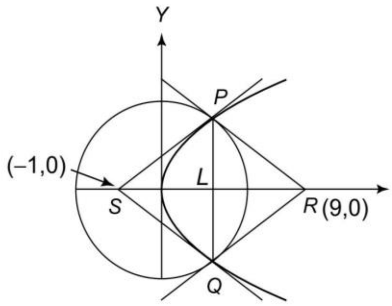

Fig. 18.22

Equation of PQ is $x = 1$

So $\dfrac{\text{Area of } \Delta\, PQS}{\text{Area of } \Delta\, PQR} = \dfrac{SL}{LR} = \dfrac{2}{8} = \dfrac{1}{4}$

59. Equation of any tangent of the parabola $y^2 = 8x$ is $y = mx + \dfrac{2}{m}$.

It will be a tangent to the circle $x^2 + y^2 = 2$ if

$$\frac{|2/m|}{\sqrt{1+m^2}} = \sqrt{2}$$

$\Rightarrow \dfrac{4}{m^2} = 2(1 + m^2)$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow m = \pm 1.$

Thus, two common tangents are $y = x + 2$ and $y = -x - 2$.

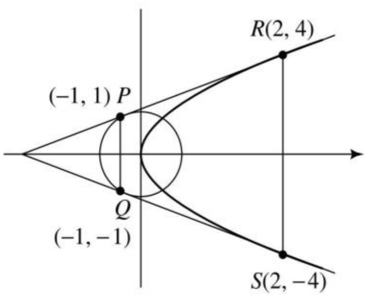

Fig. 18.23

Coordinates of P, Q, R, S are respectively $(-1, 1)$, $(-1, -1)$, $(2, 4)$, $(2, -4)$

$\therefore$ $PQ = 2$, $RS = 8$ and distance between PQ and RS is 5.

$\therefore$ area of $PQRS = \dfrac{1}{2}(2 + 8)(5) = 20(\text{unit})^2$

60. The tangents at the points $A(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$ intersect at the point $(t_1t_2,\ t_1 + t_2)$

So $t_1t_2 = -3$ and $t_1 + t_2 = 2$.

$(AB)^2 = (t_1^2 - t_2^2)^2 + 4(t_1 - t_2)^2$

$= (t_1 - t_2)^2[(t_1 + t_2)^2 + 4]$

$= [(t_1 + t_2)^2 - 4t_1t_2]\ [(t_1 + t_2)^2 + 4]$

$= (4 + 12)(4 + 4) = 16 \times 8$

$\Rightarrow AB = 8\sqrt{2}$

61. Equations of tangent and normal at point $P(at^2, 2at)$ to the parabola are

$ty = x + at^2$

$y = -tx + 2at + at^3$ respectively.

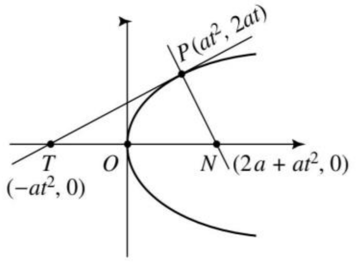

Fig. 18.24

These meet the axis of the parabola at $T(-at^2, 0)$ and $N(2a + at^2, 0)$

Let centroid of ΔPTN be (h, k), then

$h = \dfrac{1}{3}(-at^2 + 2a + at^2 + at^2) = \dfrac{1}{3}(2a + at^2)$ and

$k = \dfrac{1}{3}(0 + 0 + 2at) = \dfrac{2}{3}(at)$

Eliminating t, we get

$$\left(h - \frac{2a}{3}\right) = \frac{a}{3}\left(\frac{3k}{2a}\right)^2$$

Thus, locus of centroid is

$$y^2 = \frac{4}{3}a\left(x - \frac{2a}{3}\right)$$

Vertex of this parabola is $(2a/3, 0)$;

62. As circle with PQ as diameter passes through O, $\angle POQ = \frac{\pi}{2}$.

Let coordinates of P be $\left(t^2, \sqrt{2}t\right)$, where $t > 0$, and let coordinates of Q be $\left(t_1^2, \sqrt{2}t_1\right)$.

Let m_1 = slope of $OP = \frac{\sqrt{2}}{t}$

m_2 = slope of $OQ = \frac{\sqrt{2}}{t_1}$

As $\angle POQ = \frac{\pi}{2}$, $m_1m_2 = -1$

$\Rightarrow tt_1 = -2$.

Area of $OPQ = \frac{1}{2}|x_1y_2 - x_2y_1|$

$= \frac{1}{2}\left|t^2\sqrt{2}t_1 - t_1^2\sqrt{2}t\right|$

$= \frac{\sqrt{2}}{2}(2)|t - t_1|$ $\quad [\because tt_1 = -2]$

$= \sqrt{2}\left|t+\frac{2}{t}\right| = \sqrt{2}\left(t+\frac{2}{t}\right)$

$\Rightarrow 3\sqrt{2} = \sqrt{2}\left(t+\frac{2}{t}\right)$

$\Rightarrow t^2 - 3t + 2 = 0 \Rightarrow t = 1, 2$

Thus, coordinates of P are $(1, \sqrt{2})$ or $(4, 2\sqrt{2})$

63. Let $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$, $C(at_3^2, 2at_3)$ then $P(at_2t_3, a(t_2 + t_3))$ $Q(at_3t_1, a(t_3 + t_1))$ and $R(at_1t_2, a(t_1 + t_2))$

Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$

$= a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$

Area of $\Delta PQR = \frac{1}{2}\begin{vmatrix} at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \\ at_1t_2 & a(t_1+t_2) & 1 \end{vmatrix}$

$= \frac{1}{2}a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$

Hence the required ratio is 2 : 1.

64. Equation of a normal to the parabola $y^2 = 16x$ is $y = -tx + 8t + 4t^3$ for all values of t

Comparing with the given equation, we have

$-t = \cos\theta$, $8t + 4t^3 = -11\cos\theta - \cos3\theta$.

$\Rightarrow -8\cos\theta - 4\cos^3\theta = -11\cos\theta - \cos3\theta$.

$\Rightarrow 4\cos^3\theta - 3\cos\theta = \cos3\theta$.

Which is true for all values of θ.

65. Let the coordinates of P_1 be $(at_1^2, 2at_1)$, then the coordinates of Q_1 are $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ and the coordinates of P_2 be $(at_2^2, 2at_2)$, then the coordinates of Q_2 are $\left(\frac{a}{t_2^2}, \frac{-2a}{t_2}\right)$

Equation of P_1P_2 is

$$\frac{x - at_1^2}{at_2^2 - at_1^2} = \frac{y - 2at_1}{2at_2 - 2at_1}$$

or $2(x + at_1t_2) = (t_1 + t_2)y$ (1)

Similarly equation of Q_1Q_2 is

$$2\left(x + \frac{a}{t_1t_2}\right) = -\left(\frac{1}{t_1} + \frac{1}{t_2}\right)y$$

or $2(t_1t_2x + a) = -(t_1 + t_2)y$ (2)

Adding (1) and (2) we get

$2(x + a)(1 + t_1t_2) = 0 \Rightarrow x + a = 0$

Showing that P_1P_2 and Q_1Q_2 intersect on $x = -a$, which is the directrix of the parabola.

Previous Years' AIEEE/JEE Main Questions

1. Equation of the normal at $(bt_1^2, 2bt_1)$ of the parabola is

$y = -t_1x + 2bt_1 + bt_1^3$

which passes through $(bt_2^2, 2bt_2)$ if $2bt_2 = -t_1(bt_2^2) + 2bt_1 + bt_1^3$.

$\Rightarrow 2(t_2 - t_1) = -t_1(t_2^2 - t_1^2)$

$\Rightarrow t_2 = -t_1 - \frac{2}{t_1}$

2. Given curve $y^2 = 18x \Rightarrow 2y\frac{dy}{dx} = 18$

$\Rightarrow \frac{dy}{dx} = \frac{9}{y}$

We are given $\frac{dy}{dx} = 2$

$\therefore \frac{9}{y} = 2 \Rightarrow y = \frac{9}{2}$

Thus, $x = \frac{1}{18}y^2 = \frac{1}{18}\left(\frac{9}{2}\right)^2 = \frac{9}{8}$

$\therefore$ required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$.

3. The two parabolas $y^2 = 4ax$ and $x^2 = 4ay$ meet in $(0, 0)$ and $(4a, 4a)$. As these points lie on $2bx + 3cy + 4d = 0$, we get $d = 0$ and $2b(4a) + 3c(4a) + 4d = 0$

$\Rightarrow d = 0$ and $2b + 3c = 0$

Thus, $d^2 + (2b + 3c)^2 = 0$

4. Let Q be $(2t^2, 4t)$, $P(1, 0)$

Mid point of $PQ = (h, k) = \left(\frac{2t^2+1}{2}, 2t\right)$

Eliminating t we get

$h = \frac{2\left(\frac{k}{2}\right)^2+1}{2} \Rightarrow 4h = k^2 + 2$

Locus of (h, k) is $y^2 - 4x + 2 = 0$

5. $y = \frac{1}{3}a^3x^2 + \frac{1}{2}a^2x - 2a$ (1)

$\frac{dy}{dx} = \frac{2}{3}a^3x + \frac{1}{2}a^2$

At the vertex of parabola $y = ax^2 + bx + c$, $\frac{dy}{dx} = 0$

So $\frac{dy}{dx} = 0 \Rightarrow a = -\frac{3}{4x}$

Putting $a = \frac{-3}{4x}$ in (1), we get

$y = \frac{1}{3}\left(\frac{-27}{64}\frac{1}{x^3}\right)x^2 + \frac{1}{2}\left(\frac{9}{16x^2}\right)x + \frac{3}{2x}$

$\Rightarrow xy = \frac{105}{64}$.

6. Since perpendicular tangents intersect on the directrix $x = -2$, required point is $(-2, 0)$.

7. Vertex of the parabola is equidistant from the focus and the directrix on the axis of the parabola. So its coordinates are $(1, 0)$.

8. Any point on the parabola is (y^2, y).

Its distance from the line $x - y + 1 = 0$ is

$P = \left|\frac{y^2 - y + 1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}\left|\left(y - \frac{1}{2}\right)^2 + \frac{3}{4}\right| \geq \frac{3}{4\sqrt{2}}$

Hence the shortest distance is $\frac{3\sqrt{2}}{8}$ which is attained when the perpendicular is drawn from $\left(\frac{1}{4}, \frac{1}{2}\right)$ on the parabola.

9. Perpendicular tangent to the parabola $y^2 = 4x$ intersect on the directrix of the parabola i.e. $x = -1$.

10. An equation of tangent to the parabola $y^2 = 4ax$ is of the form $y = mx + a/m$.

$\therefore$ an equation of the tangent to

$y^2 = 16\sqrt{3}x$ is $y = mx + 4\sqrt{3}/m$.

Now, $y = mx + 4\sqrt{3}/m$ will be tangent to $2x^2 + y^2 = 4$ if the equation

$2x^2 + (mx + 4\sqrt{3}/m)^2 = 4$

or $(2 + m^2)x^2 + 8\sqrt{3}x + 48/m^2 - 4 = 0$

has equal roots.

$\Leftrightarrow (8\sqrt{3})^2 - 4(2 + m^2)(48/m^2 - 4) = 0$

$\Leftrightarrow m^4 + 2m^2 = 24$.

$\therefore$ Statement-2 is true.

Since $m = 2$ satisfies this equation, we get that statement-1 is also true and statement-2 is a correct reason for the statement-1.

11. An equation of tangent to $y^2 = 4\sqrt{5}x$ is

$y = mx + \frac{\sqrt{5}}{m}$ or $m^2x - my + \sqrt{5} = 0$.

This line will touch $x^2 + y^2 = 5/2$ if

$\frac{|m^2(0) - m(0) + \sqrt{5}|}{\sqrt{m^4 + m^2}} = \frac{\sqrt{5}}{\sqrt{2}}$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow m = \pm 1$. Note that $m = \pm 1$ satisfies $m^4 - 3m^2 + 2 = 0$.

For $m = 1$, $y = x + \sqrt{5}$ is a common tangent to the two curves.

12. $y^2 = 4ax$

$\Rightarrow 2y\dfrac{dy}{dx} = 4a$

$\Rightarrow \dfrac{dy}{dx} - \dfrac{2a}{y} = 0$

Which is a differential equation of degree 1 and order1 so statement-2 is true.

Slope of the tangent $m = \dfrac{dy}{dx} = \dfrac{2a}{y}$ at $P(x, y)$

$\Rightarrow m$ is inversely proportional to the ordinate of P.

So statement-1 is also true but does not follow from statement-1.

13. $y = mx + \dfrac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$

at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$, for $y^2 = -2x$, $a = -\dfrac{1}{2}$

$\Rightarrow y = mx - \dfrac{1}{2m}$ is a tangent to the parabola

at $\left(-\dfrac{1}{2m^2}, \dfrac{-1}{m}\right) \Rightarrow$ statement-2 is true.

$\Rightarrow x - 2y = 2$ is a tangent to the parabola at $(-2, -2)$

for $m = \dfrac{1}{2}$ and thus statement-1 is also true.

14. Ends of the latus rectum are $(1, \pm 2) = (t^2, 2t)$
$\Rightarrow t = \pm 1$

Equation of the normal at $(t^2, 2t)$ is $y = -tx + 2t + t^3$.

Equations of the normals for $t = \pm 1$ are $y = -x + 3$ and $y = x - 3$.

They intersect at $(3, 0)$.

15. $y = mx + \dfrac{1}{m}$ is a tangent to the parabola $y^2 = 4x$

which will touch the parabola $x^2 = -32y$ if the root of $x^2 = -32\left(mx + \dfrac{1}{m}\right)$ are equal.

$\Rightarrow 32m^3 - 4 = 0 \Rightarrow m = \dfrac{1}{2}$

16. For the points of intersection $x^2 + 8x - 9 = 0$

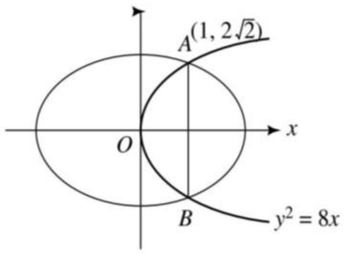

Fig. 18.25

$\Rightarrow (x - 1)(x + 9) = 0$

$\Rightarrow x = 1 \quad [\because x > 0]$

$\therefore$ Coordinate of A are $\left(1, 2\sqrt{2}\right)$

Thus, $L_1 = 4\sqrt{2}$

Also, $L_2 = 8 = (4)\ (2)$

$\therefore L_1 < L_2$

17. Let $y = mx + \dfrac{1}{m}$ be a tangent to the parabola $y^2 = 4x$.

It will pass through $(-2, -1)$, we get

$-1 = -2m + \dfrac{1}{m}$

$\Rightarrow 2m^2 - m - 1 = 0 \Rightarrow m = 1, -1/2$

$\Rightarrow |\tan\alpha| = \left|\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}}\right| = 3$

18. Let the slope of the line be m, then its equation is $y = m\left(x - \dfrac{3}{2}\right)$.

Its distance from $(0, 0)$ is

$\dfrac{|m(0-3/2)-0|}{\sqrt{1+m^2}} = \dfrac{\sqrt{5}}{2}$

$\Rightarrow (3m)^2 = 5(1 + m^2) \Rightarrow m = \pm\dfrac{\sqrt{5}}{2}$.

$\therefore$ Required slope is $\frac{\sqrt{5}}{2}$

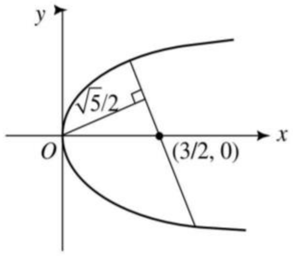

Fig. 18.26

19. Let coordinates of Q be (x', y') and that of P be (x, y), then

$x = \frac{1}{4}x', y = \frac{1}{4}y'$

As (x', y') lies on $x^2 = 8y$,

$(4x)^2 = 8(4y) \Rightarrow x^2 = 2y$

20. Let coordinates of P be $(-t^2, 2t)$, where $t > 0$, then coordinates of Q are $(-t^2, -2t)$.

If coordinates of R are (x, y), then

$x = \frac{-t^2 - 2t^2}{2+1} = -t^2$

and $y = \frac{1(2t) + 2(-2t)}{2+1} = -\frac{2}{3}t$

Eliminating t, we get

$-x = \left(-\frac{3}{2}y\right)^2$ or $9y^2 = -4x$

21. An equation of tangent to

$y - 6 = x^2$ at $(2, 10)$ is

$(y + 10) - 12 = 2(2)x$ or $4x - y + 2 = 0$ (1)

An equation of normal at the point of contact (α, β) of (1) and $x^2 + y^2 + 8x - 2y = k$ is

$(x - (-4)) + 4(y - 1) = 0$

or $x + 4y = 0$ (2)

The point (α, β) is the point of intersection of (1) and (2). Solving (1) and (2), we get

$\alpha = -\frac{8}{17}, \beta = \frac{2}{17}$.

22. Refer figure. Let $S = CP^2 = t^4 + \left(2\sqrt{2}t + 6\right)^2$

$\frac{ds}{dt} = 4t^3 + 4\sqrt{2}\left(2\sqrt{2}t + 6\right)$

$= 4\left[t^3 + 4t + 6\sqrt{2}\right]$

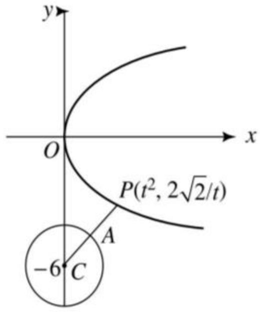

Fig. 18.27

For least value of S, $\frac{ds}{dt} = 0$

$\Rightarrow t^3 + 4t + 6\sqrt{2} = 0$

$t^3 + \sqrt{2}t^2 - \sqrt{2}t^2 - 2t + 6t + 6\sqrt{2} = 0$

$\Rightarrow \left(t + \sqrt{2}\right)\left(t^2 - \sqrt{2}t + 6\right) = 0$

$\Rightarrow t = -\sqrt{2}$ [$\because$ t is real]

Also, $\frac{d^2s}{dt^2} = 4[3t^2 + 4]$

$\Rightarrow \left.\frac{d^2s}{dt^2}\right]_{t=-\sqrt{2}} > 0$

$\Rightarrow S$ is least when $t = -\sqrt{2}$. Thus, coordinates of P are $(2, -4)$.

An equation of required circle is

$(x - 2)^2 + (y + 4)^2 = (0 - 2)^2 + (-6 + 4)^2$
or $x^2 + y^2 - 4x + 8y + 12 = 0$

23. Let coordinates of a point P on $y = x^2 - 4$ is $(t, t^2 - 4)$.

Let d = distance of P from the origin, then

$d^2 = (t - 0)^2 + (t^2 - 4 - 0)^2$

$= t^2 + t^4 + 16 - 8t^2$

$= t^4 - 7t^2 + 16$

$= \left(t^2 - \frac{7}{2}\right)^2 + \frac{15}{4}$

Note that d is least if $t^2 = 7/2$. Least value of d is $\sqrt{15}/2$.

24. An equation of normal at $P(t)$ is $y = -tx + 2t + t^3$

It passes through $Q(t_1)$ i.e. $(t_1^2, 2t_1)$ if $2t_1 = -tt_1^2 + 2t + t^3$

$\Rightarrow 2(t_1 - t) = t(t^2 - t_1^2)$

$\Rightarrow 2 = -t(t + t_1)$

$\Rightarrow t_1 = -t - \frac{2}{t}$

$\Rightarrow t_1^2 = t^2 + \frac{4}{t^2} + 4 \geq \sqrt{t^2\left(\frac{4}{t^2}\right)} + 4 = 6$ [AM ≥ GM]

Previous Years' B-Architecture Entrance Examination Questions

1. Equation of the side AB of the triangle ABC is $y = \frac{1}{\sqrt{3}}x$ which meets the parabola $y^2 = 8x$ at the point $B(24, 8\sqrt{3})$

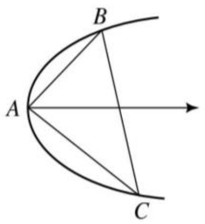

Fig. 18.28

$\Rightarrow (AB)^2 = (24)^2 + (8\sqrt{3})^2 \Rightarrow AB = 16\sqrt{3}$

2. Tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$ to the parabola $y^2 = x$ is

$y\left(\frac{1}{2}\right) = \frac{1}{2}\left(x + \frac{1}{4}\right) \Rightarrow y = x + \frac{1}{4}$

Which is parallel to $y = x + 1$, so statement-2 is true.

Any point on the parabola $y^2 = x$ is $\left(\frac{1}{4}t^2, \frac{1}{2}t\right)$

whose distance from the line $y = x + 1$ is

$\left|\frac{\frac{1}{2}t - \frac{1}{4}t^2 - 1}{\sqrt{2}}\right| = \frac{|(t-1)^2 + 3|}{4\sqrt{2}}$

Which minimum for $t = 1$. So statement-1 is also true but does not follow from statement-2.

3. Equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$ which passes through (1, 4) if $4 = m + \frac{1}{m} \Rightarrow m^2 - 4m + 1 = 0$

$\Rightarrow m_1m_2 = -1$. So the required angle is $\frac{\pi}{2}$.

4. Tangent to the parabola $x^2 = 4y$ at (4, 4) and (–4, 4) are $x(4) = 2(y + 4)$ and $x(-4) = 2(y + 4)$ which intersect on $x = 0$, i.e. y-axis.

So statement-1 is true.

Extremities of the latus rectum are (±2, 1) and the tangents at these points are $2x = 2(y + 1)$ and $(-2)x = 2(y + 1)$.

Which again intersect on y-axis, the axis of the parabola and statement-2 is also true but does justify statement-1.

5. Coordinates of L are (4, 8).

Equation of the chord through L with slope –1, is $x + y = 12$ (1)

Any point on the parabola is $(4t^2, 8t)$ which lies on (1) if

$4t^2 + 8t - 12 = 0 \Rightarrow t = 1, -3$.

$t = 1$ coordinates to L, so other end M of the chord is (36, – 24) and length of the chord is $\sqrt{(36-4)^2 + (-24-8)^2} = 32\sqrt{2}$

6. Let (h, k) be the mid point of the chord of the parabola $x^2 = 4py$ with slope m then its equation is

$hx - 2p(y - k) = h^2 - 4pk$.

$\Rightarrow m = \frac{h}{2p} \Rightarrow h = 2pm$.

Locus of (h, k) is $x = 2pm$.

Which is parallel to y-axis at a distance $|2pm|$ from it.

7. Let coordinates of P be $(t^2, 2t)$, then coordinates of Q are $\left(\frac{1}{t^2}, \frac{-2}{t}\right)$. Centre of the circle with PQ as diameter is $\left(\frac{1}{2}\left(t^2+\frac{1}{t^2}\right), t-\frac{1}{t}\right)$.

As it lies on $\sqrt{5}y + 4 = 0$,

$$t - \frac{1}{t} = \frac{-4}{\sqrt{5}}$$

Also, $PQ^2 = \left(t^2-\frac{1}{t^2}\right)^2 + 4\left(t+\frac{1}{t}\right)^2$

$$= \left(t+\frac{1}{t}\right)^2\left[\left(t-\frac{1}{t}\right)^2+4\right]$$

$$= \left[\left(t-\frac{1}{t}\right)^2+4\right]^2$$

$$\Rightarrow PQ = \left(t-\frac{1}{t}\right)^2 + 4$$

$$= \frac{16}{5} + 4 = \frac{36}{5}$$