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DETERMINANTS

CHAPTER AT A GLANCE

1. **Determinant :** To every Square matrix $A = [a_{ij}]$ of order n associated with a number (real or complex) called determinant of A. It is denoted by det (A) or |A|.

   A determinant is also denoted by Δ.

2. **Properties of Determinant:**

(i) The value of the determinant does not change when rows and columns are interchanged. The determinant obtained by interchanging the rows into corresponding columns and vice-versa is called the transpose of the determinant and is denoted by Δ^T. Thus $\Delta = \Delta^T$.

(ii) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(iii) The interchange of any two rows (or column) of the determinant changes the sign of the determinant

Thus if Δ^* is the new determinant obtained on interchanging any two rows (or columns), then

$$\Delta = -\Delta^*$$

If i-th and j-th row are interchanged then this operation is denoted by $R_i \leftrightarrow R_j$.

(iv) If each element of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant. Thus if we apply $R_i \rightarrow pR_i$, i.e, each element of i-th row is multiplied by p to a determinant Δ, then we get new determinant

$$\Delta^* = p\Delta$$

(v) If all the elements of a row (or column) are proportional (or identical) to the elements of some other row (or column) then the value of the determinant is zero.

(vi) If each element of a row (or column) of a determinant is zero then, its value is zero.

(vii) If to each element of a row (or a column) of a determinant the equimultiples of corresponding elements of other rows (or columns) are either added or subtracted then value of determinant remains the same.

(viii) If sum of all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants

For example,

$$\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3. **Area of a Triangle :** Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}\left|\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]\right|$

If area of a triangle is given, then use both positive and negative values of the determinant for calculation.

4. **Minors and Cofactors :**

(i) **Minors -** The determinant obtained by deleting the i-the row and j-th column passing through the element a_{ij} is called the minor of element a_{ij} and is denoted by M_{ij}.

(ii) **Cofactors -** The cofactor of element a_{ij} is $(-1)^{i+j}$ times the determinant obtained by deleting the i-th row and jth column passes through a_{ij} and is denoted by A_{ij} i.e. $A_{ij} = (-1)^{i+j} M_{ij}$

5. **Value of a Determinant :** The sum of the products of each element of any row (or column) by the corresponding co-factors is equal to the value of the determinant.

Let $\Delta = \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Then $\Delta = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$, if expanding along first row
$\Delta = a_{31} A_{31} + a_{32} A_{32} + a_{33} A_{33}$, if expanding along third row.
Similarly, you can find the value of the determinant by expanding it along any other row or column.

6. **Adjoint of a Matrix :** The adjoint of a square matrix A is the transpose of the matrix which is obtained by replacing each element of matrix A by their cofactors. In other words, adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by adj A

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

adj A = Transpose of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

7. If A is any given square matrix of order n, then

$A(\text{adj } A) = (\text{adj } A)\, A = |A| I$,

where I is the identity matrix of order n.

8. Singular and Non-Singular Matrix

(i) A square matrix A is said to be singular, if $|A| = 0$

(ii) A square matrix A is said to be non-singular, if $|A| \neq 0$

(iii) If A and B are two non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

9. Invertible Matrix

A square matrix A is invertible if and only if A is non-singular matrix.

10. Inverse of a Square Matrix

If two invertible matrices A and B are such that

AB = I or BA = I, where I is the identity matrix then A is called the Inverse matrix of matrix B or B is called the Inverse matrix of matrix A. Inverse of a matrix A is denoted by A^{-1}, which does not mean $\frac{1}{A}$ i.e. $A^{-1} \neq \frac{1}{A}$.

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

(i) Inverse of an invertible matrix is unique.

(ii) $A^{-1}. A = A.A^{-1} = I$, where I is the identity matrix.

11. Application of Determinants and Matrices

We use the determinants and matrices for solving the system of linear equations in two and three variables and checking the consistency of the system of linear equations.

(i) Consistent System : A system of equation is said to be consistent if its solution (one or more) exists.

(ii) Inconsistent System : A system of equation is said to be inconsistent if its solution does not exist.

(iii) Solution of System of Linear Equations using Inverse of a Matrix

Consider the system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, the system of equation can be written as $AX = B$

i.e., $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Case-I: If A a non-singular matrix, then its inverse exists.

Now $\quad AX = B \quad$ or $\quad A^{-1}(AX) = A^{-1}B$

or $\quad (A^{-1}A)X = A^{-1}B \quad$ or $\quad IX = A^{-1}B \quad$ or $\quad X = A^{-}B$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as matrix method.

Case-II: If $|A| = 0$ (i.e. A is singular matrix), then we calculate $(\text{adj } A) \cdot B$

If $(\text{adj } A)B \neq 0$, (0 being zero matrix), then solution does not exist and the system of equation is called consistent.

If $(\text{adj } A)B = 0$, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Note

In this chapter, we have to study system of linear equations having unique solutions only.

EXERCISE 4.1

1. Evaluate the following determinant : $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Sol. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2 \times (-1) - (-5) \times (4) = -2 + 20 = 18$

Notes

(a) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11} \times a_{22}) - (a_{21} \times a_{12})$.

(b) *We can only find the determinant of a square matrix.*

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ **(ii)** $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Sol. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta \cos\theta - (\sin\theta)(-\sin\theta)$

$= \cos^2\theta + \sin^2\theta = 1.$ $(\because \cos^2 A + \sin^2 A = 1)$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x - 1)(x + 1)$

$= x^3 + x^2 - x^2 - x + 1 - (x^2 - 1)^2$ $(\because (a + b)(a - b) = a^2 - b^2)$

$= x^3 + 1 - x^2 + 1 = x^3 - x^2 + 2$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$**, then show that** $|2A| = 4|A|$.

Sol. $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$\text{L.H.S} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = (2 \times 4) - (8 \times 4) = -24$

$\text{R.H.S} = 4|A| = 4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4[(1 \times 2) - (4 \times 2)] = 4(-6) = -24$

$\Rightarrow |2A| = 4|A|$.

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$**, then show that $|3A| = 27\,|A|$**

Sol. $3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$ $\quad \therefore \quad |3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$

$$= [(3\times12)-(0\times6)] - 0[(0\times12)-(0\times6)] + 3[(0\times0)-(0\times3)]$$

(Here we have expanded the determinant along 1st Row).

$= 3\,[36] = 108.$

Now; $27\,|A| = 27\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$

$= 27\{1[(1\times4)-(0\times2)] - 0[(0\times4)-(0\times2)] + 1[(0\times2)-(0\times1)]\}$

$= 27\,(4) = 108 \quad \Rightarrow \quad |3A| = 27\,|A|.$

Notes

(a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ *can be obtained by expanding along any row or any column. If we expand it along 1st row, we get :*

$$(-1)^{1+1} a_{11}\,[(a_{22}\times a_{33})-(a_{32}\times a_{23})] + (-1)^{1+2} a_{12}\,[(a_{21}\times a_{33})-(a_{31}\times a_{23})] + (-1)^{1+3} a_{13}\,[(a_{21}\times a_{32})-(a_{31}\times a_{22})].$$

or $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

(b) *In the above question we observe that :* $|3A| = 27\,|A| = 3^3\,|A|$.

Here A is a square matrix of order 3.

We can say that : $|KA| = K^n\,|A|$, *where n is the order of A & K is any orbitrary constant.*

5. Evaluate the determinants :

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Sol. (i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix}$ (expanding along R_1)

$= 3(0-5) + 1(0+3) - 2(0-0) = -15 + 3 = -12.$

(ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$ (expanding along R_1)

$= 3(1+6) + 4(1+4) + 5(3-2) = 21 + 20 + 5 = 46.$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$ (expanding along R_1)

$= 0 - 1(0-6) + 2(-3-0) = 6 - 6 = 0.$

(iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$ (expanding along R_1)

$= 2(0-5) + 1(0+3) - 2(0-6) = -10 + 3 + 12 = 5$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$.

Sol. $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix} = 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$

(expanding along R_1)

$= 1(-9+12) - 1(-18+15) - 2(8-5) = 3 + 3 - 6 = 0.$

7. Find the value of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ **(ii)** $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Sol. (i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Expanding the determinants, we get:

$\Rightarrow 2 - 20 = 2x^2 - 24 \quad \Rightarrow 2x^2 - 24 = -18 \quad \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Expanding the determinants, we get:

$\Rightarrow (2\times 5) - (4\times 3) = (5\times x) - (2x\times 3) \Rightarrow 10 - 12 = 5x - 6x \Rightarrow x = 2$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$**, then x is equal to**

(a) 6 **(b)** $\underline{+}6$ **(c)** -6 **(d)** 0

Sol. (b) $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

Expanding the determinants, we get:

$\Rightarrow x^2 - 36 = 36 - 36 \Rightarrow x^2 = 36 \Rightarrow x = \underline{+}6$

EXERCISE 4.2

Using the property of determinants and without expanding in Questions 1 to 5, prove that

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Sol. L.H.S. $= \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} \Rightarrow$ L.H.S. $= \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix}$ ...(i)

(Expressing above determinant into sum of two determinants)

Since we know that the value of determinants having equal columns (or rows) is zero. For 1st determinant : $C_1 = C_3$. For 2nd determinant : $C_2 = C_3$.

$\Rightarrow$ L.H.S. $= 0 + 0 = 0 =$ R.H.S. **Hence proved.**

Note

Alternate Solution :

$$L.H.S. = \begin{bmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{bmatrix} = \begin{bmatrix} x & a & x \\ y & b & y \\ z & c & z \end{bmatrix} \quad (Applying\ C_3 \to C_3 - C_2).$$

$$= 0 \quad (\because C_1 = C_3).$$

2. $$\begin{vmatrix} \boldsymbol{a-b} & \boldsymbol{b-c} & \boldsymbol{c-a} \\ \boldsymbol{b-c} & \boldsymbol{c-a} & \boldsymbol{a-b} \\ \boldsymbol{c-a} & \boldsymbol{a-b} & \boldsymbol{b-c} \end{vmatrix} = \mathbf{0}$$

Sol. $$\text{L.H.S.} = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{bmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{bmatrix}$$

$$(\text{Applying } C_1 \to C_1 + C_2 + C_3)$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 \qquad [\because C_1 = 0]$$

= R.H.S. **(Hence proved).**

3. $$\begin{vmatrix} \mathbf{2} & \mathbf{7} & \mathbf{65} \\ \mathbf{3} & \mathbf{8} & \mathbf{75} \\ \mathbf{5} & \mathbf{9} & \mathbf{86} \end{vmatrix} = \mathbf{0}$$

Sol. $$\text{L.H.S.} = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = \begin{vmatrix} 2 & 7 & 0 \\ 3 & 8 & 0 \\ 5 & 9 & 0 \end{vmatrix} \quad (\text{Applying } C_3 \to C_3 - C_1 - 9C_2)$$

$= 0 \qquad (\because C_3 = 0)$

= R.H.S. **(Hence proved).**

4. $$\begin{vmatrix} \mathbf{1} & \boldsymbol{bc} & \boldsymbol{a(b+c)} \\ \mathbf{1} & \boldsymbol{ca} & \boldsymbol{b(c+a)} \\ \mathbf{1} & \boldsymbol{ab} & \boldsymbol{c(a+b)} \end{vmatrix} = \mathbf{0}$$

Sol. $$\text{L.H.S.} = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = \begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ba \\ 1 & ab & ca+cb \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bc & ab+bc+ac \\ 1 & ca & bc+ca+ab \\ 1 & ab & ca+cb+ab \end{vmatrix} \quad \text{(Applying } C_3 \to C_3 + C_2\text{)}$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} \quad \text{(Taking (ab + bc + ca) common in } C_3\text{)}$$

$= (ab+bc+ca) \times 0$ $\quad (\because C_1 = C_3)$

$= 0 = \text{R.H.S.}$ **(Hence proved).**

5. $$\begin{vmatrix} \mathbf{b+c} & \mathbf{q+r} & \mathbf{y+z} \\ \mathbf{c+a} & \mathbf{r+p} & \mathbf{z+x} \\ \mathbf{a+b} & \mathbf{P+q} & \mathbf{x+y} \end{vmatrix} = 2\begin{vmatrix} \mathbf{a} & \mathbf{p} & \mathbf{x} \\ \mathbf{b} & \mathbf{q} & \mathbf{y} \\ \mathbf{c} & \mathbf{r} & \mathbf{z} \end{vmatrix}$$

Sol. $$\text{L.H.S.} = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \end{vmatrix}$$

(Applying $R_3 \to R_3 + R_2 + R_1$)

$$= 2\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b+c & p+q+r & x+y+z \end{vmatrix} \quad \text{(Taking common 2 in } R_3\text{)}$$

$$= 2\begin{vmatrix} -a & -p & -x \\ -b & -q & -y \\ a+b+c & p+q+r & x+y+z \end{vmatrix} \quad \text{(Applying } R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3\text{)}$$

$$= 2\begin{vmatrix} -a & -p & -x \\ -b & -q & -y \\ c & r & z \end{vmatrix} \quad \text{(Applying } R_3 \to R_3 + R_1 + R_2\text{)}$$

$$= (-1)^2 \times 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad \text{(Taking (–1) common from } R_1 \text{ \& } R_2 \text{ both).}$$

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{R.H.S.}$$ **(Hence proved).**

By using properties of determinants, in Questions 6 to 14, show that

6. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Sol. L.H.S. $= \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = \begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ ab & ac & 0 \end{vmatrix}$ (Applying $R_1 \to cR_1$ & $R_3 \to aR_3$)

$= \begin{vmatrix} -ab & 0 & -bc \\ -a & 0 & -c \\ ab & ac & 0 \end{vmatrix}$ (Applying $R_1 \to R_1 - R_3$)

Expand along C_2 :

L.H.S. $= 0 + 0 + (-1)^{3+2} ac[((-ab)(-c)) - ((-a)(-bc))] = -ac[abc - abc] = 0 =$ R.H.S.

7. $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Sol. Taking a, b, c common from R_1, R_2 & R_3 respectively, we get:

L.H.S. $= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$

Again a, b, c are taking common from C_1, C_2 & C_3 respectively, we get:

L.H.S. $= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ (Applying $R_1 \to R_1 + R_2$)

$= a^2b^2c^2.[2(1+1)]$ (expanding along R_1)
$= 4a^2b^2c^2 =$ R.H.S.

8. **(a)** $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(b) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Sol. (a) L.H.S. $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

(Applying $R_2 \Rightarrow R_2 - R_1$ and $R_3 \Rightarrow R_3 - R_1$)

$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b+a)(b-a) \\ 0 & c-a & (c+a)(c-a) \end{vmatrix}$ $(\because x^2 - y^2 = (x-y)(x+y))$

Taking $(b - a)$ & $(c - a)$ common from R_2 & R_3 respectively, we get:

L.H.S. $= (b-a)\ (c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$

Expanding along C_1, we get :

L.H.S. $= (b-a)\ (c-a)\ [1(c+a) - 1(b+a)] = (b-a)\ (c-a)\ (c+a-b-a)$

$= (b-a)(c-a)(c-b) = (-1)(a-b)(c-a)(-1)(b-c)$ $(\because x - y = -(y-x))$.

$= (a-b)\ (b-c)\ (c-a) =$ R.H.S.

(b) Let L.H.S. $= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$

(Changing rows into columns and columns in rows)

Appylying $R_2 \Rightarrow R_2 - R_1$ and $R_3 \Rightarrow R_3 - R_1$, we get

L.H.S. $= \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & (b-a)(b^2+a^2+ab) \\ 0 & c-a & (c-a)(c^2+a^2+ac) \end{vmatrix}$

$(\because x^3 - y^3 = (x-y)(x^2+y^2+xy))$

Taking $(b - a)$ & $(c - a)$ common from R_2 & R_3 respectively, we get :

L.H.S. $= (b-a)(c-a) \begin{vmatrix} 1 & 9 & a^3 \\ 0 & 1 & b^2+a^2+ab \\ 0 & 1 & c^2+a^2+ac \end{vmatrix}$

$= (b-a)\ (c-a)\ [1[(c^2 + a^2 + ac) - (b^2 + a^2 + ab)]]$ (Expanding along C_1).

$= (b-a)\ (c-a)\ [c^2 + a^2 + ac - b^2 - a^2 - ab]$

$= (b-a)(c-a)[(c^2-b^2)+(ac-ab)]$
$= (b-a)(c-a)[(c+b)(c-b)+a(c-b)]$
$= (b-a)(c-a)(c-b)(a+b+c)$
$= (-1)(a-b)(c-a)(-1)(b-c)(a+b+c)$
$= (a-b)(b-c)(c-a)(a+b+c) =$ R.H.S.

9. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

Sol. Let $\Delta = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} x-y & x^2-y^2 & yz-zx \\ y-z & y^2-z^2 & zx-xy \\ z & z^2 & xy \end{vmatrix}$

(Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$)

$= \begin{vmatrix} x-y & (x+y)(x-y) & -z(x-y) \\ y-z & (y-z)(y+z) & -x(y-z) \\ z & z^2 & xy \end{vmatrix}$ $(\because a^2-b^2=(a-b)(a+b))$

$= (x-y)(y-z)\begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$

(Taking common $(x-y)$ and $(y-z)$ from R_1 and R_2)

$= (x-y)(y-z)\begin{vmatrix} 0 & x-z & -z+x \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$ (Applying $R_1 \to R_1 - R_2$)

$= (x-y)(y-z)(z-x)\begin{vmatrix} 0 & -1 & -1 \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$ (Taking $(z-x)$ common from R_1)

Expanding along R_1, we get:
$= (x-y)(y-z)(z-x)[0\cdot\{(y+z)xy+xz^2\}]+1\cdot(xy+xz)-1(z^2-yz-z^2)]$
$= (x-y)(y-z)(z-x)(xy+zx+yz) = (x-y(y-z)(z-x)(xy+yz+zx)$
$=$ R.H.S.

10. **(a)** $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(b) $\begin{vmatrix} y+x & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

Sol. (a) L.H.S.= $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix}$

(Applying $C_1 \to C_1 + C_2 + C_3$)

$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$ (Taking $(5x+4)$ common from C_1)

$= (5x+4)\begin{vmatrix} 0 & x-4 & 0 \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$ (Applying $R_1 \to R_1 - R_2$)

$= (5x+4)\left[-(x-4)\begin{vmatrix} 1 & 2x \\ 1 & x+4 \end{vmatrix}\right]$ (Expanding along R_1)

$= (5x+4)\,[-(x-4)\,(x+4-2x)] = (5x+4)\,[-(x-4)\,(-x+4)]$
$= (5x+4)\,(4-x)\,(4-x) = (5x+4)\,(4-x)^2 = \text{R.H.S.}$

Note

In case of determinants, we can use both elementary row operations and elementary column operations to arrive at the result, as you have seen in the above question.

(b) L.H.S. = $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$

(Applying $C_1 \to C_1 + C_2 + C_3$)

$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$ (Taking $(3y+k)$ common from C_1)

$= (3y+k)\begin{vmatrix} 0 & -k & 0 \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$ (Applying $R_1 \to R_1 - R_2$)

$= (3y+k)\left[k\begin{vmatrix} 1 & y \\ 1 & y+k \end{vmatrix}\right]$ (Expanding along R_1).

$= (3y+k)\,.\,k\,[y+k-y] = (3y+k)\,k^2 = \text{R.H.S.}$

11. (a) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(b) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2\,(x+y+z)^3$

Sol. (a) L.H.S.$= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ (Applying $R_1 \to R_1 + R_2 + R_3$)

$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ Taking (a+b+c) common from R_1)

$= (a+b+c)\begin{vmatrix} 0 & 1 & 1 \\ b+c+a & b-c-a & 2b \\ 0 & 2c & c-a-b \end{vmatrix}$ (Applying $C_1 \to C_1 - C_2$)

$= (a+b+c)\left[-(b+c+a)\begin{vmatrix} 1 & 1 \\ 2c & c-a-b \end{vmatrix}\right]$ (Expanding along C_1)

$-(a+b+c)^2[-(c-a-b-2c)] - (a+b+c)^2[(a+b+c)]$

$= (a+b+c)^3 =$ R.H.S.

(b) Let $\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

$\Delta = \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$ (Applying $C_1 \to C_1 + C_2 + C_3$)

$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$ (Taking $2(x+y+z)$ common from C_1)

$$\Delta = 2(x+y+z)\begin{vmatrix} 0 & -(x+y+z) & 0 \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix} \quad (\text{Applying } R_1 \to R_1 - R_2)$$

$$= 2(x+y+z)\left[+(x+y+z)\begin{vmatrix} 1 & y \\ 1 & 2+x+2y \end{vmatrix}\right] \quad (\text{Expanding along } R_1)$$

$$= 2(x+y+z)(x+y+z)(z+x+2y-y) = 2(x+y+z)^2(x+y+z) = 2(x+y+z)^3$$

12. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

Sol. L.H.S. $= \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix}$ (Applying $C_1 \to C_1 + C_2 + C_3$)

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \quad (\text{Taking } 1+x+x^2 \text{ common from } C_1)$$

$$= (1+x+x^2)\begin{vmatrix} 1-x & x & x^2 \\ 0 & 1 & x \\ 1-x^2 & x^2 & 1 \end{vmatrix} \quad (\text{Applying } C_1 \to C_1 - C_2)$$

Taking $(1-x)$ common from C_1, we get:

$$\text{L.H.S.} = (1+x+x^2)(1-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 1+x & x^2 & 1 \end{vmatrix} \quad (\because a^2 - b^2 = (a-b)(a+b))$$

$$= (1+x+x^2)(1-x)\left[1\begin{vmatrix} 1 & x \\ x^2 & 1 \end{vmatrix} + (1+x)\begin{vmatrix} x & x^2 \\ 1 & x \end{vmatrix}\right] \quad (\text{Expanding along } C_1)$$

$= (1-x^3)[(1-x^3)+(1+x)(x^2-x^2)]$ $\quad (\because a^3 - b^3 = (a-b)(a^2+b^2+ab))$

$= (1-x^3)[(1-x^3)+0] = (1-x^3)^2 = $ R.H.S.

13. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

Sol. $$\text{L.H.S.} = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - bC_3$ and $C_2 \to C_2 + aC_3$, we get:

$$\text{L.H.S.} = \begin{vmatrix} 1+a^2-b^2+2b^2 & 2ab-2ab & -2b \\ 2ab-2ab & 1-a^2+b^2+2a^2 & 2a \\ 2b-b+a^2b+b^3 & -2a+a-a^3-b^3 & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

(Taking $(1+a^2+b^2)$ common from both C_1 & C_2)

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix} \quad (\text{Applying } R_3 \to R_3 - bR_1)$$

$= (1+a^2+b^2)^2 [1(1-a^2+b^2+2a^2)]$ (Now, expanding along C_1)

$= (1+a^2+b^2)^2 (1)(1+a^2+b^2) = (1+a^2+b^2)^3 = \text{R.H.S}$

14. $$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Sol. Let $$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = \begin{vmatrix} a^2+1 & ab+0 & ac+0 \\ ab+0 & b^2+1 & bc+0 \\ ca+0 & cb+0 & c^2+1 \end{vmatrix}$$

This may be expressed as the sum of 8 determinants

$$\therefore \Delta = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ca & bc & c^2 \end{vmatrix} + \begin{vmatrix} 1 & ab & ac \\ 0 & b^2 & bc \\ 0 & bc & c^2 \end{vmatrix} + \begin{vmatrix} a^2 & 0 & ac \\ ab & 1 & bc \\ ca & 0 & c^2 \end{vmatrix} + \begin{vmatrix} a^2 & ab & 0 \\ ab & b^2 & 0 \\ ca & bc & 1 \end{vmatrix}$$

$$+\begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ca & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & ab & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & ac \\ 0 & 1 & bc \\ 0 & 0 & c^2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\Rightarrow \Delta = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6 + \Delta_7 + \Delta_8$ (respectively)

$$\Delta_1 = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ca & bc & c^2 \end{vmatrix} = abc \begin{vmatrix} a & a & a \\ b & b & b \\ c & c & c \end{vmatrix}$$

(Taking a, b, c common from C_1, C_2, C_3 respectively).

$$= a^2b^2c^2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$ (Taking a, b, c common from R_1, R_2, R_3 respectively).

$= a^2b^2c^2 (0)$ $\quad (\because R_1 = R_2 = R_3)$.

$= 0$.

$$\Delta_2 = \begin{vmatrix} 1 & ab & ac \\ 0 & b^2 & bc \\ 0 & bc & c^2 \end{vmatrix} = 1\left(b^2c^2 - b^2c^2\right)$$ (Expanding along C_1)

$= 0$

$$\Delta_3 = \begin{vmatrix} a^2 & 0 & ac \\ ab & 1 & bc \\ ca & 0 & c^2 \end{vmatrix} = 1\left(a^2c^2 - a^2c^2\right)$$ (Expanding along C_2)

$= 0$

$$\Delta_4 = \begin{vmatrix} a^2 & ab & 0 \\ ab & b^2 & 0 \\ ca & bc & 1 \end{vmatrix} = 1\left(a^2b^2 - a^2b^2\right)$$ (Expanding along C_3)

$= 0$

$$\Delta_5 = \begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ca & 0 & 1 \end{vmatrix} = 1\left(a^2 - 0\right)$$ (Expanding along C_3)

$= a^2$

$$\Delta_6 = \begin{vmatrix} 1 & ab & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix} = 1\left(b^2 - 0\right)$$ (Expanding along C_1)

$= b^2$

$$\Delta_7 = \begin{vmatrix} 1 & 0 & ac \\ 0 & 1 & bc \\ 0 & 0 & c^2 \end{vmatrix} = 1\left(c^2 - 0\right) \qquad \text{(Expanding along } C_1\text{)}$$

$$= c^2$$

$$\Delta_8 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1-0) \qquad \text{(Expanding along } C_1\text{)}$$

$$= 1.$$

$\therefore\ \Delta = 0+0+0+0+a^2+b^2+c^2+1 = a^2+b^2+c^2+1 = \text{R.H.S.}$

15. If A be a square matrix of order 3 × 3, then $|kA|$ is equal to

(a) $k\,|A|$ **(b)** $k^2\,|A|$

(c) $k^3\,|A|$ **(d)** $3k\,|A|$

Sol. (c) Consider a matrix $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

$$\Rightarrow kA = \begin{bmatrix} ka & kd & kg \\ kb & ke & kh \\ kc & kf & ki \end{bmatrix} \Rightarrow |kA| = \begin{vmatrix} ka & kd & kg \\ kb & ke & kh \\ kc & kf & ki \end{vmatrix}$$

Taking k common from R_1, R_2 & R_3, we get :

$$|kA| = k.k.k \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = k^3 |A|.$$

16. Which of the following is correct :

(a) **Determinant is a square matrix**

(b) **Determinant is a number associated to a matrix**

(c) **Determinant is a number associated to a square matrix**

(d) **None of these**

Sol. (c) Since we know that, determinant of any matrix is defined if and only if A is a square matrix.

$\therefore$ determinant is a number associated to a square matrix.

EXERCISE 4.3

1. Find the area of the triangle with vertices at the point given in each of the following :

(i) (1,0), (6, 0) (4, 3) **(ii) (2, 7), (1, 1), (10, 8)**

(iii) (–2, – 3), (3, 2), (–1, – 8)

Hint

Area of a triangle with vertices (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *is given as:*

$$Area = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Sol. (i) Area of triangle with vertices (1, 0), (6, 0) and (4, 3) $= \frac{1}{2}\left\|\begin{matrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{matrix}\right\|$

$$= \frac{1}{2}\left|\left[1\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} - 0 + 1\begin{vmatrix} 6 & 0 \\ 4 & 3 \end{vmatrix}\right]\right| \qquad \text{(Expanding along } R_1)$$

$$= \frac{1}{2}\ |[1(0-3)+1(18-0]| = \frac{1}{2}|(-3+18)| = \frac{15}{2} = 7.5 \text{ sq. units.}$$

(ii) Area of triangle with vertices (2, 7), (1, 1) & (10, 8) $= \frac{1}{2}\left\|\begin{matrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{matrix}\right\|$

$$= \frac{1}{2}\left|\left[2\begin{vmatrix} 1 & 1 \\ 8 & 1 \end{vmatrix} - 7\begin{vmatrix} 1 & 1 \\ 10 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 10 & 8 \end{vmatrix}\right]\right| \qquad \text{(Expanding along } R_1)$$

$$= \frac{1}{2}\ |[2\ (1-8) - 7\ (1-10) + 1(8-10)]| = \frac{1}{2}\ |[-14+63-2]| = \frac{47}{2}$$

$= 23.5$ sq. units

(iii) Area of triangle with vertices (–2, –3), (3, 2) and (–1, –8) $= \frac{1}{2}\left\|\begin{matrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{matrix}\right\|$

$$\text{Area } = \frac{1}{2}\left|\left[-2\begin{vmatrix} 2 & 1 \\ -8 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ -1 & -8 \end{vmatrix}\right]\right| \qquad \text{(Expanding along } R_1)$$

$$=\frac{1}{2}\left|\left[-2(2+8)+3(3+1)+1(-24+2)\right]\right|=\frac{1}{2}\left|\left[-20+12-22\right]\right| = \frac{|-30|}{2}$$

$\Rightarrow$ area = 15 square units. $\quad (\because |-a| = a)$

Note

Since area is positive, we always use the absolute value of the determinant (Which is used for the area).

2. Show that the points A (a, b + c) , B (b, c + a) C (c, a + b) are collinear.

Hint

Use area of triangle.

Sol. Given points are (a, b + c), (b , c + a) and (c, a + b).
We know that if the area of the triangle formed by above three points is zero, then the points will lie on a line i.e., they will be collinear.
$\therefore$ area of a triangle with vertices (a, b + c), (b, c + a) and (c, a + b)

$$=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix} \quad (\text{Appling } C_1 \to C_1 + C_2)$$

$$=\frac{a+b+c}{2}\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} \quad (\text{Taking } (a+b+c) \text{ common from } C_1).$$

$$=\frac{a+b+c}{2}(0) \quad (\because C_1 = C_3)$$

= 0. $\quad \therefore$ the given points are collinear.

Note

Area of a triangle formed by three collinear points is zero.

3. Find the value of k if area of triangle is 4 square units and vertices are
(i) (*k*, 0), (4, 0), (0, 2) (ii) (–2, 0), (0, 4), (0, k).

Sol. (i) Area of triangle with vertices (k, 0), (4, 0) and (0, 2) = $\frac{1}{2}\left\|\begin{matrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{matrix}\right\|$

$$\Rightarrow \frac{1}{2}\left\|\begin{matrix} K & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{matrix}\right\| = 4 \quad (\text{given}).$$

Expanding along C_1, we get:

$\frac{1}{2}|k(0-2)-4(0-2)|=4 \quad \Rightarrow |-2k+8|=8$

$\Rightarrow \pm(-2k+8)=8 \qquad (\because\ |-a|=a\ \&\ |+a|=a)$

Case (a) : Taking positive sign : $-2k+8=8 \quad \Rightarrow k=0$

Case (b) : Taking negative sign: $+2k-8=+8 \Rightarrow 2k=16 \Rightarrow k=8.$

Hence, $k=0, 8$

(ii) The area of the triangle whose vertices are (2, 0), (0, 4), (0, k)

$$=\frac{1}{2}\begin{vmatrix}-2 & 0 & 1\\ 0 & 4 & 1\\ 0 & k & 1\end{vmatrix}=4 \quad \text{(given)}$$

Expanding along C_1, we get : $\frac{1}{2}|-2(4-k)|=4 \quad \Rightarrow |-8+k|=8.$

$\Rightarrow \quad \pm(-8+k)=8 \qquad (\because\ |-a|=a\ \&\ |+a|=a)$

Case (a) : Taking positive sign: $-8+k=8 \quad \Rightarrow k=16.$

Case (b) : Taking negative sign: $8-k=8 \quad \Rightarrow k=8-8=0$

Hence, $k=0, 8.$

Note

If area of a triangle is given, we use both positive and negative values of the determinant for calculations.

4. (i) Find the equation of line joining (1, 2) and (3, 6) using determinants.
(ii) Find the equation of line joining (3, 1), (9, 3) using determinants.

Hint

Equation of a line joining (x_1, y_1) and (x_2, y_2) is given as: $\begin{vmatrix}x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}=0.$

Sol. (i) Equation of the line joining the points (1, 2), (3, 6) is given as:

$$\begin{vmatrix}x & y & 1\\ 1 & 2 & 1\\ 3 & 6 & 1\end{vmatrix}=0$$

Expanding along R_1, we get:

$\Rightarrow \quad x(2-6)-y(1-3)+1(6-6)=0 \quad \Rightarrow \quad -4x+2y=0$

$\Rightarrow 2x-y=0$, which is required equation of the line.

(ii) Equation of the line joining the points (3, 1) and (9, 3) is given as:

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$$

Expanding along R_1, we get:

$\Rightarrow x(1-3) - y(3-9) + 1(9-9) = 0 \Rightarrow -2x + 6y = 0$

$\Rightarrow 6y = 2x \Rightarrow x - 3y = 0$, which is the required line.

5. If area of triangle is 35 sq. units with vertices (2, – 6), (5, 4) and (k, 4). Then k is

(a) 12 (b) – 2 (c) – 12, – 2 (d) 12, – 2

Sol. (d) Area of triangle with vertices (2, –6), (5, 4) and (k, 4)

$$= \frac{1}{2}\left\|\begin{matrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{matrix}\right\| = 35 \quad \text{(given)}$$

Expanding along R_1, we get: $\frac{1}{2}\,|[2(4-4) + 6(5-k) + 1(20-4k)]| = 35$

$\Rightarrow \frac{1}{2}\,|[30 - 6k + 20 - 4k]| = 35 \quad \Rightarrow \frac{1}{2}[50 - 10k] = 35 \Rightarrow \pm(25 - 5k) = 35$

Taking positive sign: $25 - 5k = 35 \Rightarrow -5k = 10 \Rightarrow k = -2$

Taking negative sign: $25 - 5k = -35 \Rightarrow 5k = 60 \Rightarrow k = 12$

Hence, $k = 12, -2$.

EXERCISE 4.4

1. Write the minors and cofactors of the elements of following determinants :

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ **(ii)** $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Sol. (i) For the determinant = $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

Minors: $M_{11} = 3$; $M_{12} = 0$; $M_{21} = -4$; $M_{22} = 2$

Cofactors: As $C_{ij} = (-1)^{i+j} M_{ij}$

$\therefore C_{11} = (-1)^2 3 = 3$; $C_{12} = (-1)^3 0 = 0$; $C_{12} = (-1)^3 4 = 4$; $C_{22} = (-1)^4 2 = 2.$

(ii) For the determinant = $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Minors : $M_{11} = d$; $M_{12} = b$; $M_{21} = c$; $M_{22} = a$.

Cofactors As $C_{ij} = (-1)^{i+j} M_{ij}$

$C_{11} = (-1)^2 d = d$; $C_{12} = (-1)^3 b = -b$; $C_{21} = (-1)^3 c = -c$; $C_{22} = (-1)^4 a = a.$

2. Write Minors and Cofactor of elements of the determinant

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Sol. (i) For the determinant $\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Minors: $M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1;$

$M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0;\ M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0;\ M_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0;$

$M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1;\ M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0;\ M_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0;$

$M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0;\ M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1.$

Cofactors : $C_{11} = (-1)^{1+1} M_{11} = (-1)^2\, 1 = 1;$
$C_{12} = (-1)^{1+2} M_{12} = (-1)^3\, 0 = 0;\ C_{13} = (-1)^{1+3} M_{13} = (-1)^4\, 0 = 0;$
$C_{21} = (-1)^{2+1} M_{21} = (-1)^3\, 1 = 0;\ C_{22} = (-1)^{2+2} M_{22} = (-1)^4\, 1 = 1;$
$C_{23} = (-1)^{2+3} M_{23} = (-1)^5\, 0 = 0;\ C_{31} = (-1)^{3+1} M_{31} = (-1)^4\, 0 = 0;$
$C_{32} = (-1)^{3+2} M_{32} = (-1)^5\, 0 = 0;\ C_{33} = (-1)^{3+3} M_{33} = (-1)^6\, 1 = 1.$

(ii) For the determinant $= \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Minors: $M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11;$

$M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6;\ M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3;$

$M_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4;\ M_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2;$

$M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1;\ M_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20;$

$M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13;\ M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5.$

Cofactors $C_{ij} = (-1)^{i+j} M_{ij};\ C_{11} = (-1)^2\, 11 = 11;\ C_{12}\ (-1)^3\, 6 = -6;$
$C_{13} = (-1)^4\, 3 = 3;\ C_{21} = (-1)^3 (-4) = 4;\ C_{22} = (-1)^4\, 2 = 2;\ C_{23} = (-1)^5\, 1 = -1;$
$C_{31} = (-1)^4 (-20) = -20;\ C_{32} = (-1)^5 (-13) = 13;\ C_{33} = (-1)^6\, 5 = 5.$

3. Using cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Sol. The elements of second row are : $a_{21} = 2, a_{22} = 0$ & $a_{23} = 1$.

Cofactor of $a_{21} = C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = -(9-16) = 7$

Cofactor of $a_{22} = C_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$

Cofactor of $a_{23} = C_{23} = (-1)^{2+3} \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = -(10-3) = -7$

Now, the value of determinant is given as:
$\Delta = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} = (2 \times 7) + (0 \times 7) + (1 \times (-7)) = 14 + 0 - 7 = 7.$

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

Sol. Elements of third column are $a_{13} = yz, a_{23} = zx$ and $a_{33} = xy$

$\therefore$ Cofactor of $a_{13} = C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$

Cofactor of $a_{23} = C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = -(z-x), = -z + x$

Cofactor of $a_{33} = C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = (y - x)$

Now, the value of determinant is given as:

$$\begin{aligned}
\Delta &= a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33} = yz(z-y) + zx(-z+x) + xy(y-x) \\
&= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y = (-y^2z + yz^2) + (xy^2 - xz^2) + (-x^2y + x^2z) \\
&= +yz(-y+z) + x(y^2 - z^2) + x^2(z-y) = -yz(y-z) + x(y+z)(y-z) - x^2(y-z) \\
&= -(y-z)[+yz - x(y+z) + x^2] = -(y-z)(yz - xz - xy + x^2) \\
&= -(y-z)[z(y-x) - x(y-x)] = (z-y)(y-x)(z-x) \\
&= (-1)(y-z)(-1)(x-y)(z-x) = (x-y)(y-z)(z-x)
\end{aligned}$$

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is the cofactors of a_{ij}, then value of Δ is given by

(a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ **(b)** $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
(c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ **(d)** $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Sol. (d) The value of the given determinant is given as:
$\Delta = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$ (Using first Row)

$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$ (Using second Row)
$\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$ (Using third Row)
$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ (Using first column)
$\Delta = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$ (Using second column)
$\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$ (Using third column)

Note

The value of a determinant is given as the sum of the products of the elements of a row (or a column) with their corresponding cofactors.

EXERCISE 4.5

Find the adjoint of each of the matrices in Exercises 1 and 2.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Hint

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow Adj\ (A)^T = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Here C_{ij} is the cofactor of a_{ij}.

Sol. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by: $C_{11} = (-1)^{1+1}(4) = 4$; $C_{12} = (-1)^{1+2}(3) = -3$;
$C_{21} = (-1)^{2+1}(2) = -2$; $C_{22} = (-1)^{2+2}(1) = 1$.

$$\therefore \quad \text{Adj A} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

2. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$,

Cofactors:

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3;\ C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2+10) = -12;$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0+6 = 6;\ C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1;$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = (1+4) = 5;\ C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2;$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5-6 = -11;\ C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1;$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5.$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

Verify A (adj A) = (adj A) · A = |A| I in Questions 3 and 4.

3. $\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{6} \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 2 & 3 \\ -4 & 6 \end{bmatrix} \Rightarrow |A| = (2\times 6) - (-4\times 3) = 12+12 = 24$

Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactor of elements of A are given by

$C_{11} = (-1)^{1+1}(6) = 6;\quad C_{12} = (-1)^{1+2}(-4) = 4;$

$C_{21} = (-1)^{2+1}(3) = -3;\quad C_{22} = (-1)^{2+2}(2) = 2;$

$$\therefore\quad \text{Adj } A = \begin{bmatrix} 6 & 4 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$A.\text{ Adj } A = \begin{bmatrix} 2 & 3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 6 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 12+12 & -6+6 \\ -24+24 & 12+2 \end{bmatrix} = \begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} = 24\,I$$

$$\text{Adj } A.\ A = \begin{bmatrix} 6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 12+12 & 18-18 \\ 8-8 & 12+12 \end{bmatrix} = \begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} = 24\,I$$

$$|A|\,I = 24\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 24\,I$$

Hence $A(\text{Adj } A) = (\text{Adj } A).A = 24\,I = |A|\,I$

4. $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{3} & \mathbf{0} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Cofactors:

$$A_{11}=(-1)^2\begin{vmatrix}0 & -2\\0 & 3\end{vmatrix}=0-0=0;\ A_{12}=(-1)^3\begin{vmatrix}3 & -2\\1 & 3\end{vmatrix}=-(9+2)=-11;$$

$$A_{13}=(-1)^4\begin{vmatrix}3 & 0\\1 & 0\end{vmatrix}=0-0=0;\quad A_{21}=(-1)^3\begin{vmatrix}-1 & 2\\0 & 3\end{vmatrix}=-(-3-0)=3;$$

$$A_{22}=(-1)^4\begin{vmatrix}1 & 2\\1 & 3\end{vmatrix}=(3-2)=1;A_{23}=(-1)^5\begin{vmatrix}1 & -1\\1 & 0\end{vmatrix}=-(0+1)=-1;$$

$$A_{31}=(-1)^4\begin{vmatrix}-1 & 2\\0 & -2\end{vmatrix}=(2-0)=2;A_{32}=(-1)^5\begin{vmatrix}1 & 2\\3 & -2\end{vmatrix}=-(-2-6)=8$$

$$\text{and } A_{33}=(-1)^6\begin{vmatrix}1 & -1\\3 & 0\end{vmatrix}=0+3=3.$$

$$\text{Now, } Adj(A)=\begin{bmatrix}0 & -11 & 0\\3 & 1 & -1\\2 & 8 & 3\end{bmatrix}^T=\begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}.$$

$$\text{Now, } A(Adj;A)=\begin{bmatrix}1 & -1 & 2\\3 & 0 & -2\\1 & 0 & 3\end{bmatrix}\begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}=\begin{bmatrix}0+11+0 & 3-1-2 & 2-8+6\\0+0+0 & 9+0+2 & 6+0-6\\0+0+0 & 3+0-3 & 2+0+9\end{bmatrix}$$

$$=\begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}=11\,I$$

$$(Adj\,A).A=\begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & -1 & 2\\3 & 0 & -2\\1 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}0+9+2 & 0+0+0 & 0-6+6\\-11+3+8 & 11+0+0 & -22-2+24\\0-3+3 & 0+0+0 & 0+2+9\end{bmatrix}=\begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}=11I.$$

Now, $|A|=1(0-0)+1(9+2)+2(0-0)=9+2=11.$

$\therefore\quad |A|\,I=11\,I\quad\Rightarrow\quad A\,(adj\,A)=(adj\,A)\,A=|A|\,I$

Find the inverse of each of the matrices (if it exists) given in the Questions 5 to 11 :

5. $\begin{bmatrix}2 & -2\\4 & 3\end{bmatrix}$

Hint

Inverse of any square matrix (A) exists if its determinant is not equal to zero.

i.e. $|A|\neq 0$ *and* $A^{-1}=\dfrac{Adj(A)}{|A|}$

Sol. Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6-(-8) = 14 \neq 0.$

So, A is a non-singular matrix and therefore it is invertible. Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by:
$C_{11} = (-1)^{1+1}(3) = 3; C_{12} = (-1)^{1+2}(4) = -4; C_{21} = (-1)^{2+1}(-2) = 2;$
$C_{22} = (-1)^{2+2}(2) = 2.$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \text{Adj. } A = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

Note

A square matric is called a singular matrix if its determinant is equal to zero, otherwise it is called non-singular matrix.

6. $\begin{bmatrix} \mathbf{-1} & \mathbf{5} \\ \mathbf{-3} & \mathbf{2} \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = -2+15 = 13 \neq 0.$

So, A is a non-singular matrix and therefore it is invertible. Let c_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by the cofactors of elements of A are given by:
$C_{11} = (-1)^{1+1}(2) = 2; C_{12} = (-1)^{1+2}(-3) = 3$
$C_{21} = (-1)^{2+1}(5) = -5; C_{22} = (-1)^{2+2}(-1) = -1.$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \text{Adj. } A = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

7. $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow |A| = 1(10-0)$ (Expanding along C_1) $= 10$

Cofactors: $C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = (10-0) = 10;$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = -(0-0) = 0; C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = (0-0) = 0;$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -(10-0) = -10;\ C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = (5-0) = 5;$

$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0;\ C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = (8-6) = 2;$

$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -(4-0) = -4$ and $C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = (2-0) = 2.$

$$\therefore \quad \text{Adj}\,A = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}^T = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, $A^{-1} = \dfrac{1}{|A|}\text{Adj. }A = \dfrac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix}$ (Expanding along R_1).

$= -3 - 0 = -3 \neq 0$

So, A is non singular therefore it is invertible.

Cofactors:

$C_{11} = (-1)^2\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = (-3-0) = -3;\ C_{12} = (-1)^3\begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = -(-3-0) = 3;$

$C_{13} = (-1)^4\begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = (6-15) = -9;\ C_{21} = (-1)^3\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = -(0-0) = 0;$

$C_{22} = (-1)^4\begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = (-1-0) = -1;\ C_{23} = (-1)^5\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -(2-0) = -2;$

$C_{31} = (-1)^4\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = (0-0) = 0;\ C_{32} = (-1)^5\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = -(0-0) = 0$ and

$C_{33} = (-1)^6\begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = (3-0) = 3.$

$$\therefore \quad \text{Adj}\,A = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}^T = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{-1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}.$$

9. $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{4} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-7} & \mathbf{2} & \mathbf{1} \end{bmatrix}$

Sol. Let $A=\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix} \Rightarrow |A|=\begin{vmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{vmatrix}$

$= 2(-1-0)-1(4-0)+3(8-7)$ (Expanding along R_1)

$=-2-4+3=-3$

So, A is non-singular matrix and therefore, it is invertible.

Cofactors: $C_{11}=(-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}=(-1-0)=-1$;

$C_{12}=(-1)^{1+2}\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix}=-(4-0)=-4$; $C_{13}=(-1)^{1+3}\begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix}=8-7=1$;

$C_{21}=(-1)^{2+1}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}=-(1-6)=5$; $C_{22}=(-1)^{2+2}\begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix}=2+21=23$;

$C_{23}=(-1)^{2+3}\begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix}=-(4+7)=-11$; $C_{31}=(-1)^{3+1}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}=(0+3)=3$;

$C_{32}=(-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix}=-(0-12)=12$; $C_{33}=(-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix}=-2-4=-6$

$$\therefore \quad \text{Adj } A=\begin{bmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{bmatrix}^T=\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\text{Hence } A^{-1}=\frac{1}{|A|}\text{Adj } A=-\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

10. $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{-3} \\ \mathbf{3} & \mathbf{-2} & \mathbf{4} \end{bmatrix}$

Sol. Let $A=\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \Rightarrow |A|=\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix}$

$=1(8-6)+1(0+9)+2(0-6)$ (Expanding along R_1).

$=2+9-12=-1\neq 0$

$\therefore$ A is invertible and $A^{-1} = \dfrac{\text{Adj } A}{|A|}$

Cofactors: $C_{11} = (-1)^2 \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 8-6=2;$

$C_{12} = (-1)^3 \begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0+9) = -9\,;\, C_{13} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = 0-6=-6;$

$C_{21} = (-1)^3 \begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = -(-4+4) = 0\,;\, C_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6=-2;$

$C_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -(-2+3) = -1\,;\, C_{31} = (-1)^4 \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = 3-4=-1;$

$C_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3-0) = 3$ and $C_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2+0=2.$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{Adj } A}{|A|} = -\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

11. $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\cos\alpha} & \boldsymbol{\sin\alpha} \\ \mathbf{0} & \boldsymbol{\sin\alpha} & \boldsymbol{-\cos\alpha} \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

$\Rightarrow \quad |A| = 1\,(-\cos\alpha\cos\alpha - \sin\alpha\sin\alpha)$ (Expanding along R_1)

$= -(\cos^2\alpha + \sin^2\alpha) = -1$ $(\because \cos^2\theta + \sin^2\theta = 1)$

Cofactors:

$C_{11} = (-1)^2 \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = (-\cos^2\alpha - \sin^2\alpha) = -(\cos^2\alpha + \sin^2\alpha) = -1$

$(\because \cos^2 A + \sin^2 A = 1)$

$C_{12} = (-1)^3 \begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = -(0-0) = 0\,;\, C_{13} = (-1)^4 \begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = (0-0) = 0;$

$C_{21} = (-1)^3 \begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -(0-0) = 0\,;$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} = -\cos\alpha - 0 = -\cos\alpha;$$

$$C_{23} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -(\sin\alpha - 0) = -\sin\alpha\ ;$$

$$C_{31} = (-1)^4 \begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -0 - 0 = 0;$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -(\sin\alpha - 0) = -\sin\alpha$$

$$\text{and } C_{33} = (-1)^6 \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = \cos\alpha - 0 = \cos\alpha.$$

$$\text{adj } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$\therefore \quad A \qquad - \qquad 1$

$$= \frac{1}{|A|} \text{adj } A = (-1) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$, **verify that** $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. Here $|A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = (3 \times 5) - (2 \times 7) = 15 - 14 = 1 \neq 0.$

Cofactors of elements of A are

$A_{11} = (-1)^2 5 = 5;\ A_{12} = (-1)^3 2 = -2;\ A_{21} = (-1)^3 7 = -7;\ A_{22} = (-1)^4 3 = 3.$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \Rightarrow A^{-1} = \text{Adj. } |A|\, A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$\text{Also, } |B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = (6 \times 9) - (7 \times 8) = 54 - 56 = -2 \neq 0.$$

Cofactors of elements of B are

$B_{11} = (-1)^2\, 9 = 9;\ B_{12} = (-1)^3\, 7 = -7;\ B_{21} = (-1)^3\, 8 = -8;\ B_{22} = (-1)^4\, 6 = 6$

$$\therefore \quad \text{Adj } B = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}^T = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} \Rightarrow B^{-1} \frac{\text{Adj.}B}{|B|} = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$\text{Now; } AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = (67 \times 61) - (87 \times 47) = 4087 - 4089 = -2$$

Cofactors of elements of AB are: $C_{11} = (-1)^2(61) = 61$;
$C_{12} = (-1)^3(47) = -47$; $C_{21} = (-1)^3(87) = -87$ and $C_{22} = (-1)^4(67) = 67$.

$$\therefore \text{Adj}(AB) = \begin{bmatrix} 61 & -47 \\ -87 & 67 \end{bmatrix}^T = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$\therefore \ (AB)^{-1} = \frac{\text{Adj}(AB)}{|AB|} = -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$\text{Now, } B^{-1}A^{-1} = \frac{-1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} = (AB)^{-1}. \text{ Hence, } (AB)^{-1} = B^{-1}A^{-1}.$$

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$. Hence Find A^{-1}.

Sol. $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow |A| = (3 \times 2) - (-1 \times 1) = 6 + 1 = 7.$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \qquad \therefore A^2 - 5A + 7I = 0$$

Multiplying by A^{-1} b/s of the above equation, we get:
$A^{-1}A^2 - A^{-1}(5A) - A^{-1}(7A) = (A^{-1}0)$
$\Rightarrow (A^{-1}A)A - 5A^{-1}A + 7A^{-1}I = 0 \Rightarrow IA - 5I + 7A^{-1} = 0$ ($\because X^{-1}X = I$ and $XI = X$)
$\Rightarrow 7A^{-1} = 5I - IA = 5I - A$ ($\because IX = X$)

$$\Rightarrow 7A^{-1} = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5-3 & 0-1 \\ 0+1 & 5-2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI^2 = 0$.

Sol. $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

Now $A^2 + aA + bI^2 = 0$

$$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 11+3a+b & 8+2a \\ 4+a & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Comparing the corresponding elements, we get:

$11 + 3a + b = 0$ ...(i)

$8 + 2a = 0$ ...(ii)

$4 + a = 0$ ...(iii)

$3 + a + b = 0$ ...(iv)

from (ii) & (iii) we get: $a = -4$

Put the value of a in (i) or (iv), we get; $b = -3a - 11$ or $b = -3 - a$

$\Rightarrow b = -3(-4) - 11$ or $b = -3 - (-4)$ $\Rightarrow b = 12 - 11$ or $b = -3 + 4$ $\Rightarrow b = 1$

Hence, $a = -4$ & $b = 1$.

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ show that $A^3 - 6A^2 + 5A + 11I_3 = 0$.

Hence find A^{-1}.

Sol. $$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \Rightarrow A^3 = A^2 \times A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 5A + 11\,I_3$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow A^3 - 6A^2 + 5A + 11\,I_3 = 0;$

Multiplying by A^{-1}

$A^{-1}A^3 - A^{-1}(6A^2) + A^{-1}(5A) + A^{-1}(11I_3) = A^{-1}O$

$\Rightarrow (A^{-1}A)A^2 - 6(A^{-1}A)A + 5A^{-1}A + 11A^{-1}I_3 = 0$ $(\because X^{-1}X = I \;\&\; XI = X)$

$\Rightarrow IA^2 - 6IA + 5I + 11A^{-1} = 0 \Rightarrow 11A^{-1} = -A^2 + 6A - 5I$ $(\because IX = X)$

$$\Rightarrow A^{-1}=\begin{bmatrix}-4 & -2 & -1\\ +3 & -8 & +14\\ -7 & +3 & -14\end{bmatrix}+6\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}+\begin{bmatrix}-5 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -5\end{bmatrix}$$

$$=\begin{bmatrix}-4+6-5 & -2+6+0 & -1+6+0\\ 3+6+0 & -8+12-5 & 14-18+0\\ -7+12+0 & 3-6+0 & -14+18-5\end{bmatrix}=\begin{bmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{bmatrix}$$

$$\Rightarrow A^{-1}=\frac{1}{11}\begin{bmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{bmatrix}$$

16. If $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$. Verify that $A^3-6A^2+9A-4I=O$ and hence, find A^{-1}.

Sol. We have $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ $\Rightarrow A^2=AA=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

$$=\begin{bmatrix}4+1+1 & -2-2-1 & 2+1+2\\ -2-2-1 & 1+4+1 & -1-2-2\\ 2+1+2 & -1-2-2 & 1+1+4\end{bmatrix}=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}12+5+5 & -6-10-5 & 6+5+10\\ -10-6-5 & 5+12+5 & -5-6-10\\ 10+5+6 & -5-10-6 & 5+5+12\end{bmatrix}=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$$

Now, $A^3-6A^2+9A-4I$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}+9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}22-36+18-4 & -21+30-9-0 & 21-30+9-0\\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0\\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\Rightarrow \quad A^3-6A^2+9A-4I=0 \quad \Rightarrow \quad 4I=A^3-6A^2+9A$

Multiplying A^{-1} b/s, we get: $A^{-1}(4I)=A^{-1}A^3-A^{-1}(6A^2)+A^{-1}(9A)$

$\Rightarrow \quad 4(A^{-1}I)=(A^{-1}A)A^2-6(A^{-1}A)A+9(A^{-1}A)$

$\Rightarrow \quad 4A^{-1}=IA^2-6IA+9I \qquad (\because X^{-1}X=I \ \& \ XI=X)$

$\Rightarrow \quad 4A^{-1}=A^2-6A+9I \qquad (\because IX=X).$

$$\Rightarrow \quad A^{-1}=\frac{1}{4}A^2-\frac{6}{4}A+\frac{9}{4}I$$

$$\Rightarrow \quad 4A^{-1}=\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}-6\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}+9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\Rightarrow \quad A^{-1}=\frac{1}{4}\begin{bmatrix}6-12+9&-5+6+0&5-6+0\\-5+6+0&6-12+9&-5+6+0\\5-6+0&-5+6+0&6-12+9\end{bmatrix}=\frac{1}{4}\begin{bmatrix}3&1&-1\\1&3&1\\-1&1&3\end{bmatrix}$$

17. Let A be a non-singular square matrix of order 3 × 3. Then |Adj A| is equal to :

(a) $|A|$ (b) $|A|^2$ (c) $|A|^3$ (d) $3|A|$

Sol. (b) $\because A^{-1}=\frac{\text{Adj } A}{|A|}. \quad \Rightarrow \quad A^{-1}A=\frac{(\text{Adj } A)A}{|A|}$ (Multiply b/s by A).

$$\Rightarrow \quad I=\frac{(\text{Adj } A)A}{|A|}. \qquad (\because X^{-1}X=I)$$

$$\Rightarrow \quad (\text{Adj } A)A=|A|I=|A|\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}|A|&0&0\\0&|A|&0\\0&0&|A|\end{bmatrix}$$

$$\Rightarrow \quad |(\text{Adj } A)|.|A|=\begin{vmatrix}|A|&0&0\\0&|A|&0\\0&0&|A|\end{vmatrix} \qquad (\because |XY|=|X||Y|)$$

$\Rightarrow \quad |\text{Adj } A|\,|A|=|A|(|A|^2-0) \qquad$ (Expanding along R_1)

$\Rightarrow \quad |\text{Adj } A|=|A|^2. \qquad$ (Dividing b/s by $|A|$).

18. If A is an invertible matrix of order 2, then det. (A^{-1}) is equal to :

(a) det. (A) (b) $\frac{1}{\text{det.}(A)}$ (c) 1 (d) 0.

Sol. (b) Since A is an invertible matrix. $\Rightarrow A^{-1}$ exists $\Rightarrow AA^{-1}=I$

Applying the determinant b/s. we get:

$\Rightarrow \quad |AA^{-1}|=|I|=1 \qquad (\because |I|=1).$

$\Rightarrow \quad |A||A^{-1}|=1 \qquad (\because |XY|=|X||Y|).$

$$\Rightarrow \quad |A^{-1}|=\frac{1}{|A|}=\frac{1}{\det\cdot(A)}.$$

EXERCISE 4.6

Examine the consistency of the system of equations in Questions 1 to 6 :

1. $\boldsymbol{x + 2y = 2; \quad 2x + 3y = 3}$

Hints

If the system of equations are represented by AX = B, then :

(a) the equations are consistent if $|A| \neq 0$.

(b) If $|A| = 0$ and if

(i) (Adj A) B = 0, then the equations are consistent.

(ii) (Adj A) B $\neq$ 0, then the equations are inconsistent.

Sol. $x + 2y = 2, 2x + 3y = 3$

Above equations can be written in the form of AX = B, where:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix} \& B = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad \therefore \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{Now } |A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$$

Hence, system of equations are consistent.

2. $\boldsymbol{2x - y = 5; \ x + y = 4}$

Sol. $2x - y = 5, x + y = 4$

Above equations can be written in the form of AX = B, where :

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix} \text{and } B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}. \quad \therefore \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\text{Now } |A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2 + 1 = 3 \neq 0$$

Hence, system of equations are consistent.

3. $\boldsymbol{x + 3y = 5; 2x + 6y = 8}$

Sol. $x + 3y - 5, 2x + 6y - 8$

Above equations can be written in the form of A X = B, where:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix} \text{and } B = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \therefore \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$\text{Now } |A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0 \quad \Rightarrow \quad A^{-1} \text{ does not exists.}$$

Now, for Adj A,

$C_{11} = (-1)^2\, 6 = 6; \quad C_{12} = (-1)^3\, 2 = -2; \quad C_{21} = (-1)^3\, 3 = -3; C_{22} = (-1)^4\, 1 = 1.$

$$\text{Adj } A = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\text{Now, Adj } A.\ B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, equations are inconsistent with no solution.

4. $\mathbf{x + y + z = 1;\ 2x + 3y + 2z = 2;\ ax + ay + 2az = 4}$

Sol. $x + y + z = 1; \quad 2x + 3y + 2z = 2; \quad x + y + 2z = \dfrac{4}{a}$

Above equations can be written in the form of AX = B, where :

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}; B = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and } C = \begin{bmatrix} 1 \\ 2 \\ \frac{4}{a} \end{bmatrix} \quad \therefore \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \frac{4}{a} \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \qquad (\text{Applying } R_1 \to R_1 - R_3)$$

$= -1(2-3)$ (Expanding along R_1)

$= +1 \neq 0.$

$\therefore$ given system of equations are consistent.

5. $\mathbf{3x - y - 2z = 2;\ 2y - z = -1;\ 3x - 5y = 3}$

Sol. $3x - y - 2z = 2; \quad 2y - z = -1; \quad 3x - 5y = 3$

Above equations can be written in the form of AX = B, where :

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \therefore \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3(0-5) + 1(0+3) - 2(0-6) \ (\text{Expanding along } R_1)$$

$= -15 + 3 + 12 = 0 \quad \therefore A^{-1}$ does not exist.

For Adj. A, $C_{11} = (-1)^2(0-5) = -5$; $C_{12} = (-1)^3(0+3) = -3$;
$C_{13} = (-1)^4(0-6) = -6$; $C_{21} = (-1)^3(0-10) = -(-10) = 10$;
$C_{22} = (-1)^4(0+6) = +6$; $C_{23} = (-1)^5(-15+3) = 12$;
$C_{31} = (-1)^4(1+4) = 5$; $C_{32} = (-1)^5(-3-0) = 3$;
$C_{33} = (-1)^6(6-0) = 6.$

$$\text{Adj. } A = \begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}^T = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\text{Now, Adj A . B} = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq 0$$

Hence equations are inconsistent with no solution.

6. $\mathbf{5x - y + 4z = 5;\ 2x + 3y + 5z = 2;\ 5x - 2y + 6z = -1}$

Sol. $5x - y + 4z = 5;\ 2x + 3y + 5z = 2;\ 5x - 2y + 6z = -1$

Above equations can be written in the form of AX = B, where :

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix};\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and } B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \quad \therefore \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

$= 5(18 + 10) + 1(12 - 25) + 4(-4 - 15)$ (Expanding along R_1)

$= 140 - 13 - 76 = 51 \neq 0$

Hence equations are consistent with a unique solution.

Solve the following of linear equations using matrix method in Exercises 7 to 14 :

7. $\mathbf{5x + 2y = 4}$ $\quad$ $\mathbf{7x + 3y = 5}$

Sol. The given system of equations can be written as AX = B

$$\text{i.e., } \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \text{ where } A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} = 15 - 14 = 1 \neq 0$$

$\Rightarrow A^{-1}$ exists and hence the given equation has a unique solution.

Cofactors of A : $A_{11} = (-1)^{1+1}(3) = 3;\ A_{12} = (-1)^{1+2}(7) = -7;$

$A_{21} = (-1)^{2+1}(2) = -2;\ A_{22} = (-1)^{2+2}(5) = +5.$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\text{Now } A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\because\ AX = B \Rightarrow X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Comparing the corresponding elements we get:

$x = 2, y = -3$

8. $\mathbf{2x - y = -2;\ \ 3x + 4y = 3}$

Sol. The given system of equations can be written as AX = B

$$\text{i.e., } \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \text{ where } A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Now $|A| = 8 + 3 = 11 \neq 0$

$\Rightarrow$ A^{-1} exists and hence the given equation has a unique solution.

Cofactors of A : $A_{11}=(-1)^{1+1}(4)=4; A_{12}=(-1)^{1+2}(3)=-3;$

$A_{21}=(-1)^{2+1}(-1)=1; A_{22}=(-1)^{2+2}(2)=2.$

$$\therefore \quad \text{Adj } A=\begin{bmatrix}4 & -3\\ 1 & 2\end{bmatrix}^T=\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix} \text{ and } A^{-1}=\frac{(\text{Adj} A)}{|A|}=\frac{1}{11}\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix}$$

Solution of given system is given by $X=A^{-1}B$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix}\begin{bmatrix}-2\\ 3\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-8+3\\ 6+6\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-5\\ 12\end{bmatrix}=\begin{bmatrix}\frac{-5}{11}\\ \frac{12}{11}\end{bmatrix}$$

$$\Rightarrow x=-\frac{5}{11}, y=\frac{12}{11}$$

9. $4x-3y=3$; $3x-5y=7$

Sol. The given system of equations can be written as $AX=B$

i.e., $\begin{bmatrix}4 & -3\\ 3 & -5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}3\\ 7\end{bmatrix}$ where $A=\begin{bmatrix}4 & -3\\ 3 & -5\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 7\end{bmatrix}$

Now $|A|=\begin{vmatrix}4 & -3\\ 3 & -5\end{vmatrix}=-20+9=-11\neq 0$

$\Rightarrow$ A^{-1} exists and hence the given equation has a unique solution.

Cofactors of A : $A_{11}=(-1)^{1+1}(-5)=-5; A_{12}=(-1)^{1+2}(3)=-3;$

$A_{21}=(-1)^{2+1}(-3)=3; A_{22}=(-1)^{2+2}(4)=4.$

$$\therefore \quad \text{Adj } A=\begin{bmatrix}-5 & -3\\ 3 & 4\end{bmatrix}^T=\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix} \text{ and } A^{-1}=\frac{(\text{Adj} A)}{|A|}=\frac{1}{-11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}$$

Solution of given system of equations is given by $X=A^{-1}B$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=-\frac{1}{11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}=\frac{-1}{11}\begin{bmatrix}-15+21\\ -9+28\end{bmatrix}=\begin{bmatrix}\frac{-6}{11}\\ \frac{-19}{11}\end{bmatrix}$$

$$\Rightarrow \quad x=-\frac{6}{11}, y=-\frac{19}{11}$$

10. $5x+2y=3$; $3x+2y=5$

Sol. The given system of equations can be written as $AX=B$

i.e., $\begin{bmatrix}5 & 2\\ 3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}3\\ 5\end{bmatrix}$ where $A=\begin{bmatrix}5 & 2\\ 3 & 2\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 5\end{bmatrix}$

Now $|A|=\begin{vmatrix}5 & 2\\ 3 & 2\end{vmatrix}=10-6=4\neq 0$

$\Rightarrow$ A^{-1} exists and hence the given equations have a unique solution.

Cofactors of A, $A_{11} = (-1)^{1+1}(2) = 2;\ A_{12} = (-1)^{1+2}(3) = -3;$

$A_{21} = (-1)^{2+1}(2) = -2;\ A_{22} = (-1)^{2+2}(5) = 5.$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \text{ and } A^{-1} = \frac{(\text{Adj}A)}{|A|} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

Solution of given system of equations is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Hence, $x = -1, y = 4$.

11. $\mathbf{2x + y + z = 1,\ x - 2y - z = 3/2,\ 3y - 5z = 9}$

Sol. The given system of equations are

$2x + y + z = 1,\ x - 2y - z = 3/2,\ 3y - 5z = 9$

These system of equations can be written in the form of AX = B, where,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ \&\ B = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$$

$$\text{Now } |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix} = 2(10+3) - 1(-5-0) + 1(3-0) \text{ (Expanding along } R_1)$$

$$= 26 + 5 + 3 = 34 \neq 0$$

$\Rightarrow A^{-1}$ exists and the given system of equations have a unique solution.

Cofactors of A :

$A_{11} = (10+3) = 13;\ A_{12} = -(-5-0) = 5;\ A_{13} = (3-0) = 3;$

$A_{21} = -(-5-3) = 8;\ A_{22} = (-10-0) = -10;\ A_{23} = -(6-0) = 6;$

$A_{31} = (-1+2) = 1;\ A_{32} = -(-2-1) = 3$ and $A_{33} = (-4-1) = -5.$

$$\therefore \text{adj}(A) = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}^T = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

Now, the solution of given system of equation is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$$

$\Rightarrow \ x = 1,\ \ y = 1/2,\ \ z = -3/2.$

12. $x - y + z = 4;\ 2x + y - 3z = 0;\ x + y + z = 2.$

Sol. The given system of equations can be written as AX = B

i.e., $\begin{bmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4\\0\\2\end{bmatrix}$ where $A = \begin{bmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B = \begin{bmatrix}4\\0\\2\end{bmatrix}$

Now, $|A| = \begin{vmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{vmatrix} = 1(1+3) + 1(2+3) + 1(2-1)$ (Expanding along R_1)

$= 4 + 5 + 1 = 10 \neq 0$

$\Rightarrow$ A^{-1} exists and hence the given equations have a unique solution.

Cofactors of A: $A_{11} = +(1+3) = 4; A_{12} = -(2+3) = -5; A_{13} = +(2-1) = 1;$
$A_{21} = = -(-1-1) = 2; A_{22} = +(1-1) = 0; A_{23} = -(1+1) = -2;$
$A_{31} = = +(3-1) = 2; A_{32} = -(-3-2) = 5$ and $A_{33} = +(1+2) = 3.$

$$\text{Adj } A = \begin{bmatrix}4 & -5 & 1\\2 & 0 & -2\\2 & 5 & 3\end{bmatrix}^T = \begin{bmatrix}4 & 2 & 2\\-5 & 0 & 5\\1 & -2 & 3\end{bmatrix}$$

and $A^{-1} = \dfrac{1}{|A|}(\text{Adj } A) = \dfrac{1}{10}\begin{bmatrix}4 & 2 & 2\\-5 & 0 & 5\\1 & -2 & 3\end{bmatrix}$

Solution of given system is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{10}\begin{bmatrix}4 & 2 & 2\\-5 & 0 & 5\\1 & -2 & 3\end{bmatrix}\begin{bmatrix}4\\0\\2\end{bmatrix} = \frac{1}{10}\begin{bmatrix}16+0+4\\-20+0+10\\4+0+6\end{bmatrix} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$\Rightarrow$ $x = 2, y = -1$ and $z = 1.$

13. $2x + 3y + 3z = 5; x - 2y + z = -4; 3x - y - 2z = 3.$

Sol. The given system of equations can be written as AX = B i.e.,

$\begin{bmatrix}2 & 3 & 3\\1 & -2 & 1\\3 & -1 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\-4\\3\end{bmatrix}$ where $A = \begin{bmatrix}2 & 3 & 3\\1 & -2 & 1\\3 & -1 & -2\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$ & $B = \begin{bmatrix}5\\-4\\3\end{bmatrix}$

Now, $|A| = \begin{vmatrix}2 & 3 & 3\\1 & -2 & 1\\3 & -1 & -2\end{vmatrix}$

$= 2(4+1) - 3(-2-3) + 3(-1+6)$ (Expanding along R_1)

$= 10 + 15 + 15 = 40 \neq 0$

$\therefore$ A^{-1} exists and hence the given equations have a unique solution.

Cofactors of A: $A_{11}=+(4+1)=5; A_{12}=-(-2-3)=5;\ A_{13}=+(-1+6)=5;$
$A_{21}=-(-6+3)=3; A_{22}=+(-4-9)=-13;\ A_{23}=-(-2-9)=11;$
$A_{31}=+(3+6)=9;\ A_{32}=-(2-3)=1; A_{33}=+(-4-3)=-7.$

$$\therefore\quad (\text{Adj}-\text{A}) = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore\quad A^{-1} = \frac{1}{|A|}(\text{Adj A}) = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Solution of given system is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x = 1, y = 2$ and $z = -1$.

14. $\boldsymbol{x - y + 2z = 7;\ 3x + 4y - 5z = -5}$
$\boldsymbol{2x - y + 3z = 12.}$

Sol. The given system of equations can be written as AX = B i.e.,

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} \text{ where } A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\&\; B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 1(12-5) + 1(9+10) + 2(-3-8)$ (Expanding along R_1)
$= 7 + 19 - 22 = 4 \neq 0$

$\therefore$ A^{-1} exists and hence the given equations have a unique solution.

Cofactors of A: $A_{11}=+(12-5)=7; A_{12}=-(9+10)=-19; A_{13}=+(-3-8)=-11;$
$A_{21}=-(-3+2)=1;\ A_{22}=+(3-4)=-1; A_{23}=-(-1+2)=-1;$
$A_{31}=+(5-8)=-3; A_{32}=-(-5-6)=11$ and $A_{33}=+(4+3)=7.$

$$\therefore (\text{Adj A}) = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^T = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore\quad A^{-1} = \frac{1}{|A|}(\text{Adj A}) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Solution of given system is given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence, $x = 2, y = 1$ and $z = 3$.

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$. Find A^{-1}. Using A^{-1}. Solve the following system of linear equations:
$2x - 3y + 5z = 11, 3x + 2y - 4z = -5, x + y - 2z = -3$.

Sol. Here $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$= 2(-4+4) + 3(-6+4) + 5(3-2)$ (Expanding along R_1)
$= 0 - 6 + 5 = -1 \neq 0 \quad \therefore\ A^{-1}$ exist.

Cofactors of A:

$A_{11} = +(-4+4) = 0; A_{12} = -(-6+4) = 2; A_{13} = +(3-2) = 1;$
$A_{21} = -(6-5) = -1; A_{22} = +(-4-5) = -9; A_{23} = -(2+3) = -5;$
$A_{31} = +(12-10) = 2; A_{32} = -(-8-15) = 23$ and $A_{33} = +(4+9) = 13$.

$$\therefore (\text{Adj} - A) = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \quad ...(i)$$

Now given equations : $2x - 3y + 5z = 11$; $3x + 2y - 4z = -5$ and $x + y - 2z = -3$ can be written in the form of :

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \text{ i,e; } AX = B \quad \Rightarrow X = A^{-1}B$$

$\therefore$ from equation (i), we get

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 1, y = 2$ and $z = 3$.

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 69. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹ 70. Find the cost of each item per kg by matrix method.

Sol. Let cost of 1 kg onion = ₹ x, cost of 1 kg wheat = ₹ y
and cost of 1 kg rice = ₹ z

$\therefore \quad 4x + 3y + 2z = 60$
$2x + 4y + 6z = 90$
$6x + 2y + 3z = 70$

Above equations can be written in the form of:

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} \text{ i.e., } AX = B$$

Now $$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$$

$= 4(12 - 12) - 3(6 - 36) + 2(4 - 24)$ (Expanding along R_1)
$= 90 - 40 = 50 \neq 0$

$\therefore$ A^{-1} exists.

Cofactos of A:

$A_{11} = +(12 - 12) = 0; A_{12} = -(6 - 36) = 30; A_{13} = +(4 - 24) = -20;$
$A_{21} = -(9 - 4) = -5; A_{22} = +(12 - 12) = 0; A_{23} = -(8 - 18) = 10;$
$A_{31} = +(18 - 8) = 10; A_{32} = -(24 - 4) = -20$ and $A_{33} = +(16 - 6) = 10.$

$$\therefore \text{ Adj } A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

As, $AX = B \Rightarrow X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -45 & +70 \\ 180 & +0 & -140 \\ -120 & +90 & +70 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\Rightarrow \quad x = 5, y = 8, z = 8$

Hence, cost of 1 kg of onion = ₹ 5, cost of 1 kg of wheat = ₹ 8
and cost of 1 kg of rice = ₹ 8.

MISCELLANEOUS EXERCISE

1. Prove that the determinant

$$\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$ **is independent of θ.**

Sol. Let $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

Expanding along R_1, we get:

$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$

$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$= -x^3 - x + x[\sin^2\theta + \cos^2\theta] = -x^3 - x + x \quad (\sin^2\theta + \cos^2\theta = 1)$

$= -x^3$, which is independent of θ.

2. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

Sol. $\text{LHS} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Multiply R_1, R_2 and R_3 by a, b and c respectively and dividing the determined by abc, we get:

$$\text{LHS} = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}.$$

Taking common abc from C_3, we get:

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = -\begin{vmatrix} a^2 & 1 & a^3 \\ b^2 & 1 & b^3 \\ c^2 & 1 & c^3 \end{vmatrix} \text{ (interchanging } C_2 \,\&\, C_3)$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \text{ (interchanging } C_1 \,\&\, C_2)$$

= RHS **(Hence proved).**

3. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$,

Taking out $\cos\alpha$ from R_1 and $\sin\alpha$ from R_3, we get:

$$\therefore \Delta = \cos\alpha\sin\alpha \begin{vmatrix} \cos\beta & \sin\beta & -\tan\alpha \\ -\sin\beta & \cos\beta & 0 \\ \cos\beta & \sin\beta & \cot\alpha \end{vmatrix} \quad \left(\because \frac{\sin\theta}{\cos\theta} = \tan\theta \text{ and } \frac{\cos\theta}{\sin\theta} = \cot\theta\right)$$

Operating $R_1 \to R_1 - R_3$, we get:

$$\Delta = \cos\alpha\sin\alpha \begin{vmatrix} 0 & 0 & -\tan\alpha - \cot\alpha \\ -\sin\beta & \cos\beta & 0 \\ \cos\beta & \sin\beta & \cot\alpha \end{vmatrix},$$

Expanding along R_1, we get:

$$\Delta = \cos\alpha\sin\alpha(-\tan\alpha - \cot\alpha)\begin{vmatrix} -\sin\beta & \cos\beta \\ \cos\beta & \sin\beta \end{vmatrix}$$

$$= -\sin\alpha\cos\alpha\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right)(-\sin^2\beta - \cos^2\beta)$$

$$\left(\because \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}\right)$$

$$= -\sin\alpha\cos\alpha\left(\frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha}\right)(-1) \qquad (\because \cos^2\theta + \sin^2\theta = 1)$$

$= 1.$

4. If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0.$

Show that either $a+b+c=0$ or $a=b=c$.

Sol. $\because \Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$

(Applying $C_1 \to C_1 + C_2 + C_3$)

$$= 2\begin{vmatrix} a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \\ a+b+c & b+c & c+a \end{vmatrix} \quad \text{(Taking 2 common from } C_1)$$

$$= 2\begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -c & -a \\ a+b+c & -a & -b \end{vmatrix} \quad \begin{bmatrix} \text{Applying } C_2 \to C_2 - C_1 \\ \text{and } C_3 \to C_3 - C_1 \end{bmatrix}$$

Taking $(a + b + c)$ common from C_1, (-1) from C_2 and C_3 both, we get:

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \Rightarrow \text{Either } (a+b+c) = 0 \text{ or } \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix} = 0 \text{ (Applying } R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3)$$

$\Rightarrow 1[(b-c)(a-b) - (c-a)(c-a)] = 0$ (Expanding along C_1)
$\Rightarrow ba - b^2 - ca + bc - c^2 - a^2 + 2ca = 0 \Rightarrow -a^2 - b^2 - c^2 + ab + bc + ca = 0$
$\Rightarrow -2a^2 - 2b^2 - 2c^2 + 2ab + 2bc + 2ca = 0$ (Multiplying b/s by 2).
$\Rightarrow -[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] = 0$
$\Rightarrow [(a^2 + b^2 - 2ab) + (a^2 + c^2 - 2ac) + (b^2 + c^2 - 2bc)] = 0$
$\Rightarrow (a-b)^2 + (a-c)^2 + (b-c)^2 = 0$
Since we know that square of any number is always positive.
$\therefore (a-b)^2 = (a-c)^2 = (b-c)^2 = 0 \Rightarrow (a-b) = (a-c) = (b-c) = 0$
$\Rightarrow a = b$; $a = c$ & $b = c \Rightarrow a = b = c$.
Hence if $\Delta = 0$ then either $a + b + c = 0$ or $a = b = c$.

5. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0.$

Sol. Let $\Delta = \begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$ and then take $(3x + a)$ common from G, we get:

$$\Delta = \begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = (3x+a)\begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix}$$

$$= (3x+a)\begin{vmatrix} 0 & -a & 0 \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} \quad (\text{Applying } R_1 \to R_1 - R_2)$$

Expanding along R_1, we get:

$$\Delta = (3x+a)\,a\begin{vmatrix} 1 & x \\ 1 & x+a \end{vmatrix} = a\,(3x+a)\,(x+a-x) = a^2\,(3x+a)$$

But, $\Delta = 0 \quad \therefore \quad a^2(3x+a) = 0 \Rightarrow \quad 3x+a=0 \qquad (\because a \neq 0)$

$$\Rightarrow \quad x = -\frac{a}{3}$$

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$

Sol. Let $\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} \Rightarrow = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$

(Taking 'a' common from C_1, b from C_2 and c from C_3)

$$\Delta = abc\begin{vmatrix} 2(a+c) & c & a+c \\ 2(a+b) & b & a \\ 2(b+c) & b+c & c \end{vmatrix} \quad (\text{Applying } C_1 \to C_1 + C_2 + C_3)$$

$$\Delta = 2\,abc\begin{vmatrix} a+c & c & a+c \\ a+b & b & a \\ b+c & b+c & c \end{vmatrix} \quad (\text{Taking 2 common from } C_1)$$

$$\Delta = 2abc\begin{vmatrix} 0 & c & a+c \\ b & b & a \\ b & b+c & c \end{vmatrix} \quad (\text{Applying } C_1 \to C_1 - C_3)$$

$$\Delta = 2abc\begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ b & b+c & c \end{vmatrix} \quad (\text{Applying } R_2 \to R_2 - R_3)$$

$$\Rightarrow \quad \Delta = 2abc\left[b\begin{vmatrix} c & a+c \\ -c & a-c \end{vmatrix}\right] \quad (\text{Expanding along } C_1)$$

$= 2ab^2c\,[c(a-c) + c(a+c)] = 2ab^2c\,[ca - c^2 + ca + c^2]$
$= 2ab^2c\,[2ca] = 4a^2b^2c^2.$

7. **If** $A^{-1}=\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}$ **and** $B=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$**, find** $(AB)^{-1}$**.**

Sol. $|B|=\begin{vmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{vmatrix}=1(3-0)-2(-1-0)-2(2-0)$ (Expanding along R_1)

$=3+2-4=1\neq 0 \;\therefore\; B^{-1}=$ exists,

Cofactors of B:

$B_{11}=+(3-0)=3;\ B_{12}=-(-1-0)=1;\ B_{13}=+(2-0)=2;$
$B_{21}=-(2-4)=2;\ B_{22}=+(1-0)=1;\ B_{23}=-(-2-0)=2;$
$B_{31}=+(0+6)=6;\ B_{32}=-(0-2)=2;\ B_{33}=+(3+2)=5.$

$$\therefore (\text{Adj}\,B)=\begin{bmatrix}3 & 1 & 2\\ 2 & 1 & 2\\ 6 & 2 & 5\end{bmatrix}^T=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix} \text{ and } B^{-1}=\frac{(\text{adj } B)}{|B|}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$$

Since we know that: $(AB)^{-1}=B^{-1}A^{-1}$

$$=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}=\begin{bmatrix}9-30+30 & -3+12-12 & 3-10+12\\ 3-15+10 & -1+6-4 & 1-5+4\\ 6-30+25 & -2+12-10 & 2-10+10\end{bmatrix}$$

$$\Rightarrow\quad (AB)^{-1}=\begin{bmatrix}9 & -3 & 5\\ -2 & 1 & 0\\ 1 & 0 & 2\end{bmatrix}$$

8. **Let** $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$**, verify that**

(i) $[\text{Adj } A]^{-1}=\text{Adj}(A^{-1})$ **(ii)** $[A^{-1}]^{-1}=A$**.**

Sol. (i) Here $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$

$|A|=1(15-1)+2(-10-1)+1(-2-3)$ (Expanding along R_1)

$=14-22-5=-13\neq 0$

$\therefore\quad A^{-1}$ exists

Cofactors of A:

$A_{11}=+(15-1)=14;\ A_{12}=-(-10-1)=11;\ A_{13}=+(-2-3)=-5;$

$A_{21} = -(-10-1) = 11$; $A_{22} = +(5-1) = 4$; $A_{23} = -(1+2) = -3$;
$A_{31} = +(-2-3) = -5$; $A_{32} = -(1+2) = -3$ and $A_{33} = +(3-4) = -1$.

$$\therefore \text{Adj A} = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}^T = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{Adj A}) = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Now, let B = Adj A. To find $[\text{Adj A}]^{-1}$, we find B^{-1}

$$\text{Now, } B = \text{Adj A} = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$|B| = 14(-4-9) - 11(-11-15) - 5(-33+20)$ (Expanding along R_1)
$= -182 + 286 + 65 = 169 \neq 0$.
Hence B^{-1} exists.

Cofactors of B :

$B_{11} = (-4-9) = -13$; $B_{12} = -(-11-15) = 26$; $B_{13} = -33+20 = -13$;
$B_{21} = -(-11-15) = 26$; $B_{22} = -14-25 = -39$; $B_{23} = -(-42+55) = -13$;
$B_{31} = -33+20 = -13$; $B_{32} = -(-42+55) = -13$; $B_{33} = 56-121 = -65$.

$$\therefore \quad \text{Adj B} = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}^T = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$\Rightarrow \quad B^{-1} = \frac{1}{|B|}\text{Adj B} = \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$\Rightarrow \quad (\text{Adj A})^{-1} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \quad \text{...(i)}$$

$$\text{Now, let } C = A^{-1} = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

Now, $C_{11} = (-1)^{1+1}\begin{vmatrix} -\frac{4}{13} & \frac{3}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = -\frac{4}{169} - \frac{9}{169} = -\frac{13}{169} = -\frac{1}{13}$

$$C_{12} = -\left(\frac{-11}{169} - \frac{15}{169}\right) = \frac{2}{13}; C_{13} = \left(\frac{-33}{169} + \frac{20}{169}\right) = \frac{-1}{13};$$

$$C_{21} = -\left(\frac{-11}{169} - \frac{15}{169}\right) = \frac{2}{13}; C_{22} = \left(\frac{-14}{169} - \frac{25}{169}\right) = \frac{-3}{13};$$

$$C_{23} = -\left(\frac{-42}{169} + \frac{55}{169}\right) = \frac{-1}{13}; C_{31} = \left(\frac{-33}{169} + \frac{20}{169}\right) = \frac{-1}{13};$$

$$C_{32} = -\left(\frac{-42}{169} + \frac{55}{169}\right) = \frac{-1}{13} \text{ and } C_{33} = \left(\frac{56}{169} - \frac{121}{169}\right) = \frac{-5}{13}.$$

$$\text{Adj}(A^{-1}) = \text{Adj. } C = \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \quad \text{...(ii)}$$

$\therefore$ from (i) and (ii) we get: $(\text{Adj. } A)^{-1} = \text{Adj. }(A^{-1})$

(ii) Given $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ and $A^{-1} = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$

.... (Proved above)

$$|A^{-1}| = \left(\frac{-1}{13}\right)^3 \{14(-4-9) - 11(-11-15) - 5(-33+20)\}$$

(Expanding along R_1 and $|KA| = k^3 |A|$ if A is 3×3 matrix)

$$= \left(\frac{-1}{13}\right)^3 (-182 + 286 + 65) = \frac{-1}{13}$$

Also $\text{Adj}(A^{-1}) = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$ **(Proved above)**

$$\therefore (A^{-1})^{-1} = \frac{\text{Adj}(A^{-1})}{|A^{-1}|} = \frac{1}{-\frac{1}{13}} \times \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$$\Rightarrow (A^{-1})^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ -1 & 1 & 5 \end{bmatrix} = A \quad \Rightarrow \quad (A^{-1})^{-1} = A.$$

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Sol. $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = \begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix}$ (Operating $C_1 \to C_1 + C_2 + C_3$)

$= 2(x+y)\begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix}$ (Taking $2(x+y)$ common from C_1)

$\Delta = 2(x+y)\begin{vmatrix} 1 & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix}$ (Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$= 2(x+y)\,[1(-x^2 + y(x-y))]$ (Expanding along C_1)

$= 2(x+y)\,[-x^2 + yx - y^2] = -2\,(x+y)\,(x^2 + y^2 - xy)$

$= -2\,(x^3 + y^3)$ $[\because a^3 + b^3 = (a+b)\,(a^2 + b^2 - ab]$

10. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get:

$\Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix} = 1(xy - 0)$ (Expanding along C_1)

$= xy.$

Using properties of determinants in exercises 11 to 15, prove that :

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

Sol. Let $\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$

(Interchanging rows and columns.)

$$= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix} \quad (\text{Applying } R_3 \to R_3 + R_1)$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} \quad (\text{Taking } (\alpha+\beta+\gamma) \text{ common to } R_3)$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \quad (\text{Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1)$$

$$= (\alpha+\beta+\gamma)\begin{bmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & (\beta-\alpha)(\beta+\alpha) & (\gamma-\alpha)(\gamma+\alpha) \\ 1 & 0 & 0 \end{bmatrix} \quad (\because\ a^2 - b^2 = (a-b)(a+b))$$

Taking common $(\beta-\alpha)$ & $(\gamma-\alpha)$ from C_2 & C_3 respectively, we get:

$$\Delta = (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, w get ;

$\Delta = (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha)$

$\Rightarrow \Delta = (\alpha+\beta+\gamma)(-1)(\alpha-\beta)(\gamma-\alpha)(-1)(\beta-\gamma)$

$\Rightarrow \Delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$ **(Hence proved).**

12. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x).$

Sol. Let $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$, then $\Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$

Applying $C_2 \leftrightarrow C_3$ in first determinant and Taking x, y, z & p common from R_1, R_2, R_3 & C_3 respectively in the second determinant, we get:

$$\Delta = -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_2$ in the first determinant, we get:

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get :

$$\Delta = (1+pxyz)\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Taking common $(x-y)$ & $(y-z)$ from R_1 & R_2 respectively, we get:

$$\Delta = (1+pxyz)(x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix} \quad (\because a^2-b^2 = (a-b)(a+b))$$

$\Rightarrow \quad \Delta = (1+pxyz)(x-y)(y-z)[1(y+z-x-y)]$ (Expanding along C_1)

$\Rightarrow \quad \Delta = (1+pxyz)(x-y)(y-z)(z-x)$. **(Hence proved).**

13. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

Sol. $= \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$

(Applying $C_1 \to C_1 + C_2 + C_3$)

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix} \qquad \text{(Taking common } (a+b+c) \text{ from } C_1)$$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix} \qquad (\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$= (a+b+c)\,[(2b+a)(2c+a) - (a-b)(a-c)]$ (Expanding along C_1)
$= (a+b+c)\,[4bc + 2ab + 2ac + a^2 - a^2 + ac + ab - bc]$
$= 3(a+b+c)(ab+ba+ca)$. **(Hence proved).**

14. $$\begin{vmatrix} \mathbf{1} & \mathbf{1+p} & \mathbf{1+p+q} \\ \mathbf{2} & \mathbf{3+2p} & \mathbf{4+3p+2q} \\ \mathbf{3} & \mathbf{6+3p} & \mathbf{10+6p+3q} \end{vmatrix} = \mathbf{1}.$$

Sol. $$\text{Let } \Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

(Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$)
$= 1(7 + 3p - 6 - 3p)$ (Expanding along C_1)
$= 1.$ **(Hence proved).**

15. $$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = \mathbf{0}$$

Sol. $$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

$(\cos(A+B) = \cos A\cos B - \sin A\sin B)$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \qquad (\text{Applying } C_3 \to C_3 + \sin\delta \times C_1 - \cos\delta \cdot C_2),$$

$= 0$ $(\because C_3 = 0)$

16. Solve the system of the following equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4;\quad \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1;\quad \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Sol. Let $\frac{1}{x}=u, \frac{1}{y}=v, \frac{1}{z}=w$

$\therefore$ System of equations becomes

$2u+3v+10w=4;\quad 4u-6v+5w=1;\quad 6u+9v-20w=2$

Above system of equation can be written in the form of $AX = B$

where, $A=\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}u\\ v\\ w\end{bmatrix}$ & $B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$

$|A|=(120-45)-3(-80-30)+10(36+36)$ (Expanding along R_1)

$=150+330+720=1200\neq 0$. Hence A^{-1} exists.

Cofactors of A:

$A_{11}=120-45=75;\ A_{12}=-(-80-30)=110;\ A_{13}=36+36=72;$

$A_{21}=-(-60-90)=150;\ A_{22}=-40-60=-100;\ A_{23}=-(18-18)=0;$

$A_{31}=15+60=75;\ A_{32}=-(10-40)=30;$ and $A_{33}=-12-12=-24$

$$\therefore\ \text{adj } A=\begin{bmatrix}75 & 110 & 72\\ 150 & -100 & 0\\ 75 & 30 & -24\end{bmatrix}^T=\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}\text{adj}(A)=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$

Now the solution of the system of equations are given as: $X=A^{-1}B$

$$\Rightarrow \begin{bmatrix}U\\ V\\ W\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}300+150+150\\ 440-100+60\\ 288+0-48\end{bmatrix}$$

$$=\frac{1}{1200}\begin{bmatrix}600\\ 400\\ 240\end{bmatrix}=\begin{bmatrix}1/2\\ 1/3\\ 1/5\end{bmatrix}$$

$$\therefore\ u=\frac{1}{2}, v=\frac{1}{3}, w=\frac{1}{5}\quad \Rightarrow x=\frac{1}{u}=2, y=\frac{1}{v}=3, z=\frac{1}{w}=5$$

Hence, $x=2, y=3, z=5$.

Choose the correct answer in Questions 17 to 19.

17. **If a, b, c are in A.P, then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is**

(a) 0 **(b)** 1 **(c)** x **(d)** 2x

Sol. **(a)** Let $\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 2x+6 & 2x+8 & 2x+4b \\ x+4 & x+5 & x+2c \end{vmatrix}$

(Multiply R_2 by 2 and divide the determinant by 2)

$$=\frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0+2(2b-a-c) \\ x+4 & x+5 & x+2c \end{vmatrix} \qquad (\text{Applying } R_2 \to R_2 - R_1 - R_3)$$

But a, b, c are in A.P $\Rightarrow 2b = a + c$ or $2b - a - c = 0$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0 \\ x+4 & x+1 & x+2c \end{vmatrix} = 0 \qquad (\because R_2 = 0).$$

18. **If x, y, z are nonzero real numbers, then the inverse of matrix**

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \text{ is}$$

(a) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ **(b)** $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(c) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ **(d)** $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol. **(a)** $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \Rightarrow |A| = x\,[yz - 0]$ (Expanding along R_1)

$\Rightarrow \quad |A| = xyz$ Hence A^{-1} exists.

The cofactors of the elements of A are

$A_{11} = yz; A_{12} = 0; A_{13} = 0; A_{21} = 0;$

$A_{22} = xz; A_{23} = 0; A_{31} = 0; A_{32} = 0$ and $A_{33} = xy$

$$\therefore \text{adj}(A) = \begin{bmatrix} yz & 0 & 0 \\ 0 & zx & 0 \\ 0 & 0 & xy \end{bmatrix}^T = \begin{bmatrix} yz & 0 & 0 \\ 0 & zx & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{Adj} - A}{|A|} = \frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & zx & 0 \\ 0 & 0 & xy \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

19. Let $\mathbf{A} = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$**, where** $0 \le \theta \le 2\pi$**. Then**

(a) $\mathbf{Det(A) = 0}$ **(b)** $\mathbf{Det(A) \in (2, \infty)}$

(c) $\mathbf{Det(A) \in (2, 4)}$ **(d)** $\mathbf{Det(A) \in [2, 4]}$

Sol. (d) $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$$\text{Det}(A) = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} = \begin{vmatrix} 2 & \sin\theta & 1 \\ 0 & 1 & \sin\theta \\ 0 & -\sin\theta & 1 \end{vmatrix} \text{(Applying } R_1 \to R_1 + R_3\text{)}.$$

$= 2(1 + \sin^2\theta)$ (Expanding along C_1)

$\because$ minimum value of $\sin^2\theta$ is 0 at $\theta = 0, \pi, 2\pi \in [0, 2\pi)$

$\therefore$ for $\theta = 0, \pi, 2\pi$: $\text{Det}(A) = 2(1 + 0) = 2$

$\because$ maximum value of $\sin^2\theta$ is 1 at $\theta = \frac{\pi}{2}, \frac{3\pi}{2} \in [0, 2\pi)$

$\therefore$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$: $\text{Det}(A) = 2 \cdot (1 + 1) = 4$