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SOLID MECHANICS

PROPERTIES OF SOLIDS

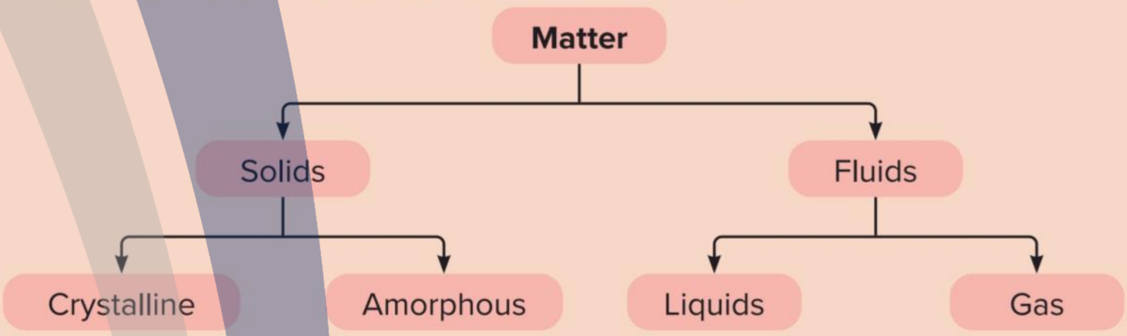
What you already know

- Dynamics of point particles
- System of particles
- Rigid body dynamics

What you will learn

- Classification of matter
- Rigid body vs real solid body
- Elasticity
- Stress
- Strain

In science, matter is generally classified as follows:

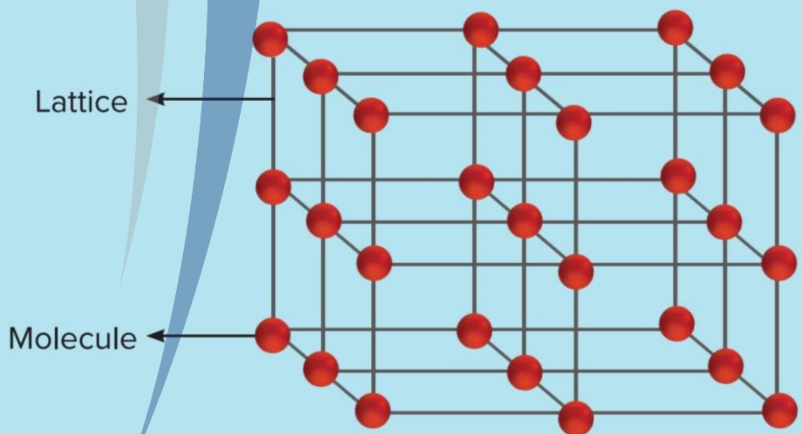


Solids

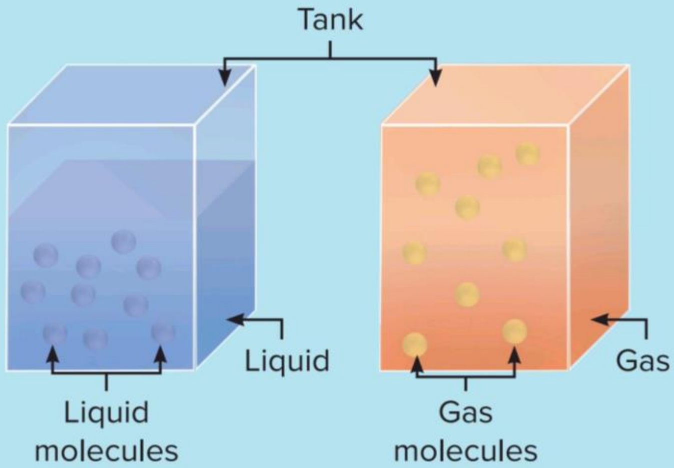
A solid can be defined as a substance that exists in the solid-state, which is one of the four fundamental states of matter.

- The solid has a definite shape and structure. This structure is known as **lattice**. In general, the lattice is symmetrical in two by two and three by three rows and columns structure.

  The red balls in the given figure represent molecules in solid. In solids, the intermolecular motion is highly restricted and high intermolecular force between solid molecules/atoms results in the definite and fixed shape of solids.

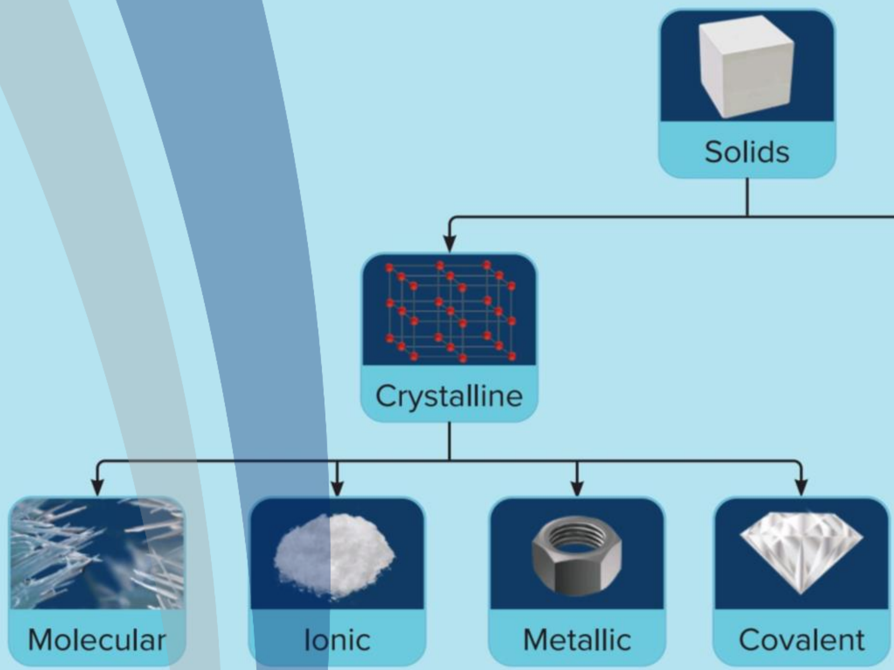


- On the contrary, fluids have random molecular motions and this motion is known as Brownian motion.



Classification of solids

Solids are further classified into different types based on their nature of intermolecular forces and chemical bonding.

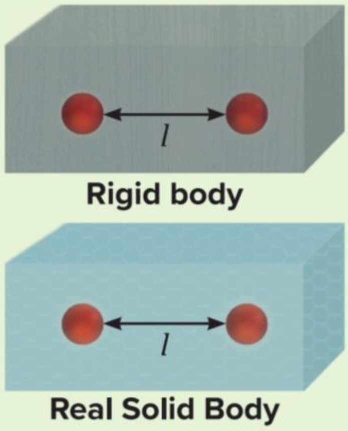


Rigid vs Real Solid Body

- In a rigid solid body, the distance between any two particles is always fixed (ideal system).
- In a real solid body, when the external forces are applied, the body may get deformed (real system).

Consider two solid cuboid-shaped bodies, one rigid and the other real.

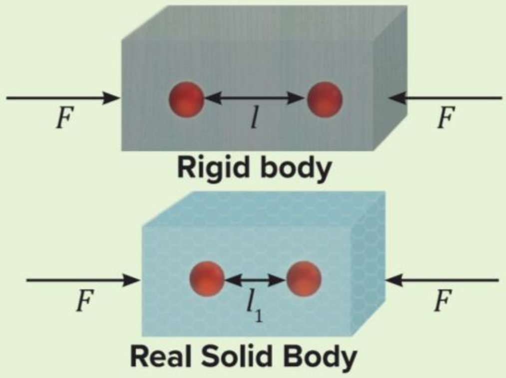
Let l be the distance between any two particles in both the bodies.



Case 1: A pair of compressive forces are acting on the cuboid shaped rigid and real bodies as shown in the figure.

In the rigid body, due to compression, the distance between the particles does not change. Therefore, the distance between the particles remains the same (l).

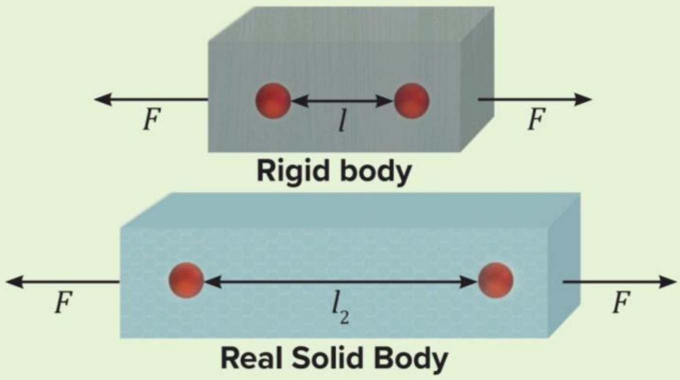
In the real solid body, due to compression, the distance between the particles becomes l_1.



Case 2: A pair of tensile forces are acting on the cuboid shaped rigid and real bodies as shown in the figure.

In the rigid body, due to stretching, the distance between the particles does not change. Therefore, the distance between the particles remains the same (l).

In the real solid body, due to stretching, the distance between the particles becomes l_2.



From case 1 and case 2, we get the following:

l_1 (Compression) < l < l_2 (Extension)

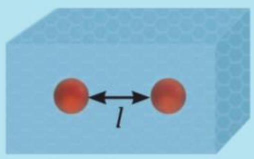
MAIN

Elasticity

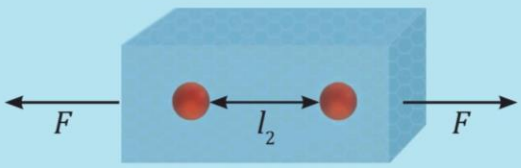
Elasticity is the property of a material to restore its natural shape or to oppose the change in its original shape.

Consider a cuboid shaped real body.

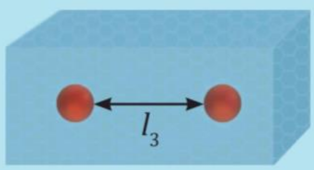
There is no external force on the body initially, and the distance between particles of the body is l.



After applying a pair of external forces on the body, as shown in the figure, the distance between the particles of the body becomes l_2.



On removing the external forces from the body, the distance between the particles of the body becomes l_3.



Perfectly elastic body

When a body completely regains its natural shape after the removal of the deforming forces, it is known as a perfectly elastic body. ($l_3 = l$)

Perfectly inelastic or plastic body

When a body remains in the deformed state and does not even partially regain its original shape after the removal of the deforming forces, it is known as a perfectly inelastic body or a plastic body. ($l_3 = l_2$)

Cause of elasticity

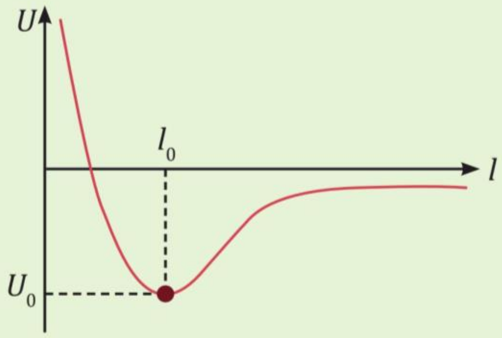
Elasticity in solids depends upon interatomic and intermolecular forces.

MAIN

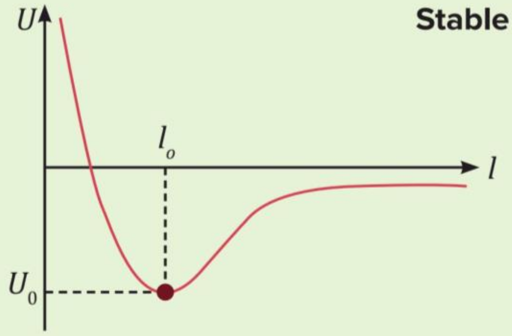
Morse Curve

The plot of potential energy (U) and intermolecular separation (l) is known as **Morse curve**.

l_0 is the intermolecular separation between the particles at which the body has minimum potential energy U_0. The state of the body at l_0 is considered stable.



Stable



Considering the cuboid shape, the real solid body is in a stable state at l_0 and the intermolecular separation has a potential energy U_0.

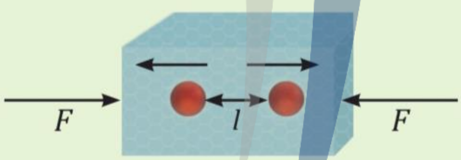
In a stable state, the net force on the body is zero.

We know that,

$$F = -\frac{\partial U}{\partial r}$$

Case 1:

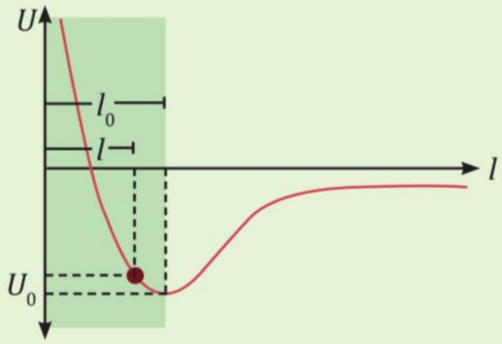
On compressing the cuboid shape of a real solid body, the distance between the particles becomes l.



Where,

$l < l_0$

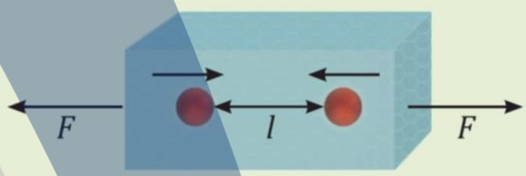
Replusion zone



If $l < l_0$, the repulsive force is greater than the attractive force. Therefore, there is a net repulsion between the particles. **This zone before l_0 is considered as a repulsion zone.**

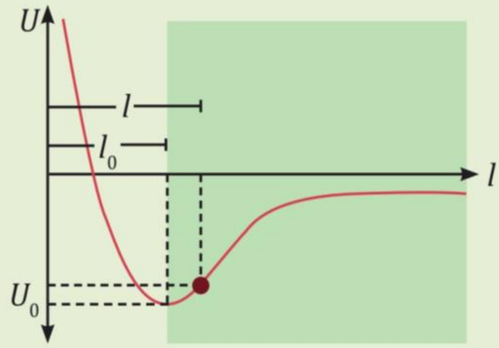
Case 2:

By extending the cuboid shape of a real solid body, the distance between the particles becomes l.



Where,

$l > l_0$



If $l > l_0$, the attractive force is greater than the repulsive force. Therefore, there is a net attraction between the particles. **This zone after l_0 is considered as an attraction zone**.

BOARDS

Stress

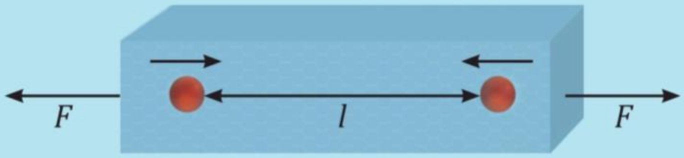
The restoring forces per unit area of the deformed body is known as **stress**.

$$\text{Stress}\,(\sigma) = \frac{\text{Restoring force}}{\text{Area}} = \frac{F}{A}$$

The SI unit of stress is Nm^{-2}.

Types of stresses

- Longitudinal/Normal stress (σ_l)
- Shear stress (σ_s)
- Volumetric stress (σ_v)



Normal stress or Longitudinal stress

If a deforming force acts normal (perpendicular) to the surface of a body, then the internal restoring force set-up per unit area of the body is known as **normal stress** or **longitudinal stress**. Normal stress is of two types.

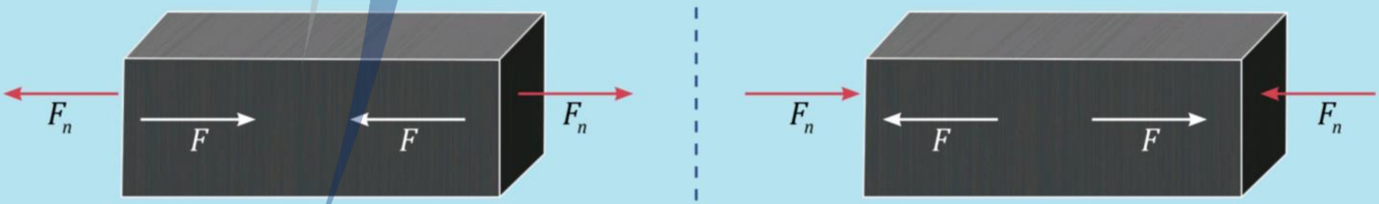
(*i*) Tensile stress

(*ii*) Compressive stress

$$\sigma_l = \frac{F_n}{A}$$

Tensile stress occurs due to the stretching force.

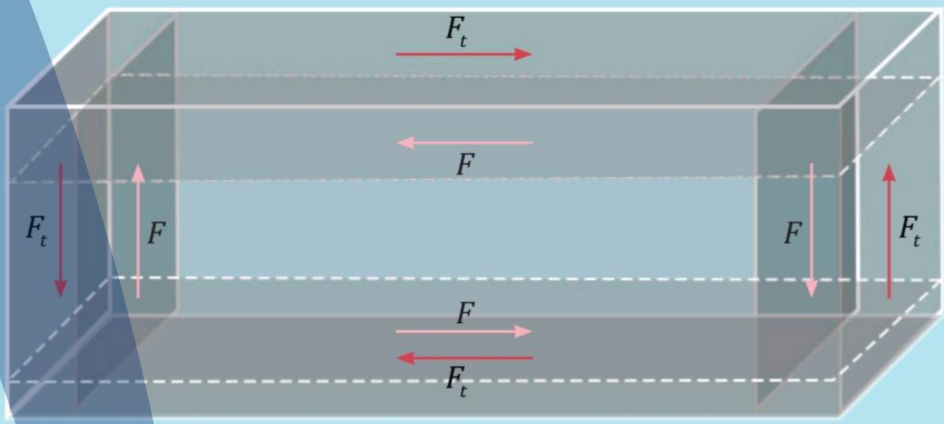
Compressive stress occurs due to the squeezing force.



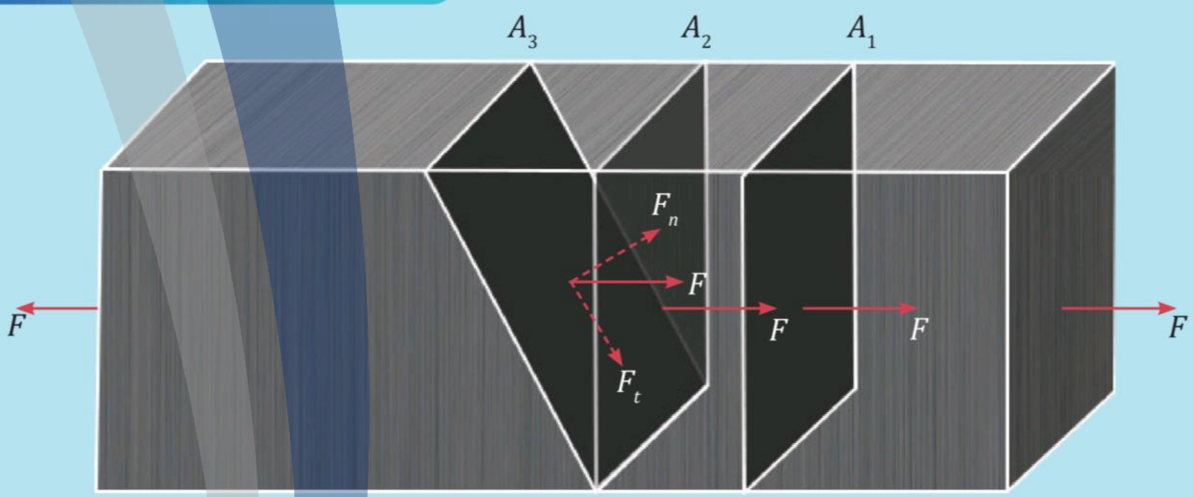
Shear stress

If a deforming force acts tangentially to the surface of a body and produces a change in the shape of the body, then the stress set-up in the body is known as **tangential stress** or **shear stress**.

$$\sigma_s = \frac{F_t}{A}$$



Stress at different cross-sections



At cross-sectional area A_1,

$F_n = F$

$F_t = 0$

Therefore,

$$\sigma_n = \frac{F_n}{A}$$

$$\Rightarrow \sigma_n = \frac{F}{A_1}$$

And,

$$\sigma_s = \frac{F_t}{A_1}$$

$$\Rightarrow \sigma_s = \frac{0}{A_1} = 0$$

At cross-sectional area A_2,

$F_n = F$

$F_t = 0$

Therefore,

$$\sigma_n = \frac{F_n}{A}$$

$$\Rightarrow \sigma_n = \frac{F}{A_2}$$

And,

$$\sigma_s = \frac{F_t}{A}$$

$$\Rightarrow \sigma_s = \frac{0}{A_2} = 0$$

At cross-sectional area A_3,

$F_n = F_n$

$F_t = F_t$

Therefore,

$$\sigma_n = \frac{F_n}{A}$$

$$\Rightarrow \sigma_n = \frac{F_n}{A_3}$$

And,

$$\sigma_s = \frac{F_t}{A}$$

$$\Rightarrow \sigma_s = \frac{F_t}{A_3}$$

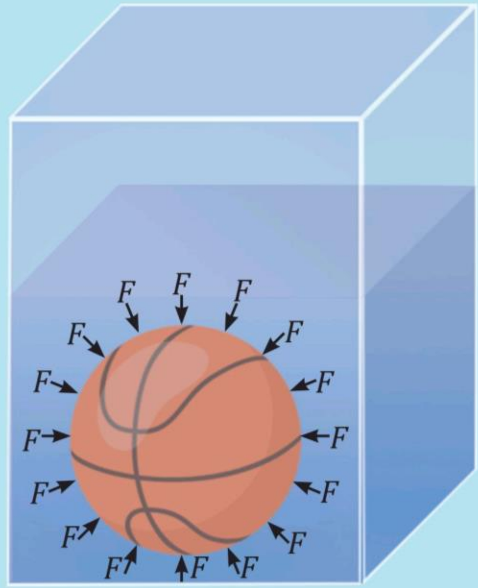
Volumetric stress

When the deforming force acts normally from all the directions resulting in the change in the volume of the object, then it is known as **volumetric stress** or **bulk stress**.

Consider a ball submerged in water.

- Liquid exerts a force on the ball normal to its surface.
- The magnitude of the force exerted by the water on any small surface area of the ball is proportional to the area on which the force is acting.

$$\sigma_v = \frac{F}{A}$$



BOARDS

Strain

The ratio of the change in configuration (i.e., shape, length, or volume) to the original configuration of the body is known as **strain**.

$$\text{Strain, } \varepsilon = \frac{\text{Change in configuration}}{\text{Original Configuration}}$$

Types of strains

Corresponding to the three types of stress, there are three types of strains.

- Longitudinal strain (ε_l)
- Shear strain (ε_s)
- Volumetric strain (ε_v)

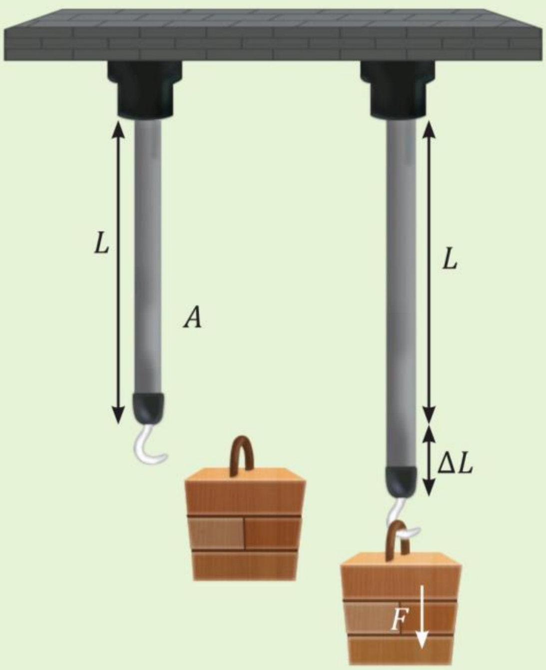
Longitudinal strain

This type of strain is produced when the deforming force causes a change in the length of the body.

It is defined as the ratio of the change in length to the original length of the body.

$$\text{Longitudinal strain, } \varepsilon_l = \frac{\text{Change in length}}{\text{Original length}}$$

Consider a rod of length (L) having an area of cross-section (A). Weight is added at the free end of the rod that applies force (F) on the rod. Initially, the rod is hinged from the roof and the weight is not added. After the weight is added to the rod, the length of the rod gets extended.



Original length = L

Area of cross-section = A

Applied force due to weight = F

Final length = $L + \Delta L$

Change in length = $L + \Delta L - L = \Delta L$

Longitudinal strain is given by,

$$\varepsilon_l = \frac{\text{Change in the length}}{\text{Original length}} = \frac{\Delta L}{L}$$

Shear strain

This type of strain is produced when the deforming force causes a change in the shape of the body.

It is defined as the ratio of the displacement of a layer to its distance from the fixed layer.

Let the height of the body be l.

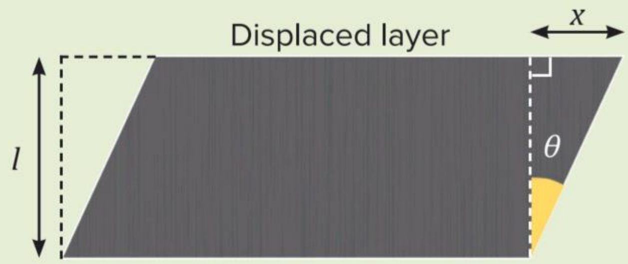
Let the displacement of the layer be x.

Shear strain, $\varepsilon_s = \dfrac{x}{l}$

$\tan\theta = \dfrac{x}{l}$

For a very small angle θ, we get the following:

$\theta \approx \dfrac{x}{l}$



Volumetric strain

Volumetric strain is produced when the deforming force causes a change in the volume of the body.

It is defined as the ratio of the change in volume to its original volume.

Volumetric strain, $\varepsilon_V = \dfrac{\text{Change in volume}}{\text{Original volume}}$

Consider the ball of volume V on which force F *is* applied normally to the surface of the ball from all the directions.

After applying the force, the volume of the ball becomes,

$V - \Delta V$

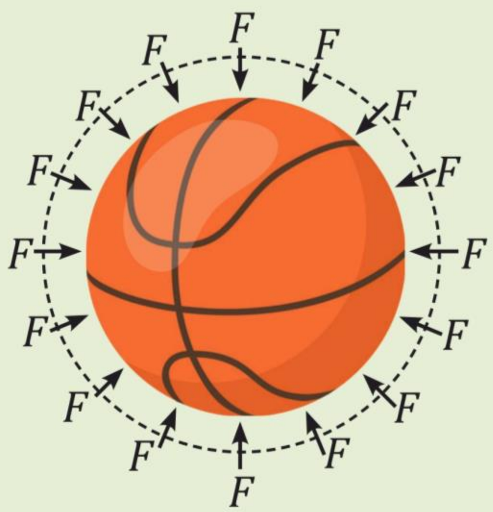
Therefore,

Change in volume = $(V - \Delta V) - V = \Delta V$

By putting this value in the equation for volumetric strain, we get,

Volumetric strain, $\varepsilon_V = \dfrac{\text{Change in volume}}{\text{Original volume}}$

Volumetric strain, $\varepsilon_V = \dfrac{\Delta V}{V}$



SOLID MECHANICS

STRESS-STRAIN ANALYSIS

What you already know

- Classification of matter
- Rigid vs real solid body
- Elasticity
- Stress
- Strain

What you will learn

- Hooke's law
- Stress-strain curve
- Types of moduli of elasticity
- Poisson's ratio
- Elastic potential energy stored in a stretched rod
- Elongation in a rod due to self-weight

Hooke's Law

If the deformation is small, the stress (σ) in a body is proportional to the corresponding strain (ε). Mathematically, it is written as follows:

$$\sigma \propto \varepsilon$$

$$\Rightarrow \sigma = k\varepsilon$$

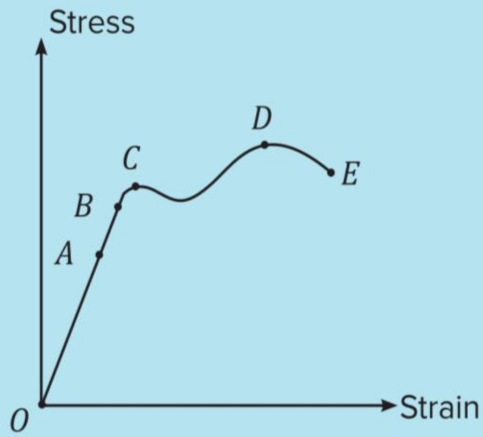
Where, k is the proportionality constant, and it is known as the modulus of elasticity.

BOARDS

Stress vs Strain Curve

The points shown in the graph are known by certain names as follows:

A : Limit of proportionality
B : Elastic limit
C : Yield point
D : Ultimate stress point
E : Fracture point



Definition of Hooke's law: Within the region of elastic limit (i.e., region OA), stress is proportional to strain ($\sigma \propto \varepsilon$).

Types of Moduli of Elasticity

There are three different types of strains. Depending upon the strain, there are three types of moduli of elasticity accompanied with each type of strain.

- Young's modulus of elasticity (Y)
- Modulus of rigidity (G or η)
- Bulk modulus of elasticity (B)

Young's modulus

It is defined as the ratio of the normal stress to the longitudinal strain.

Normal stress or longitudinal stress is defined as, $\sigma = \frac{F}{A}$

Longitudinal strain is defined as, $\varepsilon_l = \frac{\Delta L}{L}$

Hence, Young's modulus of elasticity is written as,

$\sigma_n \propto \varepsilon_l \Rightarrow \sigma_n = Y\varepsilon_l$

$$Y = \frac{\sigma_n}{\varepsilon_l} = \frac{FL}{A\Delta L}$$

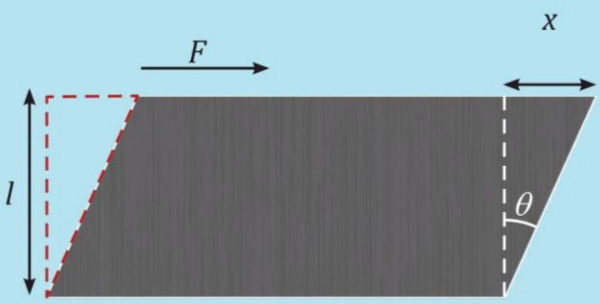
It is predominantly used only for solids.

Modulus of rigidity

It is defined as the ratio of the tangential stress to the shear strain.

Now, tangential stress is, $\sigma_s = \frac{F_{tangential}}{A}$, and shear strain is, $\varepsilon_s = \tan\theta = \frac{x}{l} \simeq \theta$

Consider a cube whose lower face is fixed, and a tangential force F acts on the upper face that has area A, as shown in the figure. Then, the modulus of rigidity is defined as follows,



Side view of cube

$$\eta = \frac{\text{Tangential stress } (\sigma_s)}{\text{Shear strain } (\varepsilon_s)}$$

$$\Rightarrow \eta = \frac{\left(\frac{F}{A}\right)}{\theta} = \frac{F}{A\theta}$$

Bulk modulus

It is defined as the ratio of the normal stress to the volumetric strain. It is important to remember that it has dominance only on the fluids (i.e., on liquids and gases).

Normal stress or longitudinal stress is given by the following equation,

$$\frac{F}{A} = P_{applied}$$

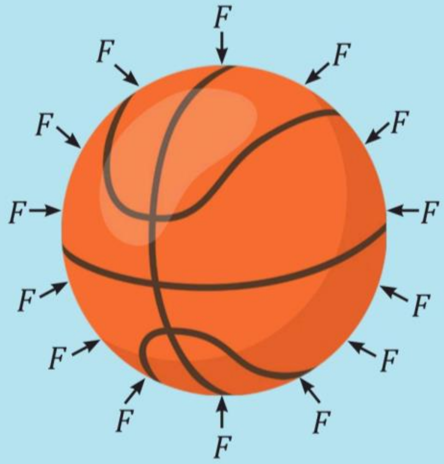
Volumetric strain = $\pm\frac{\Delta V}{V}$

(+ sign is for a case of expansion of volume, and – sign is for a case of compression of volume.)

Hence, bulk modulus of elasticity is defined as follows,

$$B = \pm\frac{P_{applied}}{\left(\frac{\Delta V}{V}\right)}$$

$$\Rightarrow B = \pm V\left(\frac{P_{applied}}{\Delta V}\right)$$



Compressibility

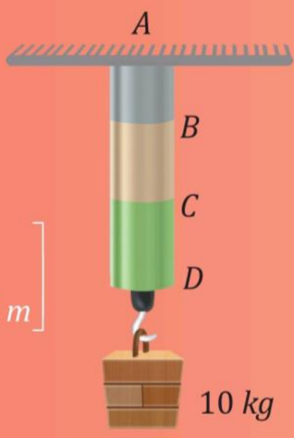
The reciprocal of bulk modulus (B) is known as compressibility (K).

$$K = \frac{1}{B}$$

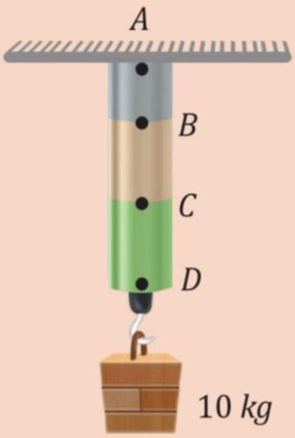
It is difficult to compress the gases that have a higher value of bulk modulus. In other words, higher the value of compressibility, easier it is to compress the gases as, $K = \frac{1}{B}$

A system of three massless rods is given in the figure. A block of mass 10 kg is attached at the end of the lowest rod. Find the shift in points B, C, and D after the mass is attached.

$$\left[\begin{array}{l} Y_{AB} = 2.5\times10^{10}\ Nm^{-2}, Y_{BC} = 4\times10^{10}\ Nm^{-2}, Y_{CD} = 1\times10^{10}\ Nm^{-2} \\ \text{Length of the three rods are, } AB = 0.1\ m, BC = 0.2\ m \text{ and } CD = 0.15\ m \end{array}\right]$$



Solution



The given three rods are massless and the applied force on each rod will be equal in magnitude.
Consider the area of cross section (A) of the given three rods to be unity (i.e., 1 m^2) and gravitational acceleration, $g = 10\ ms^{-2}$

Therefore, the force on each rod (AB, BC, and CD) is,

$F = mg$

$\Rightarrow F = 100\ N$

Hence, the stress on each rod is,

$$\sigma = \frac{F}{A} = 100\ Nm^{-2}$$

Now,

$$\sigma = Y\varepsilon = Y\frac{\Delta L}{L}$$

$$\Rightarrow \Delta L = \frac{\sigma L}{Y}$$

Therefore, the change in the length of each rod is as follows:

$$\Delta L_{AB} = \frac{(100)(0.1)}{(2.5\times 10^{10})} = 4\times 10^{-10}\ m$$

$$\Delta L_{BC} = \frac{(100)(0.2)}{(4\times 10^{10})} = 5\times 10^{-10}\ m$$

$$\Delta L_{CD} = \frac{(100)(0.15)}{(1\times 10^{10})} = 15\times 10^{-10}\ m$$

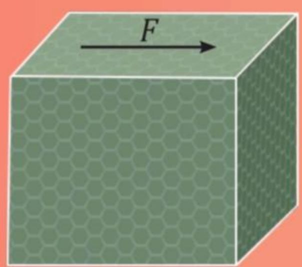
Finally,

The shift in point B is, $B_{shift} = \Delta L_{AB} = 4\times 10^{-10}\ m$

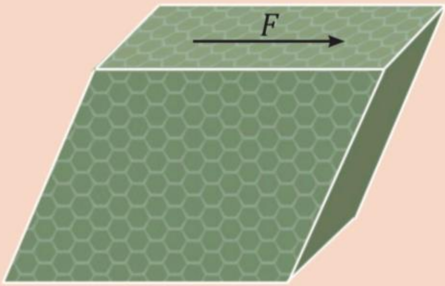
The shift in point C is, $C_{shift} = \Delta L_{AB} + \Delta L_{BC} = 9\times 10^{-10}\ m$

The shift in point D is, $D_{shift} = \Delta L_{AB} + \Delta L_{BC} + \Delta L_{CD} = 24\times 10^{-10}\ m$

A rubber cube of side 5 cm has one side fixed while a tangential force equal to 1800 N is applied to the opposite face. Find the shearing strain and the lateral displacement of the strained face. The modulus of rigidity of rubber is $2.4\times 10^6\ Nm^{-2}$.



Solution



We know that shearing stress and shearing strain are given as follows:

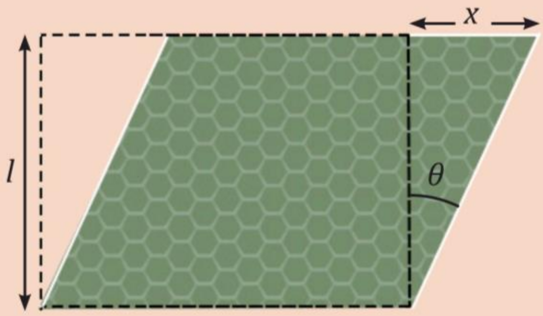
$\varepsilon_s = \tan\theta = \frac{x}{L} \simeq \theta$ and, $\sigma_s = \frac{F_{tangential}}{A}$

$\sigma_s = \eta\varepsilon_s$, where η is the modulus of rigidity.

Given,
The length of one side of the cube (L) is 5 cm
Tangential force, $F_{tangential} = 1800\ N$
Modulus of rigidity, $\eta = 2.4 \times 10^6\ Nm^{-2}$

So, the area of one face of the cube is the following:

$A = 25\ cm^2$
$\Rightarrow A = 25 \times 10^{-4}\ m^2$



Therefore,

$$\sigma_s = \frac{F_{tangential}}{A} = \frac{1800}{\left(25 \times 10^{-4}\right)} = \frac{18 \times 10^6}{25}\ Nm^{-2}$$

$$\varepsilon_s = \frac{\sigma_s}{\eta} = \frac{\left(\frac{18 \times 10^6}{25}\right)}{2.4 \times 10^6} = 0.3\ rad$$

Now, since $\varepsilon_s = \frac{x}{L}$ and $L = 5\ cm$, the lateral displacement of the strained face is the following:

$x = \varepsilon_s L = 0.3 \times 5$
$\Rightarrow x = 1.5\ cm$

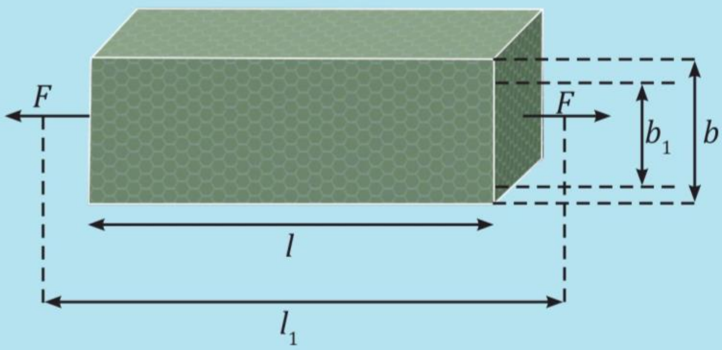
MAIN

Poisson's Ratio

Poisson's ratio is related to the simultaneous strain in the transverse (lateral) direction and longitudinal (axial) direction of an object due to an external force. It is defined as the negative ratio of the transverse (or lateral) strain to the longitudinal (or axial) strain.

$$\text{Poisson's ratio}(\nu) = -\frac{\text{Transverse strain}(\varepsilon_t)}{\text{Longitudinal strain}(\varepsilon_l)}$$

Consider a rectangular bar of length l and breadth b. Due to an external force F, which acts in the outward direction in two opposite faces of the bar as shown in the figure (just like the bar is stretched along its length), the length of the bar will be increased and there will be a consequent decrease in the breadth.



So, if the change in length is, $\Delta l = l_1 - l$, and the change in breadth is, $\Delta b = b_1 - b$, then the transverse strain is, $\varepsilon_t = \frac{\Delta b}{b}$ (where Δb is a negative quantity since the breadth has decreased) and the longitudinal strain is, $\varepsilon_l = \frac{\Delta l}{l}$

So,

$$\text{Poisson's ratio, } \nu = -\frac{\varepsilon_t}{\varepsilon_l} = -\frac{\frac{\Delta b}{b}}{\frac{\Delta l}{l}}$$

Rod-Spring Analogy

Young's modulus (Y) of a rod of length L and cross-sectional area A is defined as, $Y = \frac{FL}{A\Delta L}$.

So, the applied force on the rod is, $F = \left(\frac{YA}{L}\right)\Delta L$.

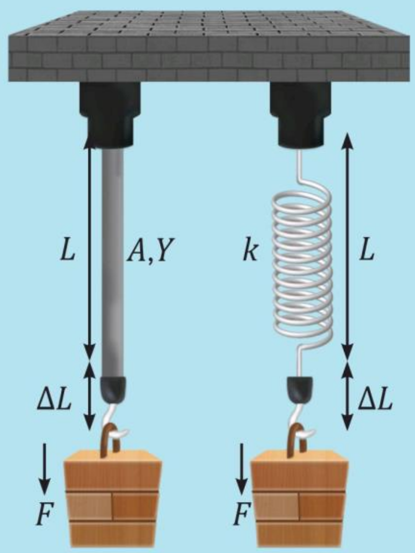
Where, ΔL is the elongation in the length of the rod. In the case of a spring, if it is elongated by an amount of ΔL due to force F, then the spring constant is defined as follows:

$$k = \frac{F}{\Delta L}$$

So, the applied force on the spring is, $F = k\Delta L$

$$F = \left(\frac{YA}{L}\right)\Delta L \quad \text{.......................}(i)$$

$$F = k\,\Delta L \quad \text{.......................}(ii)$$



So, on comparing these two equations, it can be said that $k = \frac{YA}{L}$. Hence, the term $k = \frac{YA}{L}$ is known as the **equivalent spring constant (k_{eq})**, and it depends on the type of material and the geometry of solid.

We know that, $k = \frac{YA}{L}$. So, from this, it can be said that $k \propto \frac{1}{L} \Rightarrow kL = \text{Constant}$. Hence, if we cut a spring of length L and spring constant k into two equal parts of length $\frac{L}{2}$, then from the condition (kL = Constant), the spring constant of each part of the spring will be $2k$.

MAIN

Elastic Potential Energy Stored in a Stretched Rod

We know that the elastic potential energy stored in a stretched spring is, $U_{spring} = \frac{1}{2}k(\Delta L)^2$. In the similar manner, the elastic potential energy stored in a stretched rod is,

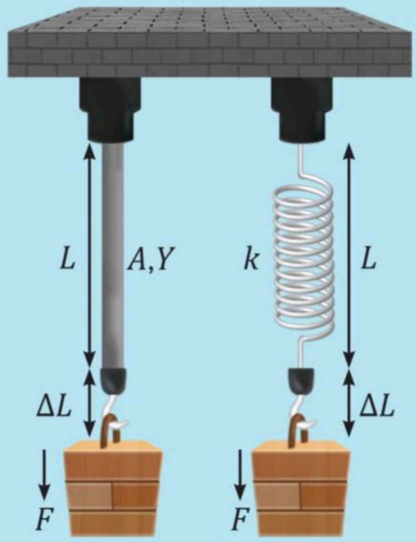
$$U_{rod} = \frac{1}{2}\left(\frac{YA}{L}\right)(\Delta L)^2$$

Since stress, $\sigma = \frac{F}{A}$, strain, $\varepsilon = \frac{\Delta L}{L}$ and, $\sigma = Y\varepsilon$, we get,

$$U_{rod} = \frac{1}{2}\left(\frac{F}{A}\right)\left(\frac{\Delta L}{L}\right)(AL)$$

$$U_{rod} = \frac{1}{2}(\text{Stress})(\text{Strain})(\text{Volume})$$

$$U_{rod} = \frac{1}{2}\sigma\varepsilon V$$



This is the potential energy stored in volume V of the stretched rod. Therefore, the potential energy per unit volume or the potential energy density is given by the following:

$$u = \frac{U_{rod}}{V} = \frac{1}{2}(\text{Stress})(\text{Strain}) = \frac{\sigma\varepsilon}{2}$$

A wire with the area of cross section as 3.0 mm^2 and a natural length of 50 cm is fixed at one end, and a mass of 2.1 kg is hung from its other end. Find the elastic potential energy stored in the wire in the steady state. Young's modulus of the material of the wire is 1.9×10^{11} Nm^{-2}. (Take $g = 10\ ms^{-2}$)

Solution MAIN

Given,

Area of cross section of the wire, $A = 3\ mm^2 = 3 \times 10^{-6}\ m^2$

Natural length of the wire, $L = 50\ cm = 0.5\ m$

Young's modulus of the wire, $Y = 1.9 \times 10^{11}\ Nm^{-2}$

Mass that is hanged to the other end of the wire, $m = 2.1\ kg$

Now, the force applied on the rod is the following:

$F = mg = (2.1\ kg) \times (10\ ms^{-2}) = 21\ N$

The elastic potential energy stored in the wire at the steady state is the following:

$$U = \frac{1}{2} \times \sigma \times \varepsilon \times (\text{Volume})$$

$$\Rightarrow U = \frac{1}{2} \times \sigma \times \left(\frac{\sigma}{Y}\right) \times (A \times L)$$

$$\Rightarrow U = \frac{\sigma^2 AL}{2Y}$$

$$\Rightarrow U = \frac{F^2 L}{2AY}$$

Now, on putting the values in the equation, we get,

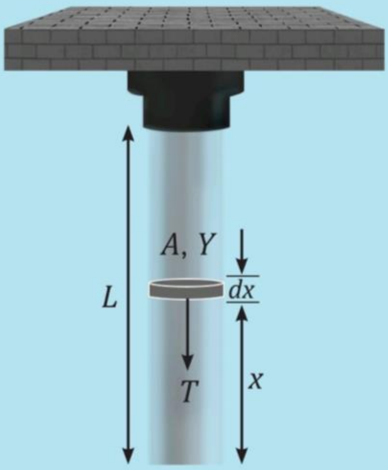
$$U = \frac{(21)^2 \times (0.5)}{2 \times (3 \times 10^{-6}) \times (1.9 \times 10^{11})} = 1.9 \times 10^{-4}\ J$$

BOARDS

Elongation in a Rod due to Self-Weight

Consider a rod having length L, weight W, area of cross section as A, and Young's modulus of elasticity as Y. Consider a thin, disc-shaped element of thickness dx at distance x from the lower end of the rod. The net force on the element is the following:

$$T(x) = \left(\frac{W}{L}\right)x$$



Here, $T(x)$ means that T is dependent on x.

We have, $\sigma = Y\varepsilon \Rightarrow \left(\frac{F}{A}\right) = Y\left(\frac{\Delta L}{L}\right)$

For small element of length dx, consider $\delta(dx)$ be elongation. Thus, for this small element, above equation can be written as follows:

$$\Rightarrow \frac{T(x)}{A} = Y\frac{\delta(dx)}{dx}$$

$$\Rightarrow \frac{Wx}{AL} = Y\frac{\delta(dx)}{(dx)}$$

$$\Rightarrow \delta(dx) = \frac{Wx}{YAL}dx$$

$$\Rightarrow \sum \delta(dx) = \Delta x = \int_0^L \frac{Wx}{YAL}dx$$

$$\Rightarrow \Delta x = \frac{WL}{2AY}$$

Hence, the net elongation in the rod due to its self-weight is, $\Delta x = \frac{WL}{2AY}$.